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Asymptotic behaviour of a Brownian motion on exterior domains

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Abstract. We study the asymptotic behaviour of the transition density of a Brownian motion in $\mathscr{D}$, killed at $\partial\mathscr{D}$, where $\mathscr{D}^c$ is a compact non polar set. Our main result concern dimension $d = 2$, where we show that the transition density $p_t^{\mathscr{D}}(x, y)$ behaves, for large t, as $\frac{2}{\pi}u(x)u(y)(t(\log t)^2)^{-1}$ for $x, y \in \mathscr{D}$, where u is the unique positive harmonic function vanishing on $(\partial\mathscr{D})^r$, such that $u(x) \sim \log|x|$.

1. Introduction and main results

Let $\mathscr{D} \subset I\!R^d,\ d \geq 2$ be a domain with compact complement $K = \mathscr{D}^c$. In this work we study the asymptotic behaviour of the transition density of a Brownian motion in $\mathscr{D}$, killed at the boundary $\partial\mathscr{D}$.

Let us fix some notations. X_t denotes a Brownian motion in $I\!R^d$ and $\mathbb{P}_x$ the probability distribution of X_t starting from x. For $x, y \in \mathscr{D},\ t \geq 0,\ p_t^{\mathscr{D}}(x, y)$ denotes the transition density of X_t, killed at $\partial\mathscr{D}$ and $T_{\mathscr{D}} = \inf\{t : X_t \in \partial\mathscr{D}\}$ is the killing time. By $\mathscr{L}$ we mean the infinitesimal generator for the process killed at $\partial\mathscr{D}$. For smooth functions φ vanishing at the boundary we have $\mathscr{L}\varphi = \frac{1}{2}\Delta\varphi$. Existence and regularity of the transition density of killed Brownian motion are fully studied in Chung and Zhao [CZ]. Also there is a large literature for killed diffusions. For one dimensional diffusions see Mandl [M] and Collet et al. [CMS1]. For diffusions constrained to compact domains see Pinsky [P], and for some planar domains with finite volume see Bañuelos and Davis [BD]. We have recently obtained ratio limit results for a Brownian motion killed at the boundary on some unbounded domains [CMS2], namely Benedicks' domains.

If K is a polar set then $\mathbb{P}_x\{T_{\mathscr{D}} < \infty\} = 0$ for all $x \in \mathscr{D}$ (see [D] 2.IX.5) and $p_t^{\mathscr{D}}(x, y)$ is the density of a standard Brownian motion for $t \geq 0, x, y \in \mathscr{D}$. Therefore the problem is interesting only when K is non polar. Hence we assume K is a compact non polar set and we call these type of domains **exterior domains**. We denote by $(\partial\mathscr{D})^r$ the regular points of $\partial\mathscr{D}$ (for definition see [CZ], section 1.6).

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An important case in this work is $\mathscr{U}$ the complement of unit ball $\mathscr{B} = \{x \in I\!R^d : |x| \le 1\}$ for dimension $d = 2$ and 3, where explicit results are available.

Positive harmonic functions will play a major role in our study, in particular their special behaviour at infinity. In general the set of positive harmonic functions vanishing at the boundary is well known only for some special domains. For example see the work of Ancona [A1,A2], Benedicks [Be] and the recent work of Ioffe and Pinsky [IP]. For exterior domains harmonic functions can be well described, see Brelot [Br]. In the next result we fix an harmonic function for each planar exterior domain.

Lemma 1.1. *Let $\mathscr{D}$ be an exterior domain in the plane. Then there is a unique non negative non trivial harmonic function u tending to zero on $(\partial\mathscr{D})^r$ and such that*

$$\lim_{|x|\to\infty} \frac{u(x)}{\log|x|} = 1 \quad . \tag{1.1}$$

Moreover any non negative non trivial harmonic function v tending to zero on $(\partial\mathscr{D})^r$ is proportional to u

Proof. For an exterior domain $\mathscr{D}$ there is at least one non negative non trivial harmonic function tending to zero on $(\partial\mathscr{D})^r$ (see [D]). Moreover, each such function behaves for large x like $a\log|x| + b + o(1)$ (see [Br]), where $a > 0$. Assume u_1 and u_2 are harmonic functions satisfying (1.1). In particular $a = 1$ for both of them. Then $u^* = u_1 - u_2$ is a bounded harmonic function tending to zero on $(\partial\mathscr{D})^r$. We have $I\!P_x\{X_{T_\mathscr{D}} \notin (\partial\mathscr{D})^r, T_\mathscr{D} < \infty\} = 0$ (see [C]), then

$$|u^*(x)| \le \mathbb{E}_x\big(|u^*\big(X_t\big)|, T_\mathscr{D} > t\big) \le \|u^*\|_\infty \, \mathbb{P}_x\big(T_\mathscr{D} > t\big) \ ,$$

which converges to zero as t tends to ∞, because K is a Borel non polar set (see [D] 2.IX.10). Uniqueness of u follows at once. Finally, let v be a non negative non trivial harmonic function tending to zero on $(\partial\mathscr{D})^r$. Then v has the asymptotic $a\log|x| + b + o(1)$ and the result follows by noticing that $\frac{1}{a}v$ satisfies the same properties as u, and therefore $v = au$. □

In dimension $d = 2$, for each exterior domain $\mathscr{D}$ we denote by $u^\mathscr{D}$, or simply by u if there is no possible confusion, the unique function defined in the above Lemma.

The asymptotic behaviour of the transition kernel $p_t^\mathscr{D}$ is described in the following result whose proof will be given in section 3.

