BOUNDARY LAYERS IN THE HOMOGENIZATION OF A SPECTRAL PROBLEM IN FLUID–SOLID STRUCTURES

GRÉGOIRE ALLAIRE† AND CARLOS CONCA‡

Abstract. This paper is devoted to the asymptotic analysis of the spectrum of a mathematical model that describes the vibrations of a coupled fluid–solid periodic structure. In a previous work [Arch. Rational Mech. Anal., 135 (1996), pp. 197–257] we proved by means of a Bloch wave homogenization method that, in the limit as the period goes to zero, the spectrum is made of three parts: the macroscopic or homogenized spectrum, the microscopic or Bloch spectrum, and a third component, the so-called boundary layer spectrum. While the two first parts were completely described as the spectrum of some limit problem, the latter was merely defined as the set of limit eigenvalues corresponding to sequences of eigenvectors concentrating on the boundary. It is the purpose of this paper to characterize explicitly this boundary layer spectrum with the help of a family of limit problems revealing the intimate connection between the periodic microstructure and the boundary of the domain. We therefore obtain a “completeness” result, i.e., a precise description of all possible asymptotic behaviors of sequences of eigenvalues, at least for a special class of polygonal domains.

Key words. homogenization, Bloch waves, spectral analysis, boundary layers, fluid–solid structures

AMS subject classification. 35B40

PHI. S0036141096304328

1. Introduction.

1.1. Setting of the problem. This paper is devoted to the study of some boundary layer phenomena which arise in the asymptotic analysis of the spectrum of a mathematical model describing the vibrations of a coupled periodic system of solid tubes immersed in a perfect incompressible fluid. This simple model is due to Planchard, who studied it intensively (see [31], [32]). Since we introduced it at length in section 1.2 of our previous work [3] we content ourselves with briefly recalling the statement of this problem.

We consider a periodic bounded domain Ωε obtained from a fixed bounded open set Ω in \( \mathbb{R}^N \) by removing a collection of identical, periodically distributed holes \((T_p)_{1 \leq p \leq n(\epsilon)}\). The distance between adjacent holes as well as their size are both of the order of \( \epsilon \), the size of the period which is a small parameter going to zero. Correspondingly, the number of holes \( n(\epsilon) \) is of the order of \( \epsilon^{-N} \), where \( N \) is the spatial dimension. More precisely, let us first define the standard unit cell \( Y = (0; 1)^N \) which, upon rescaling to size \( \epsilon \), becomes the period in \( \Omega \). Let \( T \) be a smooth, simply connected, closed subset of \( Y \), assumed to be strictly included in \( Y \) (i.e., \( T \) does not touch the boundary of \( Y \)). The set \( T \) represents the reference tube (or rod) and the unit fluid cell is defined as

\[ Y^* = Y \setminus T. \]

*Received by the editors May 28, 1996; accepted for publication (in revised form) December 10, 1996. The second author is partially supported by the Chilean programme of presidential chairs in science and by Fondecyt under grant 197-0734.
†Commissariat à l’Energie Atomique, DRN/DMT/SERMA, C.E. Saclay, 91191 Gif sur Yvette, France and Laboratoire d’Analyse Numérique, Université Paris 6 (allaire@ann.jussieu.fr).
‡Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170/3, Correo 3, Santiago, Chile (cconca@dim.uchile.cl).
For each value of the small positive parameter $\epsilon$, the fluid domain $\Omega_\epsilon$ is obtained from the reference domain $\Omega$ by removing a periodic arrangement of tubes $\epsilon T$ with period $\epsilon Y$. Denoting by $(T^\epsilon_p)$ the family of all translates of $\epsilon T$ by vectors $\epsilon p$ (where $p$ is a multi-index in $\mathbb{Z}^N$) and by $(Y^\epsilon_p)$ the corresponding family of cells, we define

$$\Omega_\epsilon = \Omega \setminus \bigcup_{p=1}^{n(\epsilon)} T^\epsilon_p.$$  

Although $p$ is a multi-index in $\mathbb{Z}^N$, for simplicity we denote its range by $1 \leq p \leq n(\epsilon)$. To obtain the fluid domain $\Omega_\epsilon$ in (1), we remove from the original domain $\Omega$ only those tubes $T^\epsilon_p$ which belong to a cell $Y^\epsilon_p$ completely included in $\Omega$. This has the effect that no tube meets the boundary $\partial \Omega$. Analogously, $(\Gamma^\epsilon_p)$ denotes the family of tubes boundaries ($\partial T^\epsilon_p$).

We are interested in the following spectral problem in $\Omega$: find the eigenvalues $\lambda_\epsilon$ and the corresponding normalized eigenvectors $u_\epsilon$, solutions of

$$\begin{cases}
-\Delta u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\
\lambda_\epsilon \frac{\partial u_\epsilon}{\partial n} = \epsilon^{-N} \vec{n} \cdot u_\epsilon \vec{n} ds & \text{on } \Gamma^\epsilon_p \text{ for } 1 \leq p \leq n(\epsilon), \\
u_\epsilon = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\vec{n}$ denotes the exterior unit normal to $\Omega_\epsilon$.

The homogenization of this model has already attracted the attention of several authors (see [1], [14], [16], [17]). Even though it is a spectral problem involving the Laplace operator, it is easily seen to admit only finitely many eigenvalues, exactly $N n(\epsilon)$ (the number of tubes times the number of degrees of freedom in their displacements). To this end, a finite-dimensional operator $S_\epsilon$ is introduced, which acts on the family of tube displacements $\vec{s} = (\vec{s}_p)_{1 \leq p \leq n(\epsilon)}$ with $\vec{s}_p \in \mathbb{R}^N$,

$$S_\epsilon : \mathbb{R}^{N n(\epsilon)} \rightarrow \mathbb{R}^{N n(\epsilon)},$$

where the fluid potential $u_\epsilon$ is now the unique solution in $H^1(\Omega_\epsilon)$ of

$$\begin{cases}
-\Delta u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\
\frac{\partial u_\epsilon}{\partial n} = \vec{s}_p \cdot \vec{n} & \text{on } \Gamma^\epsilon_p \text{ for } 1 \leq p \leq n(\epsilon), \\
u_\epsilon = 0 & \text{on } \partial \Omega.
\end{cases}$$

According to [17], $S_\epsilon$ is self-adjoint, positive definite, and its spectrum, denoted by $\sigma(S_\epsilon)$, coincides with the set of eigenvalues of (2). Of course, since $S_\epsilon$ acts in a finite-dimensional space, $\sigma(S_\epsilon)$ is made up of $N n(\epsilon)$ real numbers. It has been further proved that all eigenvalues of $S_\epsilon$ are uniformly bounded away from zero and from infinity (see, e.g., Proposition 1.2.1 and Lemma 1.2.2 in [3]). As the period $\epsilon$ goes to zero, $\sigma(S_\epsilon)$, considered as a subset of $\mathbb{R}^+$, converges to a limit set $\sigma_\infty$ which, by definition, is the set of all cluster points of (sub)sequences of eigenvalues of $S_\epsilon$,

$$\sigma_\infty = \{ \lambda \in \mathbb{R}^+ \mid \exists \text{ a subsequence } \lambda_\epsilon \in \sigma(S_\epsilon) \text{ such that } \lambda_\epsilon \rightarrow \lambda \}.$$  

Finding an adequate characterization of the limit set $\sigma_\infty$ was the main goal of our previous paper [3]. A positive answer to this problem is given in the present article for a special class of polygonal domains.
1.2. Survey of the previous results. The characterization of $\sigma_\infty$ amounts to studying the asymptotic behavior of the spectral problem (2), or, in other words, to homogenize (2) as the parameter $\epsilon$ goes to zero. To our knowledge, this can be done, at least, using two different approaches: the classical homogenization process for periodic structures (see, e.g., the reference books [7], [8], [24], [28], [35]) or the so-called Bloch wave method (also called the nonstandard homogenization procedure in [16]; see [8], [33], [34], [36] for an introduction to Bloch waves in spectral analysis). The former naturally yields the homogenized or macroscopic spectrum of (2), while the latter is associated with the so-called Bloch or microscopic spectrum.

Historically the second approach was the first applied to problem (2) by C. Conca, M. Vanninathan, and their coworkers [1], [15], [16], [17]. The key point in this method is to rescale the $\epsilon$-network of tubes to size 1 and, therefore, as $\epsilon$ goes to zero, to obtain an infinite limit domain containing a periodic array of unit tubes. Then, the limit problem is amenable to the celebrated Blochwave decomposition (also known as the Floquet decomposition; see the original work of F. Bloch [11] or the first mathematical papers [19], [30], [36] or the books [8], [33]). The spectrum of this limit problem is called the Bloch spectrum.

Although it seems the easiest to apply, the first approach (i.e., the classical homogenization) has only been recently applied to problem (2) in our previous article [3]. By homogenizing the operator $S_\epsilon$ with the help of the two-scale convergence (see [2], [29]), a homogenized equation is obtained in the domain $\Omega$. Its spectrum is called the homogenized spectrum. It turns out that the homogenized spectrum is completely different from the Bloch spectrum, and therefore both approaches are complementary. This is possible since in neither case the underlying sequences of linear operators converge uniformly to their limit which are noncompact operators. In addition to this homogenization result, our paper [3] provides a unified theory for both approaches (see also [4], [5]), and we simply recall our main results.

The homogenization of model (2) amounts to analyzing the convergence of the sequence of operators $S_\epsilon$. Since these operators are defined on a space which varies with $\epsilon$, we extend them to the fixed space $[L^2(\Omega)^N]^K$ , where $K$ is an arbitrary positive integer. Denoting by $S^K_\epsilon$ this extension, it will be amenable to a standard asymptotic analysis, while keeping essentially the same spectrum as $S_\epsilon$. Following the lead of Planchard [32], the reference cell of our homogenization procedure is $KY$ instead of simply $Y$ (this technique is referred to as homogenization by packets in [32]). To give a precise definition of $S^K_\epsilon$ we introduce two linear maps: a projection $P^K_\epsilon$ from $[L^2(\Omega)^N]^K$ into $\mathbb{R}^{Nn(\epsilon)}$ and an extension $E^K_\epsilon$ from $\mathbb{R}^{Nn(\epsilon)}$ into $[L^2(\Omega)^N]^K$ such that $S^K_\epsilon = E^K_\epsilon S_\epsilon P^K_\epsilon$. To do so, some notation is required concerning the two indices $p$ (indexing constant vectors in $\mathbb{R}^{Nn(\epsilon)}$) and $j$ (indexing vector functions in $[L^2(\Omega)^N]^K$).

**Definition 1.1.** Let $KY$ be the reference cell $(0,K)^N$ which is made of $K^N$ subcells $Y_j$ of the type $(0,1)^N$ containing a single tube $T_j$. The multi-integer $j = (j_1, \ldots, j_N)$ which enumerates all the tubes in $KY$ takes its values in $\{0,1,\ldots,K-1\}^N$ (we use the notation $0 \leq j \leq K - 1$). Let $p = (p_1, \ldots, p_N)$ be the multi-integer which enumerates all the tubes in $\Omega_\epsilon$ (see (1)). We define a third multi-integer $\ell = (\ell_1, \ldots, \ell_N)$ which enumerates all the periodic reference cells $\epsilon(KY)$ in $\Omega_\epsilon$ (its range is denoted by $1 \leq \ell \leq n_K(\epsilon)$). These three indices are assumed to be related by the
following one-to-one map:

\[ \ell_m = E \left( \frac{pm}{K} \right), \quad j_m = p_m - K \ell_m \quad \forall m = 1, \ldots, N, \]

where \( E(\cdot) \) denotes the integer-part function.

Then, \( P^K_\epsilon \) and \( E^K_\epsilon \) are defined by

\[ P^K_\epsilon : [L^2(\Omega)^N]^{K^N} \rightarrow \mathbb{R}^{Nn(\epsilon)}, \]

\[ (s_j(x))_{0 \leq j \leq K - 1} \rightarrow \left( s_p = \frac{1}{|\epsilon KY|} \int_{\epsilon KY} s_j(x) dx \right)_{1 \leq p \leq n(\epsilon)}, \]

\[ E^K_\epsilon : \mathbb{R}^{Nn(\epsilon)} \rightarrow [L^2(\Omega)^N]^{K^N}, \]

\[ (s^*_p)_{1 \leq p \leq n(\epsilon)} \rightarrow \left( s_j(x) = \sum_{\ell} \chi_{\epsilon KY}(x) s^*_p \right), \]

where \( p \) is related to \((\ell, j)\) by formula (5). One can easily check that the adjoint \((P^K_\epsilon)^*\) of \( P^K_\epsilon \) is nothing but \((\epsilon K)^{-N} E^K_\epsilon\) and that \( P^K_\epsilon E^K_\epsilon \) is equal to the identity in \( \mathbb{R}^{Nn(\epsilon)} \). Therefore, \( S^K_\epsilon \) is also self-adjoint compact and its spectrum is exactly that of \( S_\epsilon \), plus the new eigenvalue 0 which has infinite multiplicity.

The homogenization of the extended operator \( S^K_\epsilon \) is now amenable to the two-scale convergence method [2], [29]. However, the limit operator \( S^K \) has a complicated form which can be simplified by using the following discrete Bloch wave decomposition (see [1]).

**Lemma 1.2.** For any family \((s_j)_{0 \leq j \leq K - 1}\) of vectors in \( \mathbb{C}^N \), let \( \bar{s}(y) \) be the following \( KY \)-periodic function, piecewise constant in each subcell \( Y_j \):

\[ \bar{s}(y) = \sum_{j=0}^{K-1} s_j \chi_{Y_j}(y) \quad \forall y \in KY. \]

There exists a unique family of constant vectors \((\bar{t}_j)_{0 \leq j \leq K - 1}\) in \( \mathbb{C}^N \) such that

\[ \bar{s}(y) = \sum_{j=0}^{K-1} \bar{t}_j e^{2\pi i \frac{j}{K} E(y)} \quad \forall y \in KY, \]

where \( E(\cdot) \) denotes the integer-part function. Moreover, the Bloch wave decomposition operator \( \mathcal{B} \), defined by \( \mathcal{B}(s_j) = K^{N/2} \bar{t}_j \), is an isometry on \( (\mathbb{C}^N)^{K^N} \).

The first main result in [3] (see Theorem 3.2.1) is the following theorem.

**Theorem 1.3.** The sequence \( S^K_\epsilon = E^K_\epsilon S_\epsilon P^K_\epsilon \) converges strongly to a limit \( S^K \); i.e., for any family \((s_j(x))_{0 \leq j \leq K - 1}\), \( S^K_\epsilon(s_j) \) converges strongly to \( S^K(s_j) \) in \([L^2(\Omega)^N]^{K^N}\). Furthermore, the limit operator \( S^K \) is given by

\[ S^K = \mathcal{B}^* T^K \mathcal{B}, \quad \text{with} \quad T^K = \text{diag} \left( [T^K_{0 \leq j \leq K - 1}] \right), \]

where the entries \( T^K_{0 \leq j \leq K - 1} \) are self-adjoint continuous but noncompact operators in \( L^2(\Omega)^N \), defined by

\[ T^K_{0 \leq j \leq K - 1} = \left\{ \begin{array}{ll} (A(0) - I) \nabla u - (A(0) - |Y^*|) \bar{t}_0 & \text{if } j = 0, \\ \left( A(\frac{k}{k^N}) \bar{t}_j \right) & \text{if } j \neq 0, \end{array} \right. \]
where \( I \) is the identity matrix and \( u \) is the unique solution of the homogenized problem

\[
\begin{aligned}
\begin{array}{ll}
- \text{div}(A(0) \nabla u) &= \text{div}((I - A(0)) \vec{r}_0) \quad &\text{in } \Omega, \\
u &= 0 &\text{on } \partial \Omega,
\end{array}
\end{aligned}
\tag{11}
\]

and, for \( \theta \in [0, 1]^N \), \( A(\theta) \) is the Bloch homogenized matrix with components \((A_{mm'}(\theta))_{1 \leq m, m' \leq N}\) defined by

\[
\bar{A}_{mm'}(\theta) = \int_{Y^*} \nabla w_m^\theta(y) \cdot \nabla \bar{w}_{m'}^\theta(y) dy,
\tag{12}
\]

where \((w_m^\theta)_{1 \leq m \leq N}\) are solutions of the so-called cell problem at the Bloch frequency \( \theta \):

\[
\begin{aligned}
- \Delta w_m^\theta &= 0 &\text{in } Y^*, \\
(\nabla w_m^\theta - \vec{c}_m) \cdot \vec{n} &= 0 &\text{on } \partial T, \\
y \to e^{-2\pi i \theta \cdot y} w_m^\theta(y) &\text{Y'-periodic.}
\end{aligned}
\tag{13}
\]

The first component \( T^K_0 \) of the limit operator \( T^K \) is the same for all \( K \) and is denoted by \( S \) in what follows. It is called the macroscopic or homogenized limit of \( S_\epsilon \). (11) is also called the homogenized equation. The spectrum \( \sigma(S) \) is essential and has been explicitly characterized in Theorems 2.1.4 and 2.1.5 of [3]. The other components of \( T^K_0 \) are simple linear multiplication operators that represent the microscopic or Bloch limit behavior of the sequence \( S^K_\epsilon \).

