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Identification of immersed obstacles via boundary measurements

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Abstract
We study the following inverse problem: an inaccessible rigid body \( D \) is immersed in a viscous fluid, in such a way that \( D \) plays the role of an obstacle around which the fluid is flowing in a greater bounded domain \( \Omega \), and we wish to determine \( D \) (i.e., its form and location) via boundary measurement on the boundary \( \partial \Omega \). Both for the stationary and the evolution problem, we show that under reasonable smoothness assumptions on \( \Omega \) and \( D \), one can identify \( D \) via the measurement of the velocity of the fluid and the Cauchy forces on some part of the boundary \( \partial \Omega \). We also show that the dependence of the Cauchy forces on deformations of \( D \) is analytic, and give some stability results for the inverse problem.

1. Introduction

Let \( \Omega \) be a smooth bounded open set in \( \mathbb{R}^N \), filled with an incompressible fluid, and let \( D \subset \subset \Omega \) be an unknown rigid body immersed in it. Let \( \varphi \in C^1([0, T]; H^{3/2}(\partial \Omega)^N) \) be non-homogeneous Dirichlet boundary data satisfying the standard flux compatibility condition

\[
\int_{\partial \Omega} \varphi \cdot n \, ds = 0,
\]
and let \((v, p) \in L^2(0, T; H^1(\Omega \setminus \overline{D})^N) \times L^2(0, T; L^2(\Omega \setminus \overline{D}))\) be the unique solution of the Stokes (when \( \varepsilon_* = 0 \)) or Navier–Stokes (when \( \varepsilon_* = 1 \)) system of equations,
Figure 1.

\[
\begin{aligned}
\frac{\partial v}{\partial t} - \text{div}(\sigma(v, p)) + \varepsilon_* \text{div}(v \otimes v) &= 0 \quad \text{in} \quad (\Omega \setminus \overline{D}) \times (0, T) \\
\text{div}(v) &= 0 \quad \text{in} \quad (\Omega \setminus \overline{D}) \times (0, T) \\
v(0) &= 0 \quad \text{in} \quad \Omega \setminus \overline{D} \\
v &= \varphi \quad \text{on} \quad \partial \Omega \times (0, T) \\
v &= 0 \quad \text{on} \quad \partial D \times (0, T),
\end{aligned}
\]

(1.2)

with \( \sigma \) being the stress tensor defined as follows,

\[
\sigma(v, p) = -pI + 2\nu e(v),
\]

(1.3)

where \( I \) is the identity matrix, \( \nu > 0 \) is a given constant representing the kinematic viscosity of the liquid and \( e(v) \) is the linear strain tensor defined by

\[
e(v) = \frac{1}{2}(\nabla v + \nabla^T v).
\]

(1.4)

The corresponding Poincaré–Steklov operator \( \Lambda_D \) is defined by

\[
\Lambda_D(\varphi) := \sigma(v, p) n \quad \text{on} \quad \Gamma \times (0, T),
\]

(1.7)
which maps \( \varphi \in C^1([0, T]; H^{1/2}(\partial\Omega)^N) \) (with \( \int_{\partial\Omega} \varphi \cdot \mathbf{n} \, ds = 0 \)) into the Cauchy forces \( \sigma(v, p) \in L^2(0, T; H^{-1/2}(\Gamma)^N) \), where \((v, p)\) is the unique solution of the Stokes system (1.2) and \( \Gamma \) is a relatively open subset of the boundary \( \partial\Omega \). In the case of the stationary version of the problem, the operator \( \Lambda_D \) has to be considered as acting between the spaces \( H^{1/2}(\partial\Omega) \) into \( H^{-1/2}(\Gamma) \), and in this case \((v, p)\) is a solution of (1.5).

Our first result concerns the identifiability of \( D \): it states that given fixed non-homogeneous Dirichlet boundary data \( \varphi \), two different admissible geometries \( D_0 \neq D_1 \in \mathcal{D}_{ad} \), yield two different Steklov–Poincaré operators \( \Lambda_{D_0} \neq \Lambda_{D_1} \).

**Theorem 1.1.** Let \( T > 0 \) and \( \Omega \subseteq \mathbb{R}^N, N = 2 \) or \( N = 3 \), be a bounded \( C^{1,1} \) domain, and \( \Gamma \) be a non-empty open subset of \( \partial\Omega \). Let \( D_0, D_1 \in \mathcal{D}_{ad} \) and \( \varphi \in C^1([0, T]; H^{3/2}(\partial\Omega)^N) \) with \( \varphi \neq 0 \), satisfying the flux condition (1.1). For \( \varepsilon_* = 0 \) or \( \varepsilon_* = 1 \), let \((v_j, p_j)\) for \( j = 0, 1 \), be a solution of

\[
\begin{align*}
\frac{\partial v_j}{\partial t} - \text{div}(\sigma(v_j, p_j)) + \varepsilon_* \text{div}(v_j \otimes v_j) &= 0 & \text{in} & (\Omega \setminus D_j) \times (0, T) \\
\text{div}(v_j) &= 0 & \text{in} & (\Omega \setminus D_j) \times (0, T) \\
v_j(x, 0) &= 0 & \text{for} & x \in \Omega \setminus D_j \\
v_j(s, t) &= \varphi(s, t) & \text{for} & (s, t) \in \partial\Omega \times (0, T) \\
v_j(s, t) &= 0 & \text{for} & (s, t) \in \partial D_j \times (0, T).
\end{align*}
\]

Assume that \((v_j, p_j)\) are such that

\( \sigma(v_0, p_0)\mathbf{n} = \sigma(v_1, p_1)\mathbf{n} \) on \( \Gamma \times (0, T) \).

Then \( D_0 \equiv D_1 \).

The same identifiability result holds for the stationary problem:

**Theorem 1.2.** Let \( \Omega \subseteq \mathbb{R}^N, N = 2 \) or \( N = 3 \), be a bounded Lipschitz domain, and \( \Gamma \) a non-empty open subset of \( \partial\Omega \). Let \( D_0, D_1 \in \mathcal{D}_{ad} \) and \( \varphi \in H^{3/2}(\partial\Omega)^N \) with \( \varphi \neq 0 \), satisfying the flux condition (1.1). For \( \varepsilon_* = 0 \) or \( \varepsilon_* = 1 \), let \((v_j, p_j)\) for \( j = 0, 1 \), be a solution of

\[
\begin{align*}
-\text{div}(\sigma(v_j, p_j)) + \varepsilon_* \text{div}(v \otimes v) &= 0 & \text{in} & \Omega \setminus D_j \\
\text{div}(v_j) &= 0 & \text{in} & \Omega \setminus D_j \\
v_j(s) &= \varphi(s) & \text{for} & s \in \partial\Omega \\
v_j(s) &= 0 & \text{for} & s \in \partial D_j.
\end{align*}
\]

Assume that \((v_j, p_j)\) are such that

\( \sigma(v_0, p_0)\mathbf{n} = \sigma(v_1, p_1)\mathbf{n} \) on \( \Gamma \).

Then \( D_0 \equiv D_1 \).

The identifiability of a sufficiently smooth inclusion within a given domain \( \Omega \subseteq \mathbb{R}^N \) has been well studied for scalar elliptic and parabolic equations. The interested reader is referred to the papers by Beretta and Vessella [3], Canuto and Kavian [7], Canuto and Vessella [6], Canuto [5] for more details. We refer also to the paper by Alessandrini et al [1, 2] for estimates on the volume of the subdomain \( D \) immersed in a perfect fluid filling \( \Omega \); note that in this case, it is assumed that the dynamics of the fluid is governed by a velocity potential which satisfies a scalar Laplace equation. The proof by Beretta and Vessella [3] of injectivity and stability of the corresponding boundary map is mainly based on some structural properties of the Laplace equation like the maximum principle and Harnack’s inequality. Since these properties are not
valid anymore in our case, we were led to a different proof, namely we obtain our identifiability result by a suitable application of the unique continuation property for the Stokes equations due to Fabre and Lebeau [8].

Studying the stability of the identifiability is reduced to studying the continuity properties of the inverse of the boundary map. Given a background admissible configuration $\Omega_0 := \Omega \setminus \overline{D}_0$, let us consider a smooth perturbation of $D_0$, called $D_1$, and a bijective Lipschitz mapping $\Psi : \overline{\Omega} \to \overline{\Omega}$, such that $\Psi^{-1}$ is also Lipschitz and $\Psi = I$ in a neighbourhood of the boundary $\partial \Omega$, with

$$\Psi(D_0) = D_1 \quad \text{and} \quad \Psi(\Omega_0) = \Omega_1 = \Omega \setminus \overline{D}_1.$$  

Clearly we need $\Omega_1$ to be also admissible, that is, $D_1 \in \mathcal{D}_{\text{ad}}$ such that the new rigid body $D_1$, as well as $D_0$, is both Lipschitz and included in a fixed open set $\mathcal{O}$ satisfying

$$D_0 \cup D_1 \subset \subset \mathcal{O} \subset \subset \Omega.$$  

In this setting, studying the stability of the inverse problem can be viewed as studying continuity properties for the mapping $\Psi \mapsto \Lambda_{\Psi^{-1}}$. In practice, it suffices to prove continuity at $\Psi = I$, where $I$ is the identity map, that is to say, for any given $\varphi \in C^1(0, T; H^{3/2}(\Omega)^N)$ (respectively $\varphi \in H^{3/2}(\partial \Omega)$ for the stationary problem), when

$$\|\Lambda_{D_0}(\varphi) - \Lambda_{\Psi^{-1}(D_0)}(\varphi)\|_{L^2(0, T, H^{-1/2}(\Gamma))}$$

(respectively $\|\Lambda_{D_0}(\varphi) - \Lambda_{\Psi^{-1}(D_0)}(\varphi)\|_{H^{-1/2}(\Gamma)}$) is small, then the norm $\|f - \Psi\|_{W^{1,\infty}(\Omega_{10} \setminus \mathcal{O}, \mathbb{R}^N)}$ is small.

