A POINTWISE SPECTRUM AND REPRESENTATION OF OPERATORS

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(Communicated by Palle E. T. Jorgensen)

Abstract. For a self-adjoint operator \( A : H \to H \) commuting with an increasing family of projections \( \mathcal{P} = (P_t : t \in \mathbb{R}) \) we study the multifunction \( t \mapsto \Gamma^\mathcal{P}(t) = \cap \{ \sigma_I : I \text{ an open set of the topology } \mathcal{T} \text{ containing } t \} \), where \( \sigma_I \) is the spectrum of \( A \) on \( P_I H \). Let \( m_P \) be the measure of maximal spectral type. We study the condition that \( \Gamma^\mathcal{P} \) is essentially a singleton, \( m_P \{ t : \Gamma^\mathcal{P}(t) \text{ is not a singleton} \} = 0 \). We show that if \( \mathcal{T} \) is the density topology and if \( m_P \) satisfies the density theorem, in particular if it is absolutely continuous with respect to the Lebesgue measure, then this condition is equivalent to the fact that \( A \) is a Borel function of \( P \). If \( \mathcal{T} \) is the usual topology then the condition is equivalent to the fact that \( A \) is approximated in norm by step functions \( \sum_{n \in \mathbb{N}} \Gamma^\mathcal{P}(\alpha_n) \langle P_{I_n} f, f \rangle \), where the set of intervals \( \{ I_n : n \in \mathbb{N} \} \) covers the set where \( \Gamma^\mathcal{P} \) is a singleton.

1. Introduction

Let \( H \) be a real separable Hilbert space and \( \mathcal{P} = (P_t : t \in \mathbb{R}) \) be an increasing right continuous family of projections, with \( P_{-\infty} = 0 \), \( P_\infty = I \) the identity. If \( B \) is a Borel real set we denote by \( P_B \) the projection \( P_B = \int_B dP_t \).

For \( f \in H \) denote by \( m_f \) the Borel measure induced by the spectral family \( \mathcal{P} \): \( m_f(B) = \langle P_B f, f \rangle \) for any Borel set \( B \).

There exists a measure \( m_P \), associated to some element of \( H \) of maximal spectral type, i.e. \( m_f \ll m_P \) for all \( f \in H \). We shall assume that \( m_P(\mathbb{R}) = 1 \).

We shall denote by \( \mathcal{L}_P \) the completion of the Borel \( \sigma \)-field with respect to \( m_P \). If \( G \in \mathcal{L}_P \) there is a Borel set \( B \) such that \( m_P(G \Delta B) = 0 \), and we define \( P_G := P_B \).

This definition is consistent because if \( B_1 = B_2 \) \( m_P \)-a.e., then \( P_{B_1} = P_{B_2} \); in fact \( \|(P_{B_1} - P_{B_2}) f\|^2 = \int_{B_1 \Delta B_2} d\langle P_t f, f \rangle = \int_{B_1 \Delta B_2} \frac{d\langle P_t f, f \rangle}{dm_P(t)} dm_P(t) = 0 \).

We shall denote by \( \mathcal{F}_P \) the support of \( m_P \); that is, \( \mathcal{F}_P \) is the smallest closed set of full measure.
Let $A$ be a self-adjoint operator commuting with $P$. Then $A$ acts on the space $P_{\Lambda}H$ for any $\Lambda \in \mathcal{L}_P$, i.e. $AP_{\Lambda}H \subset P_{\Lambda}H$. Denote

$$\sigma_{\Lambda} = \text{spectra of } AP_{\Lambda} \text{ on } P_{\Lambda}H$$

Then $\sigma_{\Lambda}$ is a compact set included in $[-\|A\|,\|A\|]$. If $\Lambda_1 \supseteq \Lambda_2$ and $m_{\mathcal{P}}(\Lambda_2) > 0$, then $\sigma_{\Lambda_1} \supseteq \sigma_{\Lambda_2}$. In fact, if $\lambda \in \sigma_{\Lambda_2}$, then there is a sequence $(f_n) \subseteq P_{\Lambda_2}(H)$ with $\|f_n\| = 1$ such that $\langle AP_{\Lambda_2}f_n, f_n \rangle \to \lambda$, from which the property follows.

