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Homogenization of a Transmission Problem in Solid Mechanics

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In this paper we study a simplified model of the behavior of a 3-D solid made from two elastic homogeneous materials separated by a rapidly oscillating interface. We study the *asymptotic behavior* of the solution of such model using homogenization tools and a compactness result. We obtain the homogenized equation, and by studying its coefficients, we find some properties of the limiting material.

Key Words: boundary homogenization; transmission problems

1. INTRODUCTION

In this paper we study a simplified mathematical model that describes the behavior of a 3-D solid body made from two elastic materials which are separated by a periodically oscillating interface with period $\epsilon > 0$ and a constant amplitude. This work is related to the study of *transmission problem* posed in a bounded domain with a *rapidly oscillating interface* (see [3]).

The model involves the classical system of linear elasticity in a bounded domain $\Omega \subset \mathbb{R}^3$, with an interior boundary Γ^ϵ that represents the interface between both elastic materials. We consider homogeneous conditions on the external boundary of Ω and use continuity boundary conditions on Γ^ϵ. In order to simplify the elasticity system, we consider a particular kind of elastic homogeneous material.

The solution $\vec{u}^\epsilon$ of such a system represents the amount of deformation induced on the solid by the action of an external force. Our aim is to study the asymptotic behavior (as ϵ tends to zero) of the solution $\vec{u}^\epsilon$. We use classical homogenization tools (see, e.g., the books by A. Bensoussan, J.-L. Lions, G. Papanicolaou [2] and E. Sánchez-Palencia [8]) and a F. Murat's

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[6] compactness result to prove that the sequence $\vec{u}^\epsilon$ converges to $\vec{u}^0$ as ϵ tends to zero, where $\vec{u}^0$ is the solution of an elliptic boundary value system (the *homogenized equation*) that we are to explicitly find. Our study generalizes previous results obtained by R. Brizzi in the case of an elliptic scalar equation (see [3]). From a physical point of view, this limiting analysis can be seen as a mixing process of materials in the region near the rapidly oscillating interface.

The homogenized equation corresponds to a generalized elasticity system whose coefficients, arranged in a fourth-order tensor, can be seen as the elasticity coefficients of the limiting material. Studying the homogenized operator, we can conclude that the new material behaves, in the region near the interface, like a nonhomogeneous anisotropic elastic material. In the regions where no mixing process occurs, the new material still behaves like the original materials, i.e., with the same elastic coefficients.

The homogenized problem can also be deduced using the *two scales asymptotic development method* (see [2]) which consists in proposing an *asymptotic ansatz* of the solution $\vec{u}^\epsilon$ using functions that depend upon two variables, the microscopic and the macroscopic scales. With this expansion and by means of a formal calculus, the homogenized problem can be found. An important difference between the problem treated in this paper and other problems in homogenization theory is that the microscopic scale has one spatial dimension less than the macroscopic one.

Let us remark that the 2-D case is a straightforward consequence of the 3-D case studied in this paper. However, in the 2-D case it is possible to explicitly obtain the coefficients in the homogenized problem and to check technical hypotheses (see [1]).

In Section 2, we present the geometry of the domain with its rapidly oscillating interface and we formulate our model. From its variational formulation, we obtain a first result concerning the convergence of $\{\vec{u}^\epsilon\}_{\epsilon > 0}$ and $\{\sigma^\epsilon\}_{\epsilon > 0}$, the stress tensor. In Section 3, we formulate the homogenized problem and we present our main convergence result. Section 4 is devoted to proving this result. Finally, in Appendix A, we prove a result concerning the homogenized coefficients—which is necessary for the existence and uniqueness of the homogenized problem—and in Appendix B we briefly expose the 2-D case.

2. PRESENTATION OF THE MODEL

2.1. *The Geometry of the Problem*

In this section, we describe the domain $\Omega \subset \mathbb{R}^3$ with its rapidly oscillating interface. To this end, let $Y =]0, T_1[\times]0, T_2[\subset \mathbb{R}^2$, $T_i > 0$ $(i = 1, 2)$, and let $h\colon \bar{Y} \to \mathbb{R}$ be a smooth function such that

(i) $h|_{\partial Y} = h_1$, where $h_1 = \max\{h(y) | y \in \bar{Y}\}$ and $h_1 > 0$.

(ii) There exists $y_0 \in Y$ such that $h(y_0) = 0$ and $\nabla_y h(y_0) = 0$.

Let $z_0 > h_1 > 0$ be given. We define

$$\Omega_1^1 = \{(y, z) \in \mathbb{R}^2 \times \mathbb{R} | h(y) < z < z_0, y \in Y\},$$

$$\Omega_2^1 = \{(y, z) \in \mathbb{R}^2 \times \mathbb{R} | -z_0 < z < h(y), y \in Y\}.$$

Let Γ^1 be the interface between these sets, i.e.,

$$\Gamma^1 = \{(y, z) \in \mathbb{R}^2 \times \mathbb{R} | h(y) = z, y \in Y\},$$

Using these elements, the so-called reference cell Ω is built up as follows (see Fig. 1, left):

$$\Omega = \Omega_1^1 \cup \Gamma^1 \cup \Omega_2^1 \, (= Y \times]-z_0, z_0[).$$

If we intersect Ω with the hyperplane $[Z = z]$, where $0 < z < h_1$, we obtain Y and its following decomposition (Fig. 1, right)

$$Y = Y^*(z) \cup \gamma(z) \cup O(z)$$

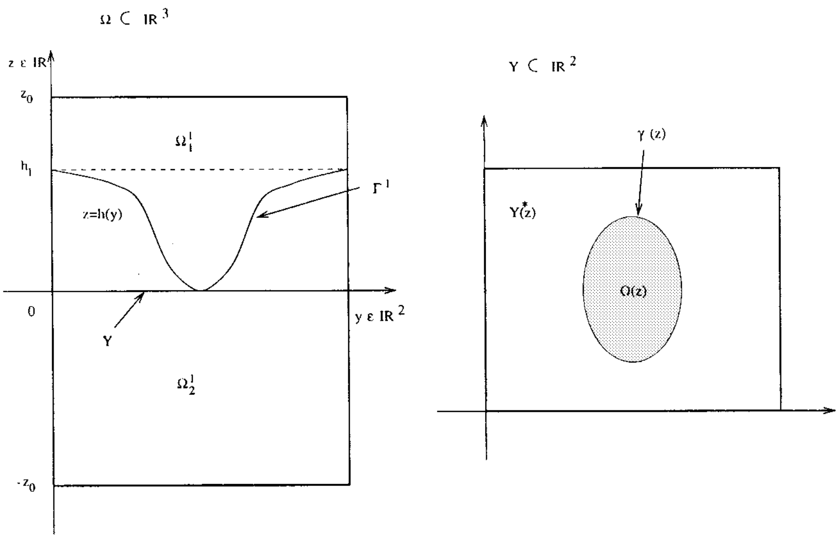

FIG. 1. The reference cell Ω (left) and the decomposition of Y (right).

where $O(z)$ (resp., $Y^*(z), \gamma(z)$) is defined as $\{y \in Y | h(y) < (\text{resp.}, >, =) z\}$. Note that $Y^*(z) = Y - \overline{O(z)}$.

Let ϵ be a positive parameter. Extending h by Y-periodicity[1] we can introduce

$$\Omega_1^\epsilon = \left\{(x,z) \in \mathbb{R}^2 \times \mathbb{R} | h\left(\frac{x}{\epsilon}\right) < z < z_0, x \in Y\right\}$$

$$\Omega_2^\epsilon = \left\{(x,z) \in \mathbb{R}^2 \times \mathbb{R} | -z_0 < z < h\left(\frac{x}{\epsilon}\right), x \in Y\right\}$$

and the rapidly oscillating interface is therefore defined by

$$\Gamma^\epsilon = \left\{(x,z) \in \mathbb{R}^2 \times \mathbb{R} | h\left(\frac{x}{\epsilon}\right) = z, x \in Y\right\} \text{ (see Fig. 2, left).}$$

Finally, as in Fig. 2 right, we set $\Omega_1 = Y \times]h_1, z_0[$, $\Omega_m = Y \times]0, h_1[$ and $\Omega_2 = Y \times]-z_0, 0[$. We used Ω_m to denote the region near the interface, because it is in this region where we obtain *mixed* material in the limit as ϵ goes to zero.