Theorem 1.2. *Let $\mathscr{D}$ be an exterior domain of $I\!R^2$. For x, y in $\mathscr{D}$ we have*

$$\lim_{t\to\infty} t\,(\log t)^2 p_t^\mathscr{D}(x, y) = \frac{2}{\pi}\, u(x)\, u(y).$$

The convergence is uniform on compact subsets of $\mathscr{D}$. □

Since K is non polar the Green function $G^\mathscr{D}(x, y) = \int_0^\infty p_t^\mathscr{D}(x, y)dt$ is finite for every $x, y \in \mathscr{D}$. Moreover, from [PS] Proposition 4.4 the following limit exists and defines a harmonic function

$$\forall\, x \in \mathscr{D} \quad \lim_{|y|\to\infty} G^\mathscr{D}(x, y) = \mathscr{W}_\mathscr{D}(x) \ .$$

This function $\mathscr{W}_{\mathscr{D}}$ is also related to the asymptotic behaviour of the survival probability $\mathbb{P}_x\big(T_{\mathscr{D}} > t\big)$. In fact, Theorem 4.16 in [PS] asserts that

$$\lim_{t\to\infty} \log t\ \mathbb{P}_x\big(T_{\mathscr{D}} > t\big) = 2\pi \mathscr{W}_{\mathscr{D}}(x)\ \ .$$

We shall give an elementary proof for the asymptotic behaviour of the survival probability which also shows that $u(x) = \pi \mathscr{W}_{\mathscr{D}}(x)$.

Proposition 1.3. *Let $\mathscr{D}$ be an exterior domain of $\mathbb{R}^2$. For any $x \in \mathscr{D}$ we have*

$$\lim_{t\to\infty} \log t\ \mathbb{P}_x\big(T_{\mathscr{D}} > t\big) = 2\ u(x)\ \ .$$

The convergence is uniform on compact subsets of $\mathscr{D}$.

Proof. Using the behaviour of $u(y)$ for large y, for $x \in \mathscr{D}$ and t large enough it is verified

$$\begin{aligned}
u(x) = \int_{\mathscr{D}} p_t^{\mathscr{D}}(x,y)\, u(y)\, dy &\geq \int_{|y|>\sqrt{t}/\log t} p_t^{\mathscr{D}}(x,y)\, u(y)\, dy \\
&\geq (1-o(1))(\log\sqrt{t} - \log\log t) \int_{|y|>\sqrt{t}/\log t} p_t^{\mathscr{D}}(x,y)\, dy \\
&\geq (1-o(1))(\log\sqrt{t} - \log\log t)\mathbb{P}_x\big(T_{\mathscr{D}} > t\big) \\
&\quad -(1-o(1))(\log\sqrt{t} - \log\log t) \int_{|y|\leq\sqrt{t}/\log t} p_t^{\mathscr{D}}(x,y)\, dy \\
&\geq (1-o(1))(\log\sqrt{t} - \log\log t)\mathbb{P}_x\big(T_{\mathscr{D}} > t\big) - o(1),
\end{aligned}$$

where the last inequality follows from the Gaussian bound $p_t^{\mathscr{D}}(x,y) \leq (2\pi t)^{-1}$ $e^{-\frac{1}{2t}|x-y|^2}$. We conclude that

$$\limsup_{t\to\infty}\ \log t\ \mathbb{P}_x\big(T_{\mathscr{D}} > t\big) \leq 2\ u(x)\ .$$

Similarly we get

$$\begin{aligned}
u(x) &\leq \int_{|y|<\sqrt{t}\log t} p_t^{\mathscr{D}}(x,y)\, u(y)\, dy + o(1) \\
&\leq (1+o(1))\log(\sqrt{t}\log t) \int_{|y|<\sqrt{t}\log t} p_t^{\mathscr{D}}(x,y)\, dy + o(1) \\
&\leq (1+o(1))\log(\sqrt{t}\log t)\mathbb{P}_x\big(T_{\mathscr{D}} > t\big) + o(1)\ .
\end{aligned}$$

It follows at once that

$$\liminf_{t\to\infty} \log t\ \mathbb{P}_x\big(T_{\mathscr{D}} > t\big) \geq 2\ u(x)\ ,$$

and the assertion is proven. The uniform convergence on compact sets follows from the uniformity of the Gaussian bounds. □

For $d \geq 3$ it is easy to see that

$$u(x) = \lim_{t\to\infty} \mathbb{P}_x(T_{\mathscr{D}} > t)$$

is a non-trivial positive harmonic function, vanishing in $(\partial\mathscr{D})^r$ and satisfying $\lim_{|x|\to\infty} u(x) = 1$. Moreover it follows from [Br] that any bounded harmonic function has a constant limit at infinity. Therefore u is the unique positive harmonic function vanishing in $(\partial\mathscr{D})^r$ and converging to 1 at infinite. u will denote this function for exterior domains in $\mathbb{R}^d$ for $d \geq 3$. For the complement of the unit ball this harmonic function is $u(x) = \left(1 - |x|^{2-d}\right)$. We shall prove the following result.

Theorem 1.4. *Let $\mathscr{D}$ be an exterior domain of $\mathbb{R}^2$, $d \geq 3$. For any $x, y \in \mathscr{D}$ we have*

$$\lim_{t\to\infty} t^{\frac{d}{2}} p_t^{\mathscr{D}}(x, y) = (2\pi)^{-\frac{d}{2}} u(x)u(y) \ .$$

The convergence is uniform on compact subsets of $\mathscr{D}$. □

Remarks. For planar exterior domains the conditional distribution of X_t has the asymptotic behaviour (see Theorem 1.2 and Proposition 1.3)

$$\forall\, y \in \mathscr{D} \quad \lim_{t\to\infty} t \log t \ \mathbb{P}_x(X_t \in dy \mid T_{\mathscr{D}} > t) = \mathscr{W}_{\mathscr{D}}(y)dy \ .$$

On the other hand, Theorem 1.4 can be written in the equivalent form

$$\lim_{t\to\infty} \frac{p_t^{\mathscr{D}}(x, y)}{p_t^{\mathbb{R}^d}(x, y)} = \mathbb{P}_x(T_{\mathscr{D}} = \infty)\mathbb{P}_y(T_{\mathscr{D}} = \infty) \ .$$

As it was pointed out by the referee, this limit has the following probabilistic interpretation: the probability that a bridge from x to y in time t does not hit the compact non polar set K in $\mathbb{R}^d$ $(d \geq 3)$ tends, as t goes to ∞, to the probability that two independent Brownian motions starting from x and y do not hit K.

