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A POINTWISE SPECTRUM AND REPRESENTATION OF OPERATORS

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ABSTRACT. For a self-adjoint operator $A : H \to H$ commuting with an increasing family of projections $\mathcal{P} = (P_t)$ we study the multifunction $t \to \Gamma^{\mathcal{T}}(t) = \bigcap\{\sigma_I : I$ an open set of the topology $\mathcal{T}$ containing $t\}$, where σ_I is the spectrum of A on $P_I H$. Let $m_{\mathcal{P}}$ be the measure of maximal spectral type. We study the condition that $\Gamma^{\mathcal{T}}$ is essentially a singleton, $m_{\mathcal{P}}\{t : \Gamma^{\mathcal{T}}(t)$ is not a singleton$\} = 0$. We show that if $\mathcal{T}$ is the density topology and if $m_{\mathcal{P}}$ satisfies the density theorem, in particular if it is absolutely continuous with respect to the Lebesgue measure, then this condition is equivalent to the fact that A is a Borel function of $\mathcal{P}$. If $\mathcal{T}$ is the usual topology then the condition is equivalent to the fact that A is approched in norm by step functions $\sum_{n\in\mathbb{N}} \Gamma^{\mathcal{T}}(\alpha_n)\langle P_{I_n} f, f\rangle$, where the set of intervals $\{I_n : n \in \mathbb{N}\}$ covers the set where $\Gamma^{\mathcal{T}}$ is a singleton.

1. Introduction

Let H be a real separable Hilbert space and $\mathcal{P} = (P_t : t \in \mathbb{R})$ be an increasing right continuous family of projections, with $P_{-\infty} = 0$, $P_\infty = I$ the identity. If B is a Borel real set we denote by P_B the projection $P_B = \int_B dP_t$.

For $f \in H$ denote by m_f the Borel measure induced by the spectral family $\mathcal{P}$:

$$m_f(B) = \langle P_B f, f\rangle \text{ for any Borel set } B.$$

There exists a measure $m_{\mathcal{P}}$, associated to some element of H of maximal spectral type, i.e. $m_f << m_{\mathcal{P}}$ for all $f \in H$. We shall assume that $m_{\mathcal{P}}(\mathbb{R}) = 1$.

We shall denote by $\mathcal{L}_{\mathcal{P}}$ the completion of the Borel σ-field with respect to $m_{\mathcal{P}}$. If $G \in \mathcal{L}_{\mathcal{P}}$ there is a Borel set B such that $m_{\mathcal{P}}(G\Delta B) = 0$, and we define

$$P_G := P_B$$

This definition is consistent because if $B_1 = B_2$ $m_{\mathcal{P}}$-a.e., then $P_{B_1} = P_{B_2}$; in fact

$$\|(P_{B_1} - P_{B_2})f\|^2 = \int_{B_1\Delta B_2} d\langle P_t f, f\rangle = \int_{B_1\Delta B_2} \frac{d\langle P_t f, f\rangle}{dm_{\mathcal{P}}(t)} dm_{\mathcal{P}}(t) = 0.$$

We shall denote by $F_{\mathcal{P}}$ the support of $m_{\mathcal{P}}$; that is, $F_{\mathcal{P}}$ is the smallest closed set of full measure.

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Let A be a self-adjoint operator commuting with $\mathcal{P}$. Then A acts on the space $P_\Lambda H$ for any $\Lambda \in \mathcal{L}_\mathcal{P}$, i.e. $AP_\Lambda H \subset P_\Lambda H$. Denote

$$\sigma_\Lambda = \text{ spectrum of } AP_\Lambda \text{ on } P_\Lambda H$$

Then σ_Λ is a compact set included in $[-\|A\|, \|A\|]$. If $\Lambda_1 \supseteq \Lambda_2$ and $m_\mathcal{P}(\Lambda_2) > 0$, then $\sigma_{\Lambda_1} \supseteq \sigma_{\Lambda_2}$. In fact, if $\lambda \in \sigma_{\Lambda_2}$, then there is a sequence $(f_n) \subseteq P_{\Lambda_2}(H)$ with $\|f_n\| = 1$ such that $\langle AP_{\Lambda_2} f_n, f_n \rangle \to \lambda$, from which the property follows.

Our work is concerned with integral representation of a self-adjoint operator A commuting with $\mathcal{P}$. In this context a set of necessary and sufficient conditions is given in [1] and [5].