2.2. *Setting the Model Up*

Let us fill the regions $\Omega_1^\epsilon \subset \Omega$ and $\Omega_2^\epsilon \subset \Omega$ with two different elastic materials. Let $\vec{u}_i^\epsilon \colon \Omega_i^\epsilon \to \mathbb{R}^3$, $i = 1, 2$, be the functions that represent the

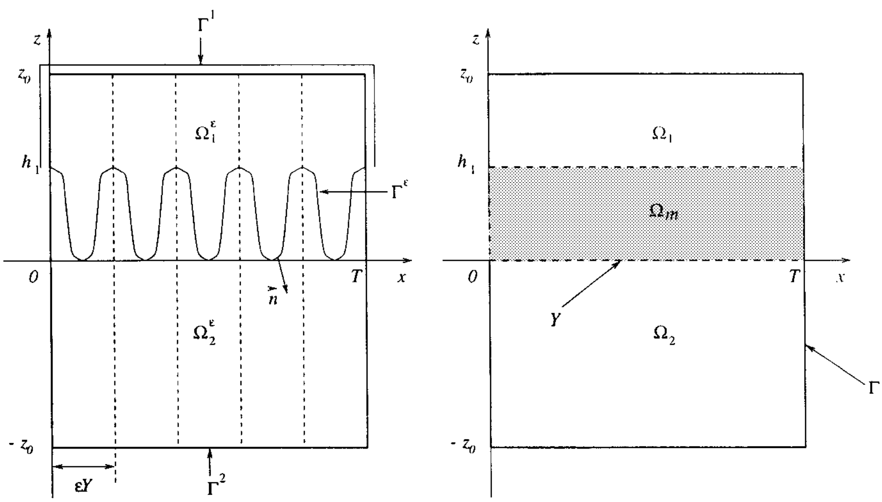

FIG. 2. The rapidly oscillating interface (left) and its homogenized version (right).

[1] A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is Y-periodic if $f(y + \sum_{i=1}^{2} k_i T_i \underset{\sim}{e}_i) = f(y)$ for all $y \in Y$ and for all $k_i \in \mathbb{Z}$ $(i = 1, 2)$, where $\underset{\sim}{e}_1 = (1,0)^T$ and $\underset{\sim}{e}_2 = (0,1)^T$.

small deformations inside both materials. By $e_{x,z}(\vec{u}_i^\epsilon)$, $i = 1, 2$, we mean the linear strain tensor associated with $\vec{u}_i^\epsilon$, i.e.,

$$e_{x,z}\left(\vec{u}_i^\epsilon\right) = \frac{1}{2}\left(\nabla_{x,z}\left(\vec{u}_i^\epsilon\right) + \nabla_{x,z}\left(\vec{u}_i^\epsilon\right)^T\right),$$

and ${\sigma_i}^\epsilon$, $i = 1, 2$, stands for the stress tensor, i.e.,

$${\sigma_i}^\epsilon = \lambda_i \operatorname{div}_{x,z} \vec{u}_i^\epsilon I + 2\mu_i e_{x,z}\left(\vec{u}_i^\epsilon\right),$$

where $\lambda_i \geq 0$ and $\mu_i > 0$ are the *Lamé coefficients* of the ith material, $i = 1, 2$. To simplify, we consider the case $\lambda_i = 0$. This mathematical simplification implies that *Poisson's ratio*[2] for each material is equal to zero. The assumption $\lambda_i = 0$ is possible since Poisson's ratio can vary between -1 and $1/2$, although this physical parameter is in practice always strictly positive. More details on the theory of elasticity can be found in the book by L. D. Landau and E. M. Lifschitz [5]. Therefore, the stress tensor in our model is now given by

$${\sigma_i}^\epsilon = 2\mu_i e_{x,z}\left(\vec{u}_i^\epsilon\right) \qquad i = 1, 2.$$

The functions $\vec{u}_i^\epsilon$ must satisfy the following system

$$(P)_\epsilon \begin{cases} -\operatorname{div}_{x,z}\left(2\mu_i e_{x,z}\left(\vec{u}_i^\epsilon\right)\right) = \vec{f}_i & \text{in } \Omega_i^\epsilon \\ \vec{u}_i^\epsilon = \vec{0} & \text{on } \Gamma_i,\ i = 1, 2 \\ \vec{u}_1^\epsilon = \vec{u}_2^\epsilon & \text{on } \Gamma^\epsilon \\ 2\mu_1 e_{x,z}\left(\vec{u}_1^\epsilon\right)\vec{n} = 2\mu_2 e_{x,z}\left(\vec{u}_2^\epsilon\right)\vec{n} & \text{on } \Gamma^\epsilon \end{cases}$$

where $\vec{f}_i \in L^2(\Omega)^3$ represents the density of external forces acting on the solid body, and $\vec{n}$ means the exterior normal vector to Ω_1^ϵ. The second equation in $(P)_\epsilon$ is the homogeneous condition on the external boundaries of Ω_1^ϵ and Ω_2^ϵ, the third and fourth equations in $(P)_\epsilon$ are the continuity boundary conditions on the interface Γ^ϵ for the deformations $\vec{u}_i^\epsilon$, and for the stresses ${\sigma_i}^\epsilon\vec{n}$.

Let $\vec{u}^\epsilon = \vec{u}_1^\epsilon \chi_{\Omega_1^\epsilon} + \vec{u}_2^\epsilon \chi_{\Omega_2^\epsilon}$, and $\vec{f}^\epsilon, \mu^\epsilon$ defined in a similar way, where $\chi_{\Omega_i^\epsilon}$ stands for the characteristic function of Ω_i^ϵ, $i = 1, 2$. Then, the variational formulation of $(P)_\epsilon$ is

$$(PV)_\epsilon \begin{cases} \text{Find } \vec{u}^\epsilon \in H_0^1(\Omega)^3 \text{ such that} \\ \int_\Omega 2\mu^\epsilon e_{x,z}(\vec{u}^\epsilon) : e_{x,z}(\vec{v})\, dx\, dz = \int_\Omega \vec{f}^\epsilon \cdot \vec{v}\, dx\, dz \quad \forall \vec{v} \in H_0^1(\Omega)^3. \end{cases}$$

[2] Poisson's ratio is the ratio of the transverse compression to the longitudinal extension.

It is easy to prove that this problem has a unique solution $\vec{u}^\epsilon \in H_0^1(\Omega)^3$ and the sequence $\{\vec{u}^\epsilon\}_{\epsilon > 0}$ is bounded in $H_0^1(\Omega)^3$ (see [7]). Therefore, the sequence $\sigma^\epsilon = 2\mu^\epsilon e_{x,z}(\vec{u}^\epsilon)$ is bounded in $\mathscr{L}^2_{3,s}(\Omega)$, the space of 3×3 symmetric second-order tensor with coefficients in $L^2(\Omega)$. Then the following proposition holds.

PROPOSITION 2.1. (a) *There exists $\vec{u}^* \in H_0^1(\Omega)^3$ and a subsequence of $\{\vec{u}^\epsilon\}_{\epsilon > 0}$, which we still denote by ϵ, such that*

$$\vec{u}^\epsilon \rightharpoonup \vec{u}^* \text{ in } H_0^1(\Omega)^3\text{—weakly}, \quad \text{and in } L^2(\Omega)^3\text{—strongly}.$$

(b) *There exists $\sigma^* \in \mathscr{L}^2_{3,s}(\Omega)$ and a subsequence of $\{\sigma^\epsilon\}_{\epsilon > 0}$, which we still denote by ϵ, such that*

$$\sigma^\epsilon \rightharpoonup \sigma^* \text{ in } \mathscr{L}^2_{3,s}(\Omega)\text{—weakly}.$$