Our work is concerned with integral representation of a self-adjoint operator $A$ commuting with $P$. In this context a set of necessary and sufficient conditions is given in [1] and [5].

First let us reduce the problem to the case of continuous $\mathcal{P}$. If $A$ commutes with $\mathcal{P}$ then it commutes with any $\Delta P_t = P_t - P_{t-} \neq 0$. A necessary condition in order that the operator $A$ can be written in the form $A = g(\int sdP_\sigma)$ is that $A\Delta P_t = \lambda_t \Delta P_t$ for some real $\lambda_t$. Denote by $L$ the Hilbert space generated by $\{\Delta P_t : t \in \mathbb{R}\}$, and let $H' = H\Theta L$. The projections $P_t$ and $P_{t'}$ commute, the family of projections $\mathcal{P}' = (P_t = P_tP_{t'} : t \in \mathbb{R})$ is continuous, and the restriction $A' = AP_{H'}$ commutes with $\mathcal{P}'$. We have that $A = g(\int sdP_\sigma)$ iff $A\Delta P_t = \lambda_t \Delta P_t$ for all $t$, and $A' = g(\int sdP_\sigma)$. Then we can assume $\mathcal{P} = (P_t : t \in \mathbb{R})$ is continuous.

Since $\mathcal{P}$ is continuous, the measure of maximal spectral type $m_{\mathcal{P}}$ is non-atomic.

Following [6], a sequence $\{K_n\}$ of $\mathcal{L}_\mathcal{P}$ sets is said to converge to $t \in \mathbb{R}$ if: $t \in \bigcap\{K_n, m_{\mathcal{P}}(K_n) > 0\}$. Let $\mathcal{K}$ be a collection of sequences in $\mathcal{L}_\mathcal{P}$.

We denote by $\mathcal{K}(t)$ the family of sequences in $\mathcal{K}$ converging to $t$. We assume that $\mathcal{K}(t) \neq \emptyset$ for every $t \in \mathbb{R}$.

The upper outer density of $E \subseteq \mathbb{R}$ at the point $t$ is defined by

$$\bar{D}^*(E, t) = \sup \left\{ \lim_n \frac{m_{\mathcal{P}}(E \cap K_n)}{m_{\mathcal{P}}(K_n)} \right\} / \{K_n \in \mathcal{K}(t)\}.$$ 

The lower outer density $\underline{D}^*(I, t)$ is defined as

$$\underline{D}^*(E, t) = \inf \left\{ \lim_n \frac{m_{\mathcal{P}}(E \cap K_n)}{m_{\mathcal{P}}(K_n)} \right\} / \{K_n \in \mathcal{K}(t)\}.$$ 

If $E$ is $\mathcal{L}_\mathcal{P}$-measurable and $\bar{D}^*(E, t) = \underline{D}^*(E, t) = 1$, then we say that $t$ is an (outer) density point for $E$. The density topology $\mathcal{T}_D$ is the set of all $I \subseteq \mathbb{R}$ such that $\forall t \in I \quad \bar{D}^*(I^c, t) = 0$. In this context also see [3], [7] and [8].

We shall assume that $m_{\mathcal{P}}$ satisfies a density theorem; this means that there exists a collection $\mathcal{K}$ such that, for every set $A \subseteq \mathbb{R}$, almost every point with respect to $m_{\mathcal{P}}$ of $A$ is an outer density point for $A$. Under this condition every open set $I \in \mathcal{T}_D$ is $\mathcal{L}_\mathcal{P}$-measurable, and every point $t \in I$ is a density point of $I$. Moreover a function $\varphi \in \mathcal{L}_\mathcal{P}$ is $\mathcal{T}_D/\mathcal{T}_0$-continuous $m_{\mathcal{P}}$-a.e., where $\mathcal{T}_0$ is the usual topology on $\mathbb{R}$. Therefore, if $\varphi$ is a bounded $\mathcal{L}_\mathcal{P}$-measurable function, then for $m_{\mathcal{P}}$-a.e. $t$, if $\{K_n\} \in \mathcal{K}(t)$

$$\lim_{n \to \infty} \frac{\int_{K_n} \varphi \ dm_{\mathcal{P}}}{m_{\mathcal{P}}(K_n)} = \varphi(t).$$

