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HOMOGENIZATION OF PERIODIC STRUCTURES VIA BLOCH DECOMPOSITION*

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Abstract. In this paper, the classical problem of homogenization of elliptic operators in arbitrary domains with periodically oscillating coefficients is considered. Using Bloch wave decomposition, a new proof of convergence is furnished. It sheds new light and offers an alternate way to view the classical results. In a natural way, this method leads us to work in the Fourier space and thus in a framework dual to the one used by L. Tartar [*Problèmes d'Homogénéisation dans les Equations aux Dérivées Partielles*, Cours Peccot au Collège de France, 1977] in his method of homogenization. Further, this technique offers a nontraditional way of calculating the homogenized coefficients which is easy to implement in the computer.

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1. Introduction.

As is well known, homogenization process in classical examples is concerned with the study of the behavior of solutions of elliptic boundary value problems when the coefficients are periodic with small period $\varepsilon > 0$. Such a situation models, for instance, the elastic behavior of a medium with large number of heterogeneities. For a nice introduction to this subject, the reader is referred to the book of A. Bensoussan, J.L. Lions and G. Papanicolaou (1978) ([B-L-P, 1978]). The main result says that the (weak) limit of such solutions resolves a suitable boundary value problem which has constant coefficients that represent what is known as homogenized medium. There are many ways to obtain the homogenized coefficients, and there is a vast body of work in the literature which justifies the limiting procedure. Let us mention some of them. The basic book just quoted, [B-L-P, 1978], presents an application of the method of multiple scale expansion to homogenization, and this technique is the easiest way to obtain the homogenized medium. Justification of this method is usually done by Tartar's method which he developed in large part in association with F. Murat; see Tartar (1977), Murat (1977–78), or, of course, [B-L-P, 1978]. This method is very general and it goes beyond the case of periodically oscillating coefficients. However, if the medium is periodically heterogeneous, there is an alternate procedure to pass to the limit by using the notion of two-scale weak convergence. For the implementation of this method, one can see Nguetseng (1989) and Allaire (1992). If the solution of the boundary value problem is realized as a minimum of a suitable energy functional, then the convergence of the solution sequence can be reduced to Γ-convergence of the energy functional. This idea has been expounded in Dal Maso (1993). Those readers who are interested in applications

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of homogenization theory in physics or mechanics should consult Sánchez-Palencia (1980).

In this paper, we suggest a different approach based on Fourier analysis. If the medium is homogeneous, i.e., if the coefficients are constants, then plane waves $e^{i\xi\cdot x}$ serve as an effective tool transforming the differential equation into a set of algebraic equations. If the medium is periodic (which is true in the present case) then the Bloch waves, to be introduced in section 2, serve the same purpose. These waves were originally introduced in solid state physics in the context of propagation of electrons in a crystal; see Bloch (1928). Several questions and properties of periodic media can be translated in terms of Bloch waves. Exploiting this idea, we have been advocating for a *nonstandard homogenization* in unbounded domains. The reader may refer to our book Conca, Planchard, and Vanninathan (1995) ([C-P-V, 1995]) for a wide variety of applications in the vibrations of fluid-solid structures (see also Conca, Planchard, Thomas, and Vanninathan (1994)). Additional references on Bloch waves are [B-L-P, 1978], Wilcox (1978), Reed and Simon (1978).

When speaking of the Fourier analytic approach to homogenization problems, one must also mention the work by Morgan and Babuška (1991) where they prove, among other results, the classical homogenization theorem for periodic coefficients in $\mathbb{R}^N$. However, their idea differs from ours: while we work exclusively with Bloch waves, they mix both Bloch and Fourier analysis to obtain a very useful representation formula for the solution, which they exploit.

One fundamental difficulty in homogenization problems is to pass to the limit in the product of two sequences both of which converge weakly and identify the limit. Due to oscillations, the limit of the product is not equal to the product of the limits, and so this problem is nontrivial. In order to analyze these oscillations in the physical space represented by the (slow) variable x, one introduces the so-called fast variable $y = x/\varepsilon$, and one produces cleverly suitable test functions which are then used as multipliers in the original equation. This is the underlying common feature in all classical methods mentioned in the first paragraph.

One natural question which arises then is to know what happens to periodic oscillations if we work in the Fourier space. Without going into details let us see this heuristically. To this end, denote by ξ and η the variables dual to x and y in the Fourier sense. Since the Fourier transform of a function depending on x/ε is a function of $\varepsilon\xi$, we have the relation $\eta = \varepsilon\xi$ (which may be referred to as Heisenberg relation). This indicates that the fast oscillations in the medium give place to the dependence on the slow variable η. On the other hand, if we replace each derivative $\partial/\partial x_j$, as is usual in Fourier analysis, by ξ_j which is equal to $\varepsilon^{-1}\eta_j$, we see that we accumulate negative powers of ε. Thus, heuristically speaking, there will be slow variations combined with singularities in the Fourier space. Such a structure can be fruitfully exploited to our advantage by Taylor's formula provided we establish some regularity properties. Thus we see that the original difficulty is transformed to establishing some regularity properties in the Fourier space. Further, since differential operators disappear giving rise to algebraic equations, the passage to the limit, as we shall see, is more direct, and one avoids completely the clever manipulations with test functions. (Test functions appear in a natural way if one wishes to identify the homogenized operator of the present method with the one obtained by classical ones.) The purpose of this work is to convince the reader that the above heuristic picture is indeed true.

In this article, we use Bloch waves to reestablish some classical results on homogenization in arbitrary domains bounded or not. Our principal result is stated in Theorem 3.1. Although the results are not new, the method sheds new light and offers an alternative way to view the classical results. The method, a version of which appeared in Santosa and Symes (1991) ([S-S, 1991]), works in the following way: the original problem is transformed to an equivalent problem in the Bloch space which is then completely analyzed in section 3. It is shown that all Bloch harmonics corresponding to $m \geq 2$ can be neglected in the homogenization process. This explains why oscillations present in the solution are not well approximated by the homogenized one. Next we establish that the Bloch waves representing the periodic medium approach Fourier waves representing the homogenized medium. This is easily interpreted as a result of homogenization in the Fourier space.

These results enable us to pass to the limit in a straightforward manner in the transformed equation if we work in the entire space. (In the case of arbitrary domains, localization is involved, and this complicates the analysis a little bit.) The passage to the limit demands, as argued earlier, certain regularity of the dominant Bloch eigenvalue and of the corresponding eigenvector in a neighborhood of the origin which is proven in section 2. After passage to the limit, we identify the limiting equation with the classical homogenized equation in the Fourier space. (These calculations were somewhat hidden in [B-L-P, 1978, pp. 633–638] and redone in [S-S, 1991].) A new characterization of the homogenized matrix is obtained via the present method. Indeed, it coincides with the Hessien of the dominant Bloch eigenvalue at the origin which is shown to be a *critical point* on the torus representing the periodic structure. This reveals an intrinsic character of the homogenized matrix which is not evident in earlier works because of the explicit use of the coordinate variables. Further, our results clearly show that very little of the information contained in the Bloch spectrum is really what is needed for homogenization.

Being a spectral method, the present method requires the operators involved to be self-adjoint, which means that the coefficients matrix representing the periodic medium is symmetric. This is in contrast to Tartar's method which homogenizes even nonsymmetric coefficients.

Let us conclude this Introduction by citing the works of Allaire and Conca (1996), (1995a,b), where a combination of two-scale convergence and Bloch wave method is used to study homogenization problems in bounded domains. Their relation to the present article remains to be explored in the future.

Finally, a word about the notation adapted in this work. Summation with respect to the repeated indices is understood throughout this paper. The constants appearing in various estimates independent of ε are generically denoted by c.