3. THE MAIN CONVERGENCE RESULT

3.1. *The Homogenized Problem*

Let us consider the following periodic system for $\underline{\chi}_{kl} = ((\chi_{kl})_1, (\chi_{kl})_2)^T$ in Y,

$$(P)_{kl} \begin{cases} -\operatorname{div}_y\big(2\mu e_y(\underline{\chi}_{kl})\big) = \operatorname{div}_y(2\mu M_{kl}) \text{ in } Y \\ \underline{\chi}_{kl} \quad Y\text{-periodic} \end{cases} \quad 1 \le k, l \le 2$$

where M_{kl}, $1 \le k, l \le 2$, is the 2×2 matrix defined by

$$[M_{kl}]_{ij} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Since $\mu(y,z) = \mu_1 \chi_{O(z)}(y) + \mu_2 \chi_{Y^*(z)}(y)$, then $(P)_{kl}$ is parameterized by $z \in]0, h_1[$. For a fixed z, it is easy to show that problem $(P)_{kl}$ has a unique solution in $(H^1_\#(Y)/\mathbb{R})^2$, where $H^1_\#(Y)/\mathbb{R}$ is the space of Y-periodic $H^1_{\text{loc}}(\mathbb{R}^2)$ functions defined up to an additive constant (proofs of existence and uniqueness for similar periodic problems can be found in C. Conca [4] and E. Sanchez-Palencia [8]). We assume that the solution of $(P)_{kl}$, as function of $(y,z) \in Y \times]0, h_1[$ $(i = 1, 2)$, satisfies the following regularity hypothesis:

$$(H1) \begin{cases} \text{(a) } \underline{\chi}_{kl} \in L^2_{\text{loc}}\big(0, h_1; H^1_\#(Y)^2\big) \cap \big(L^2_{\text{loc}}(]0, h_1[\times \mathbb{R}^2)\big)^2 \\ \text{(b) } \dfrac{\partial}{\partial z}\big((\underline{\chi}_{kl})_i\big) \in L^2_{\text{loc}}\big(0, h_1; L^2_\#(Y)\big) \cap L^2_{\text{loc}}(]0, h_1[\times \mathbb{R}^2) \\ \quad 1 \le i \le 2 \end{cases}$$

Now, we consider the following periodic scalar problem in Y

$$(P)_k \begin{cases} -\operatorname{div}_y(\mu \nabla_y \varphi_k) = \operatorname{div}_y(2\mu \underline{e}_k) & \text{in } Y \\ \varphi_k & Y\text{-periodic} \end{cases} \quad 1 \leq k \leq 2$$

where $\underline{e}_k$ is the kth vector of the canonical basis of $\mathbb{R}^2$. $(P)_k$ is also parameterized by $z \in]0, h_1[$ and, for a fixed z, this equation has a unique solution in $H^1_\#(Y)/\mathbb{R}$. We assume that the following regularity hypothesis holds for its solution:

$$(H2) \begin{cases} \text{(a)} \ \varphi_k \in L^2_{\text{loc}}\big(0, h_1; H^1_\#(Y)\big) \cap L^2_{\text{loc}}\big(]0, h_1[\times \mathbb{R}^2\big) \\ \text{(b)} \ \dfrac{\partial \varphi_k}{\partial z} \in L^2_{\text{loc}}\big(0, h_1; L^2_\#(Y)\big) \cap L^2_{\text{loc}}\big(]0, h_1[\times \mathbb{R}^2\big). \end{cases}$$

Using $\underline{\chi}_{kl}$ and φ_k we construct $\mathscr{A}(z)$, a fourth-order tensor whose coefficients are defined by

$$a_{ijkl}(z) = \begin{cases} 2\mu_1 [M_{kl}]_{ij} & h_1 < z < z_0 \\ a^m_{ijkl} & 0 < z < h_1 \\ 2\mu_2 [M_{kl}]_{ij} & -z_0 < z < 0 \end{cases} \quad 1 \leq i, j, k, l \leq 3,$$

where the coefficients in the region Ω_m are defined by

$$a^m_{ijkl} = \begin{cases} \dfrac{1}{|Y|} \displaystyle\int_Y 2\mu\Big([M_{kl}]_{ij} + \big[e_y(\underline{\chi}_{kl})\big]_{ij}\Big)\, dy & \text{if } \begin{cases} 1 \leq i, & j \leq 2 \\ 1 \leq k, & l \leq 2 \end{cases} \\ \dfrac{1}{2} \dfrac{1}{|Y|} \displaystyle\int_Y \mu \left(2\delta_{ki} + \dfrac{\partial \varphi_k}{\partial y_i}\right) dy & \text{if } \begin{cases} 1 \leq i \leq 2, & j = 3 \\ 1 \leq k \leq 2, & l = 3 \end{cases} \\ \dfrac{1}{2} \dfrac{1}{|Y|} \displaystyle\int_Y \mu \left(2\delta_{li} + \dfrac{\partial \varphi_l}{\partial y_i}\right) dy & \text{if } \begin{cases} 1 \leq i \leq 2, & j = 3 \\ k = 3, & 1 \leq l \leq 2 \end{cases} \\ \dfrac{1}{2} \dfrac{1}{|Y|} \displaystyle\int_Y \mu \left(2\delta_{kj} + \dfrac{\partial \varphi_k}{\partial y_j}\right) dy & \text{if } \begin{cases} i = 3, & 1 \leq j \leq 2 \\ 1 \leq k \leq 2, & l = 3 \end{cases} \\ \dfrac{1}{2} \dfrac{1}{|Y|} \displaystyle\int_Y \mu \left(2\delta_{lj} + \dfrac{\partial \varphi_l}{\partial y_j}\right) dy & \text{if } \begin{cases} i = 3, & 1 \leq j \leq 2 \\ k = 3, & 1 \leq l \leq 2 \end{cases} \\ \dfrac{1}{|Y|} \displaystyle\int_Y 2\mu\, dy & \text{if } \begin{cases} i = j = 3 \\ k = l = 3 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

($|Y|$ denote the Lebesgue's measure of the set Y). This tensor satisfies

PROPOSITION 3.1. *The coefficients of* $\mathscr{A}$ *are such that*:

(a) $a_{ijkl}(z) = a_{klij}(z) = a_{ijlk}(z)$, $\forall 1 \le i, j, k, l \le 3$, $\forall z \in]-z_0, z_0[$.

(b) *There exists* $\beta > 0$ *such that for all* ξ, 3×3 *symmetric second-order tensor*,

$$(\mathscr{A}(z)\xi) : \xi \ge \beta \xi : \xi, \quad \forall z \in]-z_0, z_0[.$$

Proof. See Appendix A. ∎

Let us now introduce the homogenized problem

$$(PV)_H \begin{cases} \text{Find } \vec{u}^0 \in H_0^1(\Omega)^3 \text{ such that} \\ \displaystyle\int_\Omega (\mathscr{A}e_{x,z}(\vec{u}^0)) : e_{x,z}(\vec{v})\, dx\, dz = \int_\Omega \vec{f} \cdot \vec{v}\, dx\, dz \quad \forall \vec{v} \in H_0^1(\Omega)^3, \end{cases}$$

where $\vec{f}$ is the weak limit of $\vec{f}^\epsilon$; see (3) below. Using Proposition 3.1, we conclude that $(PV)_H$ has a unique solution $\vec{u}^0 \in H_0^1(\Omega)^3$ (see [7]).

Remark 3.1. The homogenized problem $(PV)_H$ can be deduced using the *two-scale asymptotic development method* (see [2, 4, or 8] for more details). In this case the scales to be considered are the microscopic one, $y = x/\epsilon \in \mathbb{R}^2$, and the macroscopic scale, $(x, z) \in \mathbb{R}^3$. The ansatz to be used is

$$\vec{u}^\epsilon(x, z) = \vec{u}^0(x, z, x/\epsilon) + \epsilon \vec{u}^1(x, z, x/\epsilon) + \epsilon^2 \vec{u}^2(x, z, x/\epsilon) + \cdots,$$

where $\vec{u}^k : \Omega \times Y \to \mathbb{R}^3$, $k = 0, 1, 2, \ldots$, are functions such that $\vec{u}^k(x, z, \cdot)$ is Y-periodic, for all (x, z) in Ω. A detailed use of this method can be found, e.g., in C. Conca [4]. ∎

Remark 3.2. The homogenized equation $(PV)_H$ is the variational formulation of a generalized elasticity system where the stress tensor and the linear strain tensor are related by

$$[\sigma]_{ij} = \sum_{k,l=1}^{3} a_{ijkl}(z)\left[e_{x,z}(\vec{u})\right]_{kl}, \qquad 1 \le i, j \le 3, \tag{1}$$

or $\sigma = \mathscr{A}e_{x,z}(\vec{u})$. The coefficients a_{ijkl} of $\mathscr{A}$ can be seen as the *elasticity coefficients* of the homogenized material. From the definition of $\mathscr{A}$, we observe that in the regions Ω_1 and Ω_2 the homogenized material still behaves as the original materials did, with *Lamé's coefficients* μ_1 and μ_2, respectively. Instead, in Ω_m the coefficients depend upon $z \in]0, h_1[$. Therefore, the homogenized material in the region near the interface behaves like a nonhomogeneous anisotropic elastic material (see [7]). ∎

3.2. *The Convergence Result*

The main convergence theorem is

THEOREM 3.1. *If $(H1)$, $(H2)$ and $(H3)$ hold, then the sequence of solutions $\{\vec{u}^\epsilon\}_{\epsilon>0}$ of $(PV)_\epsilon$ converges to $\vec{u}^0$ in $H_0^1(\Omega)^3$—weakly, where $\vec{u}^0$ is the unique solution of the homogenized problem $(PV)_H$.*

Hypothesis $(H3)$ in Theorem 3.1 is a technical hypothesis that we present in Section 4.3.