A similar interpretation can be made for planar exterior domains, but stated in the ratio limit form

$$\lim_{t\to\infty} \frac{p_t^{\mathscr{D}}(x, y)/p_t^{\mathbb{R}^2}(x, y)}{\mathbb{P}_x(T_{\mathscr{D}} > t)\mathbb{P}_y(T_{\mathscr{D}} > t)} = 1 \ .$$

2. Preliminary results

In the following lemmas we establish monotone properties and ratio limits for the heat kernel. A basic tool we use is the parabolic Harnack's inequality (see [T]) which allows us to compare the kernel at different points of the domain. We state it in the following form. For any compact set $A \subset \mathscr{D}$ there is a constant $C = C(A)$ such that for any $\delta \leq d(A, \partial\mathscr{D})/2$, for any points z, z', z'' in A satisfying $d(z, z') < \delta$ and $d(z', z'') < \delta$, and for any $t > \delta^2$ it is verified

$$p_t^{\mathscr{D}}(z, z') \leq C p_{t+\delta^2}^{\mathscr{D}}(z', z') \leq C^3 p_{t+3\delta^2}^{\mathscr{D}}(z, z'') \ .$$

The next technical lemma allows us to get convergence along subsequences.

Lemma 2.1. *Assume $r_t(x) > 0$ satisfies the heat equation in a domain $\mathscr{D} \subseteq I\!R^d$. Let $a_t > 0$ satisfies $\sup_{t \geq t_0, |s| \leq 2} a_{t+s}/a_t < \infty$ for some $t_0 > 0$ and such that $(a_t r_t(x) : t \geq t_0)$ is bounded on compact sets. Then $(a_t r_t(x) : t \geq t_0 + 1)$ is equicontinuous on compact sets.*

Proof. Fix a compact set A contained in $\mathscr{D}$. We take A' another compact set in $\mathscr{D}$ whose interior contains A (A' is a security region around A in the use of Harnack's inequality). Take $t > t_0$ and $m = [t]$ the integer part of t. The function $a_m r_s(x)$ satisfies the heat equation for $(s, x) \in (m-1, m+2) \times \mathscr{D}$. On the other hand the function $a_{[t]} r_s(x)$ with $(t, s, x) \in [t_0 + 1, \infty) \times ([t] - \frac{1}{2}, [t] + \frac{3}{2}) \times A'$, is bounded, with a bound only depending on A'. From [T] Theorem 2.2, we get the existence of constants $C, \delta \in (0, \infty)$ only depending on A' such that

$$|a_m r_t(x) - a_m r_t(y)| \leq C|x - y|^\delta \ ,$$

for any $x, y \in A$ and for any $t \geq t_0 + 1$. From the hypothesis we conclude that $(a_t r_t(x) : t \geq t_0 + 1)$ is Hölder continuous on A with constants only depending on A', from which the result follows. □

Lemma 2.2. *Let $\mathscr{D}$ be an exterior domain of $I\!R^d$. For $x, y \in \mathscr{D}$ the ratio $p^{\mathscr{D}}_{t+s}(y, y)/p^{\mathscr{D}}_t(y, y)$ increases towards 1, as t tends to ∞. In particular $p^{\mathscr{D}}_t(y, y)$ is decreasing with t. Also the ratio $p^{\mathscr{D}}_{t+s}(x, y)/p^{\mathscr{D}}_t(x, y)$ converges to 1, as t tends to ∞.*

Proof. We only give a sketch of the proof. For more details see [CSM2] Lemmas 2.1-4. If φ is a nonnegative nonzero function belonging to $C_0^\infty(\mathscr{D})$, then the function $\langle \varphi, e^{t\mathscr{L}} \varphi \rangle$ is log-convex. In fact, from the spectral theorem for bounded self adjoint semigroups, there is a positive finite measure μ (which depends on φ) such that

$$\langle \varphi, e^{t\mathscr{L}} \varphi \rangle = \int_{-\infty}^{0} e^{\lambda t} \, d\mu(\lambda) \ . \tag{2.1}$$

Therefore

$$\frac{\partial_t \langle \varphi, e^{t\mathscr{L}} \varphi \rangle}{\langle \varphi, e^{t\mathscr{L}} \varphi \rangle} = \frac{\int_{-\infty}^0 e^{\lambda t} \lambda \, d\mu(\lambda)}{\int_{-\infty}^0 e^{\lambda t} \, d\mu(\lambda)} \quad \text{and} \quad \frac{\partial_t^2 \langle \varphi, e^{t\mathscr{L}} \varphi \rangle}{\langle \varphi, e^{t\mathscr{L}} \varphi \rangle} = \frac{\int_{-\infty}^0 e^{\lambda t} \lambda^2 \, d\mu(\lambda)}{\int_{-\infty}^0 e^{\lambda t} \, d\mu(\lambda)} \ .$$

Schwarz's inequality implies that $\partial_t^2 \log \left(\langle \varphi, e^{t\mathscr{L}} \varphi \rangle \right) \geq 0$. From the equality

$$\partial_t \log \left(\frac{\langle \varphi, e^{(t+s)\mathscr{L}} \varphi \rangle}{\langle \varphi, e^{t\mathscr{L}} \varphi \rangle} \right) = \int_t^{t+s} \partial_\tau^2 \log \left(\langle \varphi, e^{\tau \mathscr{L}} \varphi \rangle \right) d\tau \quad ,$$

we get that the ratio $\langle \varphi, e^{(t+s)\mathscr{L}} \varphi \rangle / \langle \varphi, e^{t\mathscr{L}} \varphi \rangle$ increases in t for any fixed $s \geq 0$. From (2.1) it is deduced that this ratio is bounded by 1. It is easy to see that the limit of this ratio is of the form $e^{-\lambda_0 s}$ for some $\lambda_0 \geq 0$, which may depend on φ. Using Harnack's inequality one proves that in fact λ_0 does not depend on φ. Since the spectral measure charges any small interval to the left of zero, it is deduced that $\lambda_0 = 0$. Taking a sequence of smooth functions tending to the Dirac measure concentrated on y, we prove that the ratio $p^{\mathcal{D}}_{t+s}(y, y)/p^{\mathcal{D}}_t(y, y)$ is monotone increasing. Again by Harnack's inequality we conclude that the limit of this ratio is 1. The second limit follows by a polarization argument. □

Observe that our theorems imply $\lim_{t\to\infty} p^{\mathcal{D}}_t(x, y)/p^{\mathcal{D}}_t(y, y) = u(x)/u(y)$, for any x and y in $\mathcal{D}$ and uniformly on compact sets. Since we need this result for the proof of Theorem 1.2, we establish it in this special case.