According to Proposition 3.2.6 in [3], the matrix \( A(\theta) \) is Hermitian and positive definite for any value of \( \theta \). Furthermore, it is a continuous function of \( \theta \), except at the origin \( \theta = 0 \). Nevertheless, it is continuous at the origin along rays of constant direction (see Proposition 3.4.4 in [3]). Denoting by \( 0 < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \cdots \leq \lambda_N(\theta) \) its eigenvalues, we can define the so-called Bloch spectrum by

\[
\sigma_{\text{Bloch}} = \bigcup_{m=1}^N \overline{\lambda_m([0, 1]^N)},
\]

where \( \overline{\lambda_m([0, 1]^N)} \) denotes the closure of the image of \([0, 1]^N\) under the maps \( \lambda_m(.) \).

We deduce our second main result.

**Theorem 1.4.** The strong convergence of \( S^K_\epsilon \) to the limit operator \( S^K \) implies the lower semicontinuity of the spectrum

\[
\sigma(S^K) \subset \lim_{\epsilon \to 0} \sigma(S^K_\epsilon).
\]

By letting \( K \) go to infinity, we obtain

\[
\sigma(S) \cup \sigma_{\text{Bloch}} \subset \lim_{\epsilon \to 0} \sigma(S_\epsilon).
\tag{14}
\]

**Remark 1.5.** As a matter of fact, the Bloch spectrum \( \sigma_{\text{Bloch}} \) and the homogenized spectrum \( \sigma(S) \) do not coincide. Therefore, both type of limit problems (macroscopic (11) and microscopic (13)) are complementary. As already mentioned, the Bloch spectrum has already been characterized by C. Conca and M. Vanninathan in [17] by means of a different method, the so-called nonstandard homogenization procedure (see also the book [16]).
The question is now to see whether the inclusion in (14) is actually an equality, i.e., if our asymptotic analysis is complete. It turns out that the homogenized and the Bloch spectra are usually not enough to describe $\sigma_\infty$ because the interaction between the boundary $\partial\Omega$ and the microstructure is not taken into account in our analysis. More precisely, there may well exist sequences of eigenvectors of (2) which concentrate near the boundary $\partial\Omega$ of $\Omega$. They behave as boundary layers in the sense that they converge strongly to zero locally inside the domain. Clearly the oscillations of these eigenvectors cannot be captured by the usual homogenization method; neither are they filtered in the Bloch spectrum which is insensitive to the boundary.

Nevertheless, the third main result of our previous paper [3] shows that for any other type of sequences of eigenvectors (not concentrating on the boundary), the limits of the corresponding sequences of eigenvalues belong to $\sigma(S) \cup \sigma_{\text{Bloch}}$. More exactly, introducing the subset of $\sigma_\infty$

$$\sigma_{\text{boundary}} = \{ \lambda \in \mathbb{R} \mid \exists (\lambda', \tilde{s}') \text{ such that } S_1^{\lambda'} \tilde{s}' = \lambda' \tilde{s}', \lambda' \rightarrow \lambda, \| \tilde{s}' \|_{L^2(\Omega)^N} = 1, \text{ and } \forall \omega \text{ with } \omega \subset \Omega, \| \tilde{s}' \|_{L^2(\omega)^N} \rightarrow 0 \},$$

where $\epsilon'$ is a subsequence of $\epsilon$ and $S_1^{\lambda'}$ is the extension to $L^2(\Omega)^N$ of $S_\epsilon$, we proved the following theorem (see Theorem 3.2.9 in [3]).

**Theorem 1.6.** The limit set of the spectrum of the operator $S_\epsilon$ is precisely made of three parts: the homogenized, the Bloch, and the boundary layer spectrum

$$\lim_{\epsilon \rightarrow 0} \sigma(S_\epsilon) = \sigma_\infty = \sigma(S) \cup \sigma_{\text{Bloch}} \cup \sigma_{\text{boundary}}.$$

The proof of this completeness result is the focus of section 3.4 in [3]. It involves a new type of default measure for weakly converging sequences of eigenvectors of $S_\epsilon$, the so-called Bloch measures which quantify its amplitude and direction of oscillations.

Of course the definition of $\sigma_{\text{boundary}}$ is not satisfactory, since it does not characterize that part of the limit set $\sigma_\infty$ as the spectrum of some limit operator associated with the boundary $\partial\Omega$. In particular, it is not clear whether $\sigma_{\text{boundary}}$ is empty or included in $\sigma(S) \cup \sigma_{\text{Bloch}}$. It is the purpose of the present paper to characterize explicitly $\sigma_{\text{boundary}}$, at least for special rectangular domains $\Omega$ and associated sequences of parameters $\epsilon$.

**Remark 1.7.** By their very definitions, the limit spectrum $\sigma_\infty$ and the boundary layer spectrum $\sigma_{\text{boundary}}$ depend a priori on the choice of the sequence of small parameters $\epsilon$. On the contrary, the homogenized spectrum $\sigma(S)$ and the Bloch spectrum $\sigma_{\text{Bloch}}$ are independent of the sequence $\epsilon$. We believe that $\sigma_{\text{boundary}}$ is actually strongly dependent on the sequence $\epsilon$. In particular, we shall characterize it only for a specific sequence $\epsilon$. We thank C. Castro and E. Zuazua for clarifying discussions on this topic [12].

### 1.3. Presentation of the main new results.

There are mainly two new results in this paper which correspond to the next two sections. First, in section 2 we introduce a new class of limit problems involving the interaction between the tubes array and the domain boundary. We assume that the domain $\Omega$ is cylindrical;

$$(16) \quad \Omega = \Sigma \times ]0; L[,$$

where $\Sigma$ is an open bounded set in $\mathbb{R}^{N-1}$ and $L > 0$ is a positive length. A generic point $x$ in $\mathbb{R}^N$ is denoted by $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$ ($x_N$ is the coordinate along the axis of $\Omega$). Let us define a semi-infinite band

$$G = Y' \times ]0; +\infty[,$$
where $Y' = [0, 1]^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. This new “boundary layer” limit problem takes place in the fluid part of $G$, denoted by $G^*$ and defined by

$$G^* = G \setminus \bigcup_{q \geq 1} T_q,$$

where $(T_q)$ is the infinite collection of tubes periodically disposed in $G$. With each tube $T_q$ is associated a displacement $\tilde{s}_q \in \mathbb{R}^N$. We denote by $\ell^2$ the space of families $(\tilde{s}_q)_{q \geq 1}$ such that $\sum_{q \geq 1} |\tilde{s}_q|^2$ is finite. Introducing a Bloch parameter $\theta' \in [0, 1]^{N-1}$, we define a “boundary layer” operator $d_{\theta'}$ by

$$d_{\theta'} : \ell^2 \longrightarrow \ell^2, \quad (\tilde{s}_q)_{q \geq 1} \mapsto \left( \int_{\Gamma_q} u_{\theta'} \tilde{n} ds \right)_{q \geq 1},$$

where $u_{\theta'}(y)$ is the unique solution of

$$\begin{cases}
-\Delta u_{\theta'} = 0 & \text{in } G^*, \\
\frac{\partial u_{\theta'}}{\partial n} = \tilde{s}_q \cdot \tilde{n} & \text{on } \Gamma_q, \ q \geq 1, \\
u_{\theta'} = 0 & \text{if } y_N = 0, \ y' \mapsto e^{-2\pi i \theta' \cdot y'} u_{\theta'}(y', y_N) \ Y'-\text{periodic.}
\end{cases}$$

Our first result (see Theorem 2.18) is concerned with the continuity of the spectrum of $d_{\theta'}$, considered as a subset of $\mathbb{R}$, with respect to the Bloch parameter $\theta'$.

**Theorem 1.8.** For all $\theta' \in [0, 1]^{N-1}$, $d_{\theta'}$ is a self-adjoint continuous but non-compact operator in $\ell^2$. Its spectrum $\sigma(d_{\theta'})$ depends continuously on $\theta'$, except at $\theta' = 0$. Defining the boundary layer spectrum associated with the surface $\Sigma$

$$\sigma_{\Sigma} \defeq \bigcup_{\theta' \in [0, 1]^{N-1}} \sigma(d_{\theta'}) \cup \sigma(d_0),$$

we have

$$\sigma_{\Sigma} \subset \lim_{\epsilon \to 0} \sigma(S_{\epsilon}).$$

In general, $\sigma(d_{\theta'})$ is not included in the previously found limit spectrum $\sigma(S) \cup \sigma_{\text{Bloch}}$ (see Proposition 2.17). Therefore, the new class of limit problems defined by (17) is not redundant with the homogenized or the Bloch limit problems. Our main tool for proving this theorem is a variant of the two-scale convergence adapted to boundary layers, using test functions which oscillate periodically in the directions parallel to the boundary $\Sigma$ and decay asymptotically fast in the normal direction to $\Sigma$ (see section 2.1). Remark that the above result holds for any cylindrical domain of the type (16) and for any sequence of periods $\epsilon$ going to zero.

Section 3 is devoted to our second main result which requires additional assumptions on the geometry of the domain and on the sequence of periods $\epsilon$. More precisely, we now assume that $\Omega$ is a rectangle with integer dimensions

$$\Omega = \prod_{i=1}^N (0; L_i[ \ and \ L_i \in \mathbb{N}^*$$
and that the sequence $\epsilon$ is exactly

$$\epsilon_n = \frac{1}{n}, \quad n \in \mathbb{N}^*.$$ 

These assumptions imply that, for any $\epsilon_n$, the domain $\Omega$ is the union of a finite number of entire cells of size $\epsilon_n$. Then, the above analysis of the boundary layer spectrum $\sigma_\Sigma$ can be achieved for any face $\Sigma$ of the rectangle $\Omega$. Of course a completely similar analysis can be done for all the lower dimensional manifolds (edges, corners, etc.) of which the boundary of $\Omega$ is made up. For each type of manifold, a different family of limit problems arise which are straightforward generalizations of (17). For example, in two space dimensions, the corners of $\Omega$ give rise to a limit problem in the quarter of space $\mathbb{R}^+ \times \mathbb{R}^+$ filled with a periodic array of tubes (see section 3.3). Finally, we prove a completeness result (see Theorem 3.1).

**Theorem 1.9.** The limit set of the spectrum of the operator $S_{\epsilon_n}$ is precisely made of three parts; the homogenized, the Bloch, and the union of all boundary layer spectra, as defined in Theorem 1.8,

$$\lim_{\epsilon_n \to 0} \sigma(S_{\epsilon_n}) = \sigma(S) \cup \sigma_{\text{Bloch}} \cup \sigma_{\partial \Omega},$$

with the notation

$$\sigma_{\partial \Omega} = \bigcup_{\Sigma \subset \partial \Omega} \sigma_\Sigma,$$

where the union is over all hypersurfaces and lower dimensional manifolds composing the boundary $\partial \Omega$.

**Remark 1.10.** The difference between the above completeness theorem and Theorem 1.6 is that, here, the boundary layer spectrum $\sigma_{\partial \Omega}$ is explicitly defined for the specific sequence of parameters $\epsilon_n$ as the spectrum of a family of limit operators, while, in our previous result, the boundary layer spectrum $\sigma_{\text{boundary}}$ was indirectly defined for any sequence $\epsilon$ but not explicitly characterized.

We conclude this introduction by giving a few references to related works on boundary layers in homogenization and by a short discussion on numerical studies concerning problem (2). Apart from the classical books [7, Chapter 7] and [26], we refer mainly to the papers [6], [9], [10], and [27]. Planchard’s model has already been studied numerically. The Bloch eigenvalues $\lambda_i(\theta)$ were computed by F. Aguirre in a two-dimensional example. A brief account of his work is given in [1]. On the other hand, direct numerical computations of the entire spectrum $\sigma(S_\epsilon)$ (for a fixed value of $\epsilon$, and without using homogenization) have been reported in [23]. To our knowledge, these are the only available numerical results concerning a large tube array (see also [21], [22]). Of course, these results are consistent with Theorem 1.9 describing the asymptotic behavior of $\sigma(S_\epsilon)$. In particular, some vibration modes displayed in [23] are numerical evidence that $\sigma_{\partial \Omega}$ is not empty; i.e., there exist eigenvectors which are localized near the boundary or the corners of $\Omega$.

2. Boundary layer homogenization. In this section we assume that $\Omega$ is a cylindrical bounded open set in $\mathbb{R}^N$ in the sense that it is defined by

$$\Omega = \Sigma \times ]0; L[,$$

where $\Sigma$ is an open bounded set in $\mathbb{R}^{N-1}$ and $L > 0$ is a positive length. With no loss of generality, we assume that the axis of the cylindrical domain $\Omega$ is parallel to the $N$th
canonical direction. Therefore, a generic point $x$ in $\Omega$ is denoted by $x = (x', x_N)$ with $x' \in \Sigma$ and $x_N \in [0, L]$. The goal of this section is to analyze the asymptotic behavior of that part of the spectrum $\sigma(S_\epsilon)$ which corresponds to eigenvectors concentrating on the boundary $\Sigma \times \{0\}$, under the sole geometric assumption (19) (in particular, no restrictions are made on the sequence $\epsilon$ which goes to zero).

2.1. Two-scale convergence for boundary layers. We begin by adapting the classical two-scale convergence method of Allaire [2] and Nguetseng [29] to the case of boundary layers, that is, sequences of functions in $\Omega$ which concentrate near the boundary $\Sigma \times \{0\}$. This method of “two-scale convergence for boundary layers” will allow us to understand this phenomenon of concentration of oscillations near the boundary. The usual two-scale convergence relies on periodically oscillating test functions with a unit period $Y = [0,1]^N$. Here, we use test functions which oscillate only in the directions parallel to the boundary $\Sigma$ (with period $Y' = [0,1]^{N-1}$) and which simply decay in the $N$th direction orthogonal to $\Sigma$.

Let us define a semi-infinite band $G = Y' \times [0; +\infty[$, where $Y' = [0,1]^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. A generic point $y$ is denoted by $y = (y', y_N)$ with $y' \in Y'$ and $y_N \in [0; +\infty[$. We introduce the space $L^2_{\#}(G)$ of square integrable functions in $G$ which are periodic in the $(N-1)$ first variables, i.e.,

$$L^2_{\#}(G) = \{ \phi(y) \in L^2(G) \mid y' \mapsto \phi(y', y_N) \text{ is } Y'-\text{periodic} \}.$$

We also denote by $C(\Sigma)$ the space of continuous functions on the closure of $\Sigma$, a compact set in $\mathbb{R}^{N-1}$.

Combining the concentration effect in $y_N$ and the periodic oscillations in $Y'$, the following convergence result is obtained for a sequence $\phi(\xi/\epsilon)$ when $\phi$ belongs to $L^2_{\#}(G)$ (further modulated by $x' \in \Sigma$).

**Lemma 2.1.** Let $\varphi(x', y) \in L^2_{\#}(G; C(\Sigma))$. Then

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega} \left| \varphi \left( x', \frac{x}{\epsilon} \right) \right|^2 dx = \frac{1}{|Y'|} \int_{\Sigma} \int_{G} |\varphi(x', y)|^2 dx' dy.$$

**Remark 2.2.** Remark that, in the left-hand side of the above equation, the second argument of $\varphi$ is $x'/\epsilon$ and not only $x'/\epsilon$. This implies that there is a concentration effect near 0 in the $x_N$ variable since $\varphi$ is not periodic in this direction. This, in turn, explains the $1/\epsilon$ scaling in front of the left-hand side, in order to get a nonzero limit.

As usual in the context of two-scale convergence, the above result is not specific to the space $L^2_{\#}(G; C(\Sigma))$, which could be replaced, for example, by $L^2(\Sigma; C_{c\#}(G))$, where $C_{c\#}(G)$ is the space of compactly supported functions in $G$, periodic in $y'$ of period $Y'$, and with bounded support in $y_N$.

In view of Lemma 2.1, we define a notion of “two-scale convergence for boundary layers.”

**Definition 2.3.** Let $(u_\epsilon)_{\epsilon>0}$ be a sequence in $L^2(\Omega)$. It is said to two-scale converge in the sense of boundary layers on $\Sigma$ if there exists $u_0(x', y) \in L^2(\Sigma \times G)$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega} u_\epsilon(x) \varphi \left( x', \frac{x}{\epsilon} \right) dx = \frac{1}{|Y'|} \int_{\Sigma} \int_{G} u_0(x', y) \varphi(x', y) dx' dy$$

for all smooth functions $\varphi(x', y)$ defined in $\Sigma \times G$ such that $y' \mapsto \varphi(x', y', y_N)$ is $Y'$-periodic and $\varphi$ has a bounded support in $\Sigma \times G$. 

This definition makes sense because of the following compactness theorem which generalizes the usual two-scale convergence compactness theorem in [2], [29].