Actually, our result concerning stability is weaker than the above desired continuity property: we are able to prove only a sort of directional continuity, or hemicontinuity of the mapping $\Psi \mapsto \Lambda_{\Psi^{-1}}$. More precisely, we will consider admissible deformations of the type $\Psi_t = I + t \Psi_1$, where $t$ is a real parameter, and for some integer $m \geq 1$ and a positive constant $C > 0$, both depending on $\Psi_1 \neq 0$, provided $\Psi_t(D_0) \neq D_0$, we prove that

$$\|\Lambda_{D_0}(\varphi) - \Lambda_{\Psi_t^{-1}(D_0)}(\varphi)\| \geq C |t|^m. \quad (1.10)$$

To be more specific, let us consider the linear stationary case (whose understanding is in fact the main point for all other cases). For each change of variables $\Psi : \Omega_0 \to \Omega_\tau := \Psi_t(\Omega_0)$, let $(t_\tau, p_\tau) \in H^1(\Omega_\tau)^N \times L^2(\Omega_\tau)$ denote the unique solution of the Stokes system in the deformed domain, that is,

$$\begin{cases}
-\text{div}(\sigma(v_\tau, p_\tau)) + \varepsilon \text{div}(v_\tau v) = 0 & \text{in } \Omega_\tau \\
\text{div} v_\tau = 0 & \text{in } \Omega_\tau \\
v_\tau(s) = \varphi(s) & \text{for } s \in \partial \Omega \\
v_\tau(s) = 0 & \text{for } s \in \partial D_0.
\end{cases} \quad (1.11)$$

(For $t = 0$ note that $\Psi_0 = I$ and $\Omega_\tau = \Omega_0$.) We will show that, under suitable assumptions on the function $\Psi_1$, the mapping

$$\Psi_t \mapsto \sigma(v_\tau, p_\tau) n =: \Lambda_{\Psi_t^{-1}D_0}$$

is analytic in the open set of $W^{1,\infty}$-diffeomorphisms’ (by an abuse of language we shall say that $\Psi$ is a $W^{1,\infty}$-diffeomorphism when both $\Psi$ and $\Psi^{-1}$ are in $W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, and restricted respectively to the domains $\Omega \setminus \overline{\Omega}_0$ and $\Psi(\Omega \setminus \overline{\Omega}_0)$). In particular, it can be differentiated with respect to $\Psi_t$; as a matter of fact, the corresponding derivative is the so-called shape differentiation of the solution of (1.11) with respect to the geometry.

The following stability (directional continuity) result is proven in section 4.

**Theorem 1.3.** Let $D_0 \in \mathcal{D}_{\text{ad}}$, $\Psi_0 := I$ and $\Psi_1 \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ such that $\Psi_1 \equiv 0$ in a neighbourhood of the boundary $\partial \Omega$ and $\Psi_1 \neq 0$ on $D_0$. Denote by $\Psi_t := I + t \Psi_1$ and $(v_\tau, p_\tau)$ the solution of (1.11) (including for $\tau = 0$), and by $\tau_1 > 0$ a positive number such
that \( \tau \mapsto \sigma(v, p) \mathbf{n} \) is analytic on \((-\tau_1, \tau_1)\). Assume that for some \( \tau_0 \in (-\tau_1, \tau_1) \) one has \( \Psi_{\tau_0}(D_0) \neq D_0 \). Then there exist a strictly positive constant \( C = C(\Psi_1, \Omega, D_0, \varphi) \) and a positive integer \( m = m(\Psi_1, \Omega, D_0, \varphi) \) such that for some \( \tau_0 > 0 \) and all \( \tau \in [-\tau_0, \tau_0] \), we have

\[
\| \Lambda_{\partial \tau_0}(\varphi) - \Lambda_{\Psi_{\tau_0}(D_0)}(\varphi) \|_{H^{-\frac{1}{2}}(\Gamma)} \geq C|\tau|^m.
\]

where \( \Lambda_{\Psi_{\tau_0}(D_0)}(\varphi) := \sigma,v, p,v \mathbf{n} \) on \( \Gamma \).

The main ingredient of the proof is to write the equation satisfied by \((v, p, \tau)\) in the fixed domain, using a change of variables, and then to observe that the operators involved and their inverses depend in a smooth manner on \( \Psi \).

The same stability result can be established for the linear or nonlinear evolution problem. To conclude this section, here is how this paper is organized. In section 2 we recall a few results on unique continuation properties for the Stokes or Navier–Stokes systems which are essential in the proof of the identifiability result. First we mention a unique continuation result for the Stokes equation due to Fabre and Lebeau [8, p 576].

In this section, we gather some preliminary results about unique continuation properties, which are essential in the proof of the identifiability result. First we mention a unique continuation result for the evolution problem as well as the stationary one. Section 4 is devoted to establishing the analyticity of the mapping \( \Psi_{\tau} \mapsto \Lambda_{\Psi_{\tau}(D_0)} \), for the linear stationary Stokes system, and there we prove the stability result mentioned above. We determine also the first derivative of this mapping, which is necessary in order to apply an optimization algorithm studied in an optimal control approach. In section 5 we present analyticity results for the linear or nonlinear evolution problems and the corresponding stability results.

2. Preliminary results

In this section, we gather some preliminary results about unique continuation properties, which are essential in the proof of the identifiability result. First we mention a unique continuation result for the Stokes equation due to Fabre and Lebeau [8, p 576].

**Proposition 2.1.** Let \( \Omega_0 \subseteq \mathbb{R}^N, N \geq 2 \), be a bounded domain and \( \omega_0 \) an open subset of \( \Omega_0 \). If \( a \in L^\infty(\Omega_0)^N \) and \( (v, p) \in H^1_{\text{loc}}(\Omega_0)^N \times L^2_{\text{loc}}(\Omega_0) \) is a solution of

\[
\begin{cases}
-\Delta v + (a \cdot \nabla)v + \nabla p = 0, & \text{in } \Omega_0 \\
\nabla \cdot v = 0, & \text{in } \Omega_0,
\end{cases}
\]

with \( v = 0 \) in \( \omega_0 \), then \( v = 0 \) in \( \Omega_0 \) and \( p \) is constant in \( \Omega_0 \).

A direct consequence of the above result is the following unique continuation property.

**Corollary 2.2.** Let \( \Omega_0 \subseteq \mathbb{R}^N, N \geq 2 \), be a Lipschitz domain. If \( (v, p) \in H^1(\Omega_0)^N \times L^2(\Omega_0) \) is a solution of

\[
\begin{cases}
-\text{div } \sigma(v, p) = 0 & \text{in } \Omega_0 \\
\text{div } v = 0 & \text{in } \Omega_0 \\
v = 0 & \text{on } \partial \Omega_0
\end{cases}
\]

satisfying \( \sigma(v, p) \mathbf{n} = 0 \) on \( \Gamma_0 \), where \( \Gamma_0 \subset \partial \Omega_0 \) is a relatively open non-empty subset, then \( v = 0 \) and \( p \) is constant in \( \Omega_0 \).

Analogously to the stationary case we recall the following unique continuation result due to Fabre and Lebeau [8, p 574], for the non-steady Stokes problem. Let \( \Omega_0 \subseteq \mathbb{R}^N \) be an open connected set, \( N \geq 2 \) and \( T > 0 \). We consider an open non-empty subset \( O \) of \( \Omega_0 \times (0, T) \). Let us define the horizontal component of \( O \) as

\[
C(O) = \{ (x, t) \in \Omega_0 \times (0, T) : \exists x_0 \in \Omega_0, (x_0, t) \in O \}.
\]
Theorem 2.3. Let $\Omega_0 \subset \mathbb{R}^N$ be a connected open set, $N \geq 2$ and $T > 0$. Let $a \in L^\infty_{\text{loc}}(\Omega_0 \times (0, T))^N$ and $c \in C([0, T]; L^1_{\text{loc}}(\Omega_0, M_{N \times N}))$ be a matrix-valued function with $r > N$. If $(v, p) \in L^2(0, T; H^1_0(\Omega_0))^N \times L^2(\Omega_0 \times (0, T))$ is a solution of

$$
\begin{align*}
\frac{\partial v}{\partial t} - \Delta v + (a \cdot \nabla)v + cv + \nabla p &= 0 & \text{in } \Omega_0 \times (0, T) \\
\text{div}(v) &= 0 & \text{in } \Omega_0 \times (0, T),
\end{align*}
$$

(2.3)

with $v = 0$ in $O$, then $v \equiv 0$ in $C(O)$.

The following result is also a consequence of the above theorem:

Corollary 2.4. Let $\Omega_0 \subset \mathbb{R}^N$ be a Lipschitz domain, $N \geq 2$. For $\varepsilon_* = 0$ or $\varepsilon_* = 1$, if $(v, p) \in L^2(T_1, T_2; H^1(\Omega_0))^N \times L^2(\Omega_0 \times (T_1, T_2))$ (with $0 \leq T_1 < T_2$) is a solution of

$$
\begin{align*}
\frac{\partial v}{\partial t} - \text{div}(\sigma(v, p)) + \varepsilon_* \text{div}(v \otimes v) &= 0 & \text{in } \Omega_0 \times (T_1, T_2) \\
\text{div}(v) &= 0 & \text{in } \Omega_0 \times (T_1, T_2) \\
v &= 0 & \text{on } \Gamma_0 \times (0, T),
\end{align*}
$$

(2.4)

satisfying $\sigma(v, p)n = 0$ on $\Gamma_0 \times (0, T)$, where $\Gamma_0 \subset \partial \Omega_0$ is a relatively open non-empty subset, then $v = 0$ in $\Omega_0 \times (0, T)$.

Remark. Fabre and Lebeau prove the above results in the case $c \equiv 0$, using appropriate local Carleman inequalities. As a matter of fact, a careful examination of their arguments shows that if $c \in L^\infty_{\text{loc}}(\Omega_0, M_{N \times N})$ is a matrix-valued function with $r > N$, then assuming that $(v, p) \in H^1_{\text{loc}}(\Omega_0) \times L^2_{\text{loc}}(\Omega_0)$ satisfies

$$
\begin{align*}
\frac{\partial v}{\partial t} - \Delta v + (a \cdot \nabla)v + cv + \nabla p &= 0 & \text{in } \Omega_0 \\
\nabla \cdot v &= 0 & \text{in } \Omega_0,
\end{align*}
$$

(2.5)

and $v = 0$ in $\partial \Omega_0$ (an open subdomain of $\Omega_0$) then $v \equiv 0$ in $\Omega_0$ and $p$ is constant. In particular, if $\Omega_0$ is smooth enough and if $(v, p)$ satisfies (2.5) and $v = 0$ on $\partial \Omega_0$ and if $\sigma(v, p)n = 0$ on an open subset $\Gamma \subset \partial \Omega_0$, then $v \equiv 0$ in $\Omega_0$ and $p$ is constant (see also Fernández-Cara [9]).

3. The proof of identifiability results

We begin with the proof of the identifiability result for the non-steady case, that is, the proof of theorem 1.1.

Consider first the linear case, that is, $\varepsilon_* = 0$. Let $v_0, v_1$ be solutions of system (1.8) for $j = 0, 1$, verifying

$$
\sigma(v_0, p_0)n = \sigma(v_1, p_1)n \quad \text{on } \Gamma \times (0, T),
$$

and define

$$
v = v_0 - v_1, \quad p = p_0 - p_1 \quad \text{and} \quad D = D_0 \cup D_1
$$

(see figure 2).

It is straightforward to see that $(v, p)$ is a solution of

$$
\begin{align*}
\frac{\partial v}{\partial t} - \text{div } \sigma(v, p) &= 0 & \text{in } (\Omega \setminus \overline{D}) \times (0, T) \\
\text{div}(v) &= 0 & \text{in } (\Omega \setminus \overline{D}) \times (0, T) \\
v(0) &= 0 & \text{in } \Omega \setminus \overline{D} \\
v &= 0 & \text{on } \partial \Omega \times (0, T) \\
v &= 0 & \text{on } \partial D \times (0, T).
\end{align*}
$$

(3.1)
Therefore, from the unique continuation property for the Stokes equation (corollary 2.4), we have that \( v = 0 \) in \( \Omega \setminus \overline{D} \times (0, T) \) and then \( v_0 = v_1 \) in \( \Omega \setminus \overline{D} \times (0, T) \).