4. PROOF OF THE CONVERGENCE RESULT

The proof of Theorem 3.1 consists of several steps. First, using Proposition 2.1, we find the equation that satisfies σ^*, the weak limit of $\{\sigma^\epsilon\}_{\epsilon>0}$. The next steps are devoted to proving—using classical homogenization techniques and a compactness result—that σ^* and $\vec{u}^*$, the weak limit of $\{\vec{u}^\epsilon\}_{\epsilon>0}$, are related by (1). Toward that end, we identify each component of the limit tensor of $\{\sigma^\epsilon\}_{\epsilon>0}$ separately. Finally, using the equation for σ^*, we conclude that $\vec{u}^*$ is the solution of the homogenized problem $(PV)_H$.

4.1. *The Equation Satisfied by σ^**

We begin the homogenization process passing to the limit in equation $(PV)_\epsilon$. To this end, we first identify the weak limit of $\vec{f}^\epsilon$:

Since

$$\chi_{\Omega_1^\epsilon} \rightharpoonup \theta \text{ and } \chi_{\Omega_2^\epsilon} \rightharpoonup (1-\theta) \text{ in } L^\infty(\Omega)\text{—}weakly\ *, \tag{2}$$

where

$$\theta(x,z) = \begin{cases} 1 & \text{in } \Omega_1 \\ \dfrac{|O(z)|}{|Y|} & \text{in } \Omega_m \\ 0 & \text{in } \Omega_2. \end{cases}$$

Then

$$\vec{f}^\epsilon \rightharpoonup \vec{f}_1\theta + \vec{f}_2(1-\theta) \text{ in } L^2(\Omega)^3\text{—}weakly. \tag{3}$$

Using Proposition 2.1 and (3) we pass to the limit $\epsilon \to 0$ in $(PV)_\epsilon$ and we obtain the following variational equation for σ^*:

$$\int_\Omega \sigma^* : e_{x,z}(\vec{v}) = \int_\Omega \vec{f}\cdot\vec{v} \quad \forall \vec{v} \in H_0^1(\Omega)^3, \tag{4}$$

where $\vec{f} = \vec{f}_1\theta + \vec{f}_2(1-\theta)$.

4.2. *Identification of* $[\sigma^*]_{kl}$, $1 \le k, l \le 2$, *in* Ω_m

We shall prove

PROPOSITION 4.1. *If* $(H1)$ *holds, then* $[\sigma^\epsilon]_{kl} \rightharpoonup [\sigma^*]_{kl}$ *in* $L^2(\Omega_m)$*—weakly (up to a subsequence), for* $1 \le k, l \le 2$, *where*

$$[\sigma^*]_{kl} = \frac{1}{|Y|} \sum_{i,j=1}^{2} \left\{ \int_Y 2\mu\Big([M_{ij}]_{kl} + \big[e_y(\underline{\chi}_{ij})\big]_{kl} \Big)\, dy \right\} \big[e_{x,z}(\vec{u}^*)\big]_{ij}.$$

Proof. Let $\underline{p}_{kl}: Y \to \mathbb{R}^2$, $1 \le k, l \le 2$, be the polynomial defined by

$$\underline{p}_{11}(y) = \begin{bmatrix} y_1 \\ 0 \end{bmatrix}, \qquad \underline{p}_{12}(y) = \underline{p}_{21}(y) = \begin{bmatrix} y_2/2 \\ y_1/2 \end{bmatrix}, \qquad \underline{p}_{22}(y) = \begin{bmatrix} 0 \\ y_2 \end{bmatrix}.$$

Note that $e_y(\underline{p}_{kl}) = M_{kl}$, $1 \le k, l \le 2$. Let $\underline{w}_{kl} = \underline{p}_{kl} + \underline{\chi}_{kl}$ and $\sigma_{kl} = 2\mu e_y(\underline{w}_{kl})$ (remark that σ_{kl} is a 2×2 matrix), and define the functions

$$\underline{w}^\epsilon_{kl}(x,z) = \epsilon \underline{w}_{kl}\left(\frac{x}{\epsilon}, z\right) \qquad (x,z) \in \Omega_m$$

and

$$\sigma^\epsilon_{kl}(x,z) = \sigma_{kl}\left(\frac{x}{\epsilon}, z\right) \qquad (x,z) \in \Omega_m$$

(do not mistake the 2×2 matrix σ^ϵ_{kl} for $[\sigma^\epsilon]_{kl}$, the (k,l)-element of tensor σ^ϵ). As $\underline{\chi}_{kl}$ is the solution of $(P)_{kl}$, we have

$$\operatorname{div}_x \sigma^\epsilon_{kl} = 0 \quad \text{in } \Omega_m. \tag{5}$$

Since $(H1)$ is fulfilled, we can use classical arguments about the convergence of periodic functions, to conclude that for all $\Omega' \subset\subset \Omega_m$ open sets

$$\underline{w}^\epsilon_{kl} \to \underline{p}_{kl} \quad \text{in } L^2(\Omega')^2\text{—}\textit{strongly}, \tag{6}$$

$$\frac{\partial}{\partial z}\big((\underline{w}^\epsilon_{kl})_i\big) \to 0 \quad \text{in } L^2(\Omega')\text{—}\textit{strongly}, \qquad (1 \le i \le 2), \tag{7}$$

$$[\sigma^\epsilon_{kl}]_{ij} \rightharpoonup \big[\overline{\sigma_{kl}}\big]_{ij} \quad \text{in } L^2(\Omega')\text{—}\textit{weakly}, \qquad (1 \le i, j \le 2), \tag{8}$$

where $\overline{\sigma_{kl}}$ is the average in Y of σ_{kl}, i.e.,

$$\overline{\sigma_{kl}} = \frac{1}{|Y|}\int_Y 2\mu\big(M_{kl} + e_y(\underline{\chi}_{kl})\big)\,dy.$$

Remark 4.1. From (5) and (8), we conclude that

$$\operatorname{div}_x\big(\overline{\sigma_{kl}}\big) = 0 \quad \text{in } \Omega_m. \tag{9}$$

∎

Now we can define the function $\vec{w}^{\epsilon}_{kl} = (\underline{w}^{\epsilon}_{kl}, 0)^T$. Let $\phi \in \mathscr{D}(\Omega_m)$. If we put $\vec{v} = \phi\vec{w}^{\epsilon}_{kl}$ in $(PV)_\epsilon$, after elementary calculations, we obtain

$$\begin{aligned}
&\int_{\Omega_m} \sigma^{\epsilon}\nabla_{x,z}\phi\cdot\vec{w}^{\epsilon}_{kl}\,dx\,dz + \int_{\Omega_m} 2\mu^{\epsilon}e_x(\underline{u}^{\epsilon}) : e_x(\underline{w}^{\epsilon}_{kl})\,\phi\,dx\,dz\\
&\qquad + \sum_{j=1}^{2}\int_{\Omega_m} 2\mu^{\epsilon}\big[e_{x,z}(\vec{u}^{\epsilon})\big]_{3j}\frac{\partial}{\partial z}(w^{\epsilon}_{kl})_j\phi\,dx\,dz\\
&\qquad = \int_{\Omega_m} \vec{f}^{\epsilon}\cdot\big(\phi\vec{w}^{\epsilon}_{kl}\big)\,dx\,dz,
\end{aligned} \tag{10}$$

where $\underline{u}^{\epsilon}$ denotes the first two components of $\vec{u}^{\epsilon}$. On the other hand, if we use $\phi\underline{u}^{\epsilon}$ as a test function in (5), we have

$$\int_{\Omega_m} \sigma^{\epsilon}_{kl}\nabla_x\phi\cdot\underline{u}^{\epsilon}\,dx\,dz + \int_{\Omega_m} 2\mu^{\epsilon}e_x(\underline{w}^{\epsilon}_{kl}) : e_x(\underline{u}^{\epsilon})\,\phi\,dx\,dz = 0. \tag{11}$$