**Theorem 2.4.** Let \((u_\epsilon)_{\epsilon > 0}\) be a sequence in \(L^2(\Omega)\) such that there exists a constant \(C\), independent of \(\epsilon\), for which

\[
\frac{1}{\sqrt{\epsilon}} \|u_\epsilon\|_{L^2(\Omega)} \leq C.
\]

There exists a subsequence, still denoted by \(\epsilon\), and a limit function \(u_0(x', y) \in L^2(\Sigma \times G)\) such that

\[
\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \int_{\Omega} u_\epsilon(x) \varphi \left( x', \frac{x}{\epsilon} \right) dx = \frac{1}{|\Sigma|} \int_{\Sigma} \int_{G} u_0(x', y) \varphi(x', y) dx' dy
\]

for all functions \(\varphi(x', y) \in L^2_\# (G; C(\overline{\Sigma}))\).

Remark that Theorem 2.4 does not apply to sequences which are merely bounded in \(L^2(\Omega)\) but also converge strongly to zero in \(L^2(\Omega)\) as the square root of \(\epsilon\). Of course, this is the case for a sequence of the type \(\varphi(x', \frac{x}{\epsilon})\), where \(\varphi(x', y)\) is as in Lemma 2.1; then, the limit is nothing but \(\varphi(x', y)\) itself.

It is not difficult to check that the \(L^2\)-norm is weakly lower semicontinuous with respect to the two-scale convergence (see Proposition 1.6 in [2]); i.e., in the present situation

\[
\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \|u_\epsilon\|_{L^2(\Omega)} \geq \frac{1}{|\Sigma|^{1/2}} \|u_0\|_{L^2(\Sigma \times G)}.
\]

The next proposition asserts a corrector-type result when the above inequality is actually an equality.

**Proposition 2.5.** Let \((u_\epsilon)_{\epsilon > 0}\) be a sequence in \(L^2(\Omega)\) which two-scale converges in the sense of boundary layers to a limit \(u_0(x', y) \in L^2(\Sigma \times G)\). Assume further that it two-scale converges strongly, that is,

\[
\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \|u_\epsilon\|_{L^2(\Omega)} = \frac{1}{|\Sigma|^{1/2}} \|u_0\|_{L^2(\Sigma \times G)}.
\]

Then,

(i) for any sequence \((v_\epsilon)_{\epsilon > 0}\) in \(L^2(\Omega)\) which two-scale converges in the sense of boundary layers to a limit \(v_0(x', y) \in L^2(\Sigma \times G)\), one has

\[
\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \int_{\Omega} u_\epsilon v_\epsilon dx = \frac{1}{|\Sigma|} \int_{\Sigma} \int_{G} u_0(x', y) v_0(x', y) dx' dy;
\]

(ii) if \(u_0(x', y)\) is smooth, say \(u_0 \in L^2_{\#}(G; C(\overline{\Sigma}))\), then

\[
\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \|u_\epsilon(x) - u_0 \left( x', \frac{x}{\epsilon} \right) \|_{L^2(\Omega)} = 0.
\]

In order to investigate the convergence of sequences of functions in \(H^1_0(\Omega)\), we first have to define adequate functional spaces for the two-scale limit. Let \(C^\infty_{c\#}(G)\) be the space of smooth functions in \(\overline{G}\) which are \(Y'\)-periodic in \(y'\) and have a compact support in \(y_N\) (i.e., they vanish for sufficiently large and small \(y_N\) but not necessarily on the whole \(\partial G\)). Let \(H^1_{0\#}(G)\) be the Sobolev space obtained by completion of \(C^\infty_{c\#}(G)\) with
Bounded support in respect to the $H^1(G)$-norm. We denote by $H^1_{0\#, \text{loc}}(G)$ the space of functions which are “locally” in $H^1_{0\#}(G)$, i.e., which coincide with a function of $H^1_{0\#}(G)$ in any compact set of $\Gamma$. We define a Deny–Lions-type space (cf. [18]) $D^1_{0\#}(G)$ as the completion of $C^\infty_{c\#}(G)$ with respect to the $L^2(G)^N$-norm of the gradient

$$D^1_{0\#}(G) = \left\{ \psi(y) \in H^1_{0\#, \text{loc}}(G) \mid \exists \psi_n \in C^\infty_{c\#}(G) \text{ such that } \lim_{n \to +\infty} \| \nabla(\psi - \psi_n) \|_{L^2(G)^N} = 0 \right\}. \tag{21}$$

It is easily seen that a function in $D^1_{0\#}(G)$ vanishes when $y_N = 0$ but does not necessarily go to 0 when $y_N$ goes to infinity since $D^1_{0\#}(G)$ contains functions which grow like $y_N^\alpha$ at infinity with $\alpha < 1/2$. We are now in a position to state our next result.

**Proposition 2.6.** Let $(u_\epsilon)_{\epsilon > 0}$ be a sequence in $H^1_\#(\Omega)$ such that there exists a constant $C$, independent of $\epsilon$, for which

$$\frac{1}{\sqrt{\epsilon}} \left( \| u_\epsilon \|_{L^2(\Omega)} + \| \nabla u_\epsilon \|_{L^2(\Omega)^N} \right) \leq C.$$

Then, there exists a subsequence, still denoted by $\epsilon$, and a limit $u_0(x', y) \in L^2(\Sigma; D^1_{0\#}(G))$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int \nabla u_\epsilon(x) \cdot \psi \left( \frac{x'}{\epsilon}, \frac{y}{\epsilon} \right) dx = \frac{1}{|\Sigma|} \int \int \nabla y u_0(x', y) \cdot \psi(x', y) dx' dy$$

for any functions $\varphi \in L^2_\#(G; C(\Sigma))$ and $\psi \in L^2_\#(G; C(\Sigma)^N)$.

Remark that, in Proposition 2.6, the two-scale limit $u_0(x', y)$ does not belong to $L^2(\Sigma; H^1(G))$ as could be expected. The reason is that only $\nabla y u_0 \in L^2(\Sigma \times G)$, while $u_0$ itself has no reason to belong to $L^2(\Sigma \times G)$. Since the proofs of the above results are very similar to those of the usual two-scale convergence theory, we simply sketch the proofs of Lemma 2.1, Theorem 2.4, and Proposition 2.6.

**Proof of Lemma 2.1.** Let us first assume that $\varphi(x', y) \in L^2_\#(G; C(\Sigma))$ has bounded support in $y_N$; i.e., there exists $M > 0$ such that

$$\varphi(x', y) = 0 \text{ if } y_N \geq M.$$

Then, by the change of variables $y_N = x_N/\epsilon$ and for sufficiently small $\epsilon$, we have

$$\frac{1}{\epsilon} \int \varphi \left( \frac{x'}{\epsilon}, \frac{x_N}{\epsilon} \right) dx = \frac{1}{\epsilon} \int_{y_N}^L \int_{y_N}^L \varphi \left( \frac{x'}{\epsilon}, \frac{x_N}{\epsilon} \right) dx' dx_N = \int_{y_N}^L \int_{y_N}^L \varphi \left( \frac{x'}{\epsilon}, \frac{x_N}{\epsilon}, y_N \right) dx' dy_N = \int_M^L \int_{y_N}^L \varphi \left( \frac{x'}{\epsilon}, \frac{x_N}{\epsilon}, y_N \right) dx' dy_N. \tag{22}$$

The usual convergence result for oscillating functions in $\mathbb{R}^{N-1}$ (see, e.g., [2] and references therein) yields that for almost everywhere $y_N \in (0; M)$

$$\lim_{\epsilon \to 0} \int \varphi \left( \frac{x'}{\epsilon}, \frac{x_N}{\epsilon}, y_N \right) \left| \left( \frac{x'}{\epsilon}, \frac{x_N}{\epsilon}, y_N \right) \right|^2 dx' = \frac{1}{|\Sigma|} \int \int \varphi \left( \frac{x'}{\epsilon}, \frac{x_N}{\epsilon}, y_N \right) \left| \left( \frac{x'}{\epsilon}, \frac{x_N}{\epsilon}, y_N \right) \right|^2 dx' dy'.$$
and that
\[
\int_\Omega \left| \varphi \left( x', \frac{x'}{\epsilon}, y_N \right) \right|^2 dx' \leq |\Sigma| \int_{y', x' \in \Sigma} \max |\varphi(x', y', y_N)|^2 dy'.
\]

Therefore, applying the Lebesgue theorem, we deduce that
\[
\lim_{\epsilon \to 0} \int_\Omega \int_\Sigma \left| \varphi \left( x', \frac{x'}{\epsilon}, y_N \right) \right|^2 dx' dy_N = \frac{1}{|Y'|} \int_\Sigma \int_G |\varphi(x', y', y_N)|^2 dx' dy.
\]

The density of such functions \( \varphi(x', y) \) in \( L^2_{\#}(G; C(\Sigma)) \) implies the desired result for any function in \( L^2_{\#}(G; C(\Sigma)) \).

**Proof of Theorem 2.4.** Using the assumed uniform bound on \( u_\epsilon \), by the Schwarz inequality we obtain
\[
\left| \frac{1}{\epsilon} \int_\Omega u_\epsilon(x) \varphi \left( x', \frac{x'}{\epsilon} \right) dx \right| \leq C \left( \frac{1}{\epsilon} \int_\Omega \left| \varphi \left( x', \frac{x'}{\epsilon} \right) \right|^2 dx \right)^{\frac{1}{2}}.
\]

Passing to the limit, up to a subsequence, which may depend on \( \varphi \) in the left-hand side and using Lemma 2.1 in the right-hand side, yield
\[
|\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega u_\epsilon(x) \varphi \left( x', \frac{x'}{\epsilon} \right) dx | \leq C \left( \int_\Sigma \int_G |\varphi(x', y)^2 dx' dy \right)^{\frac{1}{2}}.
\]

Since \( L^2_{\#}(G; C(\Sigma)) \) is separable, varying \( \varphi \) over a dense countable subset, by a standard diagonalization process, we can extract a subsequence of \( \epsilon \) such that (23) is valid for all functions \( \varphi \) in this subset. By density, we conclude that the limit in the left side of (23), as a function of \( \varphi \), defines a continuous linear form in \( L^2(\Sigma \times G) \). Then, the classical Riesz representation theorem immediately implies the existence of a function \( u_0(x, y) \in L^2(\Sigma \times G) \) which satisfies (20). This finishes the proof of Theorem 2.4.

**Proof of Proposition 2.6.** By application of Theorem 2.4, up to a subsequence, there exist two limits \( u(x', y) \in L^2(\Sigma \times G) \) and \( \xi^0(x', y) \in L^2(\Sigma \times G)^N \) such that \( u_\epsilon \) and \( \nabla u_\epsilon \) two-scale converge in the sense of boundary layers to these respective limits; i.e.,
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega u_\epsilon(x) \varphi \left( x', \frac{x'}{\epsilon} \right) dx = \frac{1}{|\Sigma|} \int_\Sigma \int_G u(x', y) \varphi(x', y) dx' dy,
\]
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega \nabla u_\epsilon(x) \cdot \psi \left( x', \frac{x'}{\epsilon} \right) dx = \frac{1}{|\Sigma|} \int_\Sigma \int_G \xi_0(x', y) \cdot \psi(x', y) dx' dy
\]
for any functions \( \varphi \in L^2_{\#}(G; C(\Sigma)) \) and \( \psi \in L^2_{\#}(G; C(\Sigma)^N) \). Integrating by parts in (25), we obtain
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega \nabla u_\epsilon(x) \psi \left( x', \frac{x'}{\epsilon} \right) dx = 0.
\]
In view of (24), this implies that
\[
\frac{1}{|\Sigma|} \int_\Sigma \int_G u(x', y) \psi(x', y) dx' dy = 0.
\]
Another integration by parts yields that \( u(x',y) \) does not depend on \( y \). On the other hand, it belongs to \( L^2(\Sigma \times G) \) and \( G \) is unbounded. Since the only constant which belongs to \( L^2(G) \) is zero, we deduce that \( u = 0 \). Now, specializing (25) to test functions \( \psi \) such that \( \text{div}_y \psi = 0 \) and integrating by parts, we also obtain that

\[
\frac{1}{|Y'|} \int_{\Sigma} \int_{G} \xi_0(x',y) \cdot \psi(x',y) dx' dy = 0.
\]

As is well known, the orthogonal of divergence-free fields is exactly the set of gradients (see Proposition 1.14 in [2] for a precise statement and references). Therefore, there exists a function \( u_0(x',y) \) in \( L^2(\Sigma; D^1_{0\#}(G)) \) such that \( \xi_0 = \nabla_y u_0 \) (we use the space \( D^1_{0\#}(G) \) since \( u_0 \) has no reason to belong to \( L^2(\Sigma \times G) \)).

### 2.2. Convergence analysis.

Recall that the original operator \( S_\epsilon \), defined by (3), acts in the space \( \mathbb{R}^{Nn(\epsilon)} \) which depends on \( \epsilon \) and that our strategy was to extend \( S_\epsilon \) to a fixed space where a convergence analysis is possible. So far, the domain \( \Omega = \Sigma \times ]0,L[ \) was considered periodic of period \( \epsilon Y' \). Nevertheless, from now on, \( \Omega \) is seen as a periodic domain with a new period \( G^K_\epsilon \) defined by

\[
G^K_\epsilon \overset{\text{def}}{=} [0; \epsilon K [N-1] \times ]0; L[,
\]

with \( K \) an integer larger than 1. We shall construct an extension of \( S_\epsilon \) well suited for the previous two-scale convergence “in the sense of boundary layers” with such a period \( G^K_\epsilon \).

**Remark 2.7.** As already mentioned, we make no special hypothesis on the sequence of small parameters \( \epsilon \). However, the periodic arrangement of tubes in \( \Omega \) is required to be aligned with \( \Sigma \) in such a way that the first row of periodic cells \( \epsilon Y \) has a boundary which coincides with \( \Sigma \times \{0\} \). In other words, the first layer of tubes close to \( \Sigma \) is at a fixed distance \( \frac{\epsilon}{2} \) of \( \Sigma \times \{0\} \) (see Figure 1).

By a rescaling of ratio \( \epsilon \), this new period \( G^K_\epsilon \) corresponds to a finite length truncation of the new reference cell

\[
G^K \overset{\text{def}}{=} KG = [0; K [N-1] \times ]0; +\infty[ = KY' \times ]0; +\infty[.
\]

In the reference cell \( G^K \) (see Figure 2) we put infinitely many layers of tubes in the \( n \)th direction, each layer being made of \( K^{N-1} \) tubes. The tubes in \( G^K \) are denoted by \( T_j \), where \( j = (j', j_N) \) is a multi-index such that \( j_N \geq 1 \) is an integer, which labels the corresponding layer in \( G^K \), and \( j' \) is a multi-index in \( \{0, 1, \ldots, K-1\}^{N-1} \), which locates the tube \( T_j \) in its layer \( j_N \). The fluid part in \( G^K \) is denoted by \( G^{\ast K} \), i.e.,

\[
G^{\ast K} = G^K \setminus \bigcup_{0 \leq j' \leq K-1 \atop 1 \leq j_N} T_j.
\]

To each tube \( T_j \) in \( G^K \) we associate the subcell \( Y_j \) and the fluid subcell \( Y_j^\ast = Y_j \setminus T_j \) analogous to \( Y \) and \( Y^\ast \), respectively (see Figure 2). The main idea is to attach to each tube \( T_j \) in \( G^K \) a different displacement function \( \vec{s}(x') \), depending only on the variable \( x' \in \Sigma \), such that the family \( (\vec{s}_j(x'))_{0 \leq j' \leq K-1 \atop 1 \leq j_N} \) belongs to the space \( L^2(\Sigma; \ell^2_K) \), where \( \ell^2_K \) is the Hilbert space defined by

\[
\ell^2_K = \left\{ (\vec{s}_j)_{0 \leq j' \leq K-1 \atop 1 \leq j_N} \mid \vec{s}_j \in \mathbb{C}^N, \quad \sum_{0 \leq j' \leq K-1 \atop 1 \leq j_N} |\vec{s}_j|^2 < +\infty \right\}.
\]
Remark that this definition of $\ell^2_K$ implies a decay of the displacement function $\bar{s}_j$ as $j_N$ goes to $+\infty$. Note also that each family $(\bar{s}_j(x')) \in L^2(\Sigma; \ell^2_K)$ can be identified with a function $\bar{s}(x', y) \in L^2(\Sigma \times G^K)$ which is constant in each subcell $Y_j$.

We now introduce the extended operator $B^K$ defined in $L^2(\Sigma; \ell^2_K)$ by

$$B^K = E^K S \rho^K,$$

where $P^K$ and $E^K$ are, respectively, projection and extension operators between $\mathbb{R}^{Nn(\epsilon)}$ and $L^2(\Sigma; \ell^2_K)$. To define precisely $P^K$ and $E^K$ we need the following notation.