If $m_{\mathcal{P}}$ is absolutely continuous with respect to the Lebesgue measure, then $m_{\mathcal{P}}$ satisfies the density theorem with respect to $\mathcal{K}(t) = \text{class of all regular intervals converging to } t$. 
In the next four lemmas we shall consider a topology $\mathcal{T}$ on $F_P$. Mainly we are interested in the traces of $T_D$ and $T_0$ over $F_P$, which we still denote by $T_D$ and $T_0$. We shall assume that $\mathcal{T}$ fulfills the following two conditions:

(i) $\mathcal{T} \subseteq \mathcal{L}_P$;

(ii) $\forall \emptyset \neq I \in \mathcal{T}$ $m_P(I) > 0$.

These two conditions are satisfied by $T_D$ and $T_0$. For $T_D$ they follow respectively from Corollary 4.4 and Corollaries 4.12 and 4.13 of [6]. As for $T_0$, (i) is immediate and (ii) follows from the definition of $F_P$.

We denote by $\mathcal{I}^T(t) = \{I \in T : t \in I\}$ the set of open neighbourhoods of $t \in F_P$.

Define the pointwise spectrum of $A$ with respect to $\mathcal{P}$ and the topology $\mathcal{T}$ by

$$\Gamma^T(t) = \bigcap_{I \in \mathcal{I}^T(t)} \sigma_I,$$

for $t \in F_P$.

In the proof of the lemmas we abbreviate $\Gamma(t) = \Gamma^T(t)$ and $\mathcal{I}^T(t) = \mathcal{I}(t)$

**Lemma 1.** $\Gamma^T(t)$ is a non-empty $T_0$-compact set included in $[-\|A\|, \|A\|]$.\[Proof.\] The sets $\sigma_I$ are non-empty and compact, so it suffices to show that the family $(\sigma_I)_{I \in \mathcal{I}(t)}$ has the finite intersection property. This holds because this family is a net with respect to the order induced by inclusion; in fact if $I_1, \ldots, I_n$ belong to $\mathcal{I}(t)$ the open set $\bigcap_{i=1}^n I_i$ also belongs to $\mathcal{I}(t)$, and $\bigcap_{i=1}^n \sigma_{I_i}$ contains $\sigma_{\bigcap_{i=1}^n I_i}$.\]

The mapping $\Gamma : F_P \to \text{Subsets } \mathbb{R}$ is a multifunction with compact values. We recall that for $X$ and $Y$ topological spaces, a multifunction $G : X \to \text{Subsets } Y$, is said to be upper semi-continuous if for all $t \in X$ and for all open set $V$ containing $G(t)$, there exists a neighbourhood $U$ of $t$ such that $G(s) \subset V$ for all $s \in U$ (see [2]). Now, set

$$S(G) = \{t \in F_P : G(t) \text{ is a singleton}\}$$

and assume $S(G) \neq \emptyset$. When $t \in S(G)$ we identify the singleton $G(t)$ with its unique element. If $G$ is an upper semi-continuous multifunction, then $G : S(G) \to Y$ is a continuous function.

**Lemma 2.** $\Gamma^T$ is a $T/T_0$ upper semi-continuous multifunction.\[Proof.\] Let $V \in T_0$ be an open set such that $\Gamma(t) \subset V$. Since the family of $T_0$-compact sets $(\sigma_I)_{I \in \mathcal{I}(t)}$ is a net with the order induced by inclusion, it is easy to prove that there exists $I \in \mathcal{I}(t)$ with $\sigma_I \subset V$. In particular, for all $s \in I$ we have $\Gamma(s) \subseteq \sigma_I \subseteq V$.\]

This result implies that there exist measurable selections $\gamma \in \Gamma$, i.e. an $\mathcal{L}_P$-measurable function $\gamma : F_P \to \mathbb{R}$ such that $\gamma(t) \in \Gamma^T(t)$ for all $t \in F_P$. Moreover, the functions below are two measurable selections:

$$\psi_{\Gamma^T}(t) = \max\{u \in \Gamma^T(t)\}, \quad \phi_{\Gamma^T}(t) = \min\{u \in \Gamma^T(t)\}.$$\[More precisely: \]