Now, let us suppose that \( D_0 \setminus \overline{D}_1 \) is an open non-empty subset of \( \Omega \), then we have that

\[
\frac{\partial v_1}{\partial t} - \text{div}(\sigma(v_1, p_1)) = 0 \quad \text{in} \quad D_0 \setminus \overline{D}_1 \times (0, T).
\]

Multiplying this equation by \( v_1 \) and integrating by parts in \( D_0 \setminus \overline{D}_1 \), noting that on \( \partial D_0 \) we have \( v_1 = v_0 = 0 \), while on \( \partial D_1 \) by assumption we know that \( v_1 = 0 \), we obtain that

\[
\frac{d}{dt} \int_{D_0 \setminus \overline{D}_1} |v_1(x, t)|^2 \, dx = - \int_{D_0 \setminus \overline{D}_1} |\sigma(v_1)(x, t)|^2 \, dx,
\]

which implies that the function

\[
t \mapsto E(t) = \int_{D_0 \setminus \overline{D}_1} |v_1(x, t)|^2 \, dx
\]

is a decreasing non-negative function but, since the initial data \( v_1(0) = 0 \), we conclude that \( v_1|_{D_0 \setminus \overline{D}_1} = 0 \) for all \( t \in (0, T) \). Thus, from theorem 2.3 we get that

\[
v_1 = 0 \quad \text{in} \quad \Omega \setminus \overline{D}_1 \times (0, T),
\]

which is impossible because \( v_1 = \varphi \) and \( \varphi \neq 0 \) on \( \Gamma \times (0, T) \). Therefore we have that \( D_0 \setminus \overline{D}_1 = \emptyset \). Analogously one proves that \( D_1 \setminus \overline{D}_0 = \emptyset \). Thus we have that \( D_0 = D_1 \). This completes the proof in the simpler linear case \( \varepsilon_n = 0 \).

Consider now the nonlinear case \( \varepsilon_n = 1 \). Then \((v, p) := (v_0 - v_1, p_0 - p_1)\) solves

\[
\begin{align*}
\frac{\partial v}{\partial t} - \text{div}(\sigma(v, p) + (v_0 - \nabla)v + (v \cdot \nabla)v_1) &= 0 & \text{in} & & \Omega \setminus \overline{D} \times (0, T), \\
\text{div}(v) &= 0 & \text{in} & & \Omega \setminus \overline{D} \times (0, T), \\
v(0) &= 0 & \text{on} & & \partial \Omega \times (0, T), \\
v &= 0 & \text{on} & & \partial D \times (0, T).
\end{align*}
\]

Since \( \varphi \in C^1([0, T]; H^{3/2}(\partial \Omega)) \), classical existence results for the Navier–Stokes equation (see, for instance, [13] or [14]) show that there exists \( T_* \in (0, T) \) such that a unique solution \((v_j, p_j)\) to equation (1.8) exists on the interval \([0, T_*]\) in such a way that \( v_j \in C([0, T_*]; H^3(\Omega)) \). Therefore, upon setting \( c_{k,\ell} := \partial_{\ell} v_{1,k} \), for \( 1 \leq k, \ell, N \), we have \( c_{k,\ell} \in C([0, T_*]; H^1(\Omega)) \) and by Sobolev imbedding theorems the matrix-valued function \( c := (c_{k,\ell})_{1 \leq k, \ell \leq N} \) satisfies the assumption of corollary 2.4, and since \( \sigma(v, p)n = 0 \) on \( \Gamma \times [0, T_*] \) we conclude that \( v \equiv 0 \) in \( \Omega \setminus \overline{D} \times (0, T_*) \), that is, \( v_0 = v_1 \) in \( \Omega \setminus \overline{D} \times (0, T_*) \).
If $D_0 \setminus D_1$ were non-empty, then we would have
\[
\frac{\partial v_1}{\partial t} - \text{div}(\sigma(v_1, p_1)) + (v_1 \cdot \nabla)v_1 = 0 \quad \text{in} \quad D_0 \setminus D_1 \times (0, T_\ast).
\]

Proceeding as above, multiplying this equation by $v_1$ and integrating by parts in $D_0 \setminus D_1$, noting that on $\partial D_0$ we have $v_1 = v_0 = 0$, while on $\partial D_1$ by assumption we know that $v_1 = 0$, and noting that
\[
\int_{D_0 \setminus D_1} [(v_1(x, t) \cdot \nabla)v_1(x, t)] \cdot v_1(x, t) \, dx = 0
\]
we obtain that for $0 < t < T_\ast$ we have
\[
\frac{d}{dt} \int_{D_0 \setminus D_1} |v_1(x, t)|^2 \, dx = -\int_{D_0 \setminus D_1} |e(v_1)(x, t)|^2 \, dx,
\]
which implies that the function
\[
t \mapsto E(t) = \int_{D_0 \setminus D_1} |v_1(x, t)|^2 \, dx
\]
is a decreasing non-negative function on $[0, T_\ast]$. At this point the reader is easily convinced that the remainder of the argument follows exactly the lines of the proof of the linear case seen above, and that finally this implies $D_0 \setminus D_1 = \emptyset$. In the same manner one shows that $D_1 \setminus D_0 = \emptyset$ and thus $D_0 = D_1$.

Next we present the identifiability result for the stationary case, that is,

**Theorem 3.1.** Let $\Omega \subseteq \mathbb{R}^N$, with $N \geq 2$, be a bounded Lipschitz domain and $\Gamma$ be a non-empty open subset of $\partial \Omega$. Let $\varphi \in H^{1/2}(\partial \Omega)^N$ be a given non-homogeneous Dirichlet boundary condition. Assume that for $j = 0, 1, D_j \in \mathcal{D}_{ad}$ are open sets in $\Omega$ and $(v_j, p_j)$ are the solutions of
\[
\begin{cases}
-\text{div}(\sigma(v_j, p_j)) = 0 & \text{in} \quad (\Omega \setminus D_j) \\
\text{div}(v_j) = 0 & \text{in} \quad (\Omega \setminus D_j) \\
v_j = \varphi & \text{on} \quad \partial \Omega \\
v_j = 0 & \text{on} \quad \partial D_0
\end{cases}
\]

such that
\[
\sigma(v_0, p_0)n = \sigma(v_1, p_1)n \quad \text{on} \quad \Gamma.
\]

Then $D_0 \equiv D_1$.

**Proof of theorem 3.1.** Let us define
\[
v := v_0 - v_1, \quad p = p_0 - p_1 \quad \text{and} \quad D = D_0 \cup D_1.
\]

One sees that $(v, p)$ satisfies
\[
\begin{cases}
-\text{div}(\sigma(v, p)) = 0 & \text{in} \quad \Omega \setminus \overline{D} \\
\text{div} v = 0 & \text{in} \quad \Omega \setminus \overline{D} \\
v = 0 & \text{on} \quad \Gamma \\
\sigma(v, p)n = 0 & \text{on} \quad \Gamma,
\end{cases}
\]

thus, in view of corollary 2.2, we have $v = 0$ in $\Omega \setminus \overline{D}$, and therefore $v_0 = v_1$ in $\Omega \setminus \overline{D}$.

Let us suppose that $D_0 \setminus D_1$ is an open non-empty subset of $\Omega$. We know that
\[
-\text{div}(\sigma(v_1, p_1)) = 0 \quad \text{in} \quad D_0 \setminus D_1,
\]
hence, multiplying this equation by \( v_1 \) and integrating by parts in \( D_0 \setminus D_1 \), we obtain

\[
\int_{D_0 \setminus D_1} |e(v_1)|^2 \, dx + \int_{\partial(D_0 \setminus D_1)} (\sigma(v_1, p_1)n) \cdot v_1 \, ds = 0.
\]

Now, since \( v_0 = v_1 = 0 \) on \( \partial D_0 \setminus (\partial D_1 \cap \partial D_0) \), we have that

\[
\int_{\partial(D_0 \setminus D_1)} (\sigma(v_1, p_1)n) \cdot v_1 \, ds = 0,
\]

therefore we get

\[
\int_{D_0 \setminus D_1} |e(v_1)|^2 \, dx = 0.
\]

It follows immediately from Korn’s inequality (see, for instance, [13] or [11, p 50]) that \( v_1 \equiv 0 \) in \( D_0 \setminus D_1 \), then, by proposition 2.1, \( v_1 \equiv 0 \) in \( \Omega \setminus \Omega_1 \), which is impossible because \( v_1 = \varphi \) and \( \varphi \neq 0 \) on \( \partial \Omega \). This proves that \( D_0 \setminus D_1 = \emptyset \). Analogously we prove that \( D_1 \setminus D_0 = \emptyset \). Therefore \( D_0 = D_1 \). This completes the proof. \( \square \)

4. Smooth dependence of Cauchy forces with respect to the deformation of the domain: the linear case

In this section, we consider the linear stationary Stokes equations and we prove that the velocity and the pressure depend smoothly on the deformations of the domain. For the sake of simplicity we set the viscosity to be \( \nu = 1 \). The proof of the result concerning the non-stationary case is essentially based on the results of this section and will be presented later on.

Let \( \Omega \) and \( \mathcal{O} \) be open connected sets in \( \mathbb{R}^N \) and \( D_0, D_1 \in \mathcal{P}_{ad} \) (see figure 3) such that

\[
D_0 \cup D_1 \subset \subset \mathcal{O} \subset \subset \Omega,
\]

and set

\[
\Omega_0 = \Omega \setminus D_0 \quad \text{and} \quad \Omega_1 = \Omega \setminus D_1.
\]

For the sake of clarity, we introduce some basic notation and definitions. Let \( \Psi = (\Psi^1, \ldots, \Psi^N) \in W^{1,\infty}(\Omega)^N \), then we write

\[
\Psi' := \left( \frac{\partial \Psi^i}{\partial x_j} \right)_{i,j=1}^N, \quad \text{Jac}(\Psi) := |\text{det}(\Psi')|,
\]
the Jacobian matrix and the Jacobian determinant of $\Psi$ (actually $\Psi$ is going to be a smooth bijective map $\Omega_0 \to \Omega_1$). If $A \in W^{1,\infty}(\Omega; M_{N \times N})$ is a matrix-valued function, then we define the vector $\text{div}(A)$ as being

$$
\text{div}(A) := \left( \sum_{i=1}^{N} \frac{\partial A_{ij}}{\partial x_j} \right)_{i=1}^{N}.
$$

(4.1)

For each function $f \in H^1(\Omega_1)$ and $\Psi \in W^{1,\infty}(\Omega_0; \Omega_1)$ we may set $g(x) := f(\Psi(x))$, in such a way that we have $g \in H^1(\Omega_0)$. Furthermore, when $\Psi$ is bijective and $\Psi', (\Psi')^{-1} \in L^\infty(\Omega_0, M_{N \times N})$, the mapping $f \mapsto g$ induces an isomorphism between the spaces $H^1_0(\Omega_1)^N$ and $H^1_0(\Omega_0)^N$ on the one hand, and between $L^2(\Omega_1)$ and $L^2(\Omega_0)$ on the other hand. We shall denote this isomorphism by $\tilde{\Psi}$:

$$
\tilde{\Psi} : H^1_0(\Omega_1)^N \longrightarrow H^1_0(\Omega_0)^N,
$$

$$
\tilde{\Psi}(f) := g \quad \text{with} \quad g(x) := f(\Psi(x)).
$$

If $y$ denotes the variable in $\Omega_1$ and that in $\Omega_0$ is denoted by $x$, using the change of variables $y := \Psi(x)$ and setting $g(x) := f(\Psi(x))$ we have

$$
\frac{\partial f}{\partial y_i}(y) = \frac{\partial f}{\partial y_i}(\Psi(x)) = \sum_{j=1}^{N} \frac{\partial g}{\partial x_j}(x) \frac{\partial x_j}{\partial y_i}.
$$

Let $\Psi \in W^{1,\infty}(\Omega_0, \Omega_1)$ be a diffeomorphism. We shall denote by $M \in L^\infty(\Omega_0, M_{n \times n})$ the matrix

$$
M := \left( \frac{\partial x_j}{\partial y_i} \right)_{i,j=1,N} = \left( \left( \frac{\partial \Psi_i}{\partial x_j} \right)_{i,j=1,N} \right)^* = ((\Psi')^*)^{-1},
$$

where $A^*$ denotes the transpose matrix of the matrix $A$. Note that if $\Psi$ is a small perturbation of the identity in the norm of $W^{1,\infty}$ then the matrix $M$ is invertible, and $D_1$ is in some sense a ‘small perturbation’ of $D_0$. Note that with the above notation $y := \Psi(x)$ and $g(x) := f(\Psi(x))$, we have $\nabla f(y) = M(x) \nabla g(x)$.