Lemma 2.3. *Let $\mathcal{D}$ be an exterior domain of $\mathbb{R}^2$. For $x, y \in \mathcal{D}$ we have*

$$\lim_{t\to\infty} p^{\mathcal{D}}_t(x, y)/p^{\mathcal{D}}_t(y, y) = u(x)/u(y) \ .$$

The convergence is uniform on compact subsets of $\mathcal{D}$.

Proof. Fix $y_0 \in \mathcal{D}$ large enough such that there exists an hyperplane L separating y_0 and K. We assume y_0 is to the right of L. For any x on the left of L we denote by $\bar{x}$ its reflected point with respect to L. We put $\bar{x} = x$ for points x at the right or on L. By a reflection argument (see Figure 1) for all $t > 0$

$$p^{\mathcal{D}}_t(x, y_0) \le p^{\mathcal{D}}_t(\bar{x}, y_0) \ .$$

Applying Harnack's inequality several times if necessary, we get

$$p^{\mathcal{D}}_t(x, y) \le C_1 p^{\mathcal{D}}_{t+\delta}(x, y_0) \le C_1 p^{\mathcal{D}}_{t+\delta}(\bar{x}, y_0) \ ,$$

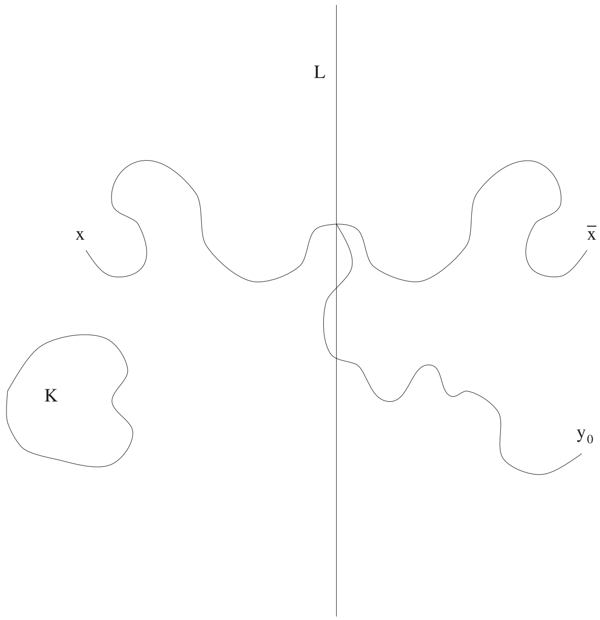

Fig. 1. Reflection used to have a control up to the boundary

where $C_1, \delta > 0$ only depend on y, y_0. Another use of Harnack's inequality allow us to obtain

$$p_t^{\mathscr{D}}(x, y) \le C_1 C_2^{|\bar{x}|+1} p_t^{\mathscr{D}}(y_0, y_0) \le C_1 C_3 C_2^{|\bar{x}|+1} p_t^{\mathscr{D}}(y, y) \ , \tag{2.2}$$

where C_2, C_3 only depend on y, y_0. Notice that in the last inequalities we have used the monotonicity of the heat kernel along the diagonal. Since $|x|$ and $|\bar{x}|$ are of the same order we conclude that

$$\forall t \ge t_0(y), \quad p_t^{\mathscr{D}}(x, y) \le C_4^{|x|+1} p_t^{\mathscr{D}}(y, y) \ , \tag{2.3}$$

for some constant C_4 which only depends on y. Hence, the family of functions, on the 'x' variable, $\left(p_t^{\mathscr{D}}(x, y)/p_t^{\mathscr{D}}(y, y) : t > t_0(y)\right)$ is bounded on compact subsets of $\mathscr{D}$, and therefore from Lemma 2.1 it is equicontinuous. We get that any sequence converging to ∞ contains a subsequence $t_n \nearrow \infty$ such that

$$\lim_{t_n \to \infty} \frac{p_{t_n}^{\mathscr{D}}(x, y)}{p_{t_n}^{\mathscr{D}}(y, y)} = V(y, x)$$

for some continuous function $V(y, \bullet)$, where the convergence is uniform on compact subsets of $\mathscr{D}$. Notice that $V(y, y) = 1$ and therefore $V(y, \bullet)$ is non trivial.

From the semigroup property we get for any $s > 0$,

$$\frac{p_{t_n+s}^{\mathscr{D}}(x, y)}{p_{t_n}^{\mathscr{D}}(y, y)} = \frac{p_{t_n}^{\mathscr{D}}(x, y)}{p_{t_n}^{\mathscr{D}}(y, y)} \frac{p_{t_n+s}^{\mathscr{D}}(x, y)}{p_{t_n}^{\mathscr{D}}(x, y)} = \int_{\mathscr{D}} \frac{p_{t_n}^{\mathscr{D}}(z, y)}{p_{t_n}^{\mathscr{D}}(y, y)} p_s^{\mathscr{D}}(x, z) dz \ .$$

Using (2.3), the Gaussian bound on $p^{\mathscr{D}}$, the Dominated Convergence Theorem and Lemma 2.2 we obtain

$$V(y, x) = \int_{\mathscr{D}} V(y, z) p_s^{\mathscr{D}}(x, z) dz \ .$$

It follows that $V(y, \bullet)$ is a non trivial harmonic and positive function vanishing at $(\partial\mathscr{D})^r$. Therefore, from Lemma 1.1 $V(y, \bullet)$ is proportional to u, that is $V(y, x) = a(y)u(x)$. Since $V(y, y) = 1$ we get $V(y, x) = u(x)/u(y)$. We conclude the limit does not depend on the particular subsequence, then the result follows. □

3. Proof of Theorem 1.2

In this section we use properties of special functions. We refer to [GR] and the notation therein for the formulas and properties of these functions.