First let us reduce the problem to the case of continuous $\mathcal{P}$. If A commutes with $\mathcal{P}$ then it commutes with any $\Delta P_t = P_t - P_{t-} \neq 0$. A necessary condition in order that the operator A can be written in the form $A = g(\int s dP_s)$ is that $A\Delta P_t = \lambda_t \Delta P_t$ for some real λ_t. Denote by L the Hilbert space generated by $\{\Delta P_t H : t \in \mathbb{R}\}$, and let $H' = H \Theta L$. The projections P_t and $P_{H'}$ commute, the family of projections $\mathcal{P}' = (P'_t = P_t P_{H'} : t \in \mathbb{R})$ is continuous, and the restriction $A' = AP_{H'}$ commutes with $\mathcal{P}'$. We have that $A = g(\int s dP_s)$ iff $A\Delta P_t = \lambda_t \Delta P_t$ for all t, and $A' = g'(\int s dP'_s)$. Then we can assume $\mathcal{P} = (P_t : t \in \mathbb{R})$ is continuous.

Since $\mathcal{P}$ is continuous, the measure of maximal spectral type $m_\mathcal{P}$ is non-atomic.

Following [6], a sequence $\{K_n\}$ of $\mathcal{L}_\mathcal{P}$ sets is said to converge to $t \in \mathbb{R}$ if: $t \in \bigcap_n K_n$, $m_\mathcal{P}(K_n) > 0$ and $m_\mathcal{P}(K_n) \underset{n}{\longrightarrow} 0$. Let $\mathcal{K}$ be a collection of sequences in $\mathcal{L}_\mathcal{P}$. We denote by $\mathcal{K}(t)$ the family of sequences in $\mathcal{K}$ converging to t. We assume that $\mathcal{K}(t) \neq \phi$ for every $t \in \mathbb{R}$.

The upper outer density of $E \subseteq \mathbb{R}$ at the point t is defined by

$$\bar{D}^*(E,t) = \sup\{\overline{\lim_n} \frac{m^*_\mathcal{P}(E \cap K_n)}{m_\mathcal{P}(K_n)} / \{K_n\} \in \mathcal{K}(t)\}.$$

The lower outer density $\underline{D}^*(I,t)$ is defined as

$$\underline{D}^*(E,t) = \inf\{\underline{\lim}_n \frac{m^*_\mathcal{P}(E \cap K_n)}{m_\mathcal{P}(K_n)} / \{K_n\} \in \mathcal{K}(t)\}.$$

If E is $\mathcal{L}_\mathcal{P}$-measurable and $\underline{D}^*(E,t) = \bar{D}^*(E,t) = 1$, then we say that t is an (outer) density point for E. The density topology $\mathcal{T}_D$ is the set of all $I \subseteq \mathbb{R}$ such that $\forall t \in I \quad \bar{D}^*(I^c, t) = 0$. In this context also see [3], [7] and [8].

We shall assume that $m_\mathcal{P}$ satisfies a density theorem; this means that there exists a collection $\mathcal{K}$ such that, for every set $A \subseteq \mathbb{R}$, almost every point with respect to $m_\mathcal{P}$ of A is an outer density point for A. Under this condition every open set $I \in \mathcal{T}_D$ is $\mathcal{L}_\mathcal{P}$-measurable, and every point $t \in I$ is a density point of I. Moreover a function $\varphi \in \mathcal{L}_\mathcal{P}$ is $\mathcal{T}_D/\mathcal{T}_0$-continuous $m_\mathcal{P}$-a.e., where $\mathcal{T}_0$ in the usual topology on $\mathbb{R}$. Therefore, if φ is a bounded $\mathcal{L}_\mathcal{P}$-measurable function, then for $m_\mathcal{P}$-a.a. t, if $\{\mathcal{K}_n\} \in \mathcal{K}(t)$

$$\lim_{n \to \infty} \frac{\int_{K_n} \varphi \, dm_\mathcal{P}}{m_\mathcal{P}(K_n)} = \varphi(t).$$

If $m_\mathcal{P}$ is absolutely continuous with respect to the Lebesgue measure, then $m_\mathcal{P}$ satisfies the density theorem with respect to $\mathcal{K}(t) =$ class of all regular intervals converging to t.