Also if $A$ and $B$ are $N \times N$ matrices, we denote their scalar product by

$$
A : B := \sum_{i,j=1}^{N} A_{ij} B_{ij},
$$

and $|A|^2 := \sum_{i,j=1}^{N} |A_{ij}|^2$.

Instead of writing equations (3.6) using the stress tensor (1.3), we use the following equivalent form (4.2) (since for $v = 1$ and $v$ such that $\text{div}(v) = 0$ one has $\text{div}(\sigma(v, p)) = \Delta v - \nabla p$; here for a matrix $A$ we denote by $\text{Tr}(A)$ its trace):

$$
\begin{cases}
-\Delta v_j + \nabla p_j = 0 & \text{in } \Omega_1 \\
\text{div}(v_j) = \text{Tr}(\nabla v_j) = 0 & \text{in } \Omega_1 \\
v_j = \varphi & \text{on } \partial \Omega \\
v_j = 0 & \text{on } \partial D_1.
\end{cases}
$$

(4.2)

Our aim is to show that $\sigma(v_1, p_1)\mathbf{n}$ and $\sigma(v_0, p_0)\mathbf{n}$ are close when $\Psi$ is close to the identity map, but since the equations are set in different domains we will change $(v_1, p_1)$ to a pair of functions $(u_0, q_0)$ defined in $\Omega_0$ which satisfy certain elliptic equations with variable $L^\infty$ coefficients, and then we show that the dependence of $(u_0, q_0)$, and thus that of $\sigma(u_0, q_0)\mathbf{n}$, on these coefficients is analytic.

Since both subdomains $D_0$ and $D_1$ have to be admissible, we will consider only those deformations $\Psi$ which respect this requirement (see figure 4):
Identification of immersed obstacles via boundary measurements

**Definition 4.1.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \). A mapping \( \Psi \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \) is called an admissible deformation of \( D_0 \subset \Omega \), if \( \Psi \) is bijective, \( \Psi^{-1} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \), \( \Psi(\Omega) = \Omega \) and there exists a Lipschitz domain \( \mathcal{O} \subset \subset \Omega \) such that
\[
D_0 \cup \Psi(D_0) \subset \subset \mathcal{O}
\]
and for some \( \delta > 0 \) one has \( \Psi = I \) on \( \Omega \setminus \overline{\mathcal{O}_\delta} \), where
\[
\mathcal{O}_\delta := \{ x \in \mathcal{O}; \text{dist}(x, \mathcal{O}) > \delta \}.
\]

Put in other terms, the class of admissible deformations \( \Psi \) we consider can be written in the form \( \Psi = I + \Psi_1 \) with \( \Psi_1 \) having a compact support contained in \( \mathcal{O} \), or \( \Psi_1 \in W^{1,\infty}_0(\mathcal{O}, \mathbb{R}^N) \) (when \( \mathcal{O} \) is Lipschitz, we may set \( \Psi_1 := \Psi \) in \( \mathcal{O} \) and \( \Psi_1 = 0 \) in \( \mathbb{R}^N \setminus \overline{\mathcal{O}} \); this yields a function \( \Psi_1 \) with compact support contained in \( \mathcal{O} \)).

When \( \varphi \) satisfies the compatibility condition
\[
\int_{\partial \Omega} \varphi(s) \cdot n(s) \, ds = 0, \tag{4.3}
\]
on one can find \( \tilde{\varphi}_1 \in H^1(\Omega)^N \) such that
\[
\text{div}(\tilde{\varphi}_1) = 0 \text{ in } \Omega, \quad \tilde{\varphi}_1 = 0 \text{ in } \mathcal{O}, \quad \tilde{\varphi}_1 = \varphi \text{ on } \partial \Omega, \tag{4.4}
\]
and then, upon writing
\[
v_1 := \tilde{\varphi}_1 + \tilde{v}_1, \quad \text{with } \tilde{v}_1 \in H^1_0(\Omega_1)^N,
\]
solving equation (4.2) with \( j = 1 \) is equivalent to solving
\[
\begin{align*}
-\Delta \tilde{v}_1 + \nabla p_1 &= \Delta \tilde{\varphi}_1, & \text{in } & \Omega_1, \\
\text{div}(\tilde{v}_1) &= \text{Tr}(\nabla \tilde{\varphi}_1) = 0, & \text{in } & \Omega_1, \\
\tilde{v}_1 &= 0 & \text{on } & \partial \Omega_1.
\end{align*}
\tag{4.5}
\]

In the following, we shall denote by \( L^2_\mathbb{R}(\Omega_1) \) the space \( L^2(\Omega_1)/\mathbb{R} \), that is,
\[
L^2_\mathbb{R}(\Omega_1) := \left\{ q \in L^2(\Omega_1); m(q) := \frac{1}{|\Omega_1|} \int_{\Omega_1} q(y) \, dy = 0 \right\}.
\]

Then equation (4.5) is equivalent to stating that \( (\tilde{v}_1, p_1) \in H^1(\Omega_1)^N \times L^2_\mathbb{R}(\Omega_1) \) and for all \( (w, p) \in H^1_0(\Omega_1)^N \times L^2_\mathbb{R}(\Omega_1) \)
\[
\int_{\Omega_1} \nabla \tilde{v}_1 : \nabla w \, dy - \int_{\Omega_1} p_1 \text{Tr}(\nabla w) \, dy + \int_{\Omega_1} \text{Tr}(\nabla \tilde{\varphi}_1) p \, dy = - \int_{\Omega_1} \nabla \tilde{\varphi}_1 : \nabla w \, dy.
\tag{4.6}
\]

Next, \( \Psi \) being admissible as in definition 4.1, we define \( (u_0, q_0) \) by setting
\[
\begin{align*}
y &:= \Psi(x), \\
u_0(x) &:= \tilde{v}_1(\Psi(x)), \\
q_0(x) &:= p_1(\Psi(x)), \\
q_0(x) &:= \tilde{\varphi}_1(\Psi(x)).
\end{align*}
\tag{4.7}
\]
Note that since \( \tilde{\varphi}_1 = 0 \) in \( \mathcal{O} \) and \( \Psi = I \) in \( \Omega \backslash \bar{\Omega}_0 \), we have \( \varphi_0(x) = \tilde{\varphi}_1(x) \) for all \( x \in \Omega \), meaning that \( \tilde{\varphi}_1 \) is invariant under the isomorphism \( \tilde{\Psi} \).

With the above notation (in particular \( M(x) := (\Psi'(x)^{-1})^* \)), one checks easily that solving the variational problem \((4.6)\) is equivalent to finding \((u_0, q_0) \in H^1(\Omega_0) \times L^2(\Omega_0)\) satisfying for all \((z, q) \in H^1_0(\Omega_0)^N \times L^2(\Omega_0)\)

\[
\int_{\Omega_0} M \nabla u_0 : M \nabla z \text{Jac}(\Psi) \, dx - \int_{\Omega_0} q_0 \text{Tr}(M \nabla z) \text{Jac}(\Psi) \, dx
+ \int_{\Omega_0} \text{Tr}(M \nabla u_0) q \text{Jac}(\Psi) \, dx = - \int_{\Omega_0} M \nabla \varphi_0 : M \nabla z \text{Jac}(\Psi) \, dx.
\]

(4.8)

This variational problem means that \((u_0, q_0)\) satisfies the following equation in \( \Omega_0 \):

\[
\begin{cases}
-\text{div}(\text{Jac}(\Psi) M^* M \nabla u_0) + \text{div}(q_0 \text{Jac}(\Psi) M^*) = \text{div}(\text{Jac}(\Psi) M^* M \nabla \varphi_0) & \text{in } \Omega_0 \\
\text{Tr}(M \nabla u_0) = 0 & \text{in } \Omega_0 \\
u_0 = 0 & \text{on } \partial \Omega_0.
\end{cases}
\]

(4.9)

Now our aim is to prove that the solution \((u_0, q_0)\) depends smoothly on \( \Psi \), in the natural \( W^{1, \infty} \) norm.

Define the space \( L^2_\Psi(\Omega_0) \) as being the space \( L^2(\Omega_0) \) endowed with the scalar product

\[
(p|q)_\Psi := \int_{\Omega_0} p(x)q(x) \text{Jac}(\Psi)(x) \, dx,
\]

(this can be viewed as the image of the usual scalar product of \( L^2(\Omega_1) \) under the isomorphism \( \Psi \) and the elliptic operator

\[
Au := -\frac{1}{\text{Jac}(\Psi)} \text{div}(\text{Jac}(\Psi) M^* M \nabla u)
\]

(4.10)

on \( H^1_0(\Omega_0)^N \), which is self-adjoint with respect to the scalar product of \( L^2_\Psi(\Omega_0) \), i.e., it is such that

\[
(Au|v)_\Psi = (u|Av)_\Psi.
\]

Define also the operator \( B \) by

\[
Bz := -\text{Tr}(M \nabla z), \quad B : H^1_0(\Omega_0)^N \rightarrow L^2_{0, \Psi}(\Omega_0),
\]

(4.11)

where \( L^2_{0, \Psi}(\Omega_0) \) is the subspace of functions \( p \) in \( L^2_\Psi(\Omega_0) \) such that

\[
\int_{\Omega_0} p(x) \text{Jac}(\Psi)(x) \, dx = 0.
\]

(Note that the fact that for \( z \in H^1_0(\Omega_0)^N \) one has \( Bz \in L^2_{0, \Psi}(\Omega_0) \) is a consequence of the fact that for \( \varepsilon \in H^1_0(\Omega_1)^N \) one has \( \text{Tr}(\nabla \varepsilon) \in L^2_{0}(\Omega_1) \)). We prove first the following lemma:

**Lemma 4.2.** The range \( R(B) \) of \( B \) is closed and the adjoint \( B^* \) of \( B \) is given by

\[
B^* q = \frac{1}{\text{Jac}(\Psi)} \text{div}(q \text{Jac}(\Psi) M^*).
\]

Moreover, the kernel \( N(B^*) \) is precisely given by the functions which are constant on \( \Omega_0 \), while \( R(B) = N(B^*)^\perp = L^2_{0, \Psi}(\Omega_0) \).