If we subtract (11) from (10), we obtain

$$\begin{aligned}
&\int_{\Omega_m} \sigma^{\epsilon}\nabla_{x,z}\phi\cdot\vec{w}^{\epsilon}_{kl}\,dx\,dz - \int_{\Omega_m} \sigma^{\epsilon}_{kl}\nabla_x\phi\cdot\underline{u}^{\epsilon}\,dx\,dz\\
&\qquad + \sum_{j=1}^{2}\int_{\Omega_m} 2\mu^{\epsilon}\big[e_{x,z}(\vec{u}^{\epsilon})\big]_{3j}\frac{\partial}{\partial z}(w^{\epsilon}_{kl})_j\phi\,dx\,dz = \int_{\Omega_m} \vec{f}^{\epsilon}\cdot\big(\phi\vec{w}^{\epsilon}_{kl}\big)\,dx\,dz.
\end{aligned} \tag{12}$$

Using Proposition 2.1 and (3), (6), (7), (8) (by considering $\Omega' \subset\subset \Omega_m$, an open set such that $\operatorname{supp}\phi \subset \Omega'$), we can pass to the limit in (12), and obtain

$$\int_{\Omega_m} \sigma^{*}\nabla_{x,z}\phi\cdot\vec{w}^{\epsilon}_{kl}\,dx\,dz - \int_{\Omega_m} \overline{\sigma_{kl}}\nabla_x\phi\cdot\underline{u}^{*}\,dx\,dz = \int_{\Omega_m} \vec{f}^{*}\cdot\phi\vec{w}^{*}_{kl}\,dx\,dz,$$

where $\vec{w}^*_{kl} = (\underline{p}_{kl}, 0)^T$. Integrating by parts the left-hand side of this equation, knowing that σ^* satisfies (4) (with $\vec{v} = \phi \vec{w}^*_{kl}$ as a test function), and $\overline{\sigma_{kl}}$ satisfies (9) (with $\phi \underline{u}^*$ as a test function), we have

$$\int_{\Omega_m} \sigma^* : e_{x,z}\left(\vec{w}^*_{kl}\right) \phi \, dx \, dz = \int_{\Omega_m} \overline{\sigma_{kl}} : e_x(\underline{u}^*) \phi \, dx \, dz.$$

We also have

$$\left[e_{x,z}\left(\vec{w}^*_{kl}\right) \right]_{ij} = \begin{cases} [M_{kl}]_{ij} & \text{if } 1 \le i, j \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Then, in the sense of $\mathscr{D}'(\Omega_m)$, we have

$$\begin{aligned} [\sigma^*]_{kl} &= \overline{\sigma_{kl}} : e_x(\underline{u}^*) \\ &= \sum_{i,j=1}^{2} \frac{1}{|Y|} \left\{ \int_Y 2\mu \left([M_{kl}]_{ij} + \left[e_y(\underline{\chi}_{kl}) \right]_{ij} \right) dy \right\} [e_x(\underline{u}^*)]_{ij}. \end{aligned}$$

Since the following symmetric property holds (see Proposition 3.1)

$$\int_Y 2\mu \left([M_{kl}]_{ij} + \left[e_y(\underline{\chi}_{kl}) \right]_{ij} \right) dy = \int_Y 2\mu \left([M_{ij}]_{kl} + \left[e_y(\underline{\chi}_{ij}) \right]_{kl} \right) dy,$$

and $[e_x(\underline{u}^*)]_{ij} = [e_{x,z}(\vec{u}^*)]_{ij}$ for $1 \le i, j \le 2$, we finally conclude that for $1 \le k, l \le 2$,

$$[\sigma^*]_{kl} = \sum_{i,j=1}^{2} \frac{1}{|Y|} \left\{ \int_Y 2\mu \left([M_{ij}]_{kl} + \left[e_y(\underline{\chi}_{ij}) \right]_{kl} \right) dy \right\} \left[e_{x,z}(\vec{u}^*) \right]_{ij};$$

hence, Proposition 4.1 is proved. ∎

4.3. *Identification of* $[\sigma^*]_{3k}$, $1 \le k \le 2$, *in* Ω_m

Let $\varphi_k \in H^1_\#(Y)/\mathbb{R}$ be the solution of the periodic problem $(P)_k$ and let $p_k(y) = y_k$, $1 \le k \le 2$—note that $\nabla_y p_k = \underline{e}_k$—and set $w_k = 2p_k + \varphi_k$ and $\underline{\xi}_k = \mu \nabla_y w_k$. We define the functions

$$w_k^\epsilon = \epsilon w_k\left(\frac{x}{\epsilon}, z\right) \qquad (x, z) \in \Omega_m$$

and

$$\underline{\xi}_k^\epsilon = \underline{\xi}_k\left(\frac{x}{\epsilon}, z\right) \qquad (x, z) \in \Omega_m.$$

Then we have

$$\operatorname{div}_x \underline{\xi}_k^\epsilon = 0 \quad \text{in } \Omega_m, \tag{13}$$

and for all $\Omega' \subset\subset \Omega_m$ (since $(H2)$ holds),

$$w_k^\epsilon \to 2p_k \quad \text{in } L^2(\Omega')\text{—}\textit{strongly}, \tag{14}$$

$$\frac{\partial}{\partial z}(w_k^\epsilon) \to 0 \quad \text{in } L^2(\Omega')\text{—}\textit{strongly}, \tag{15}$$

$$\underline{\xi}_k^\epsilon \to \overline{\underline{\xi}_k} \quad \text{in } L^2(\Omega')^2\text{—}\textit{weakly}, \tag{16}$$

where $\overline{\underline{\xi}_k}$ is the average in Y of $\underline{\xi}_k$, i.e.,

$$\overline{\underline{\xi}_k} = \frac{1}{|Y|}\int_Y \mu(2\underline{e}_k + \nabla_y \varphi_k)\, dy.$$

Remark 4.2. From (13) and (16), we conclude that $\overline{\underline{\xi}_k}$ satisfies

$$-\operatorname{div}_x \overline{\underline{\xi}_k} = 0. \tag{17}$$

▮

We also need the following compactness lemma (see F. Murat [6]):

LEMMA 4.1. *If the sequence* $\{g^\epsilon\}_{\epsilon > 0}$ *belongs to a bounded set of* $W^{-1,p}(\Omega)$ *for some* $p > 2$, *and* $g^\epsilon \geq 0$, *in the following sense*:

for all $\phi \in \mathscr{D}(\Omega)$, *such that* $\phi \geq 0$, *and* $\forall \epsilon > 0, \langle g^\epsilon, \phi\rangle \geq 0$.

Then $\{g^\epsilon\}_{\epsilon > 0}$ *belongs to a compact set of* $H^{-1}(\Omega)$.

If $\underline{\xi}_k^\epsilon = ((\xi_k^\epsilon)_1, (\xi_k^\epsilon)_2)^T$ is assumed to fulfill

$$(H3)\begin{cases} \text{(a)} \ \{(\xi_k^\epsilon)_j\}_{\epsilon>0} \subset L^p_{\text{loc}}(\Omega_m) \text{ and } \left\{\left(\underline{\xi}_k^\epsilon\right)_j\right\}_{\epsilon>0} \text{ locally bounded,} \\ \qquad \text{for some } p > 2 \\ \text{(b)} \ \dfrac{\partial}{\partial z}\left((\xi_k^\epsilon)_j\right) \geq 0 \quad \text{in the distribution sense,} \end{cases}$$

then, using Lemma 4.1, we see that for all $\Omega' \subset\subset \Omega$ $(j = 1, 2)$,

$$\frac{\partial}{\partial z}\left((\xi_k^\epsilon)_j\right) \to \frac{\partial}{\partial z}\left((\overline{\xi_k})_j\right) \text{ in } H^{-1}(\Omega')\text{—}\textit{strongly}. \tag{18}$$

Let us now prove

PROPOSITION 4.2. *If* $(H2)$ *and* $(H3)$ *hold, then* $[\sigma^\epsilon]_{3k} \to [\sigma^*]_{3k}$ *in* $L^2(\Omega)$*—weakly, for* $1 \leq k \leq 2$, *where*

$$[\sigma^*]_{3k} = \frac{1}{|Y|} \sum_{j=1}^{2} \left\{ \int_Y \mu \left(2\,\delta_{jk} + \frac{\partial \varphi_j}{\partial y_k} \right) dy \right\} \left[e_{x,z}(\vec{u}^*) \right]_{3j}.$$

Proof. Let $\phi \in \mathscr{D}(\Omega_m)$ and consider $\vec{w}_k^\epsilon = (\underline{0}, w_k^\epsilon)^T$. If we use the test function $\vec{v} = \phi \vec{w}_k^\epsilon$ in $(PV)_\epsilon$ we obtain, after algebraic developments,

$$\int_{\Omega_m} \mu^\epsilon \nabla_x u_3^\epsilon \cdot \nabla_x w_k^\epsilon \phi \, dx\, dz + \sum_{j=1}^{2} \int_{\Omega_m} \mu^\epsilon \frac{\partial u_j^\epsilon}{\partial z} \frac{\partial w_k^\epsilon}{\partial x_j} \phi \, dx\, dz$$
$$+ \int_{\Omega_m} 2\mu^\epsilon \frac{\partial u_3^\epsilon}{\partial z} \frac{\partial w_k^\epsilon}{\partial z} \phi \, dx\, dz + \int_{\Omega_m} \sigma^\epsilon \nabla_{x,z} \phi \cdot \vec{w}_k^\epsilon \, dx\, dz$$
$$= \int_{\Omega_m} \vec{f}^\epsilon \cdot \phi \vec{w}_k^\epsilon \, dx\, dz, \tag{19}$$

where u_j^ϵ $(j = 1, \ldots, 3)$ denotes the jth component of $\vec{u}^\epsilon$.