**Lemma 3.** $\psi_{\Gamma^T}$ is $T/T_0$ upper semi-continuous and $\phi_{\Gamma^T}$ is $T/T_0$ lower semi-continuous.\[Proof.\] This follows from Lemma 2. In fact, if $t \in \psi_{\Gamma^T}^{-1}(-\infty, r)$, then $\Gamma(t) \subset (-\infty, r) = V$ and there exists a neighbourhood $U$ of $t$ such that $\Gamma(s) \subset (-\infty, r)$
for all \( s \in U \), so \( U \subset \psi^{-1}_r(-\infty,r) \). Hence \( \psi^{-1}_r(-\infty,r) \in \mathcal{I} \) and \( \psi_r \) is upper semi-continuous. The proof is analogous for \( \phi_r \).

**Lemma 4.** \( S(\Gamma^\mathcal{T}) = \{ t : \Gamma^\mathcal{T}(t) \text{ is a singleton} \} \) is \( \mathcal{L}_\mathcal{P} \)-measurable.

*Proof.* This follows immediately from the equality \( S(\Gamma) = \{ t : \psi_T(t) = \phi_r(t) \} \).

**Lemma 5.** Assume \( m \) is a non-atomic Borel measure and \( \mathcal{J} = \{ J \} \) is a family of open intervals covering a Borel set \( E \). Then there exists a countable class of disjoint open intervals \( \mathcal{I} = \{ I \} \) subordinated to \( \mathcal{J} \) (i.e. for any \( I \in \mathcal{I} \) there exists \( J \in \mathcal{J} \) such that \( I \subset J \)) satisfying \( m(E \setminus \bigcup_{I \in \mathcal{I}} I) = 0 \).

*Proof.* By Lindelöf’s theorem there exists a countable class \( \{ J_n : n \in \mathbb{N} \} \subset \mathcal{J} \) such that \( E \subset \bigcup_{n \in \mathbb{N}} J_n \). Now define \( I'_1 = J_1 \), \( I'_k = J_k \setminus \bigcup_{i<k} J_i \) for \( k > 1 \). It is easily shown that each set \( I'_k \) is a finite union of disjoint intervals \( \{ I''_{i,k} : i = 1, \ldots, N_k \} \). Set \( I_{i,k} = \text{Interior } I''_{i,k} \); since \( m \) is non-atomic the class of sets \( \mathcal{I} = \{ I_{i,k} : i = 1, \ldots, N_k, k \in \mathbb{N} \} \) has the desired property.

2. Main results

**Theorem 1.** Assume \( m_P \) satisfies the density theorem (in particular, if it is absolutely continuous with respect to the Lebesgue measure). Then the following conditions are equivalent:

a) \( m_P \{ t : \Gamma^\mathcal{T}_P \text{ is not a singleton} \} = 0 \).

b) \( A = \int \psi(t) dP_t \) for \( \psi \) a Borel function.

If these conditions hold, then \( \psi = \psi_{\Gamma^\mathcal{T}_P, m} \) \( \mathcal{L}_\mathcal{P} \)-a.e.

*Proof.* a)\( \Rightarrow \)b). To avoid overburdened notation, we shall not make explicit the dependence on \( \mathcal{T}_P \). First observe that \( \psi_T = \phi_r \) on \( S(\Gamma) \), so the restriction \( \psi_T : S(\Gamma) \to \mathbb{R} \) is \( \mathcal{T}_D/\mathcal{T}_0 \)-continuous. Using Lemma 2, we get that \( \forall \varepsilon > 0 \) and \( \forall t \in S(\Gamma) \) there exists a \( \mathcal{T}_D \) open set \( I_t \in \mathcal{I}(t) \) such that

\[
\sigma(\mathcal{A}P_{I_t}) \subset (\psi_T(t) - \varepsilon, \psi_T(t) + \varepsilon);
\]

hence, \( \forall s \in I_t \cap S(\Gamma) : (\psi_T(s) - \varepsilon, \psi_T(s) + \varepsilon) \in S(\Gamma) \).