**Proof.** We prove first that \( R(B) \), the range of \( B \), is closed. Indeed, it is a classical result that the range of the mapping \( w \mapsto \text{div}(w) = \text{Tr}(\nabla w) \) from \( H^1_0(\Omega_1)^N \) into \( L^2(\Omega_1) \) is closed (see, for instance, [13] or [14]). Let \( \tilde{\Psi} \) be the mapping \( w \mapsto z := w \circ \Psi := \tilde{\Psi}(w) \), which is an isomorphism between the Hilbert spaces \( H^1_0(\Omega_1)^N \) and \( H^1_0(\Omega_0)^N \) on the one hand, and \( L^2(\Omega_1) \) on the other.
and $L^2_\Psi(\Omega_0)$ on the other hand, as recalled above. Now, since $Bz = -\tilde{\Psi}(\nabla \tilde{\Psi}^{-1}(z))$, one sees that $B$ has a closed range.

Next we determine the adjoint of $B$ with respect to the scalar product $(\cdot | \cdot)$. If $q \in C^{\infty}(\Omega_0)$ and $z \in C^{\infty}(\Omega_0)^N$, we have (using, in the last step of the following, the definition of the divergence, div, of a matrix, see (4.1))

$$
(B^* q, z) = \langle q, Bz \rangle = -\int_{\Omega_0} q(x) \text{Tr}(M(x) \nabla z(x)) \text{Jac}(\Psi)(x) \, dx
$$

$$
= -\sum_{i=1}^N \sum_{j=1}^N \int_{\Omega_0} M_{ij}(x) q(x) \frac{\partial z_j(x)}{\partial x_i} \text{Jac}(\Psi)(x) \, dx
$$

$$
= \sum_{i=1}^N \sum_{j=1}^N \int_{\Omega_0} \frac{\partial}{\partial x_i} [\text{Jac}(\Psi)(x) M_{ij}(x) q(x)] z_j(x) \, dx
$$

$$
= \int_{\Omega_0} \text{div}(q(x) \text{Jac}(\Psi)(x) M^*(x)) \cdot z(x) \, dx.
$$

From this, and the density of smooth functions in $L^2(\Omega_0)$ and $H^1_0(\Omega_0)^N$, we conclude that $B^* q = \frac{1}{\text{Jac}(\Psi)} \text{div}(q \text{Jac}(\Psi)(x) M^*)$.

To finish the proof of the lemma, assume that $q \in L^2(\Omega_0)$ is such that $B^* q = 0$. Then $q \in R(B)^\perp$, that is, for all $z \in H^1_0(\Omega_0)^N$ we have (upon setting $p(y) := q(\Psi^{-1}(y))$ and $w(y) := z(\Psi^{-1}(y))$)

$$
0 = (B^* q, z) = -\int_{\Omega_1} q(x) \text{Tr}(M(x) \nabla z(x)) \, dx
$$

$$
0 = -\int_{\Omega_1} p(y) \text{Tr}(\nabla w(y)) \, dy,
$$

for all $w \in H^1_0(\Omega_1)^N$. This implies that $p$ is constant in $\Omega_1$, and thus $q$ is constant in $\Omega_0$. □

Now one can see that equation (4.9) can be written in the form

$$
\begin{cases}
Au_0 + B^* q_0 = f & \text{in } \Omega_0 \\
Bu_0 = 0 & \text{in } \Omega_0 \\
u_0 = 0 & \text{on } \partial \Omega_0
\end{cases}
$$

in which we denote

$$
f := \frac{1}{\text{Jac}(\Psi)} \text{div}(\text{Jac}(\Psi) M^* M \nabla \varphi_0),
$$

and where $q_0$ is determined uniquely up to the addition of an element in $N(B^*)$, that is, up to the addition of a constant.

We are finally in a position to state and show the analytic dependence of $(u_0, q_0)$ on $\Psi$.

**Theorem 4.3.** Let $\mathcal{O} \subset \subset \Omega$ be a Lipschitz domain, and $\Psi_1 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be such that $\Psi := I + \Psi_1$ is an admissible deformation according to definition 4.1. The mapping $\Psi_1 \mapsto (u_0, q_0)$ from $W^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ into $H^1_0(\Omega_0)^N \times L^2(\Omega_0)$ is analytic in a neighbourhood of the origin.

**Proof.** We begin by pointing out that, due to the assumptions on $\Psi$, the function $f$ defined above is actually independent of $\Psi$, since where $\Psi \neq I$ we have $\varphi_0 \equiv 0$. It is elementary
to verify that the operator \( H_0^1(\Omega_0)^N \rightarrow H^{-1}(\Omega_0)^N \) is an isomorphism. Then the first equation of (4.13) implies that
\[
\begin{align*}
\mu_0 + A^{-1}B^*q_0 &= A^{-1}f, \\
\tag{4.14}
\end{align*}
\]
and upon applying \( B \) to both sides of this, and using the fact that \( Bu = 0 \), we find that \( q_0 \) is given by the equation
\[
B A^{-1}B^*q_0 = B A^{-1}f. \tag{4.15}
\]
However, due to the fact that \( A^{-1} \) is coercive, that is, for some \( \alpha > 0 \) one has
\[
\langle g, A^{-1}g \rangle \geq \alpha ||g||_{H^{-1}}^2,
\]
one checks that the operator \( B A^{-1}B^* \) is continuous and one-to-one in \( N(B^*)^\perp \). Therefore, up to the addition of a constant, \( q_0 \) is uniquely determined by
\[
q_0 = (BA^{-1}B^*)^{-1}BA^{-1}f. \tag{4.16}
\]
Note that \( q_0 \in L^2_{0,q}(\Omega_0) \), that is,
\[
\int_{\Omega} q_0(x) \text{Jac}(\Psi)(x) \, dx = 0.
\]
However, since \( N(B^*) \) is the one-dimensional subset of constant functions on \( \Omega_0 \), and thus independent of \( \Psi \), we may normalize \( q_0 \) by adding a constant so that
\[
\int_{\Omega} q_0(x) \, dx = 0.
\]
From (4.16) and (4.14) one concludes that \( \mu_0 \) is given by
\[
\mu_0 = (I - A^{-1}B^*(BA^{-1}B^*)^{-1}B)A^{-1}f. \tag{4.17}
\]
It is clear that the mappings \( \Psi_1 \mapsto A \) and \( \Psi_1 \mapsto B^* \), in a neighbourhood of the origin, are analytic from \( W_{0,\infty}^1(\mathcal{O}, \mathbb{R}^N) \) into \( \mathcal{L}(H_0^1(\Omega_0)^N; H^{-1}(\Omega_0)^N) \) and \( \mathcal{L}(L^2(\Omega_0), H^{-1}(\Omega_0)^N) \) respectively, and since so is the inversion of continuous operators, the formulae (4.16) and (4.17) show that the mapping \( \Psi_1 \mapsto (\mu_0, q_0) \) is analytic in a neighbourhood of the origin in \( W_{0,\infty}^1(\mathcal{O}, \mathbb{R}^N) \).

**Remark.** As a matter of fact, denoting by \( A_0 \) and \( B_0 \) the corresponding operators when \( \Psi_1 = 0 \), that is, \( A_{0\mu} := -\Delta u \) and \( B_{0\mu} := -\text{div}(u) \), using (4.17) one can write \( \mu_0 \) as a series. Indeed \( A = A_0 + A_1 \) and \( B = B_0 + B_1 \) where \( \|A_1\| \) and \( \|B_1\| \) are small. So we may write
\[
BA^{-1}B^* = B_0A_0^{-1}B_0^*(I - L), \quad A^{-1} = \left( \sum_{k \geq 0} (-A_0^{-1}A_1)^k \right) A_0^{-1}
\]
where \( \|L\| \) is small. Therefore
\[
(BA^{-1}B^*)^{-1} = \left( \sum_{k \geq 0} L^k \right) (B_0A_0^{-1}B_0^*)^{-1}.
\]
From these expressions one sees that for a sequence of operators, say \( (L_k)_k \), continuous from \( H^{-1}(\Omega_0)^N \) into \( H_0^1(\Omega_0)^N \), one has
\[
\mu_0 = \sum_{k \geq 0} L_k f,
\]
but, since we will not use this expression, we do not insist on determining the sequence \( (L_k)_k \) in terms of \( \Psi_1 \).
Remark. We should point out that even though the above analysis shows that the mapping \( \Psi_1 \mapsto (u_1, q_1) \) is analytic we cannot say anything about the analyticity of the mapping \( \Psi_1 \mapsto (v_1, p_1) \): indeed the only thing we can infer is that the mapping
\[
\Psi_1 \mapsto \sigma(v_1, p_1)n = \sigma(u_1, q_1)n
\]
is analytic.

The following corollary is of interest in the next section, where we show that the smooth dependence of \( \sigma(v_1, p_1)n \) on \( \Psi \) extends to the linear or nonlinear evolution equations.

Corollary 4.4. Let \( \lambda > 0 \) be a fixed parameter, let the assumptions of theorem 4.3 be satisfied, and for a given \( f \in H^1(\Omega)^N \), let \( (u_{0,\lambda}, q_{0,\lambda}) \) be the solution to the equation
\[
\begin{align*}
\begin{cases}
    u_{0,\lambda} + \lambda A u_{0,\lambda} + B^* q_{0,\lambda} &= f & \text{in } \Omega_0 \\
    B u_{0,\lambda} &= 0 & \text{in } \Omega_0 \\
    u_{0,\lambda} &= 0 & \text{on } \partial \Omega_0.
\end{cases}
\end{align*}
\]
Then we can write \( u_{0,\lambda} = R_\lambda f \) where the resolvent \( R_\lambda \) is defined by
\[
R_\lambda := (I + \lambda A)^{-1} - (I + \lambda A)^{-1} B^* (B(I + \lambda A)^{-1} B^*)^{-1} B(I + \lambda A)^{-1}.
\]
One has \( \|R_\lambda f\|_{L^2(\Omega_0)} \leq \|f\|_{L^2(\Omega_0)} \), and the mapping \( \Psi_1 \mapsto (u_{0,\lambda}, q_{0,\lambda}) \) from \( W_0^1(\Omega, \mathbb{R}^N) \) into \( H_0^1(\Omega_0)^N \times L^2(\Omega_0) \) is analytic in a neighbourhood of the origin.