We start by proving the result for $\mathscr{U}$, the complement of the unit ball. For this purpose let us consider the transition density $q_t(r, s)$ of $(|X_t|)$ killed at $\partial\mathscr{U}$, with respect to the measure sds in $(1, \infty)$. Thus, for $|x_0| = r > 1$ and $s > 1$,

$$q_t(r, s) s \, ds = \mathbb{P}_{x_0}(|X_t| \in ds, T_{\mathscr{U}} > t) \ .$$

We have

$$q_t(r, s) = \int_0^{2\pi} p_t^{\mathscr{U}}(r, e^{i\theta} s) d\theta \ .$$

Observe that $\int_0^{2\pi} p_t^{\mathscr{U}}(r, e^{i\theta}s)d\theta = \int_0^{2\pi} p_t^{\mathscr{U}}(x, e^{i\theta}s)d\theta$ for $x \in \mathscr{U}$ such that $|x| = r$. q_t is symmetric in its arguments $q_t(r,s) = q_t(s,r)$ for all $r, s > 1$, therefore the associated semigroup is self-adjoint in $L^2((1,\infty), r\,dr)$. q_t is the kernel of the semigroup $e^{t\mathscr{L}_0}$ with $\mathscr{L}_0 = \frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r)$, and Dirichlet boundary condition at $r = 1$.

Let us prove that the kernel q_t has the following asymptotic behaviour

$$\lim_{t\to\infty} t(\log t)^2 q_t(r,s) = 4\log r \log s \ . \tag{3.1}$$

From [CJ] formula 8, page 378, we have

$$q_t(r,s) = \int_0^\infty e^{-\rho^2 t/2}\varphi(\rho, r, s)\rho\, d\rho \ , \tag{3.2}$$

where

$$\varphi(\rho, r, s) = \frac{U_0(\rho r)U_0(\rho s)}{J_0^2(\rho) + N_0^2(\rho)} \ , \tag{3.3}$$

and $U_0(\rho r) = J_0(\rho r)N_0(\rho) - J_0(\rho)N_0(\rho r)$. Here $J_0,\ N_0$ are the Bessel functions of first and second kind. We separate the integration $\int_0^\infty$ in (3.2) into three pieces $\int_0^\infty = \int_0^{t^{-\frac{1}{2}-\delta}} + \int_{t^{-\frac{1}{2}-\delta}}^{t^{-\frac{1}{2}+\delta}} + \int_{t^{-\frac{1}{2}+\delta}}^\infty$, for $\delta > 0$ small. Observe that $\varphi(\rho, r, s)$ is a bounded function, then the third piece is $o(1/t(\log t)^2)$. For the first and the second piece we need to find the dominant contribution of the integrand near $\rho = 0$. Using formulas 8.402, 8.403 of [GR] and some algebra left to the reader, we get the following estimate

$$\rho\varphi(\rho, r, s) = \rho\frac{\log r \log s}{(\log\rho)^2} + \mathscr{O}\big(\rho/(\log\rho)^3\big) \ .$$

Hence the first piece behaves like $o(1/t(\log t)^2)$, and the second piece like $4\log r\log s(1/t(\log t)^2)$, proving (3.1).

Let us prove the corresponding limit for $p^{\mathscr{U}}$. Since $q_t(r,s) = \int_0^{2\pi} p_t^{\mathscr{U}}(x, se^{i\theta})d\theta$, we deduce from Harnack's inequality and Lemma 2.1 that for any sequence converging to ∞ there exists some subsequence $t_n \nearrow \infty$ such that

$$\lim_{n\to\infty} t_n(\log t_n)^2 p_{t_n}^{\mathscr{U}}(x,y) = V(x,y) \ ,$$

for some continuous strictly positive function V. From Harnack's inequality we also get that for x fixed $V(x,y)$ is comparable to $\log|y|$. As in the proof of Lemma 2.3 it follows that $V(x,y)$ is harmonic in y, it is not trivial and it vanishes at $(\partial\mathscr{D})^r$. Therefore $V(x,y) = a(x)\log|y|$. Due to the symmetry of the problem we have $V(x,y) = H\log|x|\log|y|$, for some constant $H > 0$ which may depend on the subsequence. Integration on the angle gives $\lim_{n\to\infty} t_n(\log t_n)^2 q_{t_n}(r,s) = 2\pi H\log r\log s$. From (3.1) we obtain $H = 2/\pi$. Then the limit does not depend on the subsequence and we conclude

$$\lim_{t\to\infty} t(\log t)^2 p_t^{\mathscr{U}}(x,y) = \frac{2}{\pi}\log|x|\log|y| \ ,$$

which proves Theorem 1.2 for the unit ball.

For proving Theorem 1.2 in its full generality we need the asymptotic behaviour of $\partial_t I\!P_y(T_{\mathscr{U}} > t)$, which is given in the next result.