2. Bloch eigenvalues and eigenvectors. We consider the operator

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial y_k}\left(a_{k\ell}(y)\frac{\partial}{\partial y_\ell}\right), \tag{2.1}$$

where the coefficients $a_{k\ell}$ are assumed to satisfy

$$\begin{cases} a_{k\ell} \in L^\infty_\sharp(Y), \text{ where } Y = [0, 2\pi[^N, \text{ i.e.,} \\ \text{each } a_{k\ell} \text{ is a } Y\text{-periodic bounded measurable function defined on } \mathbb{R}^N, \\ \exists \alpha > 0 \quad \text{such that} \quad a_{k\ell}(y)\xi_k\xi_\ell \geq \alpha|\xi|^2 \quad \text{(ellipticity)}, \\ a_{k\ell} = a_{\ell k} \quad \forall k, l = 1, \dots, N \quad \text{(symmetry)}. \end{cases} \tag{2.2}$$

We are interested in the spectral resolution of A in $L^2(\mathbb{R}^N)$. For this purpose, the classical method of Bloch (see Bloch (1928)) will be used, and it consists of introducing a family of spectral problems parametrized by $\eta \in \mathbb{R}^N$: find $\lambda = \lambda(\eta) \in \mathbb{R}$ and $\psi = \psi(y;\eta)$ (not identically zero) such that

$$A\psi(\cdot;\eta) = \lambda\psi(\cdot;\eta) \quad \text{in} \quad \mathbb{R}^N, \tag{2.3a}$$

$$\psi(\cdot;\eta) \quad \text{is} \quad (\eta, Y)\text{-periodic, i.e.,} \tag{2.3b}$$
$$\psi(y + 2\pi m;\eta) = e^{2\pi i m\cdot\eta}\psi(y) \quad \forall m \in \mathbb{Z}^N,\ y \in \mathbb{R}^N.$$

First of all, it is clear that the above problem remains the same if η is replaced by $\eta+m$, $m \in \mathbb{Z}^N$. So, there is no loss of generality in confining η to the cell $Y' = [0,1[^N$. (In the sequel, we shall take η in the translated cell $Y' = [-\frac{1}{2}, \frac{1}{2}[^N$, again without loss of generality.) The cell $Y' = [0,1[^N$ is dual to Y because the basis of lattice vectors $\{b_k\}$ defined by Y', and the basis $\{a_k\}$ associated with Y are dual to each other in the sense that $a_k \cdot b_\ell = 2\pi\delta_{k\ell}$ $\forall k, \ell = 1, ..., N$. In particular, $e^{ia_k\cdot b_\ell} = 1$. In fact, $b_k = \mathrm{e}_k$ and $a_k = 2\pi\mathrm{e}_k$ $\forall k = 1, ..., N$. The cell Y' can be viewed in a slightly different angle via Fourier transform. In fact, let us recall that the Fourier transform of a Y-periodic distribution on $\mathbb{R}^N$ is supported on the lattice generated by Y'. For this reason, it is justified to denote the variable in Y' by η which was introduced in the Introduction as a variable dual to y in the Fourier sense.

After the above discussion on duality, we turn to some motivating reasons for considering problem (2.3). Equation (2.3a) suggests itself because after all we are interested in finding the spectrum of A. On the other hand, the boundary condition (2.3b) is new and will be referred to as Bloch condition (see below or the paper by Aguirre and Conca (1988) for a discrete version of this kind of generalized periodic boundary condition). Solutions ψ of (2.3) are usually called Bloch waves or Bloch eigenvectors. They can be motivated in a couple of ways (see [C-P-V, 1995]). If the medium were homogeneous, i.e., if $a_{k\ell}$ are constants, then it is a classical idea to use Fourier waves (also called plane waves) $e^{i\eta\cdot y}$ to solve the problem. Bloch waves are natural generalizations of plane waves to treat periodic media. Consistency then demands that plane waves must satisfy (2.3b) which is a trivial fact.

A better way to reach (2.3b) is to regard the spectral equation $A\psi = \lambda\psi$ in $\mathbb{R}^N$ as a limit of spectral problems with periodic boundary conditions on cubes $Q_n =]-n, n[^N$:

$$\begin{cases} A\psi = \lambda\psi \quad \text{in} \quad \mathbb{R}^N, \\ \psi \text{ is } Q_n\text{-periodic.} \end{cases} \tag{2.4}$$

It can be easily observed that for each fixed n, the Q_n-periodicity condition on ψ is implied by the following condition:

$$\psi(y + 2\pi m) = \omega^m\psi(y) \quad \forall m \in \mathbb{Z}^N, \tag{2.5}$$

where $\omega \in \mathbb{C}^N$ is an N-tuple whose components are taken from the set $S_{2n} = \{\omega_1, ..., \omega_{2n}\}$ which are the $2n$th roots of unity. Expressing each ω_k in the form $e^{2\pi i\tau_k}$ for some τ_k in $[-\frac{1}{2}, \frac{1}{2}]$, we see that (2.5) coincides with (2.3b) for some $\eta \in Y'$. As $n \to \infty$, we conclude that all vectors $\eta \in Y'$ will play a role in the spectral resolution of A because the set $\bigcup_{n=1}^{\infty} S_{2n}$ is a dense subset of the unit circle. All these matters have been detailed in [C-P-V, 1995].

Periodic media in one dimension have been studied by Floquet (1883) prior to Bloch. Following his ansatz, we look for solutions of (2.3a) which are products of Y-periodic functions with solutions in the homogenized media, i.e., plane waves

$$\begin{cases} \psi(y;\eta) = e^{i\eta\cdot y}\phi(y;\eta), \\ \phi(\cdot;\eta) \text{ is } Y\text{-periodic.} \end{cases} \tag{2.6}$$

Such ψ's obviously fulfill (2.3b), and this provides a third motivation for (2.3b).

The transformation given by (2.6) maps problem (2.3) to the following one where the parameter $\eta \in Y'$ appears not in the boundary condition but rather in the operator: find $\lambda = \lambda(\eta) \in \mathbb{R}$ and $\phi = \phi(y;\eta)$ (not identically zero) such that

$$\begin{cases} A(\eta)\phi = \lambda\phi \quad \text{in} \quad \mathbb{R}^N, \\ \phi \text{ is } Y\text{-periodic.} \end{cases} \tag{2.7}$$

Here the operator $A(\eta)$ is defined by

$$A(\eta) \stackrel{\text{def}}{=} -\left(\frac{\partial}{\partial y_k} + i\eta_k\right)\left[a_{k\ell}(y)\left(\frac{\partial}{\partial y_\ell} + i\eta_\ell\right)\right], \tag{2.8}$$

and it is referred to as the shifted operator in the literature.

It is well known that, due to ellipticity and symmetry hypothesis, the above problem (2.7) admits a sequence of eigenvalues and eigenvectors in the space

$$L^2_\sharp(Y) = \{v \in L^2_{loc}(\mathbb{R}^N) \mid v \text{ is } Y\text{-periodic}\}.$$

They have the following properties (see [B-L-P, 1978]):

$$\begin{cases} \lambda_1(\eta) \le \cdots \le \lambda_m(\eta) \le \cdots \longrightarrow \infty, \\ \{\phi_m(\cdot;\eta)\}_{m=1}^\infty \quad \text{forms an orthonormal basis in } L^2_\sharp(Y). \end{cases}$$

In the literature, $\{\lambda_m(\eta)\}_{m\ge1}$ are referred to as Bloch eigenvalues and $\{\phi_m(\cdot;\eta)\}_{m\ge1}$ as Bloch eigenvectors or Bloch waves.

With the help of the above parametrized eigenvalues, one can describe the spectral resolution of A as an unbounded self-adjoint operator in $L^2(\mathbb{R}^N)$. Roughly, the results are as follows:

$$\{e^{i\eta\cdot y}\phi_m(y;\eta) \mid m \ge 1,\ \eta \in Y'\} \quad \text{forms a basis of } L^2(\mathbb{R}^N) \text{ in a generalized sense,}$$

and $L^2(\mathbb{R}^N)$ can be identified with $L^2(Y', \ell^2(\mathbb{N}))$ via Parseval's identity. The operator A corresponds to an operator with multipliers $\{\lambda_m(\eta)\}_{m\ge1}$:

$$A\left(e^{i\eta\cdot y}\phi_m(y;\eta)\right) = \lambda_m(\eta)e^{i\eta\cdot y}\phi_m(y;\eta).$$

This is the essence of the following result, a proof of which can be found in [B-L-P, 1978].