**Definition 2.8.** Let $j = (j', j_N)$ denote the multi-index which enumerates all tubes in the periodic reference cell $G^K$. We use the notation $0 \leq j' \leq K - 1$ to indicate that $j'$ varies in $\{0, 1, \ldots, K - 1\}^{N-1}$ and $j_N \geq 1$ to indicate that $j_N$ takes any positive integer value. Let $p = (p_1, \ldots, p_N)$ be the multi-integer which enumerates all the tubes in $\Omega$ (see Definition 1). The index $p$ is such that the tube $T_p$ is located...
in the cell whose origin lies at the point \( ep \in \Omega \). To describe its range we use the notation \( 1 \leq p \leq n(\epsilon) \), where \( n(\epsilon) \) is the total numbers of tubes in \( \Omega \). We define a third multi-integer \( \ell' = (\ell_1, \ldots, \ell_{N-1}) \) which enumerates all the periodic reference cells \( G_{\epsilon,\ell'} \) covering \( \Omega \) (each being identical, up to a translation, to \( G_{\ell}^{\epsilon} \)). For simplicity its range is denoted by \( 1 \leq \ell' \leq n_{K}(\epsilon) \). These three indices are assumed to be related by the following one-to-one relationship:

\[
\begin{align*}
\ell_m &= E\left(\frac{p_m}{\ell}\right), \\
j_m &= p_m - K\ell_m \quad \text{for} \quad 1 \leq m \leq N - 1, \\
j_N &= p_N,
\end{align*}
\]

where \( E(\cdot) \) denotes the integer-part function. This yields a one-to-one map between the tubes \( (T^\epsilon_p) \) and their location in the cell \( G_{\epsilon,\ell'}^{\epsilon} \) at the position \( j' \) in the layer \( j_N \).

Then, we define a projection

\[
P^K_{\epsilon} : L^2(\Sigma; \ell^{2}_{K}) \longrightarrow \mathbb{R}^{n(\epsilon)},
\]

\[
(\bar{s}_{j}(x'))_{0 \leq j' \leq K - 1} \mapsto (\bar{s}_{p})_{1 \leq p \leq n(\epsilon)}
\]

given by

\[
\bar{s}_{p} = \frac{1}{|\epsilon K^{r}|} \int_{(\epsilon K^{r})_{\ell'}} \bar{s}_{j}(x')dx',
\]

where \( (p, j, \ell') \) are related by formula (26) and \( (\epsilon K^{r})_{\ell'} \) is the cross section of the cell \( G_{\epsilon,\ell'}^{\epsilon} \).

We also define an extension

\[
E^K_{\epsilon} : \mathbb{R}^{n(\epsilon)} \longrightarrow L^2(\Sigma; \ell^{2}_{K}),
\]

\[
(\bar{s}_{p})_{1 \leq p \leq n(\epsilon)} \mapsto (\bar{s}_{j}(x'))_{0 \leq j' \leq K - 1}
\]

given by

\[
\bar{s}_{j}(x') = \sum_{\ell'} \chi_{(\epsilon K^{r})_{\ell'}}(x') \bar{s}_{p},
\]

where \( (p, j, \ell') \) are related by formula (26) and \( \chi_{(\epsilon K^{r})_{\ell'}}(x') \) is the characteristic function of \( (\epsilon K^{r})_{\ell'} \). By convention, \( \bar{s}_{p} \) is taken equal to 0 if the values of \( j \) and \( \ell' \) correspond to a cell truncated by the boundary \( \partial \Omega \) which therefore contains no tube.

One can easily check that \( P^K_{\epsilon} \) and \( E^K_{\epsilon} \) are adjoint operators (up to a multiplicative constant) and that the product \( P^K_{\epsilon}E^K_{\epsilon} \) is nothing but the identity in \( \mathbb{R}^{n(\epsilon)} \). Therefore, the spectrum of \( B^K_{\epsilon} \) consists of that of \( S_{\epsilon} \) and zero as an eigenvalue of infinite multiplicity. We summarize these results in the next lemma, the proof of which is safely left to the reader.

**Lemma 2.9.** The operators \( P^K_{\epsilon} \) and \( E^K_{\epsilon} \) satisfy the following properties;

1. \( (P^K_{\epsilon})^* = (\epsilon K)^{-N+1}E^K_{\epsilon} \),
2. \( (E^K_{\epsilon})^* = (\epsilon K)^{(N-1)}P^K_{\epsilon} \),
3. \( P^K_{\epsilon}E^K_{\epsilon} = \text{Id}_{\mathbb{R}^{n(\epsilon)}} \).

Therefore, the extended operator \( B^K_{\epsilon} = E^K_{\epsilon}S_{\epsilon}P^K_{\epsilon} \) is self-adjoint and compact in \( L^2(\Sigma; \ell^{2}_{K}) \). Its spectrum is

\[ \sigma(B^K_{\epsilon}) = \sigma(S_{\epsilon}) \cup \{0\}. \]
The convergence analysis of this sequence of extended operators $B^K$ is amenable to the two-scale convergence method in the sense of boundary layers (as introduced in the previous section). It turns out that the corresponding limit operator $B^K$ has a complicated form which can be considerably simplified by introducing the so-called Bloch wave decomposition. However, we emphasize that this decomposition will affect only the $(N - 1)$ first variables and not the last one, orthogonal to the boundary $\Sigma$.

**Lemma 2.10.** Given a family $(\vec{s}_j)_{0 \leq j' \leq K - 1}$ in $\ell^2_K$, there exists a unique family $(\vec{t}_j')_{0 \leq j' \leq K - 1}$ in $\ell^2_K$ such that, for any fixed $j_N$,  
\[
\sum_{0 \leq j' \leq K - 1} \vec{s}_j \chi_{Y_{j'}}(y') = \sum_{0 \leq j' \leq K - 1} \vec{t}_j' e^{2 \pi i \vec{t}_j' \cdot \vec{E}(y')},
\]
where $\vec{E}(\cdot)$ denotes the integer part function and $(Y_{j'})_{0 \leq j' \leq K - 1}$ is the family of subcells of $KY'$. Moreover, Parseval’s identity holds true; i.e., for any fixed $j_N$,  
\[
\sum_{0 \leq j' \leq K - 1} |\vec{s}_j|^2 = K^{N-1} \sum_{0 \leq j' \leq K - 1} |\vec{t}_j'|^2.
\]

The proof of Lemma 2.10 is standard (see, e.g., [1]). Remark that $\ell^2_K$ is isomorphic to $(\ell^2_1)^{K-1}$ by identifying an element $(\vec{s}_j)_{0 \leq j' \leq K - 1}$ of $\ell^2_K$ as a collection of $K^{N-1}$ elements $(\vec{s}_{j',j_N})_{j_N \geq 1}$ of $\ell^2_1$. Therefore, in Lemma 2.10, one could replace $\ell^2_K$ by $(\ell^2_1)^{K-1}$. Let us define a linear map $B'$
\[
B' : \ell^2_K \rightarrow (\ell^2_1)^{K-1},
\]
where the vectors $\vec{s}_j$ and $\vec{t}_j$ are related as in Lemma 2.10. This Bloch decomposition $B'$ (the prime indicates that it concerns only the first $(N - 1)$ variables) is easily seen to be an isometry from $\ell^2_K$ to $(\ell^2_1)^{K-1}$; namely, $(B')^* = (B')^{-1}$.

We are now in a position to state the main result on the asymptotic behavior of $B^K$.

**Theorem 2.11.** For each fixed $K \geq 1$, as $\epsilon$ goes to 0, the sequence $B^K$ converges strongly to a limit $B^K$ into $L^2(\Sigma; \ell^2_K)$; i.e., for any function $\vec{s}(x') \in L^2(\Sigma; \ell^2_K)$ we have  
\[
B^K \vec{s}(x') \longrightarrow B^K \vec{s}(x') \quad \text{in } L^2(\Sigma; \ell^2_K) \text{ strongly.}
\]

By using the Bloch decomposition $B'$ defined in (29), the operator $B^K$ can be diagonalized
\[
B^K = (B')^* D^K B' \quad \text{with} \quad D^K = \text{diag}(D^K_{j'})_{0 \leq j' \leq K - 1},
\]
where the entries $D^K_{j'}$ are self-adjoint continuous (but not compact) operators in $L^2(\Sigma; \ell^2_1)$ defined, for any $(\vec{s}_{j,N}(x'))_{j_N \geq 1} \in L^2(\Sigma; \ell^2_1)$, by
\[
D^K_{j'}(\vec{s}_{j,N}(x')) = \left( \int_{\Gamma_N} u_{j'} \vec{n} ds \right)_{j_N \geq 1},
\]
where \( u_j' (y) \) is the unique solution of

\[
\begin{aligned}
-\Delta_j u_j' &= 0 \quad \text{in } G^*, \\
\frac{\partial u_j'}{\partial n} &= \vec{s}_{jN} \cdot \vec{n} \quad \text{on } \Gamma_{jN}, \ j \geq 1, \\
u_j' &= 0 \quad \text{on } y_N = 0, \\
y' &\mapsto e^{-2\pi i \frac{x'}{N}} y' u_j' (y', y_N) \quad Y' - \text{periodic},
\end{aligned}
\]

where \( G^* \) is the fluid part of the semi-infinite band \( G \) (see Figure 2).

Remark 2.12. Of course, the solution \( u_j' \) of (30) depends also on the variable \( x' \in \Sigma \) since each displacement \( \vec{s}_{jN} (x') \) depends on \( x' \). Nevertheless, \( x' \) plays the role of a parameter, since (30) is a partial differential equation in the variable \( y \) only. The limit problem (30) admits a unique solution \( u_j' (x', y) \) in the space \( L^2 (\Sigma; D_{1,Y'}^1 (G^*)) \), where \( D_{1,Y'}^1 (G^*) \) is a Deny–Lions-type space. More precisely, it is defined as \( D_{1,Y'}^1 (G^*) \) in (21), the only difference being that functions in \( D_{1,Y'}^1 (G^*) \) satisfy a \( (e^{\pi i \frac{x'}{N}}, Y') \) periodicity condition in \( y' \), instead of the usual \( Y' \) periodicity. Recall that a function \( w(y) \) satisfying the periodicity condition of the limit problem (30) is said to be \( (e^{\pi i \frac{x'}{N}}, Y') \)-periodic in \( y' \) because such a function also satisfies the following (generalized) periodicity condition:

\[
w(y + (k', 0)) = e^{2\pi i \frac{k'}{N}} w(y) \quad \forall y = (y', y_N) \text{ and } \forall k' \in \mathbb{Z}^{N-1}.
\]

For more details on this class of functions, we refer to [1], [16].

The key of the proof of Theorem 2.11 is the following homogenization result for the fluid potential when the displacements of the tubes are given in terms of the

\[
\begin{aligned}
\frac{\partial u_j'}{\partial n} &= \vec{\bar{s}}_{jN} \cdot \vec{n} \quad \text{on } \Gamma_{jN}, \ j \geq 1, \\
u_j' &= 0 \quad \text{on } y_N = 0, \\
y' &\mapsto e^{-2\pi i \frac{x'}{N}} y' u_j' (y', y_N) \quad Y' - \text{periodic},
\end{aligned}
\]

and \( y' \) two-scale converges strongly, i.e.,

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega_\epsilon} |\nabla u_\epsilon|^2 \, dx = \frac{1}{|K_Y'|} \int_{\Sigma} \int_{G\epsilon} |\nabla u_0|^2 \, dx' \, dy.
\]

Moreover, if \( \vec{s}' (x') \) is a sequence which converges weakly to a limit \( \vec{s} (x') \) in \( L^2 (\Sigma; \ell_2^K) \), then the sequence of associated solutions \( u_\epsilon (\vec{s}) \) two-scale converges in the sense of boundary layers to \( 0 \) and \( \nabla u_\epsilon (\vec{s}) \) two-scale converges in the sense of boundary layers to \( \nabla u_0 (x', y) \), where \( u_0 \) is still the solution of (32).
Remark 2.14. A priori, the solution \( u_\epsilon \) of (31) is defined only in the fluid domain \( \Omega_\epsilon \) which is a varying set as \( \epsilon \) goes to 0. However, it is a standard matter (see [13]) to build an extension operator \( X_\epsilon \) acting from \( H^1(\Omega_\epsilon) \) into \( H^1(\Omega) \) such that, for any \( v \in H^1(\Omega_\epsilon) \),

\[
X_\epsilon v = v \text{ in } \Omega_\epsilon \text{ and } \|X_\epsilon v\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega_\epsilon)},
\]

where \( C \) is a positive constant independent of \( \epsilon \). In what follows, we shall always identify functions in \( H^1(\Omega_\epsilon) \) (as \( u_\epsilon \)) with their extension in \( H^1(\Omega) \) (as \( X_\epsilon u_\epsilon \)).

To prove Proposition 2.13 we need two technical lemmas.

Lemma 2.15. The extension and projection operators \( E^K_\epsilon \) and \( P^K_\epsilon \) satisfy the following estimates:

(i) \( \|P^K_\epsilon \tilde{s}(x')\|_{\mathbb{R}^{N_\epsilon(\epsilon)}} \leq C \epsilon^{-\frac{p-1}{2}} \|\tilde{s}(x')\|_{L^p(\Sigma; \ell^2_K)} \),

(ii) \( \|E^K_\epsilon (s_p)\|_{L^2(\Sigma; \ell^2_K)} \leq C \epsilon^{\frac{p-1}{2}} \|s_p\|_{1 \leq p \leq N(\epsilon)} \|\mathbb{R}^{N_\epsilon(\epsilon)} \),

where \( C \) is a constant independent of \( \epsilon \) and the norms are defined by

\[
\|s_p\|_{1 \leq p \leq N(\epsilon)} = \sum_{1 \leq p \leq N(\epsilon)} |s_p|^2,
\]

\[
\|s(x')\|_{L^2(\Sigma; \ell^2_K)} = \int_{\Sigma} \sum_{0 \leq j \leq K-1} |\tilde{s}_j(x')|^2 dx'.
\]

Proof. Let us prove (i) (the other inequality (ii) has a similar proof). By definition of \( P^K_\epsilon \),

\[
\|P^K_\epsilon \tilde{s}(x')\|_{\mathbb{R}^{N_\epsilon(\epsilon)}}^2 = \sum_{1 \leq p \leq N(\epsilon)} \left( \frac{1}{|\epsilon^{KY}|} \int_{(\epsilon^{KY})_{\ell'}} \tilde{s}_j(x') dx' \right)^2,
\]

where \( (p, j, \ell') \) are related by formula (26). Applying the Cauchy–Schwarz inequality and summing over \( \ell' \) yield

\[
\|P^K_\epsilon \tilde{s}(x')\|_{\mathbb{R}^{N_\epsilon(\epsilon)}}^2 \leq \sum_{1 \leq p \leq N(\epsilon)} \frac{1}{|\epsilon^{KY}|} \int_{(\epsilon^{KY})_{\ell'}} |\tilde{s}_j(x')|^2 dx' \leq \frac{1}{|\epsilon^{KY}|} \int_{\Sigma} \sum_j |\tilde{s}_j(x')|^2 dx',
\]

which is the desired result.

Lemma 2.16. Let \( \tilde{s}(x') \) be a sequence of functions which converges weakly to \( s(x') \) in \( L^2(\Sigma; \ell^2_K) \). Define a piecewise constant function

\[
\bar{a}^\epsilon(x) = \sum_{\ell'} \sum_j \left( \frac{1}{|\epsilon^{KY}|} \int_{(\epsilon^{KY})_{\ell'}} \tilde{s}_j(x') dx' \right) \chi_{\gamma_{j\ell'}}(x),
\]

where \( \chi_{\gamma_{j\ell'}}(x) \) is the characteristic function of the \( j \)th subcell of the periodic cell \( G^K_{\epsilon,\ell'} \). Then, \( \bar{a}^\epsilon \) two-scale converges in the sense of boundary layers to a limit \( \bar{a}^0(x, y) \in L^2(\Sigma \times G^K_{\epsilon,\ell'}) \) defined by

\[
\bar{a}^0(x, y) = \sum_j \tilde{s}_j(x') \chi_{\gamma_j}(y),
\]
where \( \chi_{Y_j}(y) \) is the characteristic function of the jth subcell of the reference cell \( G^K \). Moreover, if \( \bar{s}'(x') \) converges strongly to \( \bar{s}(x') \) in \( L^2(\Sigma; \mathbb{E}^2_K) \), then \( \bar{a}^t \) two-scale converges strongly to \( \bar{a}^0 \) in the sense of boundary layers, i.e.,

\[
\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \| \bar{a}^t(x) \|_{L^2(\Omega)} = \frac{1}{K \sqrt{\epsilon}} \| \bar{a}^0(x', y) \|_{L^2(\Sigma \times G^K)}.
\]

**Proof.** The proof is very similar to that of Lemma 3.3.2 in our previous work [3], so we briefly sketch it. Let \( \varphi(x', y) \) be a suitable smooth test function defined on \( \Sigma \times G^K \) with values in \( \mathbb{R}^N \) such that \( y' \to \varphi(x', y', y_N) \) is \( KY' \)-periodic and \( \varphi \) vanishes for sufficiently large \( y_N \). We check the definition of two-scale convergence:

\[
\frac{1}{\epsilon} \int_{\Omega} \bar{a}^t(x) \cdot \varphi(x', \frac{x}{\epsilon}) dx = \frac{1}{\epsilon} \sum_{j}(\frac{1}{|K|N}) \int_{\epsilon(KY')} \bar{s}^j(x') dx' \cdot \int_{\epsilon(KY')} \varphi(x', \frac{x}{\epsilon}) dx
\]

\[
= \frac{1}{K^N} \sum_{j} \sum_{\ell} \int_{\Sigma} \bar{s}^j(x') \cdot \sum_{p} \left( \frac{1}{|\epsilon \Gamma_p|} \int_{\epsilon(KY')} \varphi(x', \frac{x}{\epsilon}) dx \right) \chi_{\ell(KY')} (x') \) dx'.
\]

It is easily seen that for each fixed \( j \) the term between brackets converges strongly to \( \int_{Y_j} \varphi(x', y) dy \) in \( L^2(\Sigma)^N \). Remark that the sum in \( j \) is finite since \( \varphi \) has a bounded support in \( G^K \). Thus we can pass to the limit and obtain the desired result

\[
\frac{1}{K^N} \sum_{j} \int_{\Sigma} \bar{s}^j(x') \cdot \left( \int_{Y_j} \varphi(x', y) dy \right) \right) dx'.
\]

If \( \bar{s}^j \) converges strongly to \( \bar{s}_j \), the strong two-scale convergence of \( \bar{a}^t(x) \) is obtained by a similar proof, replacing in the above computation the test function \( \varphi \) by \( \bar{a}^t(x) \).