Proof. Proceeding as in the proof of theorem 4.3, one sees that up to a constant \( q_{0,\lambda} \) is given by
\[
q_{0,\lambda} = (B(I + \lambda A)^{-1} B^*)^{-1} B(I + \lambda A)^{-1} f,
\]
while \( u_0 \) is given by
\[
u_{0,\lambda} = (I + \lambda A)^{-1} f - (I + \lambda A)^{-1} B^* (B(I + \lambda A)^{-1} B^*)^{-1} B(I + \lambda A)^{-1} f.
\]
From this it is easily seen that the analytic dependence of \( (u_{0,\lambda}, q_{0,\lambda}) \) on \( \Psi_1 \) holds. \(\square\)

Remark. As a matter of fact, the mapping \( (\lambda, \Psi_1) \mapsto R_\lambda \) can be extended into an analytic function defined on \( \{\text{Re}(\lambda) > 0\} \times B(0, \rho) \mapsto \mathcal{L}(H^{-1}(\Omega)^N, H_0^1(\Omega)^N) \), where \( B(0, \rho) \) is the ball of \( W_0^1(\Omega, \mathbb{R}^N) \) centred at the origin with a radius \( \rho \) sufficiently small.

Our next result, a corollary of what we have proved in theorem 4.3 and the identifiability result of section 3, is the fact that if \( \Psi := \Psi_1 = I + \tau \Psi_1 \) for a fixed \( \Psi_1 \neq 0 \), and if for \( |\tau| \) small enough we have \( D_0 \in \mathcal{D}_\text{ad} \) and \( D_\tau := \Psi_1 D_0) \in D_\text{ad} \) then we have a lower bound for \( \|\Lambda_{D_\tau}(f) - \Lambda_{D_\tau}(\phi)\|_{H^{-1}} \) in terms of \( \tau \), that is, we have a certain directional stability:

Corollary 4.5. Let \( \Psi_1 \in W_0^1(\Omega, \mathbb{R}^N) \) such that \( \Psi_1 \neq 0 \) on \( D_0 \). Denote \( \Psi := I + \tau \Psi_1 \) for \( \tau \in \mathbb{R} \), and \( D_\tau := \Psi_1(D_0) \) and \( \Omega_\tau := \Psi_1(\Omega_0) \). Let \( \tau_1 > 0 \) be a positive number such that \( \tau \mapsto (v_1, p_1) \) is analytic on \((-\tau_1, \tau_1)\) and assume that for some \( \tau_1 \in (-\tau_1, \tau_1) \) one has \( \Psi_1(D_0) \neq D_0 \). Then there exist \( \tau_0 > 0 \), an integer \( m \geq 1 \) and a positive constant \( C \), all depending on \( \Psi_1, \phi, \Omega, \partial \Omega_0 \), such that for all \( \tau \in [-\tau_0, \tau_0] \) one has
\[
\|\Lambda_{D_\tau}(f) - \Lambda_{D_\tau}(\phi)\|_{H^{-1}(\Gamma)^N} \geq C|\tau|^m
\]
Proof. It is clear that for \( |\tau| \) smaller than some \( \tau_1 > 0 \) the set \( D_\tau \) belongs to the class of admissible subdomains \( \mathcal{D}_\text{ad} \). Now for such \( \tau \), let \( (v_1, p_1) \) be a solution of the Stokes system
\[
\begin{align*}
\begin{cases}
    -\text{div}(\sigma(v_1, p_1)) &= 0 & \text{in } \Omega \setminus \overline{D}_\tau \\
    \text{div}(v_1) &= 0 & \text{in } \Omega \setminus \overline{D}_\tau \\
    v_1 &= 0 & \text{on } \partial D_\tau \\
    v_1 &= \phi & \text{on } \partial \Omega.
\end{cases}
\end{align*}
\]
According to the analyticity result proved above, \( \tau \mapsto \sigma(v_\tau, p_\tau) \) is analytic in a neighbourhood of the origin, and therefore so is the mapping

\[
\tau \mapsto \Lambda_{D_\tau}(\psi) - \Lambda_{D_\tau}(\psi)
\]

from \([-\tau_1, \tau_1]\) into \(H^{-1/2}(\Gamma)^N\). Since for a sequence of \((F_k)_k\) in \(H^{-1/2}(\Gamma)^N\) we have

\[
\Lambda_{D_\tau}(\psi) - \Lambda_{D_\tau}(\psi) = \sum_{k=1}^{\infty} \tau^k F_k,
\]

and since, for some \(\tau_\star \neq 0\), thanks to the identifiability result of section 3 we know that \(\Lambda_{D_\tau}(\psi) - \Lambda_{D_\tau}(\psi) \neq 0\), because \(D_0 \neq D_{\tau_\star}\), all the \(F_k\) cannot be zero, and so there exists a least integer \(m \geq 1\) such that \(F_m \neq 0\). Upon choosing \(0 < \tau_0 < \tau_1\) so that

\[
\left\| \sum_{k=m+1}^{\infty} \tau^{k-m} F_k \right\|_{H^{-1/2}(\Gamma)^N} \leq \frac{1}{2} \| F_m \|_{H^{-1/2}(\Gamma)^N},
\]

it follows that for \(|\tau| \leq \tau_0\) we have

\[
\left\| \Lambda_{D_\tau}(\psi) - \Lambda_{D_\tau}(\psi) \right\|_{H^{-1/2}(\Gamma)^N} \geq \frac{1}{2} \| F_m \|_{H^{-1/2}(\Gamma)^N} \tau^m,
\]

and thus the result is proved. \(\square\)

In the remainder of this section, we show that the Gâteaux derivative of the mapping \(\Psi \mapsto \sigma(u_0, q_0)\) can be obtained quite easily. To be more specific, for \(\tau \in \mathbb{R}\) and \(\Psi_1, \Psi_2 \in W^{1,\infty}_{0}(\Omega, \mathbb{R}^N)\) consider a perturbation of \(I\) of the form

\[
\Psi := \Psi_\tau := I + \tau \Psi_1.
\]

Let us denote, for \(|\tau|\) small enough, by \(M_\tau\) the corresponding matrices \(M\) defined at the beginning of this section, namely

\[
M_0 := I, \quad M_\tau := ((\Psi_\tau')^{-1}
\]

and by \(A_\tau\) and \(B_\tau\) the corresponding operators \(A\) and \(B\) defined in (4.10) and (4.11). Also we shall denote by \((u_\tau, q_\tau)\) and \((u_\tau, q_\tau)\) the solutions of the corresponding equations (4.13) where \(A\) and \(B\) are replaced with \(A_0, B_0\) and \(A_\tau, B_\tau\) respectively. (Note that as a matter of fact, \(A_0 = -\Delta\) and \(B_0 = -\text{div}\), that is, the corresponding equations for \(u_0, q_0\) are just the classical Stokes system).

Since we know that \(\Psi_\tau \mapsto (u_\tau, q_\tau)\) is analytic for \(|\tau|\) small enough, this means that the mapping \(\tau \mapsto (u_\tau, q_\tau)\) is analytic and that we may write

\[
u_\tau = u_0 + \tau u_1 + O(\tau^2), \quad q_\tau = q_0 + \tau q_1 + O(\tau^2)
\]

where the notation \(O(\tau^2)\) refers to the norm of \(H^1_0(\Omega_0)\) in the case of \(u_\tau\) and that of \(L^2(\Omega_0)\) for \(q_\tau\). We may now state the proposition concerning the first derivative of the mapping \(\Psi_\tau \mapsto (u_\tau, q_\tau)\).

**Proposition 4.6.** With the above notation, the first derivative of the mapping which maps \(\Psi\) into the solution of equation (4.13) is given by

\[
\frac{\partial}{\partial \tau}(u_\tau, q_\tau)_{|\tau=0} = (u_1^\tau, q_1^\tau)
\]

where \((u_\tau, q_\tau)\) satisfies the equation

\[
\begin{align*}
-\Delta u_\tau^\tau + \nabla q_\tau^\tau &= F_0 \quad \text{in} \ \Omega_0 := \Omega \setminus \overline{D}_0 \\
\text{div}(u_\tau^\tau) &= G_0 \quad \text{in} \ \Omega_0 \\
u_\tau^\tau &= 0 \quad \text{on} \ \partial \Omega_0,
\end{align*}
\]

(4.22)
where
\[ F_0 := -\text{div}(\langle \Psi' \rangle^* - \text{Tr}(\Psi') I) \nabla u_0 + \nabla (q_0 \text{Tr}(\Psi')) - \text{div}(q_0 \Psi') \]
and
\[ G_0 := \text{Tr}(\langle \Psi' \rangle^* \nabla u_0). \]