Let \( \mathcal{K} \) be the class of sets with respect to which \( m_P \) satisfies the density theorem. Take \( \{ K_n \} \in \mathcal{K}(t) \). There exists \( n(\varepsilon) \) such that

\[
\forall n \geq n(\varepsilon) : \quad \frac{m_P(K_n \cap I_t)}{m_P(K_n)} \geq 1 - \varepsilon.
\]

We denote by \( \psi \) any \( \mathcal{L}_\mathcal{P} \)-measurable extension of \( \psi_T \) to \( \mathbb{R} \) bounded by \( \| A \| \). Let us take \( f \in H \) such that \( \left| \frac{d\psi}{dm_P} \right| \leq c \). We have

\[
\left| \langle P_{I_t}, f \rangle - \psi_T(t)P_{K_n \cap I_t}, f \rangle \right| \\
\leq \left| \langle P_{I_t}, f \rangle - \psi_T(t)P_{K_n \cap I_t}, f \rangle \right| + 2\| A \| \| P_{K_n \setminus I_t}f \|^2 \\
\leq \varepsilon \int_{I_t} \frac{dm}{dm_P} dm_P + 2\| A \| \int_{K_n \setminus I_t} \frac{dm}{dm_P} dm_P \\
\leq \varepsilon cm_P(K_n) + 2\| A \| \varepsilon m_P(K_n) = c\varepsilon(2\| A \| + 1)m_P(K_n).
\]
Hence:
\[
\left| \frac{1}{m_P(K_n)} \langle AP_n f, f \rangle - \psi_\Gamma(t) \frac{1}{m_P(K_n)} \langle P_n f, f \rangle \right| \leq c(2\|A\| + 1)\varepsilon.
\]
From the density theorem, we get
\[
\left| \frac{d}{dm_f} \langle AP_t f, f \rangle - \psi_\Gamma(t) \frac{d}{dm_f} \langle P_t f, f \rangle \right| \leq c(2\|A\| + 1)\varepsilon \quad m_P\text{-a.e in } t.
\]
We conclude that
\[
\frac{d}{dm_f} \langle AP_t f, f \rangle = \psi_\Gamma(t) \frac{d}{dm_f} \langle P_t f, f \rangle \quad m_P\text{-a.e on } S(\Gamma).
\]
Since \(m_P(S(\Gamma)^c) = 0\), for any \(f\) with \((\frac{dm_f}{dm_P})\) bounded we deduce that
\[
\langle Af, f \rangle = \int \psi_\Gamma(t)d\langle P_t f, f \rangle.
\]
By standard density arguments we obtain this last equality for any \(f \in H\).

b) ⇒ a). Let \(A = \int \psi(t)dP_t\) with \(\psi\) Borel measurable. \(\psi\) is \(T/T_0\)-continuous \(m_P\text{-a.e.}; that is, there exists a full \(m_P\)-measurable set \(E \subseteq F_P\) such that for any \(t \in E, \varepsilon > 0\) there exists \(I_t \subseteq I(t)\) such that \(\forall s \in I_t, |\psi(s) - \psi(t)| \leq \varepsilon\). It is easily obtained that:
\[
\sigma(AP_t) \subseteq [\psi(t) - \varepsilon, \psi(t) + \varepsilon].
\]
As this happens for any \(\varepsilon > 0\), we deduce that for any \(t \in E, \Gamma(t) = \{\psi(t)\}\). Then \(S(\Gamma^c) \supset E\), and it is a full \(m_P\)-measurable set.