Now, if we multiply (13) by the test function ϕu_3^ϵ, after carrying out an integration by parts, we have

$$\int_{\Omega_m} \left(\underline{\xi}_k^\epsilon \cdot \nabla_x \phi \right) u_3^\epsilon \, dx\, dz + \int_{\Omega_m} \left(\mu^\epsilon \nabla_x w_k^\epsilon \cdot \nabla_x u_3^\epsilon \right) \phi \, dx\, dz = 0. \tag{20}$$

We subtract (20) from (19),

$$\int_{\Omega_m} \sigma^\epsilon \nabla_{x,z} \phi \cdot \vec{w}_k^\epsilon \, dx\, dz + \int_{\Omega_m} 2\mu^\epsilon \frac{\partial u_3^\epsilon}{\partial z} \frac{\partial w_k^\epsilon}{\partial z} \phi \, dx\, dz$$
$$- \int_{\Omega_m} \left(\underline{\xi}_k^\epsilon \cdot \nabla_x \phi \right) u_3^\epsilon \, dx\, dz + \sum_{j=1}^{2} \int_{\Omega_m} \mu^\epsilon \frac{\partial u_j^\epsilon}{\partial z} \frac{\partial w_k^\epsilon}{\partial x_j} \phi \, dx\, dz$$
$$= \int_{\Omega_m} \vec{f}^\epsilon \cdot \phi \vec{w}_k^\epsilon \, dx\, dz. \tag{21}$$

Using (3), we can pass to the limit in the right-hand side of (21). Using Proposition 2.1, (14), (15), (16), we do the same in the first, second, and third terms of the left-hand side of (21). Let us rewrite the terms in the sum in the left-hand side of (21) as follows ($j = 1, 2$)

$$\int_{\Omega_m} \mu^\epsilon \frac{\partial u_j^\epsilon}{\partial z} \frac{\partial w_k^\epsilon}{\partial x_j} \phi \, dx \, dz$$
$$= -\left\langle \frac{\partial}{\partial z}\left((\xi_k^\epsilon)_j \right); \phi u_j^\epsilon \right\rangle_{H^{-1}(\Omega'), H_0^1(\Omega')} - \int_{\Omega_m} (\xi_k^\epsilon)_j u_j^\epsilon \frac{\partial \phi}{\partial z} \, dx \, dz,$$

where $\Omega' \subset\subset \Omega_m$ is such that $\operatorname{supp} \phi \subset \Omega'$. Using Proposition 2.1, (16) and (18), we can pass to the limit in the last equation and obtain

$$\int_{\Omega_m} \mu^\epsilon \frac{\partial u_j^\epsilon}{\partial z} \frac{\partial w_k^\epsilon}{\partial x_j} \phi \, dx \, dz \to \int_{\Omega_m} (\overline{\xi_k})_j \frac{\partial u_j^*}{\partial z} \phi \, dx \, dz, \qquad 1 \le j \le 2.$$

Then, passing to the limit in (21), we obtain

$$\int_{\Omega_m} \sigma^* \nabla_{x,z} \phi \cdot \vec{w}_{kl}^* \, dx \, dz - \int_{\Omega_m} \left(\overline{\underline{\xi}_k} \cdot \nabla_x \phi \right) u_3^* \, dx \, dz + \sum_{j=1}^{2} \int_{\Omega_m} (\overline{\xi_k})_j \frac{\partial u_j^*}{\partial z} \phi \, dx \, dz$$
$$= \int_{\Omega_m} \vec{f}^* \cdot \phi \vec{w}_k^* \, dx \, dz, \tag{22}$$

where $\vec{w}_k^* = (\underline{0}, 2 p_k)^T$. Integrating by parts the first and second terms of this equation, using that σ^* and $\overline{\underline{\xi}_k}$ satisfy (4) and (17), respectively, we obtain

$$-\int_{\Omega_m} \sigma^* \colon e_{x,z}\left(\vec{w}_k^* \right) \phi \, dx \, dz + \sum_{j=1}^{2} \int_{\Omega_m} (\overline{\xi_k})_j \frac{\partial u_j^*}{\partial z} \phi \, dx \, dz$$
$$+ \int_{\Omega_m} \overline{\underline{\xi}_k} \cdot \nabla_x u_3^* \phi \, dx \, dz = 0 \tag{23}$$

and in the distribution sense (using that $\vec{w}_k^* = (\underline{0}, 2 p_k)^T$ and then, $\sigma^* \colon e_{x,z}(\vec{w}_k^*) = 2[\sigma^*]_{3k}$),

$$[\sigma^*]_{3k} = \sum_{j=1}^{2} \left\{ \frac{1}{|Y|} \int_Y \mu \left(2 \delta_{kj} + \frac{\partial \varphi_k}{\partial y_j} \right) dy \right\} \left[e_{x,z}(\vec{u}^*) \right]_{3j}.$$

Finally, since $\int_Y \mu(2 \delta_{kj} + (\partial \varphi_k / \partial y_j)) \, dy = \int_Y \mu(2 \delta_{jk} + (\partial \varphi_j / \partial y_k)) \, dy$ (see Proposition 3.1), we conclude that

$$[\sigma^*]_{3k} = \sum_{j=1}^{2} \left\{ \frac{1}{|Y|} \int_Y \mu \left(2 \delta_{jk} + \frac{\partial \varphi_j}{\partial y_k} \right) dy \right\} \left[e_{x,z}(\vec{u}^*) \right]_{3j}.$$

Proposition 4.2 is therefore proved. ∎

Remark 4.3. By symmetry of the limit tensor σ^*, we also have identified the coefficients $[\sigma^*]_{k3}$, $k = 1, 2$. ∎

4.4. *Identification of* $[\sigma^*]_{33}$ *in* Ω_m

Using Lemma 4.1 we shall prove the following

PROPOSITION 4.3. *The sequence* $\{[\sigma^{\epsilon}]_{33}\}_{\epsilon>0}$ *is such that* $[\sigma^{\varepsilon}]_{33} \rightharpoonup [\sigma^*]_{33}$ *in* $L^2(\Omega_m)$*—weakly, where*

$$[\sigma^*]_{33} = \frac{1}{|Y|}\left\{\int_Y 2\mu\, dy\right\}[e_{x,z}(u^*)]_{33}.$$

Proof. We use the method introduced in R. Brizzi [3]. We know that $[\sigma^{\epsilon}]_{33} = 2\mu^{\epsilon}(\partial u_3^{\epsilon}/\partial z)$ and $\mu^{\epsilon} = \mu_1\chi_{\Omega_1^{\epsilon}\cap\Omega_m} + \mu_2\chi_{\Omega_2^{\epsilon}\cap\Omega_m}$; then

$$[\sigma^{\epsilon}]_{33} = 2\mu_1 P_1\left(\frac{\partial(u_3^{\epsilon})_1}{\partial z}\right) + 2\mu_2 P_2\left(\frac{\partial(u_3^{\epsilon})_2}{\partial z}\right),$$

where $(u_3^{\epsilon})_i = u_3^{\epsilon}|_{\Omega_i^{\epsilon}}$ $(i = 1, 2)$, and the operator $P_i(\cdot)$ represents the extension by zero in $\Omega \setminus \Omega_i^{\epsilon}$.