Lemma 3.1. *For $y \in \mathscr{U}$ we have*

$$\lim_{t\to\infty} t(\log t)^2 \partial_t I\!P_y(T_{\mathscr{U}} > t) = -2\log|y| \ .$$

Proof. From the equality $I\!P_y(T_{\mathscr{U}} > t) = \int_1^\infty q_t(|y|, r) r\, dr$ and the Dominated Convergence Theorem we get

$$\begin{aligned}\partial_t I\!P_y(T_{\mathscr{U}} > t) &= \int_1^\infty \partial_t q_t(|y|, r) r\, dr \\ &= \frac{1}{2}\int_1^\infty (\partial_r^2 + \frac{1}{r}\partial_r) q_t(|y|, r) r\, dr = -\frac{1}{2}\partial_r q_t(|y|, r)|_{r=1^+} \ .\end{aligned}$$

From formula (3.3) and the functional relations for the Bessel functions $J_0' = -J_1$, $N_0' = -N_1$ we obtain

$$\partial_r \varphi(\rho, r, s) = -\frac{\rho(J_1(\rho r)N_0(\rho) - J_0(\rho)N_1(\rho r))U_0(\rho s)}{J_0^2(\rho) + N_0^2(\rho)} \ .$$

Using as before formulas 8.402 and 8.403 combined with 8.441.2 and 8.444.2 in [GR], we obtain

$$\partial_r \varphi(\rho, r, s)|_{r=1^+} = \frac{\log s}{(\log \rho)^2} + O((1/\log\rho)^3) \ .$$

Then, by integrating over ρ and making analogous computations as above we find that

$$\lim_{t\to\infty} t(\log t)^2 \partial_r q_t(|y|, r)|_{r=1^+} = 4\log|y| \ ,$$

and the result follows. □

Let $\mathscr{D}$ be an exterior planar domain. We consider a ball large enough to contain the compact set $K = \mathscr{D}^c$ in its interior. By scaling and translation we can assume that this is the unit ball $\mathscr{B}$. The strong Markov property implies that for any x and y in $\mathscr{U}$ and for any $t > 0$ the following relation between $p^{\mathscr{D}}$ and $p^{\mathscr{U}}$ holds

$$p_t^{\mathscr{D}}(x, y) - p_t^{\mathscr{U}}(x, y) = \int_0^t ds \int_{\partial\mathscr{U}} p_s^{\mathscr{D}}(x, z) I\!P_y\left(X_{T_{\mathscr{U}}} \in dz, T_{\mathscr{U}} \in d(t-s)\right) \ . \tag{3.4}$$

In the following steps of the proof we assume that y is large enough. This assumption will be removed after obtaining estimate (3.7) below. We remind that u is the unique harmonic function given by Lemma 1.1 for the domain $\mathscr{D}$. Let $u^* = \max_{z\in\partial\mathscr{U}} u(z)$. From the growth condition on u, we can assume y is large enough such that $u^*/u(y) < 1/4$. We also assume that $|y| > 2$. From (3.4) we have

$$p_t^{\mathscr{D}}(y, y) = p_t^{\mathscr{U}}(y, y) + \int_0^t \int_{\partial\mathscr{U}} \mathbb{P}_y\left(X_{T_{\mathscr{U}}} \in dz, T_{\mathscr{U}} \in d(t-s)\right)\ p_s^{\mathscr{D}}(z, y) \ .$$

Lemmas 2.2 and 2.3 give the following estimates for large t

$$\int_{t/2}^{t}\int_{\partial\mathscr{U}} \mathbb{P}_y(X_{T_{\mathscr{U}}} \in dz, T_{\mathscr{U}} \in d(t-s))\, p_s^{\mathscr{D}}(z,y)$$
$$\leq C_1(t,y)\,\frac{u^*}{u(y)}\, p_{t/2}^{\mathscr{D}}(y,y)\ \mathbb{P}_y(T_{\mathscr{U}} < t/2) \leq C_1(t,y)\frac{u^*}{u(y)} p_{t/2}^{\mathscr{D}}(y,y)\ ,$$

where $C_1(t,y)$ can be chosen such that $C_1(t,y) \underset{t\to\infty}{\longrightarrow} 1$.

Now we study the integral over the range $[0, t/2]$. From the standard Gaussian bound we obtain that the heat kernel $p_s^{\mathscr{D}}(z,y)$ is bounded on the range $0 \leq s \leq 1, z \in \partial\mathscr{U}, |y| > 2$. Hence Lemma 3.1 gives

$$\int_{0}^{1}\int_{\partial\mathscr{U}} \mathbb{P}_y\left(X_{T_{\mathscr{U}}} \in dz, T_{\mathscr{U}} \in d(t-s)\right)\ p_s^{\mathscr{D}}(z,y)$$
$$\leq C_2\int_0^1 \left(-\partial_\zeta I\!P_y(T_{\mathscr{U}} > \zeta)|_{\zeta=t-s}\right) ds \leq C_3(t,y)\frac{\log|y|}{t(\log t)^2}\ ,$$

where C_2 is an absolute constant, and $C_3(t,y)$ can be chosen such that $C_3(t,y) \underset{t\to\infty}{\longrightarrow} C_2$. On the other hand from the Harnack's inequality we deduce the existence of some finite absolute constants $C_4, \delta > 0$ such that for any $z^* \in \partial\mathscr{U}$ the following estimate holds

$$\int_{1}^{t/2}\int_{\partial\mathscr{U}} \mathbb{P}_y\left(X_{T_{\mathscr{U}}} \in dz, T_{\mathscr{U}} \in d(t-s)\right) p_s^{\mathscr{D}}(z,y)$$
$$\leq C_4\int_1^{t/2} p_{s+\delta}^{\mathscr{D}}(z^*,y)\int_{\partial\mathscr{U}} \mathbb{P}_y\left(X_{T_{\mathscr{U}}} \in dz, T_{\mathscr{U}} \in d(t-s)\right)$$
$$= C_4\int_1^{t/2} p_{s+\delta}^{\mathscr{D}}(z^*,y)\left(-\partial_\zeta I\!P_y(T_{\mathscr{U}} > \zeta)|_{\zeta=t-s}\right) ds\ .$$

Using again Lemma 3.1 we get the existence of $C_5(t,y)$ satisfying $C_5(t,y) \underset{t\to\infty}{\longrightarrow} 2$ and

$$\int_{1}^{t/2}\int_{\partial\mathscr{U}} \mathbb{P}_y\left(X_{T_{\mathscr{U}}} \in dz, T_{\mathscr{U}} \in d(t-s)\right) p_s^{\mathscr{D}}(z,y)$$
$$\leq C_4C_5(t,y)\frac{\log|y|}{t(\log t)^2}\int_0^\infty p_s^{\mathscr{D}}(z^*,y)ds = C_4C_5(t,y)\frac{\log|y|}{t(\log t)^2}G^{\mathscr{D}}(z^*,y)\ .$$