**Theorem 2.** The following two conditions are equivalent:
a) \(m_P\{t : \Gamma^c \Gamma(t) \text{ is not a singleton}\} = 0\).
b) For any \(\varepsilon > 0\) there exists a class of disjoint open intervals \(I_{n,\varepsilon} = \{I_{n,\varepsilon} : n \in \mathbb{N}\}\) of full \(m_P\)-measure, i.e. \(m_P((\bigcup_{n \in \mathbb{N}} I_{n,\varepsilon})^c) = 0\), and a real sequence \((c_{n,\varepsilon} : n \in \mathbb{N})\) such that
\[
\forall f \in H, \quad |\langle Af, f \rangle - \sum_{n \in \mathbb{N}} c_{n,\varepsilon} \langle P_{n,\varepsilon} f, f \rangle| \leq \varepsilon\|f\|^2.
\]
Moreover, when these conditions hold we have that for any \(\varepsilon > 0\) we can choose a covering \(I_{\varepsilon}\) of \(S(\Gamma_t)^c\) such that each element of \(I_{\varepsilon}\) intersects \(S(\Gamma_t)^c\), and \(c_{n,\varepsilon} = \Gamma^c(t)\) with \(\alpha_{n,\varepsilon} \in I_{n,\varepsilon} \cap S(\Gamma_t)^c\). In particular, \(m_P((S(\Gamma_t)^c)^c) = 0\) implies \(A = \int_{S(\Gamma_t)} \Gamma(t)dP_t\).

**Proof.** As before, we shall not make explicit the dependence on \(T_0\). Assume b) holds. Take \(D = \bigcap_{k \geq 1} \bigcup_{n \in \mathbb{N}} I_{n,\frac{k}{2}}\). Then \(m_P(D^c) = 0\). Fix \(t \in D\). For any \(k \geq 1\) there exists some \(n_k \in \mathbb{N}\) such that \(t \in I_{n_k,\frac{k}{2}}\). Denote \(I = I_{n_k,\frac{k}{2}}, c = c_{n_k,\frac{k}{2}}\). Then
\[
|\langle AP_t f, f \rangle - c \langle P_t f, f \rangle| \leq \frac{1}{k}\|P_t f\|^2.
\]
Hence \(\sigma(AP_t) \subseteq (c - \frac{1}{k}, c + \frac{1}{k})\) for any \(J \subset I, J \in I_{\varepsilon}\). Then the diameter of \(\Gamma(t)\) satisfies \(\delta(\Gamma(t)) \leq \frac{\varepsilon}{2}\). We deduce that \(\Gamma(t)\) is a singleton, so \(D \subset S(\Gamma)\) and \(m_P(S(\Gamma)^c) \leq m_P(D^c) = 0\).

Now assume condition a) holds, and fix \(\varepsilon > 0\). For any \(t \in S(\Gamma)\) there exists \(\delta(t) > 0\) such that for \(J_t = (t - \delta(t), t + \delta(t))\) we have \(\sigma(AP_{J_t}) \subset (\Gamma(t) - \frac{\varepsilon}{2}, \Gamma(t) + \frac{\varepsilon}{2})\). At this point we notice that \(m_P(J_t) > 0\). Consider \(J = \{J_t : t \in S(\Gamma)\}\). By Lemma 5 there exists a countable class of disjoint open intervals \(I = \{I_n : n \in \mathbb{N}\}\)
As and \( J \) are increasing and the maximal spectral type measure \( \Gamma \) is subordinated to \( J \) and which \( mp \)-covers \( S(\Gamma) \). Since \( mp(S(\Gamma)^c) = 0 \), we may assume that \( mp(I_n \cap S(\Gamma)) > 0 \) for any \( n \in \mathbb{N} \). Take \( \alpha_n \in I_n \cap S(\Gamma) \); we have

\[
(1) \quad \left| (Af, f) - \sum_{n \in \mathbb{N}} \Gamma(\alpha_n)(P_{I_n} f, f) \right| \leq \sum_{n \in \mathbb{N}} \left| (AP_{I_n} f, f) - \Gamma(\alpha_n)(P_{I_n} f, f) \right|.
\]

Since \( \sigma(AP_{I_n}) \) is the spectrum of \( AP_{I_n} \) on \( P_{I_n} H \), we have

\[
(2) \quad \left| (AP_{I_n} f, f) - \Gamma(\alpha_n)(P_{I_n} f, f) \right| \leq \sup \{|\lambda - \Gamma(\alpha_n)\} : \lambda \in \sigma(AP_{I_n})\} \langle P_{I_n} f, f \rangle.
\]