THEOREM 2.1. *Let $g \in L^2(\mathbb{R}^N)$. The mth Bloch coefficient of g is defined as follows*:

$$\hat{g}_m(\eta) = \int_{\mathbb{R}^N} g(y)e^{-i\eta\cdot y}\bar{\phi}_m(y;\eta)dy \quad \forall m \ge 1,\ \eta \in Y'.$$

Then the following inverse formula holds:

$$g(y) = \int_{Y'} \sum_{m=1}^\infty \hat{g}_m(\eta)e^{i\eta\cdot y}\phi_m(y;\eta)d\eta.$$

Further, we have Parseval's identity:

$$\int_{\mathbb{R}^N} |g(y)|^2 dy = \int_{Y'} \sum_{m=1}^\infty |\hat{g}_m(\eta)|^2 d\eta. \qquad \square$$

2.1. Regularity of the first Bloch eigenvalue. In spite of the fact that $A(\eta)$ depends polynomially in η, it is well known that the eigenvalues $\lambda_m(\eta)$ are not, in general, smooth functions of $\eta \in Y'$ because of the possible change in the multiplicity (see the book Kato (1966)). Singularities exhibited by λ_m and ϕ_m indeed have physical significance. Qualitative properties of eigenvalues are often difficult to prove because η is a vector. The traditional perturbation arguments found, for instance in Kato's book, are not sufficient.

Let us start with a remark that one can always add a constant term $a_0 > 0$ to the operator A; the effect is that the eigenvalues $\lambda_m(\eta)$ are changed to $\lambda_m(\eta) + a_0$ whereas eigenvectors ϕ_m remain unchanged. Thus the regularity properties of the spectrum remain unaffected. The advantage of this trick is the following: let us recall that the space in which the Bloch spectral problem (2.7) was resolved is

$$\begin{cases} H^1_\sharp(Y) = \{v \in H^1_{loc}(\mathbb{R}^N) \mid v \text{ is } Y\text{-periodic}\} \quad \text{if} \quad \eta \neq 0, \\ H^1_\sharp(Y)/\mathbb{C} \quad \text{if} \quad \eta = 0. \end{cases}$$

Thus there is a discontinuity in the choice of the space at $\eta = 0$. This poses certain difficulties in the analysis which will disappear if we add $a_0 > 0$ to A. Indeed, in that case, we can work with the space $H^1_\sharp(Y)$ for all $\eta \in Y'$.

The first result which uses min-max characterization of eigenvalues is the following.

THEOREM 2.2. *For all $m \geq 1$, $\lambda_m(\eta)$ is a Lipschitz function of η.*

Proof. Let us recall the quadratic form associated with $A(\eta)$:

$$a(v, v; \eta) = \int_Y a_{k\ell}(y) \left(\frac{\partial v}{\partial y_\ell} + i\eta_\ell v \right) \overline{\left(\frac{\partial v}{\partial y_k} + i\eta_k v \right)} \, dy.$$

We notice that this can be decomposed as follows:

$$a(v, v; \eta) = a(v, v; \eta') + R(v, v; \eta, \eta'),$$

where

$$R = \int_Y a_{k\ell}(y) \frac{\partial v}{\partial y_\ell} \overline{(i\eta_k - i\eta'_k)v} dy + \int_Y a_{k\ell}(y)(i\eta_\ell - i\eta'_\ell) v \overline{\frac{\partial v}{\partial y_k}} dy \\ + \int_Y a_{k\ell}(y)(\eta_\ell \eta_k - \eta'_\ell \eta'_k)|v|^2 dy.$$

By Cauchy–Schwarz's inequality, R can be estimated. We have

$$|R| \leq c|\eta - \eta'| \int_Y \left(|\nabla v|^2 + |v|^2 \right) dy.$$

We are now in a position to use the min-max principle:

$$\lambda_m(\eta) = \min_{dim F = m} \max_{v \in F} \frac{a(v, v; \eta)}{(v, v)}.$$

Here F is an m-dimensional subspace of $H^1_\sharp(Y)$ and $(\cdot, \cdot)$ stands for the usual scalar product in $L^2_\sharp(Y)$. Using the above estimate on R, we deduce that

$$\lambda_m(\eta) \leq \lambda_m(\eta') + c|\eta - \eta'|$$

for a suitable constant c. Interchanging η and η', we get finally that

$$|\lambda_m(\eta) - \lambda_m(\eta')| \leq c|\eta - \eta'|. \qquad \square$$

Unfortunately the above regularity result is not enough for homogenization. Finer properties of the eigenvalues were obtained by Wilcox (1978) in the case of the Schrödinger equation in $\mathbb{R}^3$ with a periodic potential. He shows that $\lambda_m(\eta)$ is analytic except for a closed null set in Y'. There is no mention about the actual location of singularities. Wilcox uses the explicit expression of the fundamental solution of $(-\Delta + \gamma_0)$ in $\mathbb{R}^3$, and his proof is based on the notion of Fredholm determinants. This seems to be complicated to generalize to the present situation.

We use here another notion of determinant to prove that the first eigenvalue $\lambda_1(\eta)$ is analytic near $\eta = 0$. The crucial property to be exploited here is that when $\eta = 0$ the eigenvalue $\lambda_1(0)$ (which is in fact equal to 0) is simple both algebraically and geometrically. The corresponding eigenspace is $\mathbb{C}$. As in the finite-dimensional case, the idea is to obtain the eigenvalues as roots of a "characteristic function" of $A(\eta)$. To produce this function, denoted by $D(\eta, \lambda)$, we use a notion of determinant which is briefly recalled here along with some of its properties. This is an important tool in the theory of operators; for details, see Reed and Simon (1972–78) Vol. IV, Dunford and Schwartz (1964) Part III or Gohberg and Krein (1969).

Let T be a compact operator in a Hilbert space H. We suppose that $T \in \mathcal{I}_n$ for some $n \in \mathbb{N}$. Let us recall that $\mathcal{I}_n$ is the class of compact operators T whose sequence of singular values is in ℓ^n, the space of sequences which are nth power absolutely summable. ($\mathcal{I}_n$ was originally introduced by von Neumann and Schatten as a noncommutative version of ℓ^n.) Then $\det_n(I - T)$ is defined by

$$\det_n(I - T) = \prod_{j=1}^{\infty} (1 - \mu_j) \exp\left\{\sum_{k=1}^{n-1} \frac{1}{k}\mu_j^k\right\},$$

where $\{\mu_j\}_j$ is a listing of the nonzero elements of $\sigma(T)$, the spectrum of T. We know that each μ_j is an eigenvalue of T with finite algebraic multiplicity. We repeat it in the above listing a number of times equal to its algebraic multiplicity. Further, the listing places the eigenvalues in decreasing order in modulus $|\mu_{j+1}| \leq |\mu_j|$. In case $|\mu_{j+1}| = |\mu_j|$, then it is required that $\arg \mu_{j+1} \geq \arg \mu_j$. The agile reader will see the analogy between this definition and the Weierstrass' construction of an entire function with prescribed zeroes $\{\frac{1}{\mu_j}\}$ by means of canonical products (see Rudin (1979)).

In the following result, we collect some of the essential properties of the determinant which are highly nontrivial. For proofs, the reader is referred to the literature cited above.

THEOREM 2.3.

(i) *If $T \in \mathcal{I}_n$, the product in $\det_n(I - T)$ is convergent and it defines an analytic function on $\mathcal{I}_n$ with values in $\mathbb{C}$.*

(ii) *If $T \in \mathcal{I}_n$ then $(I - T)$ is not invertible iff $\det_n(I - T) = 0$.*

(iii) *The map $T \mapsto \det_n(I - T)(I - T)^{-1}$, which is a priori defined for $T \in \mathcal{I}_n$ such that $1 \notin \sigma(T)$, can in fact be extended to the whole space $\mathcal{I}_n$ and is analytic on $\mathcal{I}_n$ with values in $\mathcal{L}(H)$.* $\square$

COROLLARY 2.4. *If $\lambda \mapsto T(\lambda)$ is an analytic map defined on an open set of $\mathbb{C}$ with values in the Banach space $\mathcal{I}_n$, then $D(\lambda) = \det_n(I - T(\lambda))$ is an analytic function with values in $\mathbb{C}$. Further λ_0 is a zero of $D(\lambda)$ iff $1 \in \sigma(T(\lambda_0))$. Moreover, the algebraic multiplicity of λ_0 as a zero of $D(\lambda)$ is the same as the algebraic multiplicity of one as an eigenvalue of $T(\lambda_0)$.* $\square$