Since $\lim_{|y|\to\infty} G^{\mathscr{D}}(z^*,y) = \mathscr{W}_{\mathscr{D}}(z^*) < \infty$, we obtain the existence of an absolute constant C_6 such that for every large enough y and large enough t ($t > t_0(y)$) it is verified

$$p_t^{\mathscr{D}}(y,y)\ \leq p_t^{\mathscr{U}}(y,y) + C_1(t,y)\frac{u^*}{u(y)} p_{t/2}^{\mathscr{D}}(y,y)$$
$$+ (C_6C_5(t,y) + C_3(t,y))\frac{\log|y|}{t(\log t)^2}\ ,$$

then

$$\begin{aligned} t(\log t)^2 p_t^{\mathcal{D}}(y,y) \le t(\log t)^2 p_t^{\mathcal{U}}(y,y) \\ + \Big(\frac{t(\log t)^2}{\frac{t}{2}(\log\frac{t}{2})^2} C_1(t,y)\frac{u^*}{u(y)}\Big)\frac{t}{2}(\log\frac{t}{2})^2 p_{t/2}^{\mathcal{D}}(y,y) \\ + (C_6C_5(t,y)+C_3(t,y))\log|y| \ . \end{aligned} \tag{3.5}$$

Given that $2C_1(t,y)\frac{u^*}{u(y)} \underset{t\to\infty}{\longrightarrow} 2\frac{u^*}{u(y)} \le 1/2$, we obtain the upper bound

$$\limsup_{t\to\infty} t(\log t)^2 p_t^{\mathcal{D}}(y,y)\Big(1-\frac{2u^*}{u(y)}\Big) \le \frac{2}{\pi}(\log|y|)^2 + C_7\log|y| \ , \tag{3.6}$$

where for example $C_7 = 2C_6 + C_2$ will be enough. From $p_t^{\mathcal{D}}(y,y) \ge p_t^{\mathcal{U}}(y,y)$ we get the lower bound

$$\liminf_{t\to\infty} t(\log t)^2 p_t^{\mathcal{D}}(y,y) \ge \frac{2}{\pi}(\log|y|)^2 \ . \tag{3.7}$$

Now we fix some y_0 verifying the above requirements. Using Harnack's inequality and the reflection principle as in the proof of Lemma 2.3 (see (2.2)), it is easy to see that for some constant $C_8 = C_8(y)$ the following inequality holds

$$p_t^{\mathcal{D}}(y,x) \le C_8^{|x|+1} p_t^{\mathcal{D}}(y_0,y_0) \ \text{ for any } \ x \in \mathcal{D} \ .$$

Using (3.5) (with y_0 instead of y) we obtain that the family of functions $(t(\log t)^2 p_t^{\mathcal{D}}(y,\bullet) \ : t > t_0)$ is bounded on compact sets. From Lemma 2.1 we deduce that this family is equicontinuous on compact sets. Hence for any sequence increasing to ∞ there exists a subsequence $t_n \nearrow \infty$ such that

$$t_n(\log t_n)^2 p_{t_n}^{\mathcal{D}}(y,x) \underset{n\to\infty}{\longrightarrow} V(y,x)$$

uniformly on compacts sets in $\mathcal{D}\times\mathcal{D}$.

From the Dominated Convergence Theorem we get

$$V(y,x) = \int p_s^{\mathcal{D}}(x,z)V(y,z)dz \ .$$

Since $p^{\mathcal{D}} \ge p^{\mathcal{U}}$ necessarily $V(y,y) \ge \frac{2}{\pi}(\log|y|)^2 > 0$. We conclude that $V(y,x)$ is harmonic in x, it does not vanish, and goes to 0 when x approaches $(\partial\mathcal{D})^r$. From Lemma 1.1, $V(y,x) = m(y)u(x)$ for some function $m(y)$. Symmetry implies $V(y,x) = Hu(y)u(x)$ for some constant $H > 0$ (which may depend on the subsequence (t_n)).

Using this expression in (3.6) and (3.7), we obtain for large y

$$\frac{2}{\pi}(\log|y|)^2 \le H(u(y))^2 \le \frac{2}{\pi}\Big((\log|y|)^2 + C_7\log|y|\Big)\Big(1-\frac{2u^*}{u(y)}\Big)^{-1} \ .$$

Since $u(y)$ behaves asymptotically as $\log|y|$, we obtain $H = \frac{2}{\pi}$. Therefore it follows the convergence of $t(\log t)^2 p_t^{\mathcal{D}}(x,y)$ to $\frac{2}{\pi}u(x)u(y)$, uniformly on compacts sets in $\mathcal{D}\times\mathcal{D}$. □

4. Proof of Theorem 1.4

First, we prove the result for the unit ball $\mathscr{B}$. We reason by induction on the dimension $d \geq 3$ and we start by proving it for $d = 3$.

Denote by B_t a one dimensional Brownian motion and put $\tau^L = \inf\{t : B_t = L\}$. From Lemma 5.2.8 in [K], we have for $\mathscr{A} \in \sigma(|X_\bullet|)$

$$\mathbb{P}(\mathscr{A} \mid |X_0| = r, |X_t| = s) = \mathbb{P}(\mathscr{A} \mid B_0 = r, B_t = s, \tau^0 > t) \ .$$

Since $T_{\mathscr{U}} = \inf\{t : |X_t| = 1\}$ we find for $x \in \mathscr{U}$, $|x| = r$ and $s > 1$

$$\begin{aligned}
\mathbb{P}_x(T_{\mathscr{U}} > t|\ |X_t| = s) &= \mathbb{P}(T_{\mathscr{U}} > t|\ |X_0| = r, |X_t| = s)\\
&= \mathbb{P}(\tau^1 > t|B_0 = r, B_t = s, \tau^0 > t)\\
&= \frac{\mathbb{P}(\tau^0 > t|B_0 = r-1, B_t = s-1)}{\mathbb{P}(\tau^0 > t|B_0 = r, B_t = s)}\\
&= \frac{\sinh((r-1)(s-1)t^{-1})}{\sinh(rst^{-1})} \ .
\end{aligned}$$