In this case we have \( |\Gamma(\alpha_n) - \Gamma(t)| < \frac{\varepsilon}{2} \). On the other hand, for any interval \( J \) with \( mp(J) > 0 \) and \( J \subset J_t \) we have \( \sigma_J \subset \sigma_{J_t} \subset \langle (\Gamma(t) - \frac{\varepsilon}{2}, \Gamma(t) + \frac{\varepsilon}{2}) \rangle \). Then the right hand side of (2) is bounded by \( \varepsilon \langle P_{I_n} f, f \rangle \) and the right hand side of (1) is bounded by \( \varepsilon \sum_{n \in \mathbb{N}} \langle P_{I_n} f, f \rangle = \varepsilon \langle f, f \rangle \), because \( mp(S(\Gamma)^c) = 0 \). We have shown that

\[
\left| (Af, f) - \sum_{n \in \mathbb{N}} \Gamma(\alpha_n)(P_{I_n} f, f) \right| \leq \varepsilon \langle f, f \rangle.
\]

Let us show the last statement. We take \( \alpha_{n,\varepsilon}, I_{n,\varepsilon} \) instead of \( \alpha_n, I_n \) because these quantities depend on \( \varepsilon > 0 \). Since \( \Gamma \) restricted to \( S_\Gamma \) is \( T_0/T_0 \)-continuous, we get

\[
\sum_{n \in \mathbb{N}} \Gamma(\alpha_{n,\varepsilon})(I_{n,\varepsilon}, t) \xrightarrow{\varepsilon \to 0} \Gamma(t) \quad mp\text{-a.e. on } S_\Gamma.
\]

On the other hand, \( \sum_{n \in \mathbb{N}} |\Gamma(\alpha_{n,\varepsilon})(I_{n,\varepsilon}, t)| \leq \| A \| \), so by the dominated convergence theorem we get

\[
\sum_{n \in \mathbb{N}} \Gamma(\alpha_{n,\varepsilon})(P_{I_{n,\varepsilon}} f, f) \xrightarrow{\varepsilon \to 0} \int \Gamma(t) d\| P_{I_n} f, f \|.
\]

**Example.** Let \( A \) be a self-adjoint operator commuting with \( \mathcal{P} \). It is said that \( \mathcal{P} \) separates the spectrum (see [4]) of \( A \) if there exists an increasing function \( g(t) \) such that

\[
\sigma(-\infty, t] \subset (-\infty, g(t)], \quad \sigma(t, \infty) \subset [g(t), \infty).
\]

In this case we have \( \sigma_{[n, t]} \subseteq [g(s), g(t)] \). We deduce that \( \Gamma^{T_0}(t) \subseteq [g(t^-), g(t^+)] \). As \( g \) is increasing and the maximal spectral type measure \( mp \) is non-atomic, we get

\[
mp\{ t : \Gamma^{T_0}(t) \text{ is not a singleton} \} = 0
\]

Moreover, \( \Gamma^{T_0}(t) = \{ g(t) \} \) \( mp\text{-a.e.} \).

**Complement to This Paper**

In what follows we describe, using the pointwise spectrum, the least upper bound of \( A \) in \( \mathcal{A}_\mathcal{P} \): the closed algebra of symmetric operators generated by \( \mathcal{P} \). Notice that there is a unique symmetric operator \( A^+ \in \mathcal{A}_\mathcal{P} \), which satisfies

1) \( A^+ \geq A \);
2) \( \forall B \in \mathcal{A}_\mathcal{P}, \text{ if } B \geq A, \text{ then } B \geq A^+ \).
In fact since $A_F$ is a closed lattice it is easy to prove that
\[ A^+ = \int \alpha_A(t) dP_t \]
where
\[ \alpha_A = \text{ess inf}\{h/h \in \mathcal{L}_F \text{ and } \int h(t) dP_t \geq A\}. \]
Here the essinf is computed with respect to $m_F$.

The following result characterizes $\alpha_A$ in terms of $\Gamma_{A}^{T_D}$.