**Proof of Proposition 2.13.** Multiplying (31) by \( u_\epsilon \) and integrating by parts, we get

\[
\int_{\Omega} \| \nabla u_\epsilon \|^2 dx = \sum_{1 \leq p \leq n(\epsilon)} \left( P^K_p \bar{s}_p \right) \cdot \int_{\Gamma_p} u_\epsilon \tilde{u} ds
\]

\[
\leq \| \left( P^K_p \bar{s} \right) \|_{L^2(\Omega)} \| \left( \int_{\Gamma_p} u_\epsilon \tilde{u} ds \right) \|_{L^2(\Omega)}.
\]

An easy calculation (see Lemma 2.2.3 in [3] if necessary) shows that

\[
\left\| \left( \int_{\Gamma_p} u_\epsilon \tilde{u} ds \right) \right\|_{L^2(\Omega)}^2 \leq C \epsilon^N \| \nabla u_\epsilon \|_{L^2(\Omega)}^2,
\]

and hence, using Lemma 2.15 we conclude that

\[
\int_{\Omega} \| \nabla u_\epsilon \|^2 dx \leq C \epsilon \| \bar{s}(x') \|^2_{L^2(\Sigma; \mathbb{E}^2_K)}.
\]

A standard Poincaré inequality in \( \Omega \) yields the same estimate for \( u_\epsilon \) in \( L^2(\Omega) \):

\[
\int_{\Omega} |u_\epsilon|^2 dx \leq C \epsilon \| \bar{s}(x') \|^2_{L^2(\Sigma; \mathbb{E}^2_K)}.
\]
We now apply the method of two-scale convergence for the asymptotic analysis of the sequence $u_\epsilon$, using test functions with $G^K$ as the periodic cell (since we decided to consider $G^K$ to be the reference cell and not $G$). By virtue of Proposition 2.6 there exists a subsequence of $u_\epsilon$ and a limit function $u_0(x', y)$ in $L^2(\Sigma; D^1_{0\#}(G^K))$ such that $(u_\epsilon, \nabla u_\epsilon)$ two-scale converge in the sense of boundary layers to $(0, \nabla u_0)$. Let $\varphi(x', y)$ be a smooth function in $L^2(\Sigma; D^1_{0\#}(G^K))$. Multiplying the equation (31) by $\varphi(x', \frac{x}{\epsilon})$ we obtain

$$
\frac{1}{\epsilon} \int_\Omega \chi_{\Omega_\epsilon}(x) \nabla u_\epsilon \cdot \nabla_y \varphi \left(x', \frac{x}{\epsilon}\right) dx + \int_\Omega \chi_{\Omega_\epsilon}(x) \nabla u_\epsilon \nabla_x \varphi \left(x', \frac{x}{\epsilon}\right) dx
$$

$$
= \sum_{1 \leq p \leq n(\epsilon)} (P^K_{\epsilon} \tilde{s}_p) \int_{\Gamma_p} \varphi \left(x', \frac{x}{\epsilon}\right) \tilde{\nu} ds
$$

$$
= \frac{1}{\epsilon} \int_\Omega (\chi_{\Omega_\epsilon}(x) - 1) \tilde{a}(x) \cdot \left(\nabla_y \varphi \left(x', \frac{x}{\epsilon}\right) + \epsilon \nabla_x \varphi \left(x', \frac{x}{\epsilon}\right)\right) dx,
$$

where $\chi_{\Omega_\epsilon}(x)$ is the periodic characteristic function of $\Omega_\epsilon$ and $\tilde{a}(x)$ is a piecewise constant function defined as in Lemma 2.16 by

$$
\tilde{a}^e = \sum_{e'} \sum_j \left(-\frac{1}{|KY'|} \int_{(\epsilon KY')_{e'}} \tilde{s}_j(x') dx'\right) \chi_{Y_{e'}^j}(x).
$$

Remark that both terms involving $\nabla_x \varphi$ go to zero with $\epsilon$. Applying Lemma 2.16, we pass to the two-scale limit in the remaining terms to get

$$
\frac{1}{|KY'|} \int_{\Sigma \setminus (G^{*K})} \nabla_y u_0(x', y) \cdot \nabla_y \varphi(x', y) dx' dy = \frac{-1}{|KY'|} \sum_j \int_{\tilde{T}_j} \tilde{s}_j(x') \cdot \nabla_y \varphi(x', y) dx' dy
$$

which is nothing but the variational formulation of the limit equation (32). A standard application of the Lax–Milgram lemma yields uniqueness of the solution $u_0$ in $L^2(\Sigma; D^1_{0\#}(G^K))$. Thus the entire sequence $u_\epsilon$ converges to the same limit $u_0$.

The proof of the energy convergence (33) is standard by passing to the two-scale limit in the right-hand side of (35) since $\tilde{a}^e$ two-scale converges strongly in the sense of Proposition 2.5 (see Proposition 2.2.4 in [3]).

To prove the two-scale convergence of $u_\epsilon (\tilde{s}^e)$ to $u_0$, when $\tilde{s}^e$ converges weakly to $\tilde{s}$ in $L^2(\Sigma; \ell^2_K)$, it suffices to repeat the same above arguments since Lemma 2.16 asserts that $\tilde{a}^e$ two-scale converges to $\tilde{a}^0$ even if $\tilde{s}^e$ converges weakly. Note that in this case we do not have the energy convergence.

**Proof of Theorem 2.11.** Let $\tilde{s}(x') \in L^2(\Sigma; \ell^2_K)$ and $\tilde{t}^e$ be a sequence which converges weakly to $\tilde{t}$ in $L^2(\Sigma; \ell^2_K)$. Our goal is to prove that

$$
\lim_{\epsilon \to 0} \langle B^K_{\epsilon} \tilde{s}(x'), \tilde{t}^e(x') \rangle_{L^2(\Sigma; \ell^2_K)} = \langle B^K \tilde{s}(x'), \tilde{t}(x') \rangle_{L^2(\Sigma; \ell^2_K)}.
$$

The proof of the energy convergence (33) is standard by passing to the two-scale limit in the right-hand side of (35) since $\tilde{a}^e$ two-scale converges strongly in the sense of Proposition 2.5 (see Proposition 2.2.4 in [3]).

To prove the two-scale convergence of $u_\epsilon (\tilde{s}^e)$ to $u_0$, when $\tilde{s}^e$ converges weakly to $\tilde{s}$ in $L^2(\Sigma; \ell^2_K)$, it suffices to repeat the same above arguments since Lemma 2.16 asserts that $\tilde{a}^e$ two-scale converges to $\tilde{a}^0$ even if $\tilde{s}^e$ converges weakly. Note that in this case we do not have the energy convergence.

**Proof of Theorem 2.11.** Let $\tilde{s}(x') \in L^2(\Sigma; \ell^2_K)$ and $\tilde{t}^e$ be a sequence which converges weakly to $\tilde{t}$ in $L^2(\Sigma; \ell^2_K)$. Our goal is to prove that

$$
\lim_{\epsilon \to 0} \langle B^K_{\epsilon} \tilde{s}(x'), \tilde{t}^e(x') \rangle_{L^2(\Sigma; \ell^2_K)} = \langle B^K \tilde{s}(x'), \tilde{t}(x') \rangle_{L^2(\Sigma; \ell^2_K)}.
$$
By definition of $B^K$, we have

$$
\langle B^K_\epsilon \tilde{s}(x'), \tilde{t}(x') \rangle_{L^2(\Sigma, \mathcal{F}_K)} = \langle E^K_\epsilon S_\epsilon P^K_\epsilon \tilde{s}(x'), \tilde{t}(x') \rangle_{L^2(\Sigma, \mathcal{F}_K)}
$$

$$
= (\epsilon K)^{N-1} \langle S_\epsilon P^K_\epsilon \tilde{s}(x'), P^K_\epsilon \tilde{t}(x') \rangle_{\mathbb{R}^{N\times n}}
$$

$$
= (\epsilon K)^{N-1} \sum_{1 \leq p \leq n} \frac{1}{\epsilon^2} (\int_{\Gamma_p} u_\epsilon(\tilde{s}) \tilde{t}ds) \cdot (P^K_\epsilon \tilde{t})_p
$$

$$
= \frac{K^{N-1}}{\epsilon} \int_{\Omega_\epsilon} \nabla u_\epsilon(\tilde{s}) \cdot \nabla u_\epsilon(\tilde{t}) dx.
$$

By Proposition 2.13 we know that $\nabla u_\epsilon(\tilde{s})$ two-scale converges strongly in the sense of boundary layers to $\nabla u_0(\tilde{s})$ while $\nabla u_\epsilon(\tilde{t})$ two-scale converges weakly to $\nabla u_0(\tilde{t})$. By virtue of Proposition 2.5 we can pass to the limit in the product and we get

$$
\lim_{\epsilon \to 0} \langle B^K_\epsilon \tilde{s}(x'), \tilde{t}(x') \rangle_{L^2(\Sigma, \mathcal{F}_K)} = \int_\Sigma \int_{G^*K} \nabla u_0(\tilde{s}) \cdot \nabla u_0(\tilde{t}) dx'dy,
$$

where $u_0(\tilde{s})$ and $u_0(\tilde{t})$ are solutions of the homogenized problem (32) with $\tilde{s}$ and $\tilde{t}$, respectively, as the right-hand side. A simple integration by parts shows that

$$
\int_\Sigma \int_{G^*K} \nabla u_0(\tilde{s}) \cdot \nabla u_0(\tilde{t}) dx'dy = \langle B^K \tilde{s}(x'), \tilde{t}(x') \rangle_{L^2(\Sigma, \mathcal{F}_K)},
$$

where the limit operator $B^K$ is defined by

$$
(36) \quad B^K \tilde{s}(x') = \left( \int_{\Gamma_j} u_0(\tilde{s}) \tilde{t}ds \right)_{0 \leq j' \leq K-1, 1 \leq j \leq N}.
$$

This proves the strong convergence of $B^K_\epsilon$ to $B^K$ on $L^2(\Sigma, \mathcal{F}_K)$. Obviously, $B^K$ is self-adjoint and continuous but not compact since $x'$ plays the role of a parameter in the definition of $B^K$.

It remains to diagonalize $B^K$ with the help of the Bloch decomposition $B'$. This diagonalization process has already been exposed in section 3.3 of our previous paper [3] in a slightly different context. For the sake of brevity, we do not repeat this standard argument here. Let us simply indicate the three main steps of this Bloch diagonalization. First, we apply the operator $B'$ to $\tilde{s}(x') = (\tilde{s}_j(x'))_{0 \leq j' \leq K-1}$ which gives the Bloch decomposition of $\tilde{s}(x')$ with respect to the multi-index $j'$ (not including $j_N$). Secondly, plugging this Bloch decomposition in the limit equation (32) (which holds in $G^*K$) and using a similar Bloch decomposition of $u_0(\tilde{s})$, we decompose (32) in a family of $K^{N-1}$ equations defined in a single reference cell $G^*$. In a third step, applying again the Bloch decomposition $B'$ to formula (36) yields the desired diagonalization of $B^K$.

2.3. Analysis of the limit spectrum. In this section we analyze the spectrum of the limit operator $B^K$ and, from the strong convergence of $B^K_\epsilon$ to $B^K$, we deduce the lower semicontinuous convergence of the spectrum $\sigma(S_\epsilon)$ to the limit spectrum $\sigma(B^K)$. Recall that for any $K \geq 1$, the extended operator $B^K_\epsilon$ has a spectrum given by

$$
\sigma(B^K_\epsilon) = \sigma(S_\epsilon) \cup \{0\}.
$$
Since $B^K$ converges strongly to $B^K$ in $L^2(\Sigma; \ell_2^2)$, by virtue of Proposition 2.1.11 in [3], we have

$$\sigma(B^K) \subset \sigma_\infty = \lim_{\epsilon \to 0} \sigma(S_\epsilon).$$

From Rellich’s theorem, the strong convergence of the spectral family associated with $B^K_\epsilon$ to that of $B^K$ is also easily deduced (see Theorem 3.2.5 in [3]). This gives some (partial) information on the convergence of eigenvectors that we shall not use below.

In view of Theorem 2.11, $B^K = (B')^{-1} D^K B'$ with $D^K = \text{diag}(D^K_{j'})_{0 \leq j' \leq K-1}$, where each $D^K_{j'}$ is a self-adjoint continuous operator in $L^2(\Sigma; \ell_2^1)$. Since $B'$ is an isometry, we have

$$\sigma(B^K) = \bigcup_{0 \leq j' \leq K-1} \sigma(D^K_{j'}).$$

By the very definition of $D^K_{j'}$, the macroscopic variable $x' \in \Sigma$ plays the role of a parameter. Therefore, for any fixed value of $x'$, $D^K_{j'}$ can be identified with an operator $d_{j'}$ acting in $\ell_2^1$ which does not depend on $x'$. Introducing the Bloch parameter $\theta' = \frac{j'}{\pi} \in [0,1]^{N-1}$, this new operator $d_{\theta'}$ is defined by

$$d_{\theta'} : \ell_2^1 \longrightarrow \ell_2^1,$$

$$(s_q)_{q \geq 1} \mapsto \left(\int_{\Gamma_q} u_{\theta'} \cdot \vec{n} ds\right)_{q \geq 1},$$

(37)

where $u_{\theta'}(y)$ is the unique solution of

$$\begin{cases}
-\Delta u_{\theta'} = 0 & \text{in } G', \\
\partial_n u_{\theta'} = \vec{s}_q \cdot \vec{n} & \text{on } \Gamma_q, \ q \geq 1, \\
u_{\theta'} = 0 & \text{if } y_N = 0, \\
y' \mapsto e^{-2\pi i \theta' \cdot y'} u_{\theta'}(y', y_N) & \text{Y'-periodic.}
\end{cases}$$

In (37) the positive integer $q$ is nothing but the index $j_N$ introduced in Definition 2.8. Clearly, we have

$$\sigma(D^K_{j'}) = \sigma(d_{\theta'}).$$

As is well known, the spectrum of a self-adjoint operator can be decomposed in its discrete part, made of, at most, a countable number of isolated eigenvalues of finite multiplicities, and its essential part, for which the Weyl criterion applies (see, e.g., [25], [33], [34]). The next proposition characterizes the spectrum of $d_{\theta'}$.

PROPOSITION 2.17. For all $\theta' \in [0,1]^{N-1}$, $d_{\theta'}$ is a self-adjoint continuous but noncompact operator in $\ell_2^1$. Labeling the eigenvalues of the discrete spectrum $\sigma_{\text{disc}}(d_{\theta'})$ by decreasing order, each discrete eigenvalue is piecewise continuous in $\theta'$. The essential spectrum is given by

$$\sigma_{\text{ess}}(d_{\theta'}) = \bigcup_{\theta_N \in [0,1]} \sigma(A(\theta', \theta_N)), $$
where \( A(\theta', \theta_N) \) is the Bloch homogenized matrix, defined by (12), which is continuous in \( \theta \in [0,1]^N \) but discontinuous at \( \theta = 0 \). Moreover, the entire spectrum \( \sigma(d_{\theta'}) \), considered as a subset of \( \mathbb{R}^+ \), depends continuously on \( \theta' \), except at \( \theta' = 0 \).

Because we use the usual convenient labeling of the discrete eigenvalues by decreasing order, we can merely prove that they are piecewise continuous. This is due to the fact that, when \( \theta' \) varies, an analytical branch (if any) of discrete eigenvalues may merge into the essential spectrum: this yields a “jump” in the labeling of discrete eigenvalues. Therefore, one cannot hope to prove a global continuity of these eigenvalues with such an ordering.