The latter converges to $(1 - r^{-1})(1 - s^{-1})$ as $t \to \infty$. Then

$$\begin{aligned}
q_t(|x|, s)s^2 &= \int_{S^2} p_t^{\mathscr{U}}(x, s\sigma)s^2 d\sigma = \mathbb{P}_x(|X_t| = s, T_{\mathscr{U}} > t)\\
&= \frac{\sinh((|x|-1)(s-1)t^{-1})}{\sinh(|x|st^{-1})} s^2 \int_{S^2} p_t^{\mathbb{R}^3}(x, s\sigma)d\sigma \ . \qquad (4.1)
\end{aligned}$$

Since $\lim_{t\to\infty} p_t^{\mathbb{R}^3}(x, y)/p_t^{\mathbb{R}^3}(0, 0) = 1$, Harnack's inequality implies the existence of $C > 0$, such that for t large enough and (x, y) in a compact subset of $\mathscr{U} \times \mathscr{U}$

$$C \leq \frac{p_t^{\mathscr{U}}(x, y)}{p_t^{\mathbb{R}^3}(0, 0)} \leq 1 \ .$$

Lemma 2.1 asserts that any sequence increasing to ∞ has a subsequence $t_n \nearrow \infty$ for which there exists a constant $H > 0$ such that

$$\lim_{t_n\to\infty} \frac{p_{t_n}^{\mathscr{U}}(x, y)}{p_{t_n}^{\mathbb{R}^3}(0, 0)} = Hu(x)u(y) \ .$$

This last equality holds because the limiting function is a bounded harmonic function in x and y, vanishing at $\partial\mathscr{U}$. Integrating this relation on S^2 and using (4.1) gives $H = 1$. Then there is no dependence on the subsequence and we conclude $\lim_{t\to\infty} p_t^{\mathscr{U}}(x, y)/p_t^{\mathbb{R}^3}(0, 0) = u(x)u(y)$, from which the result holds for the unit ball in $\mathbb{R}^3$.

Now we make the induction on $d \geq 3$. We denote $x^d \in \mathbb{R}^d$, X_t^d the Brownian motion in $\mathbb{R}^d$, $\mathscr{U}_d$ the complement of the unit ball in $\mathbb{R}^d$, $T^d = T_{\mathscr{U}_d}$ and $p^d = p^{\mathscr{U}_d}$.

As for dimension $d+1$ we put $x^{d+1} = (x^d, x) \in I\!R^{d+1}$ and $X_t^{d+1} = (X_t^d, B_t)$, where B_t is a one dimensional Brownian motion. We have

$$\begin{aligned} p_t^{d+1}((x^d, x), (y^d, y)) &= \mathbb{P}_{(x^d,x)}(X_t^d = y^d, B_t = y, T^{d+1} > t) \\ &\geq \mathbb{P}_{(x^d,x)}(X_t^d = y^d, B_t = y, T^d > t) \\ &= \mathbb{P}_{x^d}(X_t^d = y^d, T^d > t)\mathbb{P}_x(B_t = y) \qquad (4.2) \\ &= p_t^d(x^d, y^d) p_t^{I\!R}(x, y) \end{aligned}$$

By induction, there exists $C > 0$ such that

$$C \leq \frac{p_t^{d+1}((x^d, x), (y^d, y))}{p_t^{I\!R^{d+1}}((x^d, x), (y^d, y))} \leq 1$$

Again by Lemma 2.1, any sequence converging to ∞ has a subsequence $t_n \nearrow \infty$ for which there exists H, which may depend on the subsequence, such that

$$\lim_{n\to\infty} \frac{p_{t_n}^{d+1}((x^d, x), (y^d, y))}{p_{t_n}^{I\!R^{d+1}}((x^d, x), (y^d, y))} = H\big(1 - |(x^d, x)|^{1-d}\big)\big(1 - |(y^d, y)|^{1-d}\big) \ .$$

But (4.2) implies this limit is at least $(1 - |x^d|^{2-d})(1 - |y^d|^{2-d})$. Letting $|x^d| \to \infty$, $|y^d| \to \infty$ we find $H \geq 1$. Obviously $H \leq 1$, then $H = 1$. As above we deduce that

$$\lim_{t\to\infty} \frac{p_t^{d+1}((x^d, x), (y^d, y))}{p_t^{I\!R^{d+1}}((x^d, x), (y^d, y))} = \big(1 - |(x^d, x)|^{1-d}\big)\big(1 - |(y^d, y)|^{1-d}\big) \ .$$

Therefore the induction is verified. By scaling and translation the result holds for the complement of any ball.

Now we complete the proof for any exterior domain. Let $\mathscr{B}(R)$ be a ball of radius R, centered at the origin, containing the compact set $\mathscr{D}^c$. We denote $\mathscr{U}(R) = (\mathscr{B}(R))^c$. Then

$$p^{\mathscr{U}(R)} \leq p^{\mathscr{D}} \leq p^{I\!R^d} \qquad (4.3)$$

By the same arguments as before any sequence increasing to ∞ contains a subsequence $t_n \nearrow \infty$ for which there exists $H > 0$ such that

$$\lim_{n\to\infty} \frac{p_{t_n}^{\mathscr{D}}(x, y)}{p_{t_n}^{I\!R^d}(x, y)} = Hu(x)u(y) \ .$$

Since

$$\lim_{n\to\infty} \frac{p_{t_n}^{\mathscr{U}(R)}(x, y)}{p_{t_n}^{I\!R^d}(x, y)} = \left(1 - \frac{R^{d-2}}{|x|^{d-2}}\right)\left(1 - \frac{R^{d-2}}{|y|^{d-2}}\right) \ ,$$

we conclude $H = 1$ after (4.3) and making $|x| \to \infty$, $|y| \to \infty$. □

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