**Theorem.** If $m_F$ satisfies a density theorem, then
\[ \alpha_A = \max \Gamma_{A}^{T_D} \quad m_F\text{-a.e.} \]

**Proof.** We denote $\psi = \max \Gamma_{A}^{T_D}(t)$ which is $\mathcal{L}_F$-measurable by Lemma 3. From Theorem 1 we have $\Gamma_{A}^{T_D}(t) = \{\alpha_A(t)\} \quad m_F\text{-a.e.}$ In that way for any point $t \in F_P$ of continuity for $\alpha_A$ we get
\[ \forall \varepsilon > 0 \exists I \in T^{T_D}(t) \forall s \in I \alpha_A(s) \leq \alpha_A(t) + \varepsilon. \]

Since
\[ \langle AP_t f, f \rangle \leq \langle A^+ P_t f, f \rangle = \int_I \alpha_A(s) d(P_s f, f) \leq (\alpha_A(t) + \varepsilon) \langle P_t f, f \rangle, \]
we conclude $\Gamma_{A}^{T_D}(t) \subseteq (-\infty, \alpha_A(t) + \varepsilon]$, and therefore $\psi(t) \leq \alpha_A(t)$. Almost all points in $F_P$ are continuity points for $\alpha_A$ (due to the density theorem) from which we conclude
\[ \psi \leq \alpha_A \quad m_F\text{-a.e.} \]

In order to prove the opposite inequality it is enough to show that $\int \psi(t) dP_t \geq A$. For that purpose consider $f \neq 0$, a fixed element in $H$, and introduce the following signed measure on $\mathcal{L}_F$, $\nu_f(A) = \langle AP_A f, f \rangle$. We have $\nu_f \ll m_f$ and moreover $\left| \frac{d\nu_f}{d\nu_f} \right| \leq \| A \|$. Consider also $\eta = \frac{d\nu_f}{dm_f}$.

We have that $\eta$ is $T_D/T_0$-continuous $m_F\text{-a.e.}$ Let $G = \{t \in F_P/\eta(t) > 0 \text{ and } \eta \text{ is continuous on } t\}$; then $m_f(G^c) = 0$. Now if $I \in T_D$ and $I \cap G \neq \emptyset$, then $m_f(I) > 0$. In fact let $t \in I \cap G$; since $\eta$ is continuous on $t$, there exists an open neighbourhood $J \subseteq I$ of $t$ such that $\eta(s) > 0$ for all $s \in J$. Therefore, since $m_F(J) > 0$,
\[ \nu_f(I) \geq \nu_f(J) = \int_J \eta(s) dm_F(s) > 0. \]

Almost all points $t \in G$ (with respect to $m_f$ or $m_F$) are continuity points for $\frac{d\nu_f}{d\nu_f}$. Fix one such $t \in G$; from Lemma 2 we conclude that
\[ \forall \varepsilon > 0 \exists I \in T^{T_D}(t) \text{ such that} \]
\[ \sigma(AP_t) \subseteq (-\infty, \psi(t) + \varepsilon) \text{ and} \]
\[ \forall s \in I \left| \frac{d\nu_f(s)}{dm_f} - \frac{d\nu_f(t)}{dm_f} \right| \leq \varepsilon. \]
Hence, $\forall g \in H \langle AP_t g, g \rangle \leq (\psi(t) + \varepsilon) \langle P_t g, g \rangle$. In particular, if $g = P_A f$, where $A \subseteq I$, we have
\[ \nu_f(A) = \langle AP_A f, f \rangle \leq (\psi(t) + \varepsilon) \langle P_A f, f \rangle = (\psi(t) + \varepsilon)m_f(A) \]
and therefore \( \frac{d\nu_f(s)}{dm_f} \leq (\psi(t) + \varepsilon) \) \( m_f \)-a.e. on \( s \in I \). Since \( m_f(I) > 0 \) we get that \( \frac{d\nu_f(t)}{dm_f} \leq \psi(t) + 2\varepsilon \). We have proved that \( \frac{d\nu_f}{dm_f} \leq \psi \) \( m_f \)-a.e.

Finally,

\[
\langle Af, f \rangle = \int \frac{d\nu_f(t)}{dm_f} dm_f(t) \leq \int \psi(t) d\langle P_f, f \rangle
\]

as we wanted to prove. \( \square \)

In a similar way the operator \( A^- = \int \phi(t) dP_t \) with \( \phi(t) = \min_{\Gamma_A} \) \( T_D(t) \) is the greatest lower bound of \( A \) in \( \mathcal{A}_\mathcal{P} \).

**Acknowledgments**

This work was partially supported by FONDECYT.

**References**


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