Let us postpone for a moment the proof of Proposition 2.17 and define the so-called boundary layer spectrum associated with the surface \( \Sigma \):

\begin{equation}
\sigma_{\Sigma} \overset{\text{def}}{=} \bigcup_{\theta' \in [0,1]^{N-1}} \sigma(d_{\theta'}) \cup \sigma(d_0) .
\end{equation}

By virtue of Proposition 2.17, we have

\begin{equation}
\sigma_{\text{Bloch}} \subset \sigma_{\Sigma} .
\end{equation}

Therefore \( \sigma_{\Sigma} \) also has a band structure since it includes the Bloch spectrum, but it may include new bands of eigenvalues of \( \sigma_{\text{disc}}(d_{\theta'}) \). It also contains the isolated eigenvalues of \( \sigma_{\text{disc}}(d_0) \). Therefore \( \sigma_{\Sigma} \) can contain elements which are not included in the previous limit spectrum \( \sigma(S) \cup \sigma_{\text{Bloch}} \) (see section 1.2). The continuity of \( \sigma(d_{\theta'}) \) with respect to \( \theta' \) ensures that \( \sigma_{\Sigma} \) is the closure of the union of all spectra \( \sigma(d_{\theta'}) \) with \( \theta' \) rational.

\[ \bigcup_{K \geq 1} \bigcup_{0 \leq j' \leq K-1} \sigma(d_{j'K}) = \sigma_{\Sigma} . \]

We summarize our results in the following theorem.

**Theorem 2.18.** The boundary layer spectrum associated to \( \Sigma \) is included in the limit spectrum

\[ \sigma_{\Sigma} \subset \sigma_{\infty} . \]

**Remark 2.19.** Of course \( \sigma_{\Sigma} \) is not the complete boundary layer spectrum since it is concerned only with that part of the spectrum concentrating near \( \Sigma \). A completely similar analysis has to be done for all the \((N-1)\)-dimensional surfaces and all other lower dimensional manifolds (edges, corners, etc.) of which the boundary of \( \Omega \) is made up. Then, we shall prove in the next section that the union of all these contributions, the so-called boundary layer spectrum, plus the usual homogenized spectrum and the Bloch spectrum, is equal to \( \sigma_{\infty} \), at least when \( \Omega \) is made up only of entire cells \( \epsilon Y \).

**Proof of Proposition 2.17.** Let us first prove that the essential spectrum of \( d_{\theta'} \) is included in the Bloch spectrum, and, more precisely,

\[ \sigma_{\text{ess}}(d_{\theta'}) = \bigcup_{0 \leq \theta_N \leq 1} \sigma(A(\theta', \theta_N)) , \]

where \( A(\theta) \) is the usual Bloch homogenized matrix defined in (12). In particular, this proves that \( \sigma_{\text{ess}}(d_{\theta'}) \neq \{0\} \), so \( d_{\theta'} \) is not compact.
Let $\lambda(\theta)$ be an eigenvalue of $A(\theta)$ and $u(\theta)$ be the associated potential solution of

$$
\begin{cases}
-\Delta u(\theta) = 0 & \text{in } Y^*, \\
\frac{\partial u(\theta)}{\partial n} = \lambda^{-1}(\theta) \int_{\Gamma} u(\theta) \bar{n} ds & \text{on } \Gamma, \\
y \mapsto e^{-2\pi i \theta \cdot y} u(\theta, y) & \text{in } Y.-\text{periodic}.
\end{cases}
$$

We construct a Weyl sequence $u_n$ associated with the spectral value $\lambda(\theta)$ by

$$u_n = \frac{u(\theta) \psi_n}{\|u(\theta) \psi_n\|_{L^2(G^*)}},$$

where $\psi_n(y_N)$ is a cut-off function defined by

$$\begin{align*}
\psi_n(y_N) &= y_N & \text{when } 0 \leq y_N \leq 1, \\
\psi_n(y_N) &= 1 & \text{when } 1 \leq y_N \leq n, \\
\psi_n(y_N) &= n + 1 - y_N & \text{when } n \leq y_N \leq n + 1, \\
\psi_n(y_N) &= 0 & \text{when } y_N \geq n + 1.
\end{align*}$$

By definition, $\|u_n\|_{L^2(G^*)} = 1$ and $\lim_{n \to +\infty} \|u(\theta) \psi_n\|_{L^2(G^*)} = +\infty$. Then, it is easily checked that, for any $\varphi \in D_{a\#}(G)$ (the Deny–Lions-type space defined in (21)),

$$\int_{G^*} \nabla u_n \cdot \nabla \varphi dy = \frac{1}{\lambda(\theta)} \sum_{q \geq 1} \left( \int_{\Gamma_q} u_n \bar{n} ds \right) \cdot \left( \int_{\Gamma_q} \varphi \bar{n} ds \right) + \langle r_n, \varphi \rangle,$$

where $r_n$ is a negligible remainder term in the sense that

$$\lim_{n \to +\infty} \|r_n \cdot \varphi\|_{L^2(G^*)} = 0.$$

Furthermore, $\bar{s}_n = (\int_{\Gamma_q} u_n \bar{n} ds)_{q \geq 1}$ converges weakly to 0 in $\ell_1^2$ since

$$\lim_{n \to +\infty} \|u(\theta) \psi_n\|_{L^2(G^*)} = +\infty.$$

Therefore, $\bar{s}_n$ is a Weyl sequence associated with $\lambda(\theta)$ for the operator $d_\theta'$. This proves that $\lambda(\theta) \in \sigma_{ess}(d_\theta')$. To prove the converse inclusion,

$$\sigma_{ess}(d_\theta') \subset \bigcup_{0 \leq \theta_N \leq 1} \sigma(A(\theta', \theta_N)),$$

we consider a Weyl sequence $\bar{s}_n$ for a spectral value $\lambda \in \sigma_{ess}(d_\theta')$. Let $u_n$ be the associated potential solution, i.e.,

$$
\begin{cases}
-\Delta u_n = 0 & \text{in } G^*, \\
\frac{\partial u_n}{\partial n} = (\bar{s}_n)_q \cdot \bar{n} & \text{on } \Gamma_q, \ q \geq 0, \\
u_n = 0 & \text{if } y_N = 0, \\
y' \mapsto e^{-2\pi i \theta' \cdot y'} u_n(y', y_N) & \text{in } Y'-\text{periodic}
\end{cases}
$$

(40)

Since $\|\bar{s}_n\|_{\ell_1^2} = 1$ and $\bar{s}_n \to 0$ in $\ell_1^2$ weakly, it is easily seen that $u_n$ converges to 0 weakly in $H^1(G^*)$. Furthermore, since the weak convergence to 0 of $\bar{s}_n$ implies that its components $\bar{s}_n^q \to 0$ for fixed $q$, it is not difficult to check that, for any compact set $K$ of $G^*$, $u_n$ converges strongly to 0 in $H^1(K)$ (multiply equation (40).
by \( \phi u_n \) where \( \phi \) is equal to 1 in \( K \) and is compactly supported away from infinity). Introducing a sequence

\[ v_n = \frac{\psi u_n}{\|\psi u_n\|_{L^2(G^*)}}, \]

where \( \psi(y_N) \) is a cut-off function defined by

\[
\begin{align*}
\psi(y_N) &= 0 \quad \text{for } y_N \leq 0, \\
\psi(y_N) &= y_N \quad \text{for } 0 \leq y_N \leq 1, \\
\psi(y_N) &= 1 \quad \text{for } y_N \geq 1,
\end{align*}
\]

it is straightforward to prove that

\[
\int_{B^*} \nabla v_n \cdot \nabla \varphi dx = \frac{1}{\lambda} \sum_{q \in \mathbb{Z}} \left( \int_{\Gamma_q} v_n \bar{r} ds \right) \cdot \left( \int_{\Gamma_q} \varphi \bar{r} ds \right) + \langle r_n, \varphi \rangle
\]

for any \( \varphi \in L^2(B^*) \), where \( B^* \) is the infinite band \( Y' \times ]-\infty; +\infty[ \) perforated by the periodic arrangement of tubes \( (T_q)_{q \in \mathbb{Z}} \), and \( r_n \) is another negligible remainder term such that

\[
\lim_{n \to +\infty} \frac{\langle r_n, \varphi \rangle}{\|\nabla \varphi\|_{L^2(B^*)}} = 0.
\]

Therefore,

\[
\tilde{r}_n = \left( \int_{\Gamma_q} v_n \bar{r} ds \right)_{q \in \mathbb{Z}}
\]

is a Weyl sequence for an operator similar to \( d_{\theta'} \) but defined in the whole infinite band \( B^* \) instead of the semi-infinite band \( G^* \). A standard Bloch decomposition with respect to the variable \( y_N \) yields that \( \lambda \) belongs to \( \bigcup_{0 \leq y_N \leq 1} \sigma(A(\theta', y_N)) \).

To conclude the proof of Proposition 2.17, it remains to prove that the isolated eigenvalues of finite multiplicity \( \lambda(\theta') \in \sigma_{\text{disc}}(d_{\theta'}) \) are piecewise continuous with respect to \( \theta' \). Let \( \theta'_n \) be a sequence converging to \( \theta' \) in \( ]0, 1[^{N-1} \). Obviously, the sequence of continuous operators \( d_{\theta'_n} \) uniformly converges to \( d_{\theta'} \) in \( \ell^2(\Gamma) \). Now, let us invoke a classical theorem (see, e.g., Theorem 3.1., Chapter I.3 in [20]) which states that for any closed curve \( \gamma \) in the complex plane, which encloses a finite number of eigenvalues of \( \sigma_{\text{disc}}(d_{\theta'}) \) and does not intersect \( \sigma(d_{\theta'}) \), there exists \( n_0 \) such that for any \( n \geq n_0 \), the curve \( \gamma \) contains the same number of eigenvalues (including multiplicities) of \( \sigma_{\text{disc}}(d_{\theta'}) \) and does not intersect \( \sigma(d_{\theta'_n}) \). This is nothing but the local continuity of the eigenvalues of \( \sigma_{\text{disc}}(d_{\theta'}) \) (enumerated, for example, in decreasing order). Remark that the continuity of the \( p \)th eigenvalue of \( \sigma_{\text{disc}}(d_{\theta'}) \) breaks down only when one of the previous eigenvalues (with label between 1 and \( p-1 \)) meets the essential spectrum \( \sigma_{\text{ess}}(d_{\theta'}) \) as \( \theta' \) varies. In any case, since \( \sigma_{\text{ess}}(d_{\theta'}) \) depends continuously on \( \theta' \neq 0 \), this proves that the entire spectrum \( \sigma(d_{\theta'}) \) depends also continuously on \( \theta' \neq 0 \). The lack of continuity for \( \sigma(d_{\theta'}) \) at \( \theta' = 0 \) is a phenomenon already explained in our previous work (see Proposition 3.3.4 in [3]).

Remark 2.20. When the tube \( T \) is symmetric in \( Y \) (in other words, by reflexion with respect to the hyperplane \( y_N = 0 \)), \( G^* \) yields the infinite periodic array of tubes \( B^* \), it can readily be checked that there is no isolated eigenvalue of finite multiplicity.
for $d_{\theta'}$: i.e., $\sigma_{\text{disc}}(d_{\theta'}) = 0$ for all $\theta' \in [0,1]^{N-1}$. If this were not the case, by symmetry an eigenvalue of $\sigma_{\text{disc}}(d_{\theta'})$ would also be an eigenvalue of finite multiplicity for a similar operator in the infinite band $B^*$, which is impossible since by translation there exists an infinite number of eigenvectors.

We conclude this section by proving that the eigenvectors corresponding to isolated eigenvalues of finite multiplicity of $d_{\theta'}$ are localized in the vicinity of the boundary $[y_N = 0]$ since they decay exponentially at infinity.

**Proposition 2.21.** Let $\lambda$ be an eigenvalue in $\sigma_{\text{disc}}(d_{\theta'})$ and let $(\hat{s}_q)_{q \geq 1}$ be a corresponding eigenvector. There exists a positive constant $\alpha > 0$ such that $(e^{\alpha q} \hat{s}_q)_{q \geq 1}$ belongs to $l_1^2$.

**Proof.** The argument is by contradiction of the Weyl property for eigenvalues in the essential spectrum. For $\lambda \in \sigma_{\text{disc}}(d_{\theta'})$, let $\hat{s} = (\hat{s}_q)_{q \geq 1}$ be a corresponding normalized eigenvector and $u(y)$ the corresponding potential, solution of

$$
\begin{align*}
-\Delta u &= 0 & \text{in } G^*, \\
\frac{\partial u}{\partial n} &= \hat{s}_q \cdot \bar{n} & \text{on } \Gamma_q, \ q \geq 1, \\
u &= 0 & \text{if } y_N = 0, \\
y' \mapsto e^{-2\pi \theta' \cdot y'} u(y',y_N) & \text{ } Y' \text{-periodic.}
\end{align*}
$$

(41)

By definition, for all $q \geq 1$, it satisfies

$$
\int_{\Gamma_q} u \cdot \bar{n} ds = \lambda \hat{s}_q.
$$

Let us define a sequence $(\hat{s}^n)_n \geq 0$ in $l_1^2$ by

$$
\hat{s}^n = (\hat{s}^n_q)_{q \geq 1} \text{ with } \hat{s}^n_q = \begin{cases} 0 & \text{if } q < n, \\ \frac{\hat{s}_q}{\sqrt{\sum_{p=n}^{\infty} |\hat{s}_p|^2}} & \text{if } q \geq n.
\end{cases}
$$

It is easily seen that $\hat{s}^n$ converges weakly to $0$ in $l_1^2$ with $\|\hat{s}^n\|_{l_2^2} = 1$. However, since $\lambda$ does not belong to the essential spectrum of $d_{\theta'}$, any subsequence of $\hat{s}^n$ cannot be a Weyl sequence for $\lambda$. This implies the existence of a positive constant $C$ and an integer $n_0$ such that, for any $n \geq n_0$,

$$
\|d_{\theta'} \hat{s}^n - \lambda \hat{s}^n\|_{l_1^2} \geq C > 0.
$$

(42)

As usual $u_n(y)$ is the potential associated with $\hat{s}^n$ through an equation similar to (41). We introduce a smooth cut-off function $\psi_n(y_N)$ such that $\psi_n = 0$ on all tubes $T_q$ for $q < n$, and $\psi_n = 1$ on all tubes $T_q$ for $q \geq n$. Let us denote by $\omega_n$ the bounded support of $\nabla \psi_n$, which lies between $T_{n-1}$ and $T_n$. Introducing an approximation $v_n$ of the potential $u_n$, defined by

$$
v_n(y) = \frac{\psi_n(y_N) (u(y) - c_n)}{\sqrt{\sum_{p=n}^{\infty} |\hat{s}_p|^2}} \text{ with } c_n = \frac{1}{|\omega_n|} \int_{\omega_n} u(y) dy,
$$

we write

$$
d_{\theta'} \hat{s}^n = \lambda \hat{s}^n + \left( \int_{\Gamma_q} (u_n - v_n) \cdot \bar{n} ds \right)_{q \geq 1}.
$$
From (42) we deduce
\[ \| \nabla (u_n - v_n) \|_{L^2(G^*)^N} \geq C > 0 \text{ for } n \geq n_0. \]

Using the equations for \( u \) and \( u_n \), a simple computation yields

\[ \int_{G^*} |\nabla (u_n - v_n)|^2 dy = \int_{G^*} \nabla \psi_n \cdot \frac{(u_n - v_n)\nabla u - (u - c_n)\nabla (u_n - v_n)}{\sqrt{\sum_{p=n}^\infty |\vec{s}_p|^2}} \]

Remark that the integral in the right-hand side reduces to \( \omega_n \) since \( \nabla \psi_n \) has bounded support in \( \omega_n \). Applying the Poincaré–Wirtinger inequality in \( \omega_n \) to \((u - c_n)\) and \((u_n - v_n)\) (this last term has not zero average in \( \omega_n \), but (43) is invariant by substraction of a constant to \((u_n - v_n)\)), we obtain from (43)

\[ \| \nabla (u_n - v_n) \|_{L^2(G^*)^N} \leq C \| \nabla u \|_{L^2(\omega_n)^N} \sqrt{\sum_{p=n}^\infty |\vec{s}_p|^2}, \]

which implies

\[ \sum_{p=n}^\infty |\vec{s}_p|^2 \leq C \| \nabla u \|_{L^2(\omega_n)^N}^2. \]

On the other hand, multiplying equation (41) by \( \psi_n (u - c_n) \) and integrating by parts gives

\[ \int_{G^*} \psi_n |\nabla u|^2 dy + \int_{G^*} (u - c_n) \nabla u \cdot \nabla \psi_n dy = \lambda \sum_{p=n}^\infty |\vec{s}_p|^2. \]

Applying again the Poincaré–Wirtinger inequality in \( \omega_n \) to \((u - c_n)\) yields

\[ \int_{G^*} \psi_n |\nabla u|^2 dy \leq \lambda \sum_{p=n}^\infty |\vec{s}_p|^2 + C \| \nabla u \|_{L^2(\omega_n)^N}^2. \]

Let us denote by \( G_n \) the subset of \( G^* \) defined by \( G_n = \{ y \in G^* | y_N > n \} \). From (44) and (45) we deduce

\[ \| \nabla u \|_{L^2(G_{n+1})^N}^2 \leq C \| \nabla u \|_{L^2(\omega_n)^N}^2 \leq C \left( \| \nabla u \|_{L^2(G_n)^N}^2 - \| \nabla u \|_{L^2(G_{n+1})^N}^2 \right), \]

which implies, for \( n \geq n_0 \),

\[ \| \nabla u \|_{L^2(G_n)^N}^2 \leq \left( \frac{C}{1+C} \right)^{n-n_0} \| \nabla u \|_{L^2(G_{n_0})^N}^2. \]

It is easily seen that (46) implies the desired result.