**Proof.** Indeed on the one hand we have
\[ M_\tau = I - \tau \langle \Psi' \rangle^* O(\tau^2). \]  
(4.23)
On the other hand Jac(\Psi_\tau) = \det(I + \tau \langle \Psi' \rangle^*) and therefore
\[ \text{Jac}(\Psi_\tau) = 1 + \tau \text{Tr}(\Psi') + O(\tau^2). \]  
(4.24)
Now recall that \((u_\tau, q_\tau)\) is characterized by the fact that \((u_\tau, q_\tau) \in H^1(\Omega_0) \times L^2(\Omega_0)\) and for all \((z, q) \in H^1_0(\Omega_0) \times L^2(\Omega_0)\) one has
\[
\int_{\Omega_0} M_\tau \nabla u_\tau : M_\tau \nabla z \text{Jac}(\Psi_\tau) \, dx - \int_{\Omega_0} q_\tau \text{Tr}(M_\tau \nabla z) \text{Jac}(\Psi_\tau) \, dx + \int_{\Omega_0} \text{Tr}(M_\tau \nabla u_\tau) q \text{Jac}(\Psi_\tau) \, dx
\]
\[ = - \int_{\Omega_0} M_\tau \nabla \varphi_0 : M_\tau \nabla z \text{Jac}(\Psi_\tau) \, dx. \]  
(4.25)
Using expansions (4.23) and (4.24) on the one hand and the fact that
\[(u_\tau, q_\tau) = (u_0, q_0) + \tau (u_1, q_1) + O(\tau^2),\]
on the other hand, one finds that the three integrals on the left-hand side of (4.25) have the following expansions: the first one is
\[
\int_{\Omega_0} M_\tau \nabla u_\tau : M_\tau \nabla z \text{Jac}(\Psi_\tau) \, dx = \int_{\Omega_0} \nabla u_0 : \nabla z \, dx + \tau \int_{\Omega_0} \nabla u_1 : \nabla z \, dx
\]
\[ + \tau \int_{\Omega_0} [\text{Tr}(\Psi') I - \Psi'_1 - \langle \Psi'_1 \rangle^*] \nabla u_0 : \nabla z \, dx + O(\tau^2), \]  
(4.26)
while the second one is
\[
\int_{\Omega_0} q_\tau \text{Tr}(M_\tau \nabla z) \text{Jac}(\Psi_\tau) \, dx = \int_{\Omega_0} q_0 \text{Tr}(\nabla z) \, dx + \tau \int_{\Omega_0} q_1 \text{Tr}(\nabla z) \, dx
\]
\[ + \tau \int_{\Omega_0} [\text{Tr}(\Psi'_0) q_0 \text{Tr}(\nabla z) - q_0 \text{Tr}(\langle \Psi'_1 \rangle^* \nabla z)] \, dx + O(\tau^2), \]  
(4.27)
and finally the third one is
\[
\int_{\Omega_0} \text{Tr}(M_\tau \nabla u_\tau) q \text{Jac}(\Psi_\tau) \, dx = \int_{\Omega_0} \text{Tr}(\nabla u_0) q \, dx + \tau \int_{\Omega_0} \text{Tr}(\nabla u_1) q \, dx
\]
\[ - \tau \int_{\Omega_0} \text{Tr}(\langle \Psi'_1 \rangle^* \nabla u_0) q \, dx + O(\tau^2). \]  
(4.28)
Analogously, the integral on the right-hand side of (4.25) can be expanded into
\[
\int_{\Omega_0} M_\tau \nabla \varphi_0 : M_\tau \nabla z \text{Jac}(\Psi_\tau) \, dx = \int_{\Omega_0} \nabla \varphi_0 : \nabla z \, dx
\]
\[ + \tau \int_{\Omega_0} [\text{Tr}(\Psi') I - \Psi'_1 - \langle \Psi'_1 \rangle^*] \nabla \varphi_0 : \nabla z \, dx + O(\tau^2) \]
\[ = \int_{\Omega_0} \nabla \varphi_0 : \nabla z \, dx + O(\tau^2), \]  
(4.29)
since we have \([\text{Tr}(\Psi'_1) I - \Psi'_1 - \langle \Psi'_1 \rangle^*] \nabla \varphi_0 \equiv 0 \text{ in } \Omega_0.\]
At this point recall that \((u_0, q_0)\) is the solution of a variational problem corresponding to
\[
\int_{\Omega_0} \nabla u_0 : \nabla z \, dx = \int_{\Omega_0} q_0 \text{Tr}(\nabla z) \, dx + \int_{\Omega_0} \text{Tr}(\nabla u_0) q \, dx = \int_{\Omega_0} \nabla \varphi_0 : \nabla z \, dx,
\]
that is, \(-\Delta u_0 + \nabla q_0 = \Delta \varphi_0\) and \(\text{div}(u_0) = \text{Tr}(\nabla u_0) = 0\). Therefore, after reporting the expressions in (4.26)-(4.29) into the variational equation (4.25), using the equation satisfied by \((u_0, q_0)\), dividing by \(\tau\) and letting \(\tau \to 0\), one finds that \((u^*_1, q^*_1) \in H^1_0(\Omega_0)^N \times L^2(\Omega_0)\) is characterized by the fact that for all \((z, q) \in H^1_0(\Omega_0)^N \times L^2(\Omega_0)\)
\[
\int_{\Omega_0} \nabla u^*_1 : \nabla z \, dx = \int_{\Omega_0} q^*_1 \text{Tr}(\nabla z) \, dx + \int_{\Omega_0} \text{Tr}(\nabla u^*_1) q \, dx = \langle F_0, z \rangle + \int_{\Omega_0} \text{Tr}(\Psi'_1 \nabla u_0) q \, dx,
\]
where \(F_0 \in H^{-1}(\Omega_0)^N\) is given by
\[
F_0 := -\text{div}((\Psi'_1)' - \text{Tr}(\Psi'_1) I) \nabla u_0 + \text{div}(q_0 \text{Tr}(\Psi'_1)) - \text{div}(q_0 \Psi'_1).
\]
This implies that \((u^*_1, q^*_1) \in H^1_0(\Omega_0) \times L^2(\Omega_0)\) satisfies the equation \(-\Delta u^*_1 + \nabla q^*_1 = F_0\) in \(\Omega_0\), and that \(\text{div}(u^*_1) = \text{Tr}(\Psi'_1 \nabla u_0)\) in \(\Omega_0\), and the proof is over.

5. Smooth dependence of Cauchy forces with respect to the deformation of the domain: the evolution case

In this section, we shall consider the nonlinear Navier–Stokes equation and prove that the Cauchy forces depend smoothly on the deformation of the obstacle \(D_0\). We begin by considering the linear evolution equations
\[
\left\{ \begin{array}{ll}
\frac{\partial v_j}{\partial t} - \Delta v_j + \nabla p_j = 0 & \text{in } \Omega_j \times (0, T) \\
\text{div}(v_j) = \text{Tr}(\nabla v_j) = 0 & \text{in } \Omega_j \times (0, T) \\
v_j(s, t) = \psi(s, t) & \text{on } \partial \Omega \times (0, T) \\
v_j(\cdot, 0) = 0 & \text{in } \Omega_j \\
v_j(\cdot, 0) = 0 & \text{on } \partial D_j \times (0, T)
\end{array} \right.
\]
and we show that when \(D_0\) and \(D_1\) are close, then the Cauchy forces \(\sigma(v_0, p_0) n\) and \(\sigma(v_1, p_1) n\) are close, and that there is a smooth dependence in the sense explained in the previous section, from which we use the notation.

First, observe that we may find \(\widetilde{\psi}_1 \in C^1([0, T], (H^1(\Omega))^N)\) such that
\[
\left\{ \begin{array}{ll}
\frac{\partial \widetilde{\psi}_1}{\partial t} = 0 & \text{in } \Omega \\
\text{div}(\widetilde{\psi}_1) = 0 & \text{in } \Omega \times (0, T) \\
\widetilde{\psi}_1 = 0 & \text{in } \Omega \times (0, T) \\
\widetilde{\psi}_1 = \psi & \text{on } \partial \Omega \times (0, T).
\end{array} \right.
\]

Then we define \(\widetilde{v}_1\) by setting \(\widetilde{v}_1 := v_1 - \widetilde{\psi}_1\) so that \(\widetilde{v}_1\) is a solution to
\[
\left\{ \begin{array}{ll}
\frac{\partial \widetilde{v}_1}{\partial t} - \Delta \widetilde{v}_1 + \nabla p_1 = \frac{\partial \widetilde{\psi}_1}{\partial t} - \Delta \widetilde{\psi}_1 & \text{in } \Omega_1 \times (0, T) \\
\text{div}(\widetilde{v}_1) = \text{Tr}(\nabla \widetilde{v}_1) = 0 & \text{in } \Omega_1 \times (0, T) \\
\widetilde{v}_1(s, t) = 0 & \text{on } \partial \Omega_1 \times (0, T) \\
\widetilde{v}_1(\cdot, 0) = 0 & \text{in } \Omega_1.
\end{array} \right.
\]
If \(\Psi\) is an admissible deformation as in definition 4.1, the operators \(A\) and \(B\) as in (4.10) and (4.11), and proceeding as we did in the previous section, after writing the variational
formulation of equation (5.3) (and some lengthy but straightforward elementary calculations) one checks that

\[ u_0(x, t) := \mathcal{T}_1(\Psi(x), t), \quad q_0(x, t) := p_1(\Psi(x), t) \]  

(5.4)
satisfy the equation

\[
\begin{aligned}
\frac{\partial u_0}{\partial t} + Au_0 + B^*q_0 &= f & \text{in } \Omega_0 \times (0, T) \\
Bu_0 &= 0 & \text{in } \Omega_0 \times (0, T) \\
u_0 &= 0 & \text{on } \partial \Omega_0 \times (0, T) \\
u_0(x, 0) &= 0 & \text{in } \Omega_0,
\end{aligned}
\]

(5.5)
where

\[ f(x, t) := -\frac{1}{\text{Jac}(\Psi)} \left[ \frac{\partial \phi_0}{\partial t} - \text{div}(\text{Jac}(\Psi)M^*M\nabla \phi_0) \right], \]

(5.6)
and

\[ \varphi_0(x, t) := \mathcal{T}_1(\Psi(x), t). \]

We can now establish the analytic dependence of \((u_0, q_0)\) on \(\Psi\).

**Theorem 5.1.** Let \(O \subset \subset \Omega\) be a Lipschitz domain, and \(\Psi_1 \in W^{1,\infty}(O, \mathbb{R}^N)\) be such that \(\Psi := I + \Psi_1\) is an admissible deformation according to definition 4.1. If \(\varphi \in C([0, T], H^{1/2}(\partial \Omega)^N)\) and \(\int_{\partial \Omega} \varphi(s, t) \cdot n(s) ds = 0\) for all \(t \in [0, T]\), then by mapping there exists a neighbourhood \(B(0, \rho)\) of the origin in \(W^{1,\infty}_0(O, \mathbb{R}^N)\) such that the mapping \(\Psi_1 \mapsto (u_0, q_0)\) is analytic from \(B(0, \rho)\) into \(C([0, T], H^1_0(\Omega_0) \times L^2(\Omega_0))\).

**Proof.** Note that since where \(\Psi \neq I\) we have \(\varphi_0 \equiv 0\), the function \(f\) defined above is actually independent of \(\Psi\) (note also that \(f \in C([0, T], H^{-1}(\Omega)^N)\)).

Now, for a given \(u_{\text{init}} \in H^1_0(\Omega_0)^N\) with \(Bu_{\text{init}} = 0\), it is known that there exists a unique \(u \in C((0, \infty); H^1_0(\Omega_0)^N)\) solution to the evolution equation

\[
\begin{aligned}
\frac{\partial u}{\partial t} + Au + B^*q &= 0 & \text{in } \Omega_0 \times (0, \infty) \\
Bu &= 0 & \text{in } \Omega_0 \times (0, \infty) \\
u(x, 0) &= u_{\text{init}}(x),
\end{aligned}
\]

(5.7)
and that this solution may be defined via the semi-group \(S(t)\) generated by \(A\) on \(N(B)\). More precisely, with the notation of corollary 4.4, it is a classical result of the theory of semi-groups (see, for instance, [16], chapter IX) that for any \(t > 0\) we have

\[ S(t)u_{\text{init}} := u(t) = \lim_{n \to \infty} (R_{1/n})^n u_{\text{init}}. \]

where the convergence takes place actually in the topology of \(C([0, T], L^2(\Omega_0)^N)\). As a matter of fact \(S(t)\) is a holomorphic semi-group and one can represent \(S\) by a path integral in the following way (see, for instance, [16], chapter IX, section 10). For \(\theta_0 \in (\pi/2, \pi)\) fixed, let \(\gamma\) be the path in the complex plane \(C\) defined as

\[ \gamma := \{ se^{i\theta}; s \geq 1\} \cup \{ e^{i\theta}; -\theta_0 \leq \theta \leq \theta_0\} \cup \{ se^{i\theta}; s \geq 1\}, \]

in which we assume that \(\gamma\) is oriented as a path coming from the direction \(s e^{i\theta_0}\) with \(s\) ranging from \(s = -\infty\) to \(s = 1\). Then we have