Let $\psi_i^{\epsilon} = P_i((\partial(u_3^{\epsilon})_i)/\partial z)$; we see that it is a bounded sequence in $L^2(\Omega_m)$. Then there exists $\psi_i^* \in L^2(\Omega_m)$ and a subsequence, such that

$$\psi_i^{\epsilon} \rightharpoonup \psi_i^* \quad \text{in } L^2(\Omega_m)\text{—}weakly. \tag{24}$$

It is possible to identify the functions ψ_i^*: Let $\phi \in \mathscr{D}(\Omega_m)$; then

$$\left\langle \frac{\partial}{\partial z}(\chi_{\Omega_1^{\epsilon}\cap\Omega_m}); \phi u_3^{\epsilon}\right\rangle_{H^{-1}(\Omega_m), H_0^1(\Omega_m)}$$
$$= -\int_{\Omega_m}\psi_1^{\epsilon}\phi\, dx\, dz - \int_{\Omega_m}\chi_{\Omega_1^{\epsilon}\cap\Omega_m} u_3^{\epsilon}\frac{\partial\phi}{\partial z}\, dx\, dz.$$

For the sequence $\{(\partial/\partial z)(\chi_{\Omega_1^{\epsilon}\cap\Omega_m})\}_{\epsilon>0}$ we use the Lemma 4.1 (in R. Brizzi [3] it is shown that this sequence satisfies the hypothesis required in the compactness lemma), and for the right-hand terms we have (24), Proposition 2.1 and (2). Then, passing to the limit

$$\left\langle \frac{\partial}{\partial z}\left(\frac{|O(z)|}{|Y|}\right); \phi u_3^*\right\rangle_{H^{-1}(\Omega_m); H_0^1(\Omega_m)}$$
$$= -\int_{\Omega_m}\psi_1^*\phi\, dx\, dz - \int_{\Omega_m}\frac{|O(z)|}{|Y|}u_3^*\frac{\partial\phi}{\partial z}\, dx\, dz,$$

and developing the duality product in the last equation, we obtain in the distribution sense the following identity:

$$\psi_1^* = \frac{|O(z)|}{|Y|} \frac{\partial u_3^*}{\partial z}.$$

In the same way,

$$\psi_2^* = \frac{|Y - \overline{O(z)}|}{|Y|} \frac{\partial u_3^*}{\partial z}.$$

Finally, from (24) we know that $[\sigma^*]_{33} = 2\mu_1 \psi_1^* + 2\mu_2 \psi_2^*$; then we conclude that

$$[\sigma^*]_{33} = \frac{1}{|Y|} 2\left(\mu_1 |O(z)| + \mu_2 |Y - \overline{O(z)}|\right) \frac{\partial u_3^*}{\partial z}$$
$$= \frac{1}{|Y|} \left\{ \int_Y 2\mu \, dy \right\} \frac{\partial u_3^*}{\partial z}.$$

Hence, Proposition 4.3 is proved. ∎

4.5. *Identification of* σ^* *in* Ω_1 *and* Ω_2

Finally, to identify the limit σ^* in Ω_1 and Ω_2 we prove the following.

PROPOSITION 4.4. *For* $j = 1, 2$, $\sigma^\epsilon|_{\Omega_j} \rightharpoonup \sigma_j^*$ *in* $\mathscr{L}_s^2(\Omega_j)$*—weakly, where*

$$\sigma_j^* = 2\mu_j e_{x,z}\left(\vec{u}_j^*\right) \text{ and } \vec{u}_j^* = \vec{u}^*|_{\Omega_j}.$$

Proof. From the *a priori* estimates (Proposition 2.1), we know that $\vec{u}^\epsilon \rightharpoonup \vec{u}^*$ in $H_0^1(\Omega)^3$—*weakly* and $\sigma^\epsilon \rightharpoonup \sigma^*$ in $\mathscr{L}_{3,s}^2(\Omega)$—*weakly*, so, if we consider $\vec{u}_{\Omega_j}^\epsilon = \vec{u}^\epsilon|_{\Omega_j}$ and $\sigma_{\Omega_j}^\epsilon = \sigma^\epsilon|_{\Omega_j}$ $(j = 1, 2)$, then

$$\vec{u}_{\Omega_j}^\epsilon \rightharpoonup \vec{u}_j^* \quad \text{in } H^1\left(\Omega_j\right)^3\text{—weakly} \qquad (j = 1, 2) \tag{25}$$

$$\sigma_{\Omega_j}^\epsilon \rightharpoonup \sigma_j^* \quad \text{in } \mathscr{L}_{3,s}^2\left(\Omega_j\right)\text{—weakly} \qquad (j = 1, 2), \tag{26}$$

where $\sigma_j^* = \sigma^*|_{\Omega_j}$ and $\vec{u}_j^* = \vec{u}^*|_{\Omega_j}$.

We also know, from the definition of σ^ϵ, that $\sigma_{\Omega_j}^\epsilon = 2\mu_j e_{x,z}(\vec{u}_{\Omega j}^\epsilon)$. Now, using (25) and (26), we conclude that

$$\sigma_j^* = 2\mu_j e_{x,z}\left(\vec{u}_j^*\right) \qquad (j = 1, 2).$$

Proposition 4.4 is proved. ∎

4.6. *Conclusion*

From Propositions (4.1), (4.2), (4.3), (4.4) and the definition of tensor $\mathscr{A}$, we conclude that σ^* and $\vec{u}^*$ are related by (1). Therefore, since σ^* satisfies (4), $\vec{u}^*$ is solution of $(PV)_H$.

Since all the steps shown above can be repeated for any weak accumulation point of $\{\vec{u}^\epsilon\}_{\epsilon > 0}$, we conclude that the solution of the homogenized problem $(PV)_H$ is the unique weak accumulation point of this sequence. Hence, the whole sequence converges weakly to this limit, and Theorem 3.1 is therefore proved.

APPENDIX A: PROOF OF PROPOSITION 3.1

To prove part (a) of Proposition 3.1, we first study the coefficients of tensor A with indexes $1 \le i, j, k, l \le 2$. The symmetry of these coefficients is evident when $z \in]h_1, z_0[$ and $z \in]-z_0, 0[$. To prove the symmetry when $z \in]0, h_1[$ (i.e., in the region near the interface), we use the bilinear form $b\colon (H^1_\#(Y)/\mathbb{R})^2 \times (H^1_\#(Y)/\mathbb{R})^2 \to \mathbb{R}$, defined by

$$b(\underline{\chi}, \underline{\psi}) = \frac{1}{|Y|}\int_Y 2\mu e_y(\underline{\chi}) : e_y(\underline{\psi})\, dy \quad \forall \underline{\chi}, \underline{\psi} \in \left(H^1_\#(Y)/\mathbb{R}\right)^2. \tag{27}$$

Let $\underline{w}_{kl} = \underline{p}_{kl} + \underline{\chi}_{kl}$, $1 \le k, l \le 2$, and $\underline{\chi}_{kl}$ solution of $(P)_{kl}$ and $\underline{p}_{kl}$ the polynomial defined in Section 4.2. Using properties of these functions we can easily see that

$$b(\underline{w}_{kl}, \underline{w}_{ij}) = a^m_{ijkl}. \tag{28}$$

Since $b(\cdot, \cdot)$ is a symmetric bilinear form (see [4]), then $a^m_{ijkl} = a^m_{klij}$, for $1 \le i, j, k, l \le 2$.

On the other hand, as $M_{kl} = M_{lk}$, then by uniqueness of problem $(P)_{kl}$, we have $\underline{\chi}_{kl} = \underline{\chi}_{lk}$ (up to an additive constant), and then $a^m_{ijkl} = a^m_{ijlk}$.

We now study the coefficients a_{ijkl} with $j = l = 3$ and $1 \le i, k \le 2$. From the definition of tensor $\mathscr{A}$, we have the symmetry of these coefficients when $z \in]-z_0, 0[$ and $z \in]h_1, z_0[$. To prove the symmetry in the region near the interface, we now consider the bilinear form $\hat{b}\colon H^1_\#(Y)/\mathbb{R} \times H^1_\#(Y)/\mathbb{R} \to \mathbb{R}$ defined by

$$\hat{b}(\varphi, \xi) = \frac{1}{4|Y|}\int_Y \mu \nabla_y \varphi \cdot \nabla_y \xi.$$

Let $w_k = 2p_k + \varphi_k$, where φ_k is solution of $(P)_k$ and p_k is the polynomial defined in Section 4.3. Doing elementary calculations we conclude that

$$\hat{b}(w_k, w_i) = a^m_{i3k3},$$

and by symmetry of $\hat{b}(\cdot,\cdot)$, we have $a^m_{i3k3} = a^m_{k3i3}$, $1 \leq i, j, \leq 2$. By construction of $\mathscr{A}$, we also have $a^m_{i3k3} = a^m_{i33k}$, $1 \leq i, j, \leq 2$.