In order to apply the above result to our situation, we work with the Hilbert space $H^{-1}_\sharp(Y)$ which is defined to be the dual space of $H^1_\sharp(Y)$, identifying the dual of $L^2_\sharp(Y)$ with itself. We recall briefly some essential properties of this space to make the exposition self-contained. Since smooth elements of $H^1_\sharp(Y)$ form a dense subspace, we conclude that $H^{-1}_\sharp(Y)$ can be identified with a subspace of $\mathcal{D}'_\sharp(Y)$, the space of Y-periodic distributions on $\mathbb{R}^N$ (see Schwartz (1966)). Further, the space $H^{-1}_\sharp(Y)$ can be represented as follows:

$$H^{-1}_\sharp(Y) = \left\{ T = f_0 + \sum_{j=1}^N \frac{\partial f_j}{\partial y_j} \mid f_0, f_j \in L^2_\sharp(Y),\ 1 \le j \le N \right\}.$$

For reasons mentioned earlier, we consider the operator $A_0 = A + a_0 I$ where $a_0 > 0$ is an arbitrary constant. We wish to make it a self-adjoint unbounded operator in $H^{-1}_\sharp(Y)$ with domain $H^1_\sharp(Y)$ by defining a suitable inner product in $H^{-1}_\sharp(Y)$. To this end, let us introduce eigenvalues and eigenvectors of A_0 in Y with periodic boundary conditions:

$$\begin{cases} A_0 w_m = \nu_m w_m, & w_m \in H^1_\sharp(Y), \\ (w_m, w_n) = \delta_{mn} & \forall m, n \ge 1. \end{cases} \tag{2.9}$$

Here the eigenvalues are indexed in the increasing order: $0 < \nu_1 < \nu_2 \le \nu_3 \le \cdots \to \infty$. The Fourier coefficients of an element $T \in H^{-1}_\sharp(Y)$ can be defined as follows:

$$\hat{T}_m = \langle T, w_m \rangle \quad \forall m \ge 1,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1}_\sharp(Y)$ and $H^1_\sharp(Y)$ which extends the scalar product $(\cdot, \cdot)$ in $L^2_\sharp(Y)$. We are now in a position to define the scalar product in $H^{-1}_\sharp(Y)$. Define for $T, T' \in H^{-1}_\sharp(Y)$,

$$[T, T'] = \sum_{m=1}^\infty \nu_m^{-1} \hat{T}_m \bar{\hat{T}}'_m.$$

It is easily checked that the above defines a scalar product on $H^{-1}_\sharp(Y)$ which makes it a Hilbert space. Further, the norm induced by $[\cdot, \cdot]$ is equivalent to the norm on $H^{-1}_\sharp(Y)$ as a dual space of $H^1_\sharp(Y)$. For $T \in H^1_\sharp(Y)$, we have $(\widehat{A_0 T})_m = \langle A_0 T, w_m \rangle = (T, A_0 w_m) = \nu_m \hat{T}_m$ owing to the symmetry of $(a_{k\ell})$. Using this fact, we can easily check that A_0 is a self-adjoint operator with respect to $[\cdot, \cdot]$. Moreover, the Lax–Milgram lemma implies that A_0 is invertible, and the inverse A_0^{-1} defines a bounded operator between $H^{-1}_\sharp(Y)$ and $H^1_\sharp(Y)$.

After all these preliminaries on $H^{-1}_\sharp(Y)$, let us turn our attention to the spectrum of the shifted operator $A_0(\eta)$ associated with A_0. We can split this operator as follows:

$$A_0(\eta) = A_0 + \eta_k B_k + \eta_\ell C_\ell + \eta_k \eta_\ell a_{k\ell},$$

where

$$B_k v = -i a_{k\ell}(y) \frac{\partial v}{\partial y_\ell}, \quad C_\ell v = -i \frac{\partial}{\partial y_k}(a_{k\ell} v).$$

Just as A_0, we consider these operators as unbounded operators with appropriate domains in $H_\sharp^{-1}(Y)$. For λ in the resolvent set of $A_0(\eta)$, we can write

$$A_0(\eta) - \lambda = (I + \eta_k B_k A_0^{-1} + \eta_\ell C_\ell A_0^{-1} + \eta_k \eta_\ell a_{k\ell} A_0^{-1} - \lambda A_0^{-1}) A_0,$$

$$(A_0(\eta) - \lambda)^{-1} = A_0^{-1}(I - T)^{-1}, \tag{2.10}$$

where

$$T = T(\eta, \lambda) \stackrel{\text{def}}{=} -\eta_k B_k A_0^{-1} - \eta_\ell C_\ell A_0^{-1} - \eta_k \eta_\ell a_{k\ell} A_0^{-1} + \lambda A_0^{-1}.$$

The crucial property of T required is given in the following proposition.

PROPOSITION 2.5. *The operators $B_k A_0^{-1}$, $C_\ell A_0^{-1}$, $a_{k\ell} A_0^{-1}$, A_0^{-1} (and hence T) are all in $\mathcal{I}_n$ for some n large enough.* ∎

We will take up the proof of Proposition 2.5 later. For the moment, let us use it to get the "characteristic function" of $A_0(\eta)$.

THEOREM 2.6. *There is an analytic function $D = D(\eta, \lambda)$ defined for $\eta \in \mathbb{R}^N$ and $\lambda \in \mathbb{C}$ such that for $\eta \in \mathbb{R}^N$ fixed, the zeros of $D(\eta, \lambda)$ are precisely $\{a_0 + \lambda_m(\eta)\}_{m=1}^{\infty}$.*

Proof. Indeed we take $D(\eta, \lambda) = \det_n(I - T(\eta, \lambda))$ where

$$T(\eta, \lambda) \stackrel{\text{def}}{=} -\eta_k B_k A_0^{-1} - \eta_\ell C_\ell A_0^{-1} - \eta_k \eta_\ell a_{k\ell} A_0^{-1} + \lambda A_0^{-1}.$$

Since $T(\eta, \lambda) \in \mathcal{I}_n$, $D(\eta, \lambda)$ is a well-defined analytic function defined for $\eta \in \mathbb{R}^N$ and $\lambda \in \mathbb{C}$. Next, we appeal to Corollary 2.4 to say that for $\eta \in \mathbb{R}^N$ fixed, λ_0 is a zero of $D(\eta, \lambda)$ iff $1 \in \sigma(T(\eta, \lambda_0))$ iff $(I - T)$ is not invertible. Since we have the relation $(A_0(\eta) - \lambda)^{-1} = A_0^{-1}(I - T)^{-1}$ valid in the resolvent, we conclude that λ_0 is a zero of $D(\eta, \lambda)$ iff $(A_0(\eta) - \lambda_0)^{-1}$ does not exist; i.e., λ_0 is an eigenvalue of $A_0(\eta)$ and hence $\lambda_0 = a_0 + \lambda_m(\eta)$ for some $m \in \mathbb{N}$. ∎

2.2. Proof of Proposition 2.5. *First step*. Let us take the more difficult case, namely the operator $C_\ell A_0^{-1}$. We will show that $C_\ell A_0^{-1} \in \mathcal{I}_n$ for some n. First, we can say that $C_\ell A_0^{-1} \in \mathcal{L}(H_\sharp^{-1}(Y))$. Indeed, we have for $f \in H_\sharp^{-1}(Y)$,

$$\|C_\ell A_0^{-1} f\|_{H_\sharp^{-1}(Y)} = \|C_\ell u\|_{H_\sharp^{-1}(Y)} \le \|a_{k\ell} u\|_{L_\sharp^2(Y)} \le c\|u\|_{L_\sharp^2(Y)} \le c\|f\|_{H_\sharp^{-1}(Y)},$$

where we have used $u = A_0^{-1} f$. The constant c depends on $\|a_{k\ell}\|_{L^\infty(Y)}$. The operator $C_\ell A_0^{-1}$ is compact since A_0^{-1} is compact from $H_\sharp^{-1}(Y)$ into $L_\sharp^2(Y)$.