In the next four lemmas we shall consider a topology $\mathcal{T}$ on $F_{\mathcal{P}}$. Mainly we are interested in the traces of $\mathcal{T}_D$ and $\mathcal{T}_0$ over $F_{\mathcal{P}}$, which we still denote by $\mathcal{T}_D$ and $\mathcal{T}_0$. We shall assume that $\mathcal{T}$ fulfills the following two conditions:

(i) $\mathcal{T} \subseteq \mathcal{L}_{\mathcal{P}}$;

(ii) $\forall \phi \neq I \in \mathcal{T} \quad m_{\mathcal{P}}(I) > 0$.

These two conditions are satisfied by $\mathcal{T}_D$ and $\mathcal{T}_0$. For $\mathcal{T}_D$ they follow respectively from Corollary 4.4 and Corollaries 4.12 and 4.13 of [6]. As for $\mathcal{T}_0$, (i) is inmediate and (ii) follows from the definition of $F_{\mathcal{P}}$.

We denote by $\mathcal{I}^{\mathcal{T}}(t) = \{I \in \mathcal{T} : t \in I\}$ the set of open neighbourhoods of $t \in F_{\mathcal{P}}$.

Define the pointwise spectrum of A with respect to $\mathcal{P}$ and the topology $\mathcal{T}$ by

$$\Gamma^{\mathcal{T}}(t) = \bigcap_{I \in \mathcal{I}^{\mathcal{T}}(t)} \sigma_I,$$

for $t \in F_{\mathcal{P}}$.

In the proof of the lemmas we abbreviate $\Gamma(t) = \Gamma^{\mathcal{T}}(t)$ and $\mathcal{I}^{\mathcal{T}}(t) = \mathcal{I}(t)$

Lemma 1. $\Gamma^{\mathcal{T}}(t)$ *is a non-empty* $\mathcal{T}_0$*-compact set included in* $[-\|A\|, \|A\|]$.

Proof. The sets σ_I are non-empty and compact, so it suffices to show that the family $(\sigma_I)_{I\in\mathcal{I}(t)}$ has the finite intersection property. This holds because this family is a net with respect to the order induced by inclusion; in fact if $I_1, ..., I_n$ belong to $\mathcal{I}(t)$ the open set $\bigcap_{i=1}^n I_i$ also belongs to $\mathcal{I}(t)$, and $\bigcap_{i=1}^n \sigma_{I_i}$ contains $\sigma_{\bigcap_{i=1}^n I_i}$. □

The mapping $\Gamma : F_{\mathcal{P}} \to$ Subsets $\mathbb{R}$ is a multifunction with compact values. We recall that for X and Y topological spaces, a multifunction $G : X \to$ Subsets Y, is said to be upper semi-continuous if for all $t \in X$ and for all open set V containing $G(t)$, there exists a neighbourhood U of t such that $G(s) \subset V$ for all $s \in U$ (see [2]). Now, set

$$S(G) = \{t \in F_{\mathcal{P}} : G(t) \text{ is a singleton}\}$$

and assume $S(G) \neq \varnothing$. When $t \in S(G)$ we identify the singleton $G(t)$ with its unique element. If G is an upper semi-continuous multifunction, then $G : S(G) \to Y$ is a continuous function.

Lemma 2. $\Gamma^{\mathcal{T}}$ *is a* $\mathcal{T}/\mathcal{T}_0$ *upper semi-continuous multifunction.*

Proof. Let $V \in \mathcal{T}_0$ be an open set such that $\Gamma(t) \subset V$. Since the family of $\mathcal{T}_0$-compact sets $\{\sigma_I\}_{I\in\mathcal{I}(t)}$ is a net with the order induced by inclusion, it is easy to prove that there exists $I \in \mathcal{I}(t)$ with $\sigma_I \subseteq V$. In particular, for all $s \in I$ we have $\Gamma(s) \subseteq \sigma_I \subseteq V$. □

This result implies that there exist measurable selections $\gamma \in \Gamma$, i.e. an $\mathcal{L}_{\mathcal{P}}$-measurable function $\gamma : F_{\mathcal{P}} \to \mathbb{R}$ such that $\gamma(t) \in \Gamma^{\mathcal{T}}(t)$ for all $t \in F_{\mathcal{P}}$. Moreover, the functions below are two measurable selections:

$$\psi_{\Gamma^{\mathcal{T}}}(t) = \max\{u \in \Gamma^{\mathcal{T}}(t)\}, \quad \phi_{\Gamma^{\mathcal{T}}}(t) = \min\{u \in \Gamma^{\mathcal{T}}(t)\}.$$

More precisely:

Lemma 3. $\psi_{\Gamma^{\mathcal{T}}}$ *is* $\mathcal{T}/\mathcal{T}_0$ *upper semi-continuous and* $\phi_{\Gamma^{\mathcal{T}}}$ *is* $\mathcal{T}/\mathcal{T}_0$ *lower semi-continuous.*

Proof. This follows from Lemma 2. In fact, if $t \in \psi_{\Gamma}^{-1}(-\infty, r)$, then $\Gamma(t) \subset (-\infty, r) = V$ and there exists a neighbourhood U of t such that $\Gamma(s) \subset (-\infty, r)$

for all $s \in U$, so $U \subset \psi_\Gamma^{-1}(-\infty, r)$. Hence $\psi_\Gamma^{-1}(-\infty, r) \in \mathcal{T}$ and ψ_Γ is upper semi-continuous. The proof is analogous for ϕ_Γ. □

Lemma 4. *$S(\Gamma^{\mathcal{T}}) = \{t : \Gamma^{\mathcal{T}}(t)$ is a singleton$\}$ is $\mathcal{L}_{\mathcal{P}}$-measurable.*

Proof. This follows inmediately from the equality $S(\Gamma) = \{t : \psi_\Gamma(t) = \phi_\Gamma(t)\}$. □

Lemma 5. *Assume m is a non-atomic Borel measure and $\mathcal{J} = \{J\}$ is a family of open intervals covering a Borel set E. Then there exists a countable class of disjoint open intervals $\mathcal{I} = \{I\}$ subordinated to $\mathcal{J}$ (i.e. for any $I \in \mathcal{I}$ there exists $J \in \mathcal{J}$ such that $I \subset J$) satisfying $m(E \setminus \bigcup_{I \in \mathcal{I}} I) = 0$.*

Proof. By Lindelöf's theorem there exists a countable class $\{J_n : n \in \mathbb{N}\} \subset \mathcal{J}$ such that $E \subset \bigcup_{n\in\mathbb{N}} J_n$. Now define $I'_1 = J_1, I'_k = J_k \setminus \bigcup_{i<k} J_i$ for $k > 1$. It is easily shown that each set I'_k is a finite union of disjoint intervals $\{I''_{i,k} : i = 1, \dots, N_k\}$. Set $I_{i,k}$ = Interior $I''_{i,k}$; since m is non-atomic the class of sets $\mathcal{I} = \{I_{i,k} : i = 1, ..., N_k, k \in \mathbb{N}\}$ has the desired property. □

2. Main results

Theorem 1. *Assume $m_{\mathcal{P}}$ satisfies the density theorem (in particular, if it is absolutely continuous with respect to the Lebesgue measure). Then the following conditions are equivalent:*

a) $m_{\mathcal{P}}\{t : \Gamma^{\mathcal{T}_D}$ is not a singleton$\} = 0$.

b) $A = \int \psi(t) dP_t$ for ψ a Borel function.

If these conditions hold, then $\psi = \psi_{\Gamma^{\mathcal{T}_D}}$ $m_{\mathcal{P}}$-a.e.

Proof. a)⇒b). To avoid overburdened notation, we shall not make explicit the dependence on $\mathcal{T}_D$. First observe that $\psi_\Gamma = \phi_\Gamma$ on $S(\Gamma)$, so the restriction $\psi_\Gamma : S(\Gamma) \to \mathbb{R}$ is $\mathcal{T}_D/\mathcal{T}_0$-continuous. Using Lemma 2, we get that $\forall \varepsilon > 0$ and $\forall t \in S(\Gamma)$ there exists a $\mathcal{T}_D$ open set $I_t \in \mathcal{I}(t)$ such that

$$\sigma(AP_{I_t}) \subset (\psi_\Gamma(t) - \varepsilon, \psi_\Gamma(t) + \varepsilon);$$

hence, $\forall s \in I_t \cap S(\Gamma) : \psi_\Gamma(s) \in (\psi_\Gamma(t) - \varepsilon, \psi_\Gamma(t) + \varepsilon)$.