\[ S(t)u_{\text{init}} = \frac{1}{2i\pi} \int_{\gamma} e^{i\lambda} R_{1/\lambda} u_{\text{init}} \frac{d\lambda}{\lambda} = \frac{1}{2i\pi} \int_{\gamma} e^{i\lambda(\lambda I + A)^{-1}} u_{\text{init}} d\lambda. \]
where the integral converges uniformly for \( t \in [t_0, T] \) for any \( 0 < t_0 < T \), and actually one may define \( S(t) \) for \( t \in \mathbb{C} \) with \( |\arg(t)| < (2\theta_0 - \pi)/2 \). According to corollary 4.4, in a fixed neighbourhood \( B(0, \rho) \) of the origin in \( W^{1,\infty}(\Omega, \mathbb{R}^N) \), for any \( f \in H^{-1}(\Omega_0)^N \) the mapping \( \Psi_1 \mapsto R_s f \) is analytic from \( B(0, \rho) \) into \( H^1_s(\Omega_0)^N \). In particular, if \( u_{\text{init}} \) has its support in \( \Omega_0 \), then \( Bu_{\text{init}} = 0 \) means that \( \text{div}(u_{\text{init}}) = 0 \), and thus for any \( \Psi_1 \in B(0, \rho) \) given, \( S(t)u_{\text{init}} \) is well defined and one can see from the above integral representation of \( S(t) \) that \( \Psi_1 \mapsto S(t)u_{\text{init}} \) is analytic from \( B(0, \rho) \) into \( H^1_s(\Omega_0)^N \). In particular, if \( u_{\text{init}} \) has its support in \( \Omega_0 \), then \( Bu_{\text{init}} = 0 \) means that \( \text{div}(u_{\text{init}}) = 0 \), and thus for any \( \Psi_1 \in B(0, \rho) \) given, \( S(t)u_{\text{init}} \) is well defined and one can see from the above integral representation of \( S(t) \) that \( \Psi_1 \mapsto S(t)u_{\text{init}} \) is analytic from \( B(0, \rho) \) into \( L^2(\Omega_0)^N \). However, since the semi-group \( S(t) \) is analytic, for any \( t > 0 \) and \( u_{\text{init}} \in L^2(\Omega_0)^N \) with \( \text{div}(u_{\text{init}}) = 0 \) we have actually \( S(t)u_{\text{init}} \in H^1_s(\Omega_0)^N \) (with an estimate of the form \( \|S(t)u_{\text{init}}\|_{H^1_s} \leq ct^{-1/2}\|u_{\text{init}}\|_{L^2} \)), we can infer that \( \Psi_1 \mapsto S(t)u_{\text{init}} \) is analytic from \( B(0, \rho) \) into \( C([0, T], H^1_s(\Omega_0)^N) \). In fact when \( u_{\text{init}} \in H^1_s(\Omega_0)^N \cap D(A) \), one has (with \( u(t) := S(t)u_{\text{init}} \))

\[
u \in C([0, T], H^1_s(\Omega_0)^N) \cap C^1([0, T], L^2(\Omega_0)^N)
\]

and one can conclude that the mapping \( \Psi_1 \mapsto S(t)u_{\text{init}} \) from

\[
B(0, \rho) \longrightarrow C([0, T], H^1_s(\Omega_0)^N) \cap C^1([0, T], L^2(\Omega_0)^N)
\]

is analytic.

Now \( f \) being as in (5.6), if we denote by \( f_0 \) the projection of \( f \) in the space of divergence-free functions, then for any \( \Psi_1 \in B(0, \rho) \), we also have \( f_0 \in N(B) \), and the solution \( u_0 \) of (5.5) can be written as

\[
u_0(t) = \int_0^T S(t - \tau) f_0(\tau) \, d\tau.
\]

This shows that \( \Psi_1 \mapsto u_0 \) is analytic from \( B(0, \rho) \) into \( C([0, T], H^1_s(\Omega_0)^N) \cap C^1([0, T], L^2(\Omega_0)^N) \). Using the equation satisfied by \( (u_0, q_0) \) one sees that

\[
u_0 + A^{-1}B^*q_0 = A^{-1}f - A^{-1}\frac{\partial u_0}{\partial t}
\]

and hence (because \( Bu_0 = 0 \))

\[
BA^{-1}B^*q_0 = BA^{-1}f - BA^{-1}\frac{\partial u_0}{\partial t}.
\]

From this it is straightforward to see that \( q_0 \) is determined up to a constant (in the spatial variable) which may be chosen so that \( \int_{\Omega_0} q_0(x, t) \, dx = 0 \), and that \( \Psi_1 \mapsto q_0 \), as a mapping from \( B(0, \rho) \) into \( C([0, T], L^2(\Omega_0)) \), is also analytic.

Using the same procedures, the nonlinear Navier–Stokes equation written in \( \Omega_1 \) can be transformed into a nonlinear equation written in \( \Omega_0 \). Namely, if \( (v_1, p_1) \) is the solution of equation (1.8) with \( \varepsilon = 1 \) and \( j = 1 \), one defines first \( (\tilde{v}_1, p_1) \) with \( \tilde{v}_1 := v_1 - \bar{v}_1 \) and then one applies the change of variables \( \Psi \) by setting

\[
u_0(x, t) := \tilde{v}_1(\Psi(x), t), \quad q_0(x, t) := p_1(\Psi(x), t).
\]

After some cumbersome calculations, which we may omit, one checks that \( (u_0, q_0) \) satisfies the following equation,

\[
\begin{aligned}
\frac{\partial u_0}{\partial t} + Au_0 + B^*q_0 + F(u_0) &= f & \text{in} & \Omega_0 \times (0, T) \\
Bu_0 &= 0 & \text{in} & \Omega_0 \times (0, T) \\
\nu_0 &= 0 & \text{on} & \partial\Omega_0 \times (0, T) \\
u_0(x, 0) &= 0 & \text{in} & \Omega_0,
\end{aligned}
\]

(5.9)
where the nonlinearity $F$ is defined by (for $1 \leq i \leq N$ and $u \in H^1_0(\Omega_0)^N$)

$$(F(u))_i := \sum_{j=1}^{N} u_j M_{jk} \partial_k u_i + \sum_{j=1}^{N} \varphi_{0,j} M_{jk} \partial_k \varphi_{0,i} + \sum_{j=1}^{N} u_j M_{jk} \partial_k \varphi_{0,i},$$

and the right-hand side

$$(f(x, t))_i := -\frac{1}{\text{Jac}(\Psi)} \left[ \frac{\partial \psi_0}{\partial t} - \text{div}(\text{Jac}(\Psi) M^* M \nabla \psi_0) \right]_i - \sum_{j=1}^{N} \varphi_{0,j} M_{jk} \partial_k \varphi_{0,i}, \quad (5.10)$$

and

$$\psi_0(x, t) := \tilde{\psi}_1(\Psi(x), t).$$

At this point, we know that the semi-group $S(t)$ depends in a smooth manner on $\Psi_1 \in B(0, \rho)$, and that the solution of equation (5.9) can be obtained as a fixed point for the mild version of that equation, that is,

$$u_0(t) = \int_0^t S(t - \tau) f_0(\tau) \, d\tau - \int_0^t S(t - \tau) F_0(u_0(\tau)) \, d\tau,$$

where by $f_0$ and $F_0(u)$ we denote the projection of $f$ and $F(u)$ on $N(B)$. It is known that, for some $T_\ast > 0$ small enough, the above equation admits a unique solution in $C([0, T_\ast], H^1_0(\Omega_0)^N)$. As a matter of fact, this solution is obtained as a fixed point of the mapping

$$\tilde{\Phi}(u)(t) := \int_0^t S(t - \tau) f_0(\tau) \, d\tau - \int_0^t S(t - \tau) F_0(u(\tau)) \, d\tau$$

in the space $C([0, T_\ast], H^1_0(\Omega_0)^N)$, and the fact that $\Psi_1 \mapsto \tilde{\Phi}$ is analytic implies that the mapping $\Psi_1 \mapsto u_0$ is also analytic as a mapping from $B(0, \rho)$ into $C([0, T_\ast], H^1_0(\Omega_0)^N)$. In turn, this implies that $\Psi_1 \mapsto q_0$ is analytic, and finally we can state these observations in the following:

**Corollary 5.2.** Let $\mathcal{O} \subset \subset \Omega$ be a Lipschitz domain, and $\Psi_1 \in W^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ be such that $\Psi := I + \Psi_1$ is an admissible deformation according to definition 4.1. Using the above notation, if $\varphi \in C^1([0, T], H^{1/2}(\partial \Omega)^N)$ and $\int_{\partial \Omega} \varphi(s, t) \cdot n(s) \, ds = 0$ for all $t \in [0, T]$, then there exist $T_\ast > 0$ and $\rho > 0$ such that for all $\Psi_1 \in B(0, \rho)$ equation (5.9) has a unique solution $(u_0, q_0) \in C([0, T_\ast], H^1_0(\Omega_0)^N \times L^2(\Omega_0))$ (with $\int_{\partial \Omega} q_0(x, t) \, dx = 0$). Moreover, the mapping $\Psi_1 \mapsto (u_0, q_0)$ from $B(0, \rho) \subset W^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ into $C([0, T_\ast], H^1_0(\Omega_0)^N \times L^2(\Omega_0))$ is analytic.

As we observed in corollary 4.5, the analyticity result together with the identification result established above, allows us to state the following stability result concerning the linear or nonlinear evolution problem. Let $(v_\tau, p_\tau)$ be the solution of

$$\begin{aligned}
\frac{\partial v_\tau}{\partial t} - \text{div}(\sigma(v_\tau, p_\tau)) + \varepsilon_s \text{div}(v_\tau \otimes v_\tau) &= 0 & \text{in} & & (\Omega \setminus \overline{D}_\tau) \times (0, T) \\
\text{div}(v_\tau) &= 0 & \text{in} & & (\Omega \setminus \overline{D}_\tau) \times (0, T) \\
v_\tau(x, 0) &= 0 & \text{for} & & x \in \Omega \setminus \overline{D}_\tau. \\
v_\tau(s, t) &= \varphi(s, t) & \text{for} & & (s, t) \in \partial \Omega \times (0, T) \\
v_\tau(s, t) &= 0 & \text{for} & & (s, t) \in \partial D_\tau \times (0, T),
\end{aligned}
$$

where for $\tau = 0$ the domain $D_0$ is supposed to be the unperturbed obstacle, and $D_\tau$ denotes the perturbation of $D_0$ according to the convention we have used in the above studies, and $T_\ast$ is the minimum time of existence for the solutions $(v_\tau, p_\tau)$. 

Corollary 5.3. Let $\Psi_1 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be such that $\Psi_1 \neq 0$ on $D_0$. Denote $\Psi_\tau := I + \tau \Psi_1$ for $\tau \in \mathbb{R}$, and $D_\tau := \Psi_\tau(D_0)$ and $\Omega_\tau := \Psi_\tau(\Omega_0)$, and let $(v_\tau, p_\tau)$ be the solution of equation (5.11), for some $\phi \in C^1([0, T], H^{1/2}(\partial \Omega))$ satisfying $\int_\Omega \phi(s,t) \cdot n(s) ds = 0$ for $t \in [0, T]$. Let $\tau_1 > 0$ be a positive number such that $\tau \mapsto (v_\tau, p_\tau)$ is analytic on $(-\tau_1, \tau_1)$ and assume that for some $\tau_* \in (-\tau_1, \tau_1)$ one has $\Psi_{\tau_*}(D_0) \neq D_0$. Then there exist $t_0 \geq 0$, an integer $m \geq 1$ and a positive constant $C$, all depending on $\Psi_1, \phi, \Omega, \Omega_0$, such that for all $\tau \in [-t_0, t_0]$ one has

$$
\| \Lambda_{D_\tau}(\phi) - \Lambda_{D_0}(\phi) \|_{C([0,T], H^{-1/2}(\Gamma)^N)} \geq C|\tau|^m.
$$

(5.12)

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