Let us now consider the singular values $\{s_j\}$ of $C_\ell A_0^{-1}$ arranged in the decreasing order. By the very definition of $\mathcal{I}_n$, we will have $C_\ell A_0^{-1} \in \mathcal{I}_n$ if we show that $\{s_j\} \in \ell^n$ (see Reed and Simon (1972–78) Vol. II). Thus the problem is to estimate s_j. For this, we use the min-max characterization of singular values:

$$s_j = \min_{dim F = j-1} \max_{f \perp F} \frac{\|C_\ell A_0^{-1} f\|_{H_\sharp^{-1}(Y)}}{\|f\|_{H_\sharp^{-1}(Y)}},$$

where F is a $(j-1)$-dimensional subspace of $H_\sharp^{-1}(Y)$. We already remarked that

$$\|C_\ell A_0^{-1} f\|^2_{H_\sharp^{-1}(Y)} \le c\|u\|^2_{L_\sharp^2(Y)} \quad \text{with} \quad A_0 u = f.$$

To estimate s_j, we use the eigenvalues and eigenvectors of A_0 in Y with the periodic boundary conditions introduced in (2.9). Then the equation $A_0 u = f$ is equivalent to

$$\nu_m \hat{u}_m = \hat{f}_m \quad \text{with} \quad \hat{u}_m = (u, w_m) \quad \text{and} \quad \hat{f}_m = (f, w_m).$$

We take $F = \langle w_1, ..., w_{j-1} \rangle$ as a test space in the min-max characterization given above. The condition $f \perp F$ is equivalent to $\hat{f}_m = 0$, $m = 1, ..., j-1$. Thus $\hat{u}_m = 0$ for $m = 1, ..., j-1$. Therefore,

$$\frac{\|C_\ell A_0^{-1} f\|^2_{H_\sharp^{-1}(Y)}}{\|f\|^2_{H_\sharp^{-1}(Y)}} \leq c \frac{\sum_{m=j}^{\infty} |\hat{u}_m|^2}{\sum_{m=j}^{\infty} \nu_m |\hat{u}_m|^2} \leq \frac{c}{\nu_j},$$

since we have arranged $\{\nu_m\}$ in the increasing order.

Thus we have established that

$$s_j \leq \frac{c}{\nu_j} \quad \forall j.$$

Second step. This is somewhat classical. Min-max characterization applied to $\{\nu_j\}$ shows that ν_j is bounded from below and from above by the jth eigenvalue of $(-\Delta + I)$ with periodic boundary conditions. If we replace periodic boundary condition by Neumann boundary condition on ∂Y, we decrease the value. Thus we get $\nu_j \geq c\nu'_j$, where ν'_j is the jth eigenvalue of the following problem:

$$\begin{cases} -\Delta u + u = \nu' u & \text{in} \quad Y, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial Y. \end{cases}$$

Weyl's asymptotic formula then shows that $\nu'_j \approx j^s$ for some $s > 0$. (More precisely, we can take $s = \frac{2}{N}$.)

Third step. Combining the results of the previous two steps, we get the following estimate on the singular values which shows algebraic decay:

$$s_j \leq c j^{-s} \quad \text{for some} \quad s > 0.$$

This implies, of course, that $\{s_j\} \in \ell^n$ for $n > \frac{1}{s}$.

Fourth step. The claim that $B_k A_0^{-1}$, $a_{k\ell} A_0^{-1}$, and A_0^{-1} belong to some $\mathcal{I}_n$ can be proved in a manner analogous to the earlier steps. ∎

As an immediate consequence of Theorem 2.6, we deduce the following regularity of the first Bloch eigenvalue.

THEOREM 2.7. *We assume that $a_{k\ell}$ satisfy* (2.2). *Then in a suitable neighborhood U of the origin, the first eigenvalue $\lambda_1(\eta)$ remains simple geometrically and it defines an analytic function of $\eta \in U$.*

Proof. As remarked already, when $\eta = 0$, the first eigenvalue $\lambda_1(0) = 0$ is simple geometrically since the eigenvector $\phi_1(y; 0) \equiv$ constant. Since A is a self-adjoint operator in $H_\sharp^{-1}(Y)$, $\lambda_1(0) = 0$ is simple algebraically also. Hence the first eigenvalue of $A_0 = A + a_0 I$, which is a_0, is also algebraically simple.

Let us now recall the relation

$$A_0(\eta) - \lambda I = (I - T(\eta, \lambda)) A_0. \tag{2.11}$$

When $\eta = 0$ and $\lambda = a_0$, the above relation reads as $A = (I - a_0 A_0^{-1})A_0 = A_0(I - a_0 A_0^{-1})$. Since A_0 is invertible, it follows that $\text{Ker}A^m = \text{Ker}(I - a_0 A_0^{-1})^m$ for all $m \geq 1$. We conclude therefore that 1 is an algebraically simple eigenvalue of $a_0 A_0^{-1}$ $(= T(0, a_0))$.

We are now in a position to apply Corollary 2.4 and see that $D(0, \lambda)$ has a simple root at $\lambda = a_0$, so

$$\frac{\partial D}{\partial \lambda}(0, a_0) \neq 0.$$

This implies, via *implicit function theorem* in the analytic category, that there exists a unique analytic function $\eta \mapsto \lambda(\eta)$ defined in a neighborhood of $\eta = 0$ such that $D(\eta, \lambda(\eta)) = 0$ and $\lambda(0) = a_0$. Thanks to Corollary 2.4 and the relation (2.11), this function must be of the form $\lambda(\eta) = a_0 + \lambda_1(\eta)$, where $\lambda_1(\eta)$ is the first eigenvalue of $A(\eta)$. This establishes the analyticity of $\lambda_1(\eta)$. Its geometrical simplicity follows from standard continuity arguments. ∎

Remark. The above proof shows that $1 \in \sigma(T(\eta, \lambda(\eta)))$ is an algebraically simple eigenvalue. However, we are not able to conclude that $\lambda(\eta) \in \sigma(A_0(\eta))$ is algebraically simple using (2.11). This is due to noncommutativity of the operators $T(\eta, \lambda)$ and A_0. (However, as observed in the above proof, when $\eta = 0$ and $\lambda = a_0$, these operators commute and so the above difficulty disappears.) The fact that $\lambda(\eta) \in \sigma(A_0(\eta))$ is geometrically simple is all that is required in asserting the analyticity of the first Bloch eigenvector $\phi_1(\cdot; \eta)$ (see the next paragraph).

2.3. Regularity of the first Bloch eigenvector. As will be evident from the analysis that follows, for the homogenization process we need the first eigenvector $\phi_1(y; \eta)$ to have the regularity that $\frac{\partial \phi_1}{\partial \eta}$ and $\frac{\partial^2 \phi_1}{\partial \eta^2}$ are in $L^\infty(U; L^2_\sharp(Y))$ where U is a small neighborhood of the origin in the η-space (see the proofs of Proposition 3.6 and Theorem 3.1). Since the operator $A(\eta)$ depends analytically on η, and the first eigenvalue $\lambda_1(\eta)$ does not change geometrical multiplicity (in fact, it is simple) and remains analytic with respect to η in a small neighborhood U of the origin, the choice of the first eigenvector can be made so that it depends analytically on $\eta \in U$. The easiest way to prove this is perhaps to apply some general considerations in the theory of vector bundles. We will not give any details of this theory, and we refer the reader to the literature; e.g., Dieudonné (1974) or Simon (1979). Of course, the difficulty of making such a choice is clear. Even if we fix the norm of the eigenvector which is complex valued, there is always a phase factor which is still arbitrary.

As before, we work with $A_0(\eta) = A(\eta) + a_0 I$, where $a_0 > 0$ is a constant. The eigenvectors of $A(\eta)$ and $A_0(\eta)$ are the same. We introduce the Green operator:

$$S(\eta) = A_0(\eta)^{-1} - \mu_1(\eta) I$$

with $\mu_1(\eta) = (a_0 + \lambda_1(\eta))^{-1}$.

First of all, let us remark that $A_0(\eta)^{-1}$ exists as a bounded operator on $H^{-1}_\sharp(Y)$ when η is near the origin. We are interested in $\text{Ker}[S(\eta)]$ which coincides with the first eigenspace and so is of dimension one. It will be advantageous for us to consider $S(\eta)$ as an operator acting on $L^2_\sharp(Y)$. In doing so, we are not changing $\text{Ker}[S(\eta)]$ as $\text{Ker}[S(\eta)]$ is a subspace of $L^2_\sharp(Y)$ (in fact $H^1_\sharp(Y)$). For the same reason, $A_0(\eta)^{-1}$ is compact in $L^2_\sharp(Y)$ and so $S(\eta)$, being a compact perturbation of the identity, is Fredholm. Further, $S(\eta)$ depends analytically on η in a neighborhood U of the origin since $\lambda_1(\eta)$ is already proved to be analytic.