Let $\mathcal{K}$ be the class of sets with respect to which $m_{\mathcal{P}}$ satisfies the density theorem. Take $\{K_n\} \in \mathcal{K}(t)$. There exists $n(\varepsilon)$ such that

$$\forall n \geq n(\varepsilon) : \qquad \frac{m_{\mathcal{P}}(K_n \cap I_t)}{m_{\mathcal{P}}(K_n)} \geq 1 - \varepsilon.$$

We denote by ψ any $\mathcal{L}_{\mathcal{P}}$-measurable extension of ψ_Γ to $\mathbb{R}$ bounded by $\|A\|$. Let us take $f \in H$ such that $|\frac{dm_f}{dm_{\mathcal{P}}}| \leq c$. We have

$$|\langle AP_{K_n} f, f\rangle - \psi_\Gamma(t)\langle P_{K_n} f, f\rangle|$$

$$\leq |\langle AP_{K_n \cap I_t} f, f\rangle - \psi_\Gamma(t)\langle P_{K_n \cap I_t} f, f\rangle| + 2\|A\|\|P_{K_n \setminus I_t} f\|^2$$

$$\leq \varepsilon \int\limits_{K_n \cap I_t} \frac{dm_f}{dm_{\mathcal{P}}} dm_{\mathcal{P}} + 2\|A\| \int\limits_{K_n \setminus I_t} \frac{dm_f}{dm_{\mathcal{P}}} dm_{\mathcal{P}}$$

$$\leq \varepsilon c m_{\mathcal{P}}(K_n) + 2\|A\| c\varepsilon m_{\mathcal{P}}(K_n) = c\varepsilon(2\|A\| + 1) m_{\mathcal{P}}(K_n).$$

Hence:

$$\left|\frac{1}{m_{\mathcal{P}}(K_n)}\langle AP_{K_n}f,f\rangle - \psi_\Gamma(t)\frac{1}{m_{\mathcal{P}}(K_n)}\langle P_{K_n}f,f\rangle\right| \le c(2\|A\|+1)\varepsilon.$$

From the density theorem, we get

$$\left|\frac{d}{dm_f}\langle AP_tf,f\rangle - \psi_\Gamma(t)\frac{d}{dm_f}\langle P_tf,f\rangle\right| \le c(2\|A\|+1)\varepsilon \quad m_{\mathcal{P}}\text{-a.e in } t.$$

We conclude that

$$\frac{d}{dm_f}\langle AP_tf,f\rangle = \psi_\Gamma(t)\frac{d}{dm_f}\langle P_tf,f\rangle \quad m_{\mathcal{P}}\text{-a.e. on } S(\Gamma).$$

Since $m_{\mathcal{P}}(S(\Gamma)^c)=0$, for any f with $(\frac{dm_f}{dm_{\mathcal{P}}})$ bounded we deduce that

$$\langle Af,f\rangle = \int \psi_\Gamma(t)d\langle P_tf,f\rangle.$$

By standard density arguments we obtain this last equality for any $f\in H$.

b)⇒a). Let $A=\int\psi(t)dP_t$ with ψ Borel measurable. ψ is $\mathcal{T}/\mathcal{T}_0$-continuous $m_{\mathcal{P}}$-a.e.; that is, there exists a full $m_{\mathcal{P}}$-measurable set $E\subseteq F_{\mathcal{P}}$ such that for any $t\in E,\varepsilon>0$ there exists $I_t\in\mathcal{I}(t)$ such that $\forall s\in I_t, |\psi(s)-\psi(t)|\le\varepsilon$. It is easily obtained that:

$$\sigma(AP_{I_t})\subset[\psi(t)-\varepsilon,\psi(t)+\varepsilon].$$

As this happens for any $\varepsilon>0$, we deduce that for any $t\in E$, $\Gamma(t)=\{\psi(t)\}$. Then $S(\Gamma^{\mathcal{T}_D})\supset E$, and it is a full $m_{\mathcal{P}}$-measurable set. □

Theorem 2. *The following two conditions are equivalent:*

a) $m_{\mathcal{P}}\{t:\Gamma^{\mathcal{T}_0}$ *is not a singleton*$\}=0$.

b) For any $\varepsilon>0$ *there exists a class of disjoint open intervals* $\mathcal{I}_\varepsilon=\{I_{n,\varepsilon}:n\in\mathbb{N}\}$ *of full* $m_{\mathcal{P}}$*-measure, i.e.* $m_{\mathcal{P}}((\bigcup_{n\in\mathbb{N}}I_{n,\varepsilon})^c)=0$*, and a real sequence* $(c_{n,\varepsilon}:n\in\mathbb{N})$ *such that*

$$\forall f\in H,\quad \left|\langle Af,f\rangle-\sum_{n\in\mathbb{N}}c_{n,\varepsilon}\langle P_{I_{n,\varepsilon}}f,f\rangle\right|\le\varepsilon\|f\|^2.$$