For the other nonzero terms (i.e., a_{i33l}, a_{3jk3} and a_{3j3l}), the same method can be used.

To prove part (b), we first note that coerciveness when $z \in]h_1, z_0[$ and $z \in]-z_0, 0[$ is evident since $\mu_i > 0$. When $z \in]0, h_1[$, we must show that for all symmetric second-order tensors ξ we have $(\mathscr{A}\xi) \colon \xi \geq 0$, and if $(\mathscr{A}\xi) \colon \xi = 0$, then $\xi = 0$.

Using the relationships between the coefficients of $\mathscr{A}$ and the bilinear forms b and $\hat{b}$, we obtain

$$(\mathscr{A}\xi) \colon \xi = b\left(\sum_{k,l=1}^{2} w_{kl}\,\xi_{kl}, \sum_{k,l=1}^{2} w_{kl}\,\xi_{kl}\right) + \hat{b}\left(\sum_{l=1}^{2} w_l\,\xi_{l3}, \sum_{i=1}^{2} w_i\,\xi_{i3}\right) + \hat{b}\left(\sum_{k=1}^{2} w_k\,\xi_{k3}, \sum_{j=1}^{2} w_j\,\xi_{j3}\right) + \left(\frac{1}{|Y|}\int_Y 2\mu\, dy\right)\xi_{33}^2. \tag{29}$$

From this equation, the positiveness of $\mathscr{A}$ is due to the positiveness of the bilinear forms (see [4 or 8]). When $(\mathscr{A}\xi) \colon \xi = 0$, all the terms on the right-hand side of (29) are equal to zero, which implies that $\xi_{33} = 0$ and, by coerciveness of b and $\hat{b}$, that $\xi_{kl} = 0$ $(k, l = 1, 2)$ and $\xi_{k3} = 0$, $k = 1, 2$ (see, e.g., [4]). Hence $\xi = 0$. ∎

APPENDIX B: THE CASE $\Omega \subset \mathbb{R}^2$

When $\Omega \subset \mathbb{R}^2$, it is possible to find explicit functions which are solutions of the periodic problems $(P)_{kl}$, $(P)_k$, and validate the hypotheses $(H1)$, $(H2)$, and $(H3)$ (see [1]). In the 2-D case, the periodic problems $(P)_{kl}$ and $(P)_k$ become ordinary differential equations in $]0, T[$ $(= Y)$ with periodic boundary conditions:

$$(P)_{11}\begin{cases} -\dfrac{d}{dy}\left(2\mu\dfrac{d\chi}{dy}\right) = \dfrac{d}{dy}(2\mu) \text{ in } Y \\ \chi(0) = \chi(T) \end{cases} \quad \text{and}$$

$$(P)_{1}\begin{cases} -\dfrac{d}{dy}\left(\mu\dfrac{d\varphi}{dy}\right) = \dfrac{d}{dy}(2\mu) \text{ in } Y \\ \varphi(0) = \varphi(T). \end{cases}$$

It is easy to prove the following.

PROPOSITION B.1. *For* $z \in]0, h_1[$ *fixed, let* φ *and* χ *be defined by*

$$\varphi(y) = \begin{cases} \left(\dfrac{C}{\mu_2} - 2\right) y & \text{if } 0 < y < a(z) \\ \left(\dfrac{C}{\mu_2} - \dfrac{C}{\mu_1}\right) a + \left(\dfrac{C}{\mu_1} - 2\right) y & \text{if } a(z) < y < b(z) \\ \left(\dfrac{C}{\mu_1} - \dfrac{C}{\mu_2}\right)(b - a) + \left(\dfrac{C}{\mu_2} - 2\right) y & \text{if } b(z) < y < 1 \end{cases}$$

and

$$\chi(y) = \begin{cases} \left(\dfrac{C}{2\mu_2} - 1\right) y & \text{if } 0 < y < a(z) \\ \left(\dfrac{C}{2\mu_2} - \dfrac{C}{2\mu_1}\right) a + \left(\dfrac{C}{2\mu_1} - 1\right) y & \text{if } a(z) < y < b(z) \\ \left(\dfrac{C}{2\mu_1} - \dfrac{C}{2\mu_2}\right)(b - a) + \left(\dfrac{C}{2\mu_2} - 1\right) y & \text{if } b(z) < y < 1 \end{cases}$$

where $O(z) =]a(z), b(z)[$ *and*

$$C = C(z) = 2\left(\frac{1}{|Y|}\int_Y \frac{1}{\mu}\, dy\right)^{-1}.$$

Then φ *and* χ *are the unique solutions (up to an additive constant) in* $H^1_{\#}(Y)$ *of problems* $(P)_1$ *and* $(P)_{11}$, *respectively.*

Remark B.1. Studying the regularity of the solutions of the equations $(P)_1$ and $(P)_{11}$ one can validate hypotheses $(H1)$, $(H2)$, and $(H3)$ (see [1]). ∎

In the 2-D case, the homogenized tensor has six nonzero coefficients,

$$a_{1111} = \begin{cases} 2\mu_1 & h_1 < z < z_0 \\ \dfrac{1}{|Y|}\displaystyle\int_Y 2\mu\left(1 + \frac{\partial \chi}{\partial y}\right) dy & 0 < z < h_1 \\ 2\mu_2 & z_0 < z < 0 \end{cases},$$

$$a_{2222} = \begin{cases} 2\mu_1 & h_1 < z < z_0 \\ \dfrac{1}{|Y|}\displaystyle\int_Y 2\mu\, dy & 0 < z < h_1 \\ 2\mu_2 & z_0 < z < 0 \end{cases}$$

$$a_{1212} = a_{1221} = a_{2112} = a_{2121} = \begin{cases} \mu_1 & h_1 < z < z_0 \\ \dfrac{1}{2|Y|}\displaystyle\int_Y \mu\left(2 + \dfrac{\partial\varphi}{\partial y}\right) dy & 0 < z < h_1 \\ \mu_2 & z_0 < z < 0. \end{cases}$$

Remark B.2. If we denote by μ^* and μ^+ the following positive constants,

$$\mu^* = \frac{1}{|Y|}\int_Y \mu\, dy, \qquad \mu^+ = \left(\frac{1}{|Y|}\int_Y \frac{1}{\mu}\, dy\right)^{-1}.$$

(these constants are the arithmetic and the harmonic media of μ in Y, respectively), then we note that $a_{2222} = 2\mu^*$ and $C(z) = 2\mu^+$ in Ω_m. ∎

Since we know the explicit solutions of the periodic problems, we can easily show that the homogenized coefficients in the 2-D case are:

$$a_{1111} = \begin{cases} 2\mu_1 & h_1 < z < z_0 \\ 2\mu^+ & 0 < z < h_1 \\ 2\mu_2 & z_0 < z < 0 \end{cases}, \qquad a_{2222} = \begin{cases} 2\mu_1 & h_1 < z < z_0 \\ 2\mu^* & 0 < z < h_1 \\ 2\mu_2 & z_0 < z < 0 \end{cases},$$

$$a_{1212} = a_{1221} = a_{2112} = a_{2121} = \begin{cases} \mu_1 & h_1 < z < z_0 \\ \mu^+ & 0 < z < h_1 \\ \mu_2 & z_0 < z < 0 \end{cases}.$$

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REFERENCES

1. L. Baffico and C. Conca, Homogenization of a transmission problem in 2D elasticity, *in* ‘‘Computational Science for the 21st Century’’ (M.-O. Bristeau *et al.*, Eds.), pp. 539–548, Wiley, New York, 1997.

2. A. Bensoussan, J.-L. Lions, and G. Papanicolaou, "Asymptotic Analysis for Periodic Structures," North-Holland, Amsterdam, 1978.
3. R. Brizzi, Transmission problem and boundary homogenization, *Rev. Mat. Apl.* **15** (1994), 1–16.
4. C. Conca, On the application of the homogenization theory to a class of problems arising in fluid mechanics, *J. Math. Pures Appl.* **64** (1985), 31–75.
5. L. D. Landau and E. M. Lifshitz, "Theory of Elasticity," Collection Course of Theoretical Physics, Vol. 7, Addison–Wesley, London, 1959.
6. F. Murat, L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl.* **60** (1981), 309–322.
7. P. A. Raviart and J. M. Thomas, "Introduction à l'analyse numérique des équations aux dérivees partielles," Masson, Paris, 1983.
8. E. Sánchez-Palencia, "Nonhomogeneous media and vibrating theory," Lecture Notes in Physics, Vol. 127, Springer-Verlag, Berlin, 1980.