This information is sufficient to assert that $\bigcup_{\eta\in U}\mathrm{Ker}[S(\eta)]$ forms an analytic subbundle of the trivial bundle $U\times L^2_\sharp(Y)$ over U (see Dieudonné (1974), p. 119). In particular, there are analytic sections over U (Dieudonné (1974), p. 102). This means that there is a choice of the first eigenvector $\phi_1(y;\eta)$ such that

$$\begin{cases} \eta \longmapsto \phi_1(\cdot;\eta)\in L^2_\sharp(Y) \quad \text{is analytic on } U,\\ \phi_1(y;0)=|Y|^{-\frac{1}{2}}. \end{cases}$$

Let us note that the constant $|Y|^{-\frac{1}{2}}$ is dictated by the normalization condition that the norm in $L^2_\sharp(Y)$ is unity.

3. Homogenization results. As an application of the regularity of the first Bloch eigenvalue and eigenvector, we are going to deduce a classical homogenization result in arbitrary domains. To announce the result, let us consider the coefficients $a_{k\ell}=a_{k\ell}(y)$ which are Y-periodic functions satisfying the hypotheses of section 2 (see (2.2)). Associated with these coefficients is the operator A defined in (2.1). For each $\varepsilon>0$, we consider also the operator A^ε, where

$$A^\varepsilon \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(a^\varepsilon_{k\ell}(x)\frac{\partial}{\partial x_\ell}\right) \quad \text{with} \quad a^\varepsilon_{k\ell}(x)=a_{k\ell}\left(\frac{x}{\varepsilon}\right).$$

From the theory of homogenization (see [B-L-P, 1978]), it is known that there is a corresponding homogenized operator A^* given by

$$A^* \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(q_{k\ell}\frac{\partial}{\partial x_\ell}\right).$$

The homogenized coefficients $q_{k\ell}$ are constants, and their definition can be found in the book [B-L-P, 1978, p. 17]; we will recall it later (see (3.5)). It is known that $(q_{k\ell})$ is a symmetric positive definite matrix:

$$q_{k\ell}\xi_k\xi_\ell \geq \alpha|\xi|^2 \quad \forall\xi\in\mathbb{R}^N,$$

where $\alpha>0$ is the same constant appearing in (2.2).

For the moment, let us announce our principal result.

THEOREM 3.1. *Let Ω be an arbitrary domain in $\mathbb{R}^N$. Let the coefficients $a_{k\ell}$ satisfy assumptions* (2.2). *Suppose $\{u^\varepsilon\}$ is a sequence in $H^1(\Omega)$ and $u^*\in H^1(\Omega)$, $f\in L^2(\Omega)$ are such that*

$$\begin{cases} u^\varepsilon \rightharpoonup u^* \quad in \quad H^1(\Omega)\text{-}weak,\\ A^\varepsilon u^\varepsilon = f \quad in \quad \Omega. \end{cases}$$

Then the stress vector $\sigma^\varepsilon_k \stackrel{\text{def}}{=} a^\varepsilon_{k\ell}\frac{\partial u^\varepsilon}{\partial x_\ell}$ converges in $L^2(\Omega)$ weak to the corresponding homogenized stress vector:

$$\sigma^\varepsilon_k \rightharpoonup q_{k\ell}\frac{\partial u^*}{\partial x_\ell} \quad in \quad L^2(\Omega)\text{-}weak\ \forall k=1,...,N.$$

In particular, u^* satisfies the homogenized equation; namely,

$$A^*u^* = f \quad in \quad \Omega. \qquad \square$$

In the above theorem, we have assumed the weak convergence of $\{u^\varepsilon\}$. This is because H^1-bound on $\{u^\varepsilon\}$ is not guaranteed. However, if Ω is a bounded domain and u^ε satisfies certain boundary conditions (e.g., Dirichlet) on the boundary $\partial\Omega$ in addition to the equation $A^\varepsilon u^\varepsilon = f$ in Ω, then the H^1-bound on $\{u^\varepsilon\}$ will be a consequence of ellipticity and the Poincaré inequality. In case Ω is unbounded, say $\Omega = \mathbb{R}^N$, then we do not have estimate on $\{u^\varepsilon\}$ in $H^1(\mathbb{R}^N)$ because the Poincaré inequality is not available. However, if we consider $A^\varepsilon + I$ instead of A^ε, then the bound in $H^1(\mathbb{R}^N)$ is automatic. In these cases, we would be able to deduce the usual homogenization results. For the sake of completeness, we announce them separately.

COROLLARY 3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$. Let $(a_{k\ell})$ and f be as in Theorem* 3.1. *Consider v^ε the unique solution of*

$$A^\varepsilon v^\varepsilon = f \quad in \quad \Omega, \quad v^\varepsilon \in H_0^1(\Omega).$$

Then

$$\begin{aligned} v^\varepsilon &\rightharpoonup v^* \quad in \quad H_0^1(\Omega)\text{-}weak, \\ a_{k\ell}^\varepsilon \frac{\partial v^\varepsilon}{\partial x_\ell} &\rightharpoonup q_{k\ell} \frac{\partial v^*}{\partial x_\ell} \quad in \quad L^2(\Omega)\text{-}weak \quad \forall k = 1, ..., N, \end{aligned}$$

where v^* is the unique solution satisfying

$$A^* v^* = f \quad in \quad \Omega, \quad v^* \in H_0^1(\Omega). \qquad \square$$

COROLLARY 3.3. *Let $(a_{k\ell})$ and f be as in Theorem* 3.1. *Consider w^ε the unique solution of*

$$A^\varepsilon w^\varepsilon + w^\varepsilon = f \quad in \quad \mathbb{R}^N, \quad w^\varepsilon \in H^1(\mathbb{R}^N).$$

Then

$$\begin{aligned} w^\varepsilon &\rightharpoonup w^* \quad in \quad H^1(\mathbb{R}^N)\text{-}weak, \\ a_{k\ell}^\varepsilon \frac{\partial w^\varepsilon}{\partial x_\ell} &\rightharpoonup q_{k\ell} \frac{\partial w^*}{\partial x_\ell} \quad in \quad L^2(\mathbb{R}^N)\text{-}weak \quad \forall k = 1, ..., N, \end{aligned}$$

where w^* is the unique solution of

$$A^* w^* + w^* = f \quad in \quad \mathbb{R}^N, \quad w^* \in H^1(\mathbb{R}^N). \qquad \square$$

These results have been proved by L. Tartar using his method of homogenization, and a proof is available in [B-L-P, 1978]. His proof handles even nonsymmetric coefficients $(a_{k\ell})$. We are going to re-prove Theorem 3.1 using what we call the *Bloch-wave method*. Since it is a spectral method, we are naturally led to suppose the symmetry of the coefficients $(a_{k\ell})$.

Our plan to prove Theorem 3.1 is as follows: in section 3.1, we introduce Bloch waves at ε-scale and Bloch transforms and we analyze their behavior as $\varepsilon \to 0$. The approach to homogenized medium will be very transparent in this analysis. Though it is not strictly rigorous, it will be instructive to consider the special case where $\Omega = \mathbb{R}^N$. The differential equation $A^\varepsilon u^\varepsilon = f$ in $\mathbb{R}^N$ can be easily transformed to a set of algebraic equations for the Bloch transforms (see equation (3.2)). We show next that the energy of u^ε contained in all Bloch modes except the first one goes to zero (Proposition 3.5). Thus only the first Bloch mode is excited in the homogenization process and all higher modes can be neglected. Heuristically speaking, the higher modes admit dependence on x/ε, and in view of our hypothesis that $\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^N)^N} \le$

c, we see that the energy of u^ε contained in these modes must decay. Our next aim is to pass to the limit in equation (3.2) corresponding to the first Bloch mode. We prove that the first Bloch transform tends to the usual Fourier transform in a suitable sense (Proposition 3.6). Thus we have the picture that the Bloch waves representing periodic medium tend to Fourier waves representing homogeneous medium. This can be interpreted as a result of homogenization in the Fourier space. We deduce this as a consequence of the fact that we can choose the first Bloch eigenvector $\phi_1(y;\eta)$ in a smooth way (with the condition $\phi_1(y;0) \equiv |Y|^{-\frac{1}{2}}$) when η is near the origin. All we need for the proof of Proposition 3.6 is that $\eta \to \phi_1(\cdot,\eta)$ is a Lipschitz map with values in $L^2_\sharp(Y)$ when η is in a neighborhood of the origin.