Moreover, when these conditions hold we have that for any $\varepsilon>0$ *we can choose a covering* $\mathcal{I}_\varepsilon$ *of* $S_\Gamma^{\mathcal{T}_0}$ *such that each element of* $\mathcal{I}_\varepsilon$ *intersects* $S_\Gamma^{\mathcal{T}_0}$*, and* $c_{n,\varepsilon}=\Gamma^{\mathcal{T}_0}(\alpha_{n,\varepsilon})$ *with* $\alpha_{n,\varepsilon}\in I_{n,\varepsilon}\cap S_\Gamma^{\mathcal{T}_0}$*. In particular,* $m_{\mathcal{P}}((S_\Gamma^{\mathcal{T}_0})^c)=0$ *implies* $A=\int_{S_\Gamma^{\mathcal{T}_0}}\Gamma(t)dP_t$.

Proof. As before, we shall not make explicit the dependence on $\mathcal{T}_0$. Assume b) holds. Take $D=\bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}I_{n,\frac{1}{k}}$. Then $m_{\mathcal{P}}(D^c)=0$. Fix $t\in D$. For any $k\ge1$ there exists some $n_k\in\mathbb{N}$ such that $t\in I_{n_k,\frac{1}{k}}$. Denote $I=I_{n_k,\frac{1}{k}}, c=c_{n_k,\frac{1}{k}}$. Then

$$|\langle AP_If,f\rangle - c\langle P_If,f\rangle|\le\frac{1}{k}\|P_If\|^2.$$

Hence $\sigma(AP_J)\subseteq(c-\frac{1}{k},c+\frac{1}{k})$ for any $J\subset I, J\in\mathcal{I}_t$. Then the diameter of $\Gamma(t)$ satisfies $\delta(\Gamma(t))\le\frac{2}{k}$. We deduce that $\Gamma(t)$ is a singleton, so $D\subset S(\Gamma)$ and $m_{\mathcal{P}}(S(\Gamma)^c)\le m_{\mathcal{P}}(D^c)=0$.

Now assume condition a) holds, and fix $\varepsilon>0$. For any $t\in S(\Gamma)$ there exists $\delta(t)>0$ such that for $J_t=(t-\delta(t),t+\delta(t))$ we have $\sigma(AP_{J_t})\subset(\Gamma(t)-\frac{\varepsilon}{2},\Gamma(t)+\frac{\varepsilon}{2})$. At this point we notice that $m_{\mathcal{P}}(J_t)>0$. Consider $\mathcal{J}=\{J_t:t\in S(\Gamma)\}$. By Lemma 5 there exists a countable class of disjoint open intervals $\mathcal{I}=\{I_n:n\in\mathbb{N}\}$

subordinated to $\mathcal{J}$ and which $m_{\mathcal{P}}$-covers $S(\Gamma)$. Since $m_{\mathcal{P}}(S(\Gamma)^c) = 0$, we may assume that $m_{\mathcal{P}}(I_n \cap S(\Gamma)) > 0$ for any $n \in \mathbb{N}$. Take $\alpha_n \in I_n \cap S(\Gamma)$; we have

$$(1) \qquad \left|\langle Af, f\rangle - \sum_{n\in\mathbb{N}} \Gamma(\alpha_n)\langle P_{I_n} f, f\rangle\right| \le \sum_{n\in\mathbb{N}} |\langle AP_{I_n} f, f\rangle - \Gamma(\alpha_n)\langle P_{I_n} f, f\rangle| \,.$$

Since $\sigma(AP_{I_n})$ is the spectrum of AP_{I_n} on $P_{I_n}H$, we have

$$(2) \qquad \begin{aligned} &|\langle AP_{I_n} f, f\rangle - \Gamma(\alpha_n)\langle P_{I_n} f, f\rangle| \\ &\qquad \le \sup\{|\lambda - \Gamma(\alpha_n)| : \lambda \in \sigma(AP_{I_n})\}\,\langle P_{I_n} f, f\rangle. \end{aligned}$$