3. Completeness of the boundary layer spectrum. In this section we assume that \( \Omega \) is a rectangle with integer dimensions, i.e.,

\[ \Omega = \prod_{i=1}^N [0; L_i[ \text{ and } \quad L_i \in \mathbb{N}^*. \]
The sequence of small parameters $\epsilon$ is also assumed to be
\begin{equation}
\epsilon_n = \frac{1}{n}, \quad n \in \mathbb{N}^*.
\end{equation}

Remark that all the previous results in this paper hold for any type of sequence $\epsilon$ going to zero. From now on, we restrict ourselves to the sequence $\epsilon_n$ since, for any $n \geq 1$, the domain $\Omega$ is the union of a finite number of entire periodic cells $Y^p_{\epsilon^n}$. However, to simplify the notation, we shall not indicate the dependence on $n$ and simply denote by $\epsilon$ the particular sequence defined in (48).

Remark that the assumption on the geometry of $\Omega$ can be slightly relaxed. Any polygonal domain with faces parallel to the axis (i.e., the normal is everywhere one of the basis vectors) and having vertex with integer coordinates could equally be considered.

3.1. Presentation of the main result. This section is devoted to the so-called completeness of the limit spectrum. Recall that in our previous work [3] we proved that
\begin{equation}
\sigma_{\infty} = \sigma(S) \cup \sigma_{\text{Bloch}} \cup \sigma_{\text{boundary}},
\end{equation}
where $\sigma_{\text{boundary}}$ is defined in (15). In section 2, we proved that
\begin{equation}
\sigma_{\infty} \supset \sigma_{\Sigma},
\end{equation}
where $\sigma_{\Sigma}$ is the boundary layer spectrum associated with the surface $\Sigma$, defined by (38). Remark that, due to our hypotheses on the domain $\Omega$ and on the sequence $\epsilon$, the surface $\Sigma$ can be any of the faces of $\Omega$ defined by

\[ \prod_{j=1, j \neq i}^{N} [0; L_j] \times \{0\} \quad \text{or} \quad \prod_{j=1, j \neq i}^{N} [0; L_j] \times \{L_i\} \quad \text{for } 1 \leq i \leq N. \]

Of course, the analysis of section 2 can be repeated for any other lower dimensional manifolds (edges, corners, etc.) which compose the boundary of $\Omega$. For $0 \leq m \leq N-1$, let us define the $m$-dimensional parts of $\partial \Omega$ as
\begin{equation}
\Sigma_{m,\tau} = \prod_{j=1}^{m} [0; L_{\tau(j)}] \times \prod_{j=m+1}^{N} \{x_{\tau(j)} = 0 \text{ or } L_{\tau(j)}\},
\end{equation}
where $\tau$ is any permutation of the numbers $\{1, 2, \ldots, N\}$. There are $2^{N-m}C_{N-1}^{N-m}$ $m$-dimensional manifolds of the type $\Sigma_{m,\tau}$. A simple adaptation of the two-scale convergence in the sense of boundary layers for such manifolds allows us to prove that, for any $m$ and $\tau$,
\begin{equation}
\sigma_{\infty} \supset \sigma_{\Sigma_{m,\tau}},
\end{equation}
where $\sigma_{\Sigma_{m,\tau}}$ is the spectrum of a family of limit problems posed, not in a semi-infinite band as in section 2, but rather in a periodic domain bounded in the variables $x_{\tau(1)}, \ldots, x_{\tau(m)}$ and unbounded with respect to the other variables (see section 3.3 for the case of corners in two space dimension). Eventually, defining the union of all these spectra
\begin{equation}
\sigma_{\partial \Omega} = \bigcup_{m,\tau} \sigma_{\Sigma_{m,\tau}},
\end{equation}
we deduce from Theorem 2.18 and from the geometric assumptions (47), (48) that
\[ \sigma_\infty \supset \sigma_{\partial \Omega}. \]
Comparing our results (49) and (51), a completeness result amounts to link the two definitions of the boundary layer spectrum \( \sigma_{\partial \Omega} \) and \( \sigma_{\text{boundary}} \).

**Theorem 3.1.** For the sequence \( \epsilon_n \) defined by (48), the boundary layer spectrum satisfies
\[ \sigma_{\text{boundary}} \subset \sigma_{\partial \Omega}. \]
Therefore, the limit spectrum of the sequence \( S_{\epsilon_n} \) is precisely made of three parts: the homogenized, the Bloch, and the boundary layer spectrum
\[ \lim_{\epsilon_n \to 0} \sigma(S_{\epsilon_n}) = \sigma(S) \cup \sigma_{\text{Block}} \cup \sigma_{\partial \Omega}, \]
where the boundary layer spectrum \( \sigma_{\partial \Omega} \) is explicitly defined by (50).

**Remark 3.2.** Remark that Theorem 3.1 does not state that \( \sigma_{\text{boundary}} \), defined by (15), and \( \sigma_{\partial \Omega} \) coincide. Indeed, we have shown in (39) that \( \sigma_{\partial \Omega} \) contains the Bloch spectrum. It is not clear whether \( \sigma_{\text{boundary}} \) contains the Bloch spectrum too. The comparison of \( \sigma_{\partial \Omega} \) and \( \sigma_{\text{boundary}} \) is definitely a very difficult question. We suspect that if the definition of \( \sigma_{\text{boundary}} \) is modified in such a way that it contains only limit eigenvalues corresponding to sequences of eigenvectors which decay exponentially fast away from the boundary, then it may coincide with that part of \( \sigma_{\partial \Omega} \) made of discrete eigenvalues (which also have exponentially decreasing corresponding eigenvectors).

To prove this completeness result, we need an intermediate result in the spirit of section 2.

**Theorem 3.3.** As in section 2, let \( \Omega \) be a domain defined by
\[ \Omega = \Sigma \times ]0; L[, \]
with \( \Sigma \) a bounded open set in \( \mathbb{R}^{N-1} \) and \( L > 0 \). Recall that \( S^1_\epsilon \) is the extension of \( S_\epsilon \) to \( L^2(\Omega)^N \). Consider a sequence of eigenvalues \( \lambda_\epsilon \) and eigenvectors \( \vec{s}_\epsilon \) such that
\[ S^1_\epsilon \vec{s}_\epsilon = \lambda_\epsilon \vec{s}_\epsilon \quad \text{with} \quad \| \vec{s}_\epsilon \|_{L^2(\Omega)^N} = 1 \quad \text{and} \quad \lim_{\epsilon \to 0} \lambda_\epsilon = \lambda. \]
Assume that for all subset \( \omega \) such that \( \omega \subset \Omega \), we have
\[ \lim_{\epsilon \to 0} \| \vec{s}_\epsilon \|_{L^2(\omega)^N} = 0. \]
Assume further that there exists an \( (N - 1) \)-dimensional open set \( \sigma \), with \( \sigma \subset \Sigma \), a positive number \( l \), with \( 0 < l < L \), and a positive constant \( c \) such that
\[ \lim_{\epsilon \to 0} \| \vec{s}_\epsilon \|_{L^2(\sigma \times ]0, l[)^N} \geq c > 0. \]
Then, \( \lambda \) belongs to the boundary layer spectrum associated with the surface \( \Sigma \)
\[ \lambda \in \sigma_{\Sigma}, \]
where \( \sigma_{\Sigma} \) is defined by (38).

The proof of Theorem 3.3 is the focus of the next section. If we admit it for the moment, as well as its generalizations concerning all other manifolds \( \Sigma_{m, \tau} \) making up the boundary \( \partial \Omega \), we are in a position to complete the following proof.
Proof of Theorem 3.1. Let $\lambda \in \sigma_{\text{boundary}}$. By definition there exists a subsequence (still denoted by $\epsilon$), eigenvalues $\lambda_\epsilon$, and eigenvectors $\vec{s}_\epsilon$ of $S^1_\epsilon$ such that

$$S^1_\epsilon \vec{s}_\epsilon = \lambda_\epsilon \vec{s}_\epsilon \quad \text{with} \quad \|\vec{s}_\epsilon\|_{L^2(\Omega)^N} = 1 \quad \text{and} \quad \lim_{\epsilon \to 0} \lambda_\epsilon = \lambda,$$

and, for all subset $\omega$ satisfying $\overline{\omega} \subset \Omega$,

$$\lim_{\epsilon \to 0} \|\vec{s}_\epsilon\|_{L^2(\omega)^N} = 0.$$

If there exists an $(N - 1)$-dimensional open subset $\sigma_i$, compactly embedded in $\prod_{j \neq i}^N [0, L_j]$, a positive length $0 < l_i < L_i$, a positive constant $c$, and another subsequence (still denoted by $\epsilon$) such that

$$\lim_{\epsilon \to 0} \|\vec{s}_\epsilon\|_{L^2(\sigma_i \times [0, l_i])^N} \geq c > 0 \quad \text{or} \quad \lim_{\epsilon \to 0} \|\vec{s}_\epsilon\|_{L^2(\sigma_i \times [l_i, L_i])^N} \geq c > 0,$$  

then, by application of Theorem 3.3, the limit eigenvalue belongs to $\sigma_{\partial \Omega}$ as desired.

If (54) does not hold true for any such $\sigma_i, l_i, c$, and subsequence $\epsilon$, it implies that the $L^2$-norm of $\vec{s}_\epsilon$ concentrates near the lower dimensional edges of the rectangle $\Omega$. In this case, we repeat the above argument with an $(N - 2)$-dimensional open set included in one of the set $\Sigma_{N-2, \tau}$, and so on up to the 0-dimensional set made of one of the vertices of $\Omega$. A tedious but simple induction argument on the dimension $m$ shows that there exists at least a dimension $0 \leq m \leq N - 1$, a permutation $\tau$, positive lengths $(l_{\tau(j)})_{m+1 \leq j \leq N}$, a positive constant $c$, and a subsequence $\epsilon$ such that

$$\lim_{\epsilon \to 0} \|\vec{s}_\epsilon\|_{L^2(\omega)^N} \geq c > 0,$$

with $\omega \subset \Omega$ of the type

$$\omega = \sigma \times \prod_{j=m+1}^N ([0, l_{\tau(j)}] \text{ or } [l_{\tau(j)}, L_{\tau(j)}) \quad \text{and} \quad \sigma \subset \prod_{j=1}^m [0, L_{\tau(j)}].$$

Then, applying an adequate generalization of Theorem 3.3, this proves that the limit eigenvalue belongs to $\sigma_{\partial \Omega}$.

3.2. Proof of the completeness. This section is devoted to the proof of Theorem 3.3 which is divided in several lemmas and propositions. Let us begin by recalling the definition of the associated potential $u_\epsilon$, solution of

$$\begin{cases}
-\Delta u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\
\frac{\partial u_\epsilon}{\partial n} = \vec{s}_\epsilon \cdot \vec{n} & \text{on } \Gamma^p_\epsilon, \ 1 \leq p \leq n(\epsilon), \\
u_\epsilon = 0 & \text{on } \partial \Omega.
\end{cases}$$

The spectral equation $S^1_\epsilon \vec{s}_\epsilon = \lambda_\epsilon \vec{s}_\epsilon$ implies that

$$\left( \int_{\Gamma^p_\epsilon} u_\epsilon \vec{n} \right)_{1 \leq p \leq n(\epsilon)} = \lambda_\epsilon \vec{s}_p.$$  

By assumption (52), for all subsets $\omega$ such that $\overline{\omega} \subset \Omega$, we have

$$\lim_{\epsilon \to 0} \|\vec{s}_\epsilon\|_{L^2(\omega)^N} = 0.$$
In other words, all the energy of the eigenvectors $\vec{s}$ concentrates near the boundary $\partial \Omega$. This concentration effect has important consequences on the associated potential $u_e$.

**Lemma 3.4.** The sequence $u_e$ defined in (55) converges to 0 in $H^1_0(\Omega)$ weakly and strongly in $L^2(\Omega)$. Furthermore, $u_e$ converges strongly to 0 in $H^1_{loc}(\Omega)$.

**Proof.** Multiplying equation (55) by a test function $v \in H^1_0(\Omega)$ yields

$$\int_{\Omega_e} \nabla u_e \cdot \nabla v dx = \sum_{p=1}^{n(\epsilon)} \vec{s}_p^\epsilon \cdot \left( \int_{\Gamma_p^e} v \tilde{n} ds \right) = \int_{\Omega} \vec{s}(x) \cdot \vec{z}(x) dx,$$

where

$$\vec{z}(x) = - \sum_{p=1}^{n(\epsilon)} \frac{1}{\epsilon^N} \left( \int_{\Gamma_p^e} \nabla v(x) dx \right) x Y_p^e(x).$$

It is easily seen that $\vec{s}$ converges strongly to $-\frac{T}{N} \nabla v(x)$ in $L^2(\Omega)^N$. Since $\vec{s}$ converges weakly to 0 in $L^2(\Omega)^N$, by virtue of (52), we deduce that $u_e$ converges to 0 weakly in $H^1_0(\Omega)$ and, by the Rellich theorem, strongly in $L^2(\Omega)$. Finally, for any open set $\omega$ such that $\varpi \subset \Omega$, let $\psi$ be a smooth function with compact support in $\Omega$ and equal to 1 on $\omega$. Multiplying (55) by $\psi^2 u_e$ and integrating by parts leads to

$$\int_{\Omega_e} \psi^2 |\nabla u_e|^2 dx = -2 \int_{\Omega_e} \psi u_e \nabla \psi \cdot \nabla u_e dx + \sum_{p=1}^{n(\epsilon)} \vec{s}_p^\epsilon \cdot \left( \int_{\Gamma_p^e} \psi^2 u_e \tilde{n} ds \right).$$

Since $u_e$ converges weakly to 0 in $H^1_0(\Omega)$, the first term in the right-hand side of (57) goes to 0 with $\epsilon$. In view of (56), the second term is bounded by

$$||\psi||^2_{L^\infty(\Omega)} ||\vec{s}||^2_{L^2(\text{supp}(\psi))},$$

which goes to 0 by virtue of the assumption (52). Therefore, we deduce from (57) that $\nabla u_e$ converges strongly to 0 in $L^2(\omega)^N$. This concludes the proof of Lemma 3.4.

By assumption (53), there exists an $(N-1)$-dimensional open set $\sigma$, with $\sigma \subset \Sigma$, such that the sequence of eigenvectors concentrates partly near $\sigma$. By translation, one can always assume that the origin lies inside $\sigma$. The strategy of the proof is to rescale the domain $\Omega$ by the change of variables $y = \frac{x}{\epsilon}$ and then to transform the sequence of eigenvectors $\vec{s}$ in a Weyl sequence for a limit operator. The limit domain will be $\mathbb{R}_+^N = \{ y \in \mathbb{R}^N | j_N > 0 \}$ since we have carefully chosen the origin to belong to $\sigma$. The limit fluid domain is denoted by $G^{*\infty}$, which is defined by

$$G^{*\infty} = \mathbb{R}_+^N \setminus \bigcup_{j \in \mathbb{Z}_+^N} T_j,$$

where $T_j$ denotes the tube $j$ placed in the subcell $Y_j$ (centered at the point $j = (j', j_N)$ with $j' \in \mathbb{Z}^{N-1}$ and $j_N \in \mathbb{Z}_+$). In this limit domain we define a limit operator $B^{*\infty}$, which acts from $\ell^\infty_2$ in itself, by

$$B^{*\infty} \vec{s} = \left( \int_{T_j} u \tilde{n} ds \right)_{j \in \mathbb{Z}_+^N} \quad \forall \vec{s} \in \ell^\infty_2,$$
where $u(y)$ is the unique solution in $D_0^2(G^{*\infty})$ of

$$
\begin{aligned}
\left\{
\begin{array}{ll}
-\Delta u = 0 & \text{in } G^{*\infty}, \\
\frac{\partial u}{\partial n} = \vec{s}_j \cdot \vec{n} & \text{on } \Gamma_j, \ j \in \mathbb{Z}_+^N, \\
u = 0 & \text{on } \mathbb{R}^{N-1} \times \{0\}.
\end{array}
\right.
\end{aligned}
$$

(58)

Recall that elements in $D_0^1(G^{*\infty})$ are restrictions to $G^{*\infty}$ of functions $w_n$ in $D_0^1(\mathbb{R}^N_+)$ which, in its turn, is the closure, with respect to the $L^2$-norm of the gradient, of smooth functions with compact support in $\mathbb{R}^N_+$.