The passage to the limit in equation (3.2) requires not only the smoothness of the first Bloch mode but also that of the first Bloch eigenvalue (which has been established in section 2). The negative powers of ε, alluded to in the Introduction, pose a problem. In order to overcome this, we use Taylor expansion of the first Bloch eigenvalue $\lambda_1(\eta)$. We know already that $\lambda_1(0) = 0$. To compensate the negative powers of ε and have a finite limit as $\varepsilon \to 0$, we need to show that $\lambda_1'(0)$ also vanishes. We do this in Proposition 3.7. Once done, this shows (not rigorously though) that the homogenized matrix obtained in this method is nothing but $\frac{1}{2}\lambda_1''(0)$, i.e., $\frac{1}{2}$ times the Hessien matrix of λ_1 at the origin, which is already shown to be a critical point for λ_1. Another purpose of Proposition 3.7 is to calculate the Hessien of λ_1 and identify the homogenized matrix obtained in this method with the one obtained via classical means in [B-L-P, 1978].

In order to make the passage to the limit in (3.2) more rigorous, we must localize the equation $A^\varepsilon u^\varepsilon = f$ in Ω by means of a cut-off function, as demanded by Proposition 3.6. This reduces the problem in Ω to another one in $\mathbb{R}^N$ for which our foregoing arguments apply. The details are presented in section 3.3.

3.1. Preliminaries for the proof of Theorem 3.1. The first step is to consider the case $\Omega = \mathbb{R}^N$ and express the equation $A^\varepsilon u^\varepsilon = f$ in $\mathbb{R}^N$ in an equivalent way in terms of the Bloch coefficients of u^ε and f. In order to do this, we introduce Bloch eigenvalues $\{\lambda_m^\varepsilon(\xi)\}_{m=1}^\infty$ and eigenvectors $\{\phi_m^\varepsilon(x;\xi)\}_{m=1}^\infty$ in the ε-scale which will diagonalize the operator A^ε in $L^2(\mathbb{R}^N)$ (analogous to what we did in section 2). By homothety, there exist the following relations:

$$\lambda_m^\varepsilon(\xi) = \varepsilon^{-2}\lambda_m(\eta), \quad \phi_m^\varepsilon(x;\xi) = \phi_m(y;\eta),$$

where $\lambda_m(\eta)$, $\phi_m(y;\eta)$ are already introduced in section 2, and (x,ξ) and (y,η) are related by

$$y = \frac{x}{\varepsilon}, \quad \eta = \varepsilon\xi.$$

Recall that $y \in Y = [0,2\pi[^N$ and $\eta \in Y' =]-\frac{1}{2},\frac{1}{2}[^N$. Hence $\xi \in \varepsilon^{-1}Y' = [-\frac{\varepsilon^{-1}}{2},\frac{\varepsilon^{-1}}{2}[^N$. The following fundamental result regarding the Bloch waves is proved in [B-L-P, 1978].

THEOREM 3.4. *Let $g \in L^2(\mathbb{R}^N)$ be arbitrary. One defines the Bloch coefficients of g as follows: let $m \in \mathbb{N}$, $\xi \in \varepsilon^{-1}Y'$ be given. The mth Bloch coefficient of g is defined by*

$$\hat{g}_m^\varepsilon(\xi) = \varepsilon^{-\frac{N}{2}} \int_{\mathbb{R}^N} g(x)e^{-i\xi\cdot x}\bar{\phi}_m^\varepsilon(x;\xi)dx. \tag{3.1}$$

Then the following inverse formula holds:

$$g(x) = \varepsilon^{\frac{N}{2}} \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \hat{g}_m^{\varepsilon}(\xi) e^{i\xi\cdot x} \phi_m^{\varepsilon}(x;\xi) d\xi.$$

Further, we have Parseval's identity:

$$\varepsilon^{-N} \int_{\mathbb{R}^N} |g(x)|^2 dx = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} |\hat{g}_m^{\varepsilon}(\xi)|^2 d\xi.$$

More generally, the following Plancherel identity is also valid:

$$\varepsilon^{-N} \int_{\mathbb{R}^N} g(x)\bar{h}(x) dx = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \hat{g}_m^{\varepsilon}(\xi) \overline{\hat{h}_m^{\varepsilon}}(\xi) d\xi \quad \forall g, h \in L^2(\mathbb{R}^N). \qquad \square$$

Thanks to the above result and the relation

$$A^{\varepsilon}(e^{i\xi\cdot x}\phi_m^{\varepsilon}(x;\xi)) = \lambda_m^{\varepsilon}(\xi) e^{i\xi\cdot x} \phi_m^{\varepsilon}(x;\xi),$$

we see that our equation $A^{\varepsilon}u^{\varepsilon} = f$ in $\mathbb{R}^N$ is equivalent to

$$\hat{f}_m^{\varepsilon}(\xi) = \lambda_m^{\varepsilon}(\xi)\hat{u}_m^{\varepsilon}(\xi) \quad \forall m \geq 1 \ \forall \xi \in \varepsilon^{-1}Y'. \tag{3.2}$$

Our goal is to pass to the limit in these equations. The first assertion is that one can neglect all the equations corresponding to $m \geq 2$.

PROPOSITION 3.5. *Let*

$$v^{\varepsilon}(x) = \varepsilon^{\frac{N}{2}} \int_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} \hat{u}_m^{\varepsilon}(\xi) e^{i\xi\cdot x} \phi_m^{\varepsilon}(x;\xi) d\xi.$$

Then $\|v^{\varepsilon}\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon$.

Proof. We have

$$\int_{\mathbb{R}^N} A^{\varepsilon}u^{\varepsilon}\bar{u}^{\varepsilon} = \int_{\mathbb{R}^N} f\bar{u}^{\varepsilon}.$$

By Plancherel identity, we deduce that

$$\begin{aligned} c\int_{\mathbb{R}^N} |\nabla u^{\varepsilon}|^2 &\geq \varepsilon^N \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \hat{f}_m^{\varepsilon}(\xi)\overline{\hat{u}_m^{\varepsilon}(\xi)} d\xi \\ &= \varepsilon^N \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \lambda_m^{\varepsilon}(\xi)|\hat{u}_m^{\varepsilon}(\xi)|^2 d\xi. \end{aligned}$$

At this point, we use the following lower bound which exists for all eigenvalues $\lambda_m(\eta)$, $m \geq 2$. In fact, as a simple consequence of the min-max principle, one can prove

$$\lambda_m(\eta) \geq \lambda_2(\eta) \geq \lambda_2^{(N)} > 0 \quad \forall m \geq 2 \quad \forall \eta \in Y',$$

where $\lambda_2^{(N)}$ is the second eigenvalue of the eigenvalue problem for A in the cell Y with Neumann boundary condition on ∂Y. The positivity of $\lambda_2^{(N)}$ is a consequence of the fact that the first eigenvalue $\lambda_1^{(N)}$ is simple geometrically (and hence algebraically owing to the self-adjointness of the problem). Recalling that $\lambda_m^\varepsilon(\xi) = \varepsilon^{-2}\lambda_m(\eta)$, we arrive at

$$\varepsilon^N \int_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} |\hat{u}_m^\varepsilon(\xi)|^2 d\xi \leq c\varepsilon^2.$$

By Parseval's identity, the left side is equal to $\|v^\varepsilon\|^2_{L^2(\mathbb{R}^N)}$. This finishes the proof of Proposition 3.5. $\square$

With the aim of passing to the limit in (3.2) with $m = 1$, we now prove Proposition 3.6.