□

For any $n \in \mathbb{N}$ we have $I_n \subset J_t$ for some $t \in S(\Gamma)$. Since $\alpha_n \in I_n \subset J_t$, we have $|\Gamma(\alpha_n) - \Gamma(t)| < \frac{\varepsilon}{2}$. On the other hand, for any interval J with $m_{\mathcal{P}}(J) > 0$ and $J \subset J_t$ we have $\sigma_J \subset \sigma_{J_t} \subset (\Gamma(t) - \frac{\varepsilon}{2}, \Gamma(t) + \frac{\varepsilon}{2})$. Then the right hand side of (2) is bounded by $\varepsilon\langle P_{I_n} f, f\rangle$ and the right hand side of (1) is bounded by $\varepsilon \sum_{n\in\mathbb{N}} \langle P_{I_n} f, f\rangle = \varepsilon\langle f, f\rangle$, because $m_f(S(\Gamma)^c) = 0$. We have shown that

$$\left|\langle Af, f\rangle - \sum_{n\in\mathbb{N}} \Gamma(\alpha_n)\langle P_{I_n} f, f\rangle\right| \le \varepsilon\langle f, f\rangle.$$

Let us show the last statement. We take $\alpha_{n,\varepsilon}, I_{n,\varepsilon}$ instead of α_n, I_n because these quantities depend on $\varepsilon > 0$. Since Γ restricted to S_Γ is $\mathcal{T}_0/\mathcal{T}_0$-continuous, we get

$$\sum_{n\in\mathbb{N}} \Gamma(\alpha_{n,\varepsilon}) \mathbb{1}_{I_{n,\varepsilon}}(t) \underset{\varepsilon\to 0}{\longrightarrow} \Gamma(t) \qquad m_{\mathcal{P}}\text{-a.e. on } S_\Gamma.$$

On the other hand, $|\sum_{n\in\mathbb{N}} \Gamma(\alpha_{n,\varepsilon}) \mathbb{1}_{I_{n,\varepsilon}}| \le \|A\|$, so by the dominated convergence theorem we get

$$\sum_{n\in\mathbb{N}} \Gamma(\alpha_{n,\varepsilon})\langle P_{I_{n,\varepsilon}} f, f\rangle \underset{\varepsilon\to 0}{\longrightarrow} \int_{S_\Gamma} \Gamma(t) d\langle P_t f, f\rangle. \quad \square$$

Example. Let A be a self-adjoint operator commuting with $\mathcal{P}$. It is said that $\mathcal{P}$ *separates the spectrum* (see [4]) of A if there exists an increasing function $g(t)$ such that

$$\sigma_{(-\infty,t]} \subset (-\infty, g(t)], \quad \sigma_{[t,\infty)} \subset [g(t), \infty).$$

In this case we have $\sigma_{[s,t]} \subseteq [g(s), g(t)]$. We deduce that $\Gamma^{\mathcal{T}_0}(t) \subseteq [g(t-), g(t+)]$. As g is increasing and the maximal spectral type measure $m_{\mathcal{P}}$ is non-atomic, we get

$$m_{\mathcal{P}}\{t : \Gamma^{\mathcal{T}_0}(t) \text{ is not a singleton}\} = 0$$

Moreover, $\Gamma^{\mathcal{T}_0}(t) = \{g(t)\}$ $m_{\mathcal{P}}$-a.e.

Complement to this paper

In what follows we describe, using the pointwise spectrum, the least upper bound of A in $\mathcal{A}_{\mathcal{P}}$: the closed algebra of symmetric operators generated by $\mathcal{P}$. Notice that there is a unique symmetric operator $A^+ \in \mathcal{A}_{\mathcal{P}}$, which satisfies

1) $A^+ \ge A$;
2) $\forall B \in \mathcal{A}_{\mathcal{P}}$, if $B \ge A$, then $B \ge A^+$.

In fact since $\mathcal{A}_{\mathcal{P}}$ is a closed lattice it is easy to prove that

$$A^+ = \int \alpha_A(t) dP_t$$

where

$$\alpha_A = \operatorname{ess\,inf}\{h/h \in \mathcal{L}_P \text{ and } \int h(t)dP_t \geq A\}.$$

Here the essinf is computed with respect to $m_{\mathcal{P}}$.

The following result characterizes α_A in terms of $\Gamma_A^{\mathcal{T}_D}$.