**Remark 3.5.** The limit domain $G^{*\infty}$ is nothing but the limit as $K$ goes to infinity of the domain $G^{*K}$ defined in section 2.2. By the same token, the Hilbert space $\ell_2^K$ is the limit of $\ell_2$ (it is also equal to $\ell_2(\mathbb{Z}_+^N; \mathbb{C}^N)$). In some sense the limit operator $B^{*\infty}$ is also the limit of the operator $B^K$ defined in Theorem 2.11.

Let $\varphi$ be a smooth function, equal to 1 in $\omega = \sigma \times [0, L]$, with compact support in $\Sigma \times [0, L]$ (i.e., $\varphi$ vanishes on all faces of $\Omega$ except on that defined by $x_N = 0$). We use $\varphi$ to localize the sequence of eigenvectors $\vec{s}$ in a vicinity of $\omega$. Let us define a sequence $\vec{t}$ by

$$
\vec{t} = E_1^1 P_1^1 (\varphi(x) \vec{s}(x)),
$$

where $E_1^1 P_1^1$ is the projection operator in $L^2(\Omega)^N$ on piecewise constant functions (cf. their definitions (27) and (28)).

Remark that, by assumption (53), the sequence $\vec{t}$ satisfies

$$
\lim_{\epsilon \to 0} ||\vec{t}||_{L^2(\omega)^N} \geq c > 0.
$$

Let us define $G^{*\infty}$ as $G^{*\infty}$ rescaled to size $\epsilon$. Let $v_\epsilon$ be the potential in $G^{*\infty}_\epsilon$ associated with $\vec{t}$, defined by

$$
\begin{aligned}
\left\{
\begin{array}{ll}
-\Delta v_\epsilon = 0 & \text{in } G^{*\infty}_\epsilon, \\
\frac{\partial v_\epsilon}{\partial n} = \vec{n} \cdot \vec{n} & \text{on } \Gamma_p, p \in \mathbb{Z}_+^N, \\
v_\epsilon = 0 & \text{on } \mathbb{R}^{N-1} \times \{0\}.
\end{array}
\right.
\end{aligned}
$$

(59)

**Lemma 3.6.** The sequence $v_\epsilon$ defined by (59) converges to zero in $D_0^2(\mathbb{R}^N_+)$ weakly and in $H^1_{loc}(\mathbb{R}^N_+)$ strongly.

**Proof.** The argument is similar to that of Lemma 3.4, except that the Rellich theorem applies only for compact sets in $\mathbb{R}^N_+$.

**Lemma 3.7.** The difference $w_\epsilon = v_\epsilon - \varphi u_\epsilon$ converges strongly to zero in $D_0^1(\mathbb{R}^N_+)$.  

**Proof.** A simple calculation provides the following key identity:

$$
\int_{\mathbb{R}^N_+} |\nabla w_\epsilon|^2 \leq \int_{\mathbb{R}^N_+} \nabla v_\epsilon \cdot \nabla w_\epsilon - \int_{\mathbb{R}^N_+} \nabla u_\epsilon \cdot \nabla (\varphi w_\epsilon) - \int_{\mathbb{R}^N_+} \nabla \varphi \cdot (u_\epsilon \nabla w_\epsilon - w_\epsilon \nabla u_\epsilon).
$$

(60)

By virtue of Lemmas 3.4 and 3.6, $u_\epsilon$ and $w_\epsilon$ converge to zero strongly in $L^2$ of the support of $\varphi$. Therefore, the last term in (60) goes to zero with $\epsilon$. On the other hand, an integration by parts yields

$$
\int_{G^{*\infty}_\epsilon} \nabla v_\epsilon \cdot \nabla w_\epsilon - \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla (\varphi w_\epsilon) = \sum_{p=1}^{n(\epsilon)} \left[ t_p \cdot \left( \int_{\Gamma_p} w_\epsilon \vec{n} \right) - s_p \cdot \left( \int_{\Gamma_p} \varphi w_\epsilon \vec{n} \right) \right].
$$
Since $\tilde{t}_p^\varepsilon = \frac{1}{\varepsilon^N} \int_{\Gamma_p^\varepsilon} \varphi \tilde{s}_p^\varepsilon ds$ and $|\varphi(x) - \varphi(x_p^\varepsilon)| \leq \varepsilon \|
abla \varphi\|_{L^\infty}$, where $x_p^\varepsilon$ is the center of the cube $Y_p^\varepsilon$ which contains $x$, we obtain
\[
\left| \int_{G_p^\varepsilon} \nabla v_\varepsilon \cdot \nabla w_\varepsilon - \int_{\Omega} \nabla u_\varepsilon \cdot \nabla (\varphi w_\varepsilon) \right| \leq \varepsilon \|
abla \varphi\|_{L^\infty} \|	ilde{s}_p^\varepsilon\|_{L^2(\Omega)} \|
abla w_\varepsilon\|_{L^2(\mathbb{R}^N)} ,
\]
which gives the desired result.

**Lemma 3.8.** From Lemma 3.7 we deduce the following approximation result for the displacement vector $\tilde{t}$:
\[
\lim_{\varepsilon \to 0} \sum_{p \in \mathbb{Z}^N_+} \varepsilon N \left| \frac{1}{\varepsilon^N} \int_{\Gamma_p^\varepsilon} v_\varepsilon \tilde{s}_p^\varepsilon ds - \lambda \tilde{t}_p^\varepsilon \right|^2 = 0.
\]

**Proof.** We have
\[
\varepsilon N \sum_{p \in \mathbb{Z}^N_+} \left| \frac{1}{\varepsilon^N} \int_{\Gamma_p^\varepsilon} (v_\varepsilon - \varphi u_\varepsilon) \tilde{s}_p^\varepsilon ds \right|^2 \leq \sum_{p \in \mathbb{Z}^N_+} \|
abla (v_\varepsilon - \varphi u_\varepsilon)\|_{L^2(\Omega)}^2 \leq \|
abla (v_\varepsilon - \varphi u_\varepsilon)\|_{L^2(\mathbb{R}^N)}^2,
\]
which goes to zero as $\varepsilon \to 0$ by virtue of Lemma 3.7. Furthermore,
\[
\varepsilon^N \sum_{p \in \mathbb{Z}^N_+} \left| \frac{1}{\varepsilon^N} \int_{\Gamma_p^\varepsilon} \varphi u_\varepsilon \tilde{s}_p^\varepsilon ds - \lambda \tilde{t}_p^\varepsilon \right|^2 \leq \varepsilon \|
abla \varphi\|_{L^\infty} \|
abla u_\varepsilon\|_{L^2(\Omega)}
\]
since, $\tilde{s}_p^\varepsilon$ being constant in each cell $Y_p^\varepsilon$,
\[
\frac{1}{\varepsilon^N} \int_{\Gamma_p^\varepsilon} \varphi u_\varepsilon \tilde{s}_p^\varepsilon ds = \frac{1}{\varepsilon^N} \int_{\Gamma_p^\varepsilon} \left( \varphi(s) - \frac{1}{\varepsilon^N} \int_{Y_p^\varepsilon} \varphi(t) dt \right) u_\varepsilon \tilde{s}_p^\varepsilon ds + \lambda \tilde{t}_p^\varepsilon (\varepsilon \tilde{s}_p^\varepsilon).
\]
Summing these two estimates yields the desired result.

Now, let us define a sequence $\bar{\tilde{t}}\varepsilon$ in $\ell^\infty$ by
\[
\bar{\tilde{t}}\varepsilon = \varepsilon^{N/2} (\tilde{t}_p^\varepsilon)_{p \in \mathbb{Z}^N_+},
\]
which plays the role of a Weyl sequence for the limit operator $B^\infty$.

**Proposition 3.9.** The sequence $\bar{\tilde{t}}\varepsilon$ satisfies
\[
\lim_{\varepsilon \to 0} \|\bar{\tilde{t}}\varepsilon\|_{\ell^\infty} \geq c > 0,
\]
and
\[
B^\infty \bar{\tilde{t}}\varepsilon = \lambda \bar{\tilde{t}}\varepsilon + \bar{\tilde{t}}\varepsilon,
\]
where $\bar{\tilde{t}}\varepsilon$ is a remainder term which goes to zero strongly in $\ell^\infty$.

**Proof.** A simple rescaling in (59) shows that $\tilde{v}_\varepsilon(y) = \varepsilon \tilde{t}_p^\varepsilon \varphi(\varepsilon y)$ is the unique solution in $D_0(G^\varepsilon)$ of
\[
\begin{cases} 
-\Delta \tilde{v}_\varepsilon = 0 & \text{in } G^\varepsilon, \\
\frac{\partial \tilde{v}_\varepsilon}{\partial n} = \bar{\tilde{t}}\varepsilon \cdot n & \text{on } \Gamma_p, p \in \mathbb{Z}^N_+, \\
\tilde{v}_\varepsilon = 0 & \text{on } \mathbb{R}^{N-1} \times \{0\},
\end{cases}
\]

Furthermore, $\|\nabla_y \tilde{v}_\epsilon\|_{L^2(G^*;N)} = \|\nabla_x v_\epsilon\|_{L^2(G^*;N)}$. By definition,
\[
B^\infty \vec{r}^\epsilon = \left( \int_{\Gamma_p} \tilde{v}_\epsilon \vec{r} \, ds \right)_{p \in \mathbb{Z}_+^N} = \epsilon^{-\frac{N}{2}} \left( \int_{\Gamma_p} v_\epsilon \vec{r} \, ds \right)_{p \in \mathbb{Z}_+^N}.
\]
Defining $\vec{r}^\epsilon = (\vec{r}^\epsilon_p)_{p \in \mathbb{Z}_+^N}$ by
\[
\vec{r}^\epsilon_p = \epsilon^{-\frac{N}{2}} \left( \frac{1}{\epsilon^N} \int_{\Gamma_p} v_\epsilon \vec{r} \, ds - \lambda P_{\vec{r}^\epsilon_p} \right),
\]
we get
\[
B^\infty \vec{r}^\epsilon = \lambda P_{\vec{r}^\epsilon} + \vec{r}^\epsilon,
\]
which, by virtue of Lemma 3.8, is the desired result.

To conclude the proof of Theorem 3.3, we remark that either $\vec{r}^\epsilon$ converges weakly in $\ell^\infty_2$ to a nonzero limit $\vec{r}$ (up to a subsequence) or $\vec{r}^\epsilon$ converges weakly to $\vec{0}$ in $\ell^\infty_2$. In the first case, passing to the limit as $\epsilon$ goes to 0, we obtain that $\vec{r} \neq \vec{0}$ is an eigenvector of $B^\infty$ for $\lambda$ (the limit of the sequence $\lambda_\epsilon$). In the latter case, this proves that $\vec{r}^\epsilon$ is a Weyl sequence for the spectral value $\lambda$ which belongs to the essential spectrum of $B^\infty$. Now, it is a standard matter (see, e.g., [15], [16]) to show, by a Bloch wave decomposition analogous to that of section 2.3, that the spectrum of $B^\infty$ is nothing but $\lim_{K \to +\infty} \sigma(B^K)$, i.e., the boundary layer spectrum associated with the face $\Sigma$ of $\Omega$.

**Remark 3.10.** Let us remark that Theorem 3.3 is valid for any choice of the sequence $\epsilon$ and not only for the particular sequence $\epsilon_n$ defined in (48). The interested reader will not fail to notice that the present proof of the completeness result is different from that of our previous work [3]. In this paper, we used the concept of Bloch measures in order to prove a similar completeness result by means of an energetic method. Here, we propose a new proof (in a slightly different context), based on a rescaling argument, which is simpler, although less precise, and which could equally be applied in [3].

### 3.3. Analysis of the corner spectrum.
In section 3.1 the boundary layer spectrum $\sigma_{\partial \Omega}$ was defined as the union of all spectra of the type $\sigma_{\Sigma}$, where $\Sigma$ is any lower dimensional manifold composing the boundary $\partial \Omega$. When $\Sigma$ is an $(N - 1)$-dimensional hyperplane, a complete derivation of $\sigma_{\Sigma}$ has been given in section 2. However, for lower dimensional manifolds we have been a little cavalier in saying that the analysis of section 2 can be easily generalized to the case of edges, corners, and so on. The purpose of this section is to briefly indicate some details of this generalization when analyzing the corner spectrum. Since the physical problem of interest is truly two-dimensional, we restrict ourselves to the case of corners of the plane square domain $\Omega$ (this has the advantage of simplifying the exposition).

Therefore, our domain $\Omega$ is now a rectangle with integer dimensions, i.e.,
\[
\Omega = [0; L_1] \times [0; L_2].
\]
We describe the limit spectrum associated with the corner located at the origin. We introduce the space $\ell^2_+$ of displacements defined by
\[
\ell^2_+ = \left\{ (s_j)_{j=(j_1,j_2) \quad j_1 \geq 1, j_2 \geq 1} \quad |s_j| \in \mathbb{R}^2, \quad \sum_{j_1,j_2=1}^{+\infty} |s_{j_1,j_2}|^2 < +\infty \right\}.
\]
Remark that this definition of $\ell_2^+$ implies a decay of the displacement $\vec{s}_j$ as $j_1$ or $j_2$ goes to $+\infty$.

We extend the operator $S_\epsilon$ to the larger space $\ell_2^+$ by the following formula:
$$C_\epsilon = E_\epsilon S_\epsilon P_\epsilon,$$
where $P_\epsilon$ and $E_\epsilon$ are, respectively, projection and extension operators between $\mathbb{R}^{Nn(\epsilon)}$ and $\ell_2^+$. Their definition is very simple. Recall that a tube $T_j^\epsilon$ in $\Omega$ is located in a cell $Y_j^\epsilon$ whose origin is $\epsilon j$. We denote the range of all indices $j$ such that $T_j^\epsilon$ is included in $\Omega$ by $1 \leq j \leq n(\epsilon)$. The projection is defined by
$$P_\epsilon : \ell_2^+ \to \mathbb{R}^{Nn(\epsilon)},$$
and the extension by
$$E_\epsilon : \mathbb{R}^{Nn(\epsilon)} \to \ell_2^+,$$
with $\vec{t}_j = \vec{s}_j$ if $1 \leq j \leq n(\epsilon)$ and $\vec{t}_j = 0$ otherwise.

One can easily check that $P_\epsilon$ and $E_\epsilon$ are adjoint operators and that the product $P_\epsilon E_\epsilon$ is equal to the identity in $\mathbb{R}^{Nn(\epsilon)}$. Therefore, the spectrum of $C_\epsilon$ consists of that of $S_\epsilon$ and zero as an eigenvalue of infinite multiplicity.

The convergence analysis of $C_\epsilon$ is much simpler than that in section 2 because $\ell_2^+$ is not a space of periodically oscillating displacements. There is no need to introduce any notion of two-scale convergence for corner boundary layers. A simple rescaling argument is enough. More precisely, denoting by $Q_+$ the first quadrant in the plane
$$Q_+ = [0; +\infty[\times]0; +\infty[,$$
we replace the two-scale convergence by the weak convergence in $L^2(Q_+)$: with each bounded sequence $u_\epsilon(x)$ in $L^2(\Omega)$, we associate the rescaled sequence $v_\epsilon(y)$ defined by
$$v_\epsilon(y) = \begin{cases} \epsilon^2 u_\epsilon(\epsilon y) & \text{if } \epsilon y \in \Omega, \\ 0 & \text{otherwise}, \end{cases}$$
which is also bounded in $L^2(Q_+)$. Then, a similar analysis to that of section 2 shows that the sequence of operators $C_\epsilon$ converges strongly in $L(\ell_2^+)$ to a limit operator $C_\infty$ defined by
$$C_\infty : \ell_2^+ \to \ell_2^+$$
by
$$\begin{pmatrix} \vec{s}_j \end{pmatrix}_{j=(j_1,j_2) \; j_1 \geq 1, j_2 \geq 1} \mapsto \left( \int_{T_j} u\vec{n}d\sigma \right)_{j=(j_1,j_2) \; j_1 \geq 1, j_2 \geq 1},$$
where $u(y)$ is the unique solution of
$$\begin{cases} -\Delta u = 0 & \text{in } Q_+^s = Q_+ \setminus \bigcup_j T_j, \\ \frac{\partial u}{\partial \vec{n}} = \vec{s}_j \cdot \vec{n} & \text{on } \Gamma_j, \; j_1 \geq 1, j_2 \geq 1, \\ u = 0 & \text{on } \partial Q_+, \\ \lim_{|y| \to +\infty} u(y) = 0. \end{cases}$$

Clearly, $C_\infty$ is a self-adjoint noncompact operator acting in $\ell_2^+$. As in Proposition 2.17, one can prove that the essential spectrum of $C_\infty$ is precisely the Bloch spectrum. However, the discrete spectrum of $C_\infty$ may contain new eigenvalues which correspond to eigenvectors localized in the corner of $Q_+$. 

Acknowledgments. The authors would like to thank one of the anonymous referees for several sharp comments which enabled them to improve and revise the original version.

REFERENCES