Theorem. *If $m_{\mathcal{P}}$ satisfies a density theorem, then*

$$\alpha_A = \max \Gamma_A^{\mathcal{T}_D} \quad m_{\mathcal{P}}\text{-}a.e.$$

Proof. We denote $\psi = \max \Gamma_A^{\mathcal{T}_D}(t)$ which is $\mathcal{L}_{\mathcal{P}}$-measurable by Lemma 3. From Theorem 1 we have $\Gamma_{A^+}^{\mathcal{T}_D}(t) = \{\alpha_A(t)\}$ $m_{\mathcal{P}}$-a.e. In that way for any point $t \in F_{\mathcal{P}}$ of continuity for α_A we get

$$\forall \varepsilon > 0 \, \exists I \in \mathcal{I}^{\mathcal{T}_D}(t) \, \forall s \in I \;\; \alpha_A(s) \leq \alpha_A(t) + \varepsilon.$$

Since

$$\begin{aligned} \langle AP_I f, f\rangle \leq \langle A^+ P_I f, f\rangle &= \int_I \alpha_A(s) d\langle P_s f, f\rangle \\ &\leq (\alpha_A(t) + \varepsilon)\langle P_I f, f\rangle, \end{aligned}$$

we conclude $\Gamma_A^{\mathcal{T}_D}(t) \subseteq (-\infty, \alpha_A(t) + \varepsilon]$, and therefore $\psi(t) \leq \alpha_A(t)$. Almost all points in $F_{\mathcal{P}}$ are continuity points for α_A (due to the density theorem) from which we conclude

$$\psi \leq \alpha_A \quad m_{\mathcal{P}}\text{-a.e.}$$

In order to prove the opposite inequality it is enough to show that $\int \psi(t) dP_t \geq A$. For that purpose consider $f \neq 0$, a fixed element in H, and introduce the following signed measure on $\mathcal{L}_{\mathcal{P}}, \nu_f(\Lambda) = \langle AP_\Lambda f, f\rangle$. We have $\nu_f << m_f$ and moreover $|\frac{d\nu_f}{dm_f}| \leq \|A\|$. Consider also $\eta = \frac{dm_f}{dm_{\mathcal{P}}}$.

We have that η is $\mathcal{T}_D/\mathcal{T}_0$-continuous $m_{\mathcal{P}}$-a.e. Let $G = \{t \in F_{\mathcal{P}}/\eta(t) > 0$ and η is continuous on $t\}$; then $m_f(G^c) = 0$. Now if $I \in \mathcal{T}_D$ and $I \cap G \neq \varnothing$, then $m_f(I) > 0$. In fact let $t \in I \cap G$; since η is continuous on t, there exists an open neighbourhood $J \subseteq I$ of t such that $\eta(s) > 0$ for all $s \in J$. Therefore, since $m_{\mathcal{P}}(J) > 0$,

$$\nu_f(I) \geq \nu_f(J) = \int_J \eta(s) dm_{\mathcal{P}}(s) > 0.$$

Almost all points $t \in G$ (with respect to m_f or $m_{\mathcal{P}}$) are continuity points for $\frac{d\nu_f}{dm_f}$. Fix one such $t \in G$; from Lemma 2 we conclude that

$$\begin{aligned} &\forall \varepsilon > 0 \, \exists I \in \mathcal{I}^{\mathcal{T}_D}(t) \text{ such that} \\ &\qquad \sigma(AP_I) \subseteq (-\infty, \psi(t) + \varepsilon) \text{ and} \\ &\qquad \forall s \in I \; \left| \frac{d\nu_f(s)}{dm_f} - \frac{d\nu_f(t)}{dm_f} \right| \leq \varepsilon. \end{aligned}$$

Hence, $\forall g \in H \; \langle AP_I, g\rangle \leq (\psi(t) + \varepsilon)\langle P_I g, g\rangle$. In particular, if $g = P_\Lambda f$, where $\Lambda \subseteq I$, we have

$$\nu_f(\Lambda) = \langle AP_\Lambda f, f\rangle \leq (\psi(t) + \varepsilon)\langle P_\Lambda f, f\rangle = (\psi(t) + \varepsilon) m_f(\Lambda)$$

and therefore $\frac{d\nu_f(s)}{dm_f} \leq (\psi(t)+\varepsilon)\ m_f$-a.e. on $s \in I$. Since $m_f(I) > 0$ we get that $\frac{d\nu_f(t)}{dm_f} \leq \psi(t) + 2\varepsilon$. We have proved that $\frac{d\nu_f}{dm_f} \leq \psi\ m_f$-a.e.

Finally,

$$\langle Af, f\rangle = \int \frac{d\nu_f(t)}{dm_f}\, dm_f(t) \leq \int \psi(t) d\langle P_t f, f\rangle$$

as we wanted to prove. □

In a similar way the operator $A^- = \int \phi(t) dP_t$ with $\phi(t) = \min \Gamma_A^{\mathcal{T}_D}(t)$ is the greatest lower bound of A in $\mathcal{A}_{\mathcal{P}}$.

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