PROPOSITION 3.6. *Let $\{g^\varepsilon\}$ be a sequence in $L^2(\mathbb{R}^N)$, and let g be an element of $L^2(\mathbb{R}^N)$. Let us denote by $\hat{g}_1^\varepsilon$ the first Bloch transform of g^ε defined in* (3.1)*, and let $\hat{g}$ be the usual Fourier transform of g. The following statements hold true:*

(i) *If $g^\varepsilon \rightharpoonup g$ in $L^2(\mathbb{R}^N)$-weak then $\varepsilon^{\frac{N}{2}}\hat{g}_1^\varepsilon \rightharpoonup \hat{g}$ in $L^2_{loc}(\mathbb{R}^N)$-weak provided there is a fixed compact set K such that* $\operatorname{supp} g^\varepsilon \subseteq K$, $\forall\varepsilon$.

(ii) *If $g^\varepsilon \to g$ in $L^2(\mathbb{R}^N)$-strong then $\varepsilon^{\frac{N}{2}}\hat{g}_1^\varepsilon \to \hat{g}$ in $L^2_{loc}(\mathbb{R}^N)$-strong.*

Proof. It is understood that $\hat{g}_1^\varepsilon(\xi)$, which is a priori defined for

$$\xi \in \varepsilon^{-1}Y' = \left]-\frac{\varepsilon^{-1}}{2}, \frac{\varepsilon^{-1}}{2}\right[^N,$$

is extended by zero outside $\varepsilon^{-1}Y'$. Let us start proving (i). We write

$$\text{(3.3)} \quad \begin{cases} \varepsilon^{\frac{N}{2}}\hat{g}_1^\varepsilon(\xi) = \displaystyle\int_{\mathbb{R}^N} g^\varepsilon(x)e^{-ix\cdot\xi}\bar{\phi}_1\left(\frac{x}{\varepsilon};0\right)dx \\ \qquad\qquad + \displaystyle\int_K g^\varepsilon(x)e^{-ix\cdot\xi}\left(\bar{\phi}_1\left(\frac{x}{\varepsilon};\varepsilon\xi\right) - \bar{\phi}_1\left(\frac{x}{\varepsilon};0\right)\right)dx. \end{cases}$$

Since $\phi_1(y;0) = |Y|^{-\frac{1}{2}} = (2\pi)^{-\frac{N}{2}}$, we see that the first term is nothing but the Fourier transform of g^ε and so it converges to $\hat{g}(\xi)$ in $L^2(\mathbb{R}^N)$-weak. Applying the Cauchy–Schwarz inequality, we can bound the second term from above by

$$\|g^\varepsilon\|_{L^2(\mathbb{R}^N)}\left[\int_K \left|\phi_1\left(\frac{x}{\varepsilon};\varepsilon\xi\right) - \phi_1\left(\frac{x}{\varepsilon};0\right)\right|^2 dx\right]^{\frac{1}{2}} \leq c\|\phi_1(y;\varepsilon\xi) - \phi_1(y;0)\|_{L^2(Y)}.$$

Here is where some regularity of the first Bloch mode $\eta \to \phi_1(\cdot,\eta) \in L^2_\sharp(Y)$ is required when η is near zero. Analyticity of this map is established in section 2.3. Here we use simply the fact that the above map is Lipschitz. Thus we conclude that the second term in the right side of (3.3) is bounded above by $c\varepsilon\xi$. Thus if $|\xi| \leq M$, we see that it is bounded above by $cM\varepsilon$ and so, in particular, it converges to zero in $L^\infty_{loc}(\mathbb{R}^N)$. This completes the proof of (i).

Before proving (ii), let us introduce the notation B_1^ε for the bounded operator on $L^2(\mathbb{R}^N)$ which maps g to its first Bloch transform $\hat{g}_1^\varepsilon$. Parseval's identity stated in Theorem 3.4 implies that $\varepsilon^{\frac{N}{2}}\|B_1^\varepsilon\| \leq 1$. The proof of (i) shows that if $g \in L^2(\mathbb{R}^N)$ is with compact support then $\varepsilon^{\frac{N}{2}}B_1^\varepsilon g \to \hat{g}$ in $L^2_{loc}(\mathbb{R}^N)$-strong. We can now complete the proof of (ii) by density arguments. Indeed, if $g \in L^2(\mathbb{R}^N)$ is arbitrary, we can approximate it by a function $h \in L^2(\mathbb{R}^N)$ with compact support. The desired result follows if we apply triangle inequality to the relation

$$\varepsilon^{\frac{N}{2}}B_1^\varepsilon g - \hat{g} = \varepsilon^{\frac{N}{2}}B_1^\varepsilon(g-h) + \varepsilon^{\frac{N}{2}}B_1^\varepsilon h - \hat{h} + \hat{h} - \hat{g}.$$

Finally, if $g^\varepsilon \to g$ in $L^2(\mathbb{R}^N)$-strong then the relation

$$\varepsilon^{\frac{N}{2}} B_1^\varepsilon g^\varepsilon - \hat{g} = \varepsilon^{\frac{N}{2}} B_1^\varepsilon (g^\varepsilon - g) + \varepsilon^{\frac{N}{2}} B_1^\varepsilon g - \hat{g}$$

shows that $\varepsilon^{\frac{N}{2}} B_1^\varepsilon g^\varepsilon \to \hat{g}$ in $L^2_{loc}(\mathbb{R}^N)$, thereby completing the proof of (ii). ∎

In general, one cannot assert that $\varepsilon^{\frac{N}{2}} \hat{g}_1^\varepsilon \rightharpoonup \hat{g}$ in $L^2_{loc}(\mathbb{R}^N)$-weak whenever $g^\varepsilon \rightharpoonup g$ in $L^2(\mathbb{R}^N)$-weak without any additional assumptions. In the sequel, we will see that the result stated in Proposition 3.6(i) above is sufficient for our purposes.

3.2. Identification of the homogenized coefficients. The aim of this paragraph is to give a different expression for the homogenized matrix $(q_{k\ell})$ in terms of the first Bloch eigenvalue $\lambda_1(\eta)$. Let us recall the classical expression for $(q_{k\ell})$ from [B-L-P, 1978]:

$$q_{k\ell} = \frac{1}{|Y|}\int_Y a_{k\ell}dy + \frac{1}{|Y|}\int_Y a_{km}\frac{\partial \chi_\ell}{\partial y_m}dy \quad \forall k,\ell = 1,...,N, \tag{3.4}$$

where χ_k is the unique solution (defined up to an additive constant) of the following problem with periodic boundary conditions:

$$\begin{cases} A\chi_k = \dfrac{\partial a_{k\ell}}{\partial y_\ell} \quad \text{in} \quad \mathbb{R}^N, \\ \chi_k \quad Y\text{-periodic } \forall k = 1,...,N. \end{cases} \tag{3.5}$$

We then have the following proposition.

PROPOSITION 3.7. *The origin is a critical point of the first Bloch eigenvalue*:

$$\frac{\partial \lambda_1}{\partial \eta_k}(0) = 0 \quad \forall k = 1,...,N. \tag{3.6}$$

Further, the Hessien of λ_1 at $\eta = 0$ is given by

$$\frac{1}{2}\frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_\ell}(0) = q_{k\ell} \quad \forall k,\ell = 1,...,N. \tag{3.7}$$

The derivatives of the first Bloch mode can also be calculated and they are as follows:

$$\frac{\partial \phi_1}{\partial \eta_k}(y;0) = i|Y|^{-\frac{1}{2}}\chi_k(y) \quad \forall k = 1,...,N. \tag{3.8}$$

Proof. Given the fact that $\eta \to \lambda_1(\eta)$ and $\eta \to \phi_1(y;\eta)$ are smooth, it is straightforward to compute their derivatives at $\eta = 0$. Indeed, it is enough to differentiate the eigenvalue equation $A(\eta)\phi_1(\cdot;\eta) = \lambda_1(\eta)\phi_1(\cdot;\eta)$ with respect to η twice and evaluate at $\eta = 0$. Since the computations are classical, we present only the essential steps. We obtain

$$\frac{\partial \lambda_1}{\partial \eta_k}(\eta) = \left(\frac{\partial A(\eta)}{\partial \eta_k}\phi_1(\cdot;\eta), \phi_1(\cdot;\eta)\right),$$

$$(A(\eta) - \lambda_1(\eta))\frac{\partial \phi_1}{\partial \eta_k}(\cdot;\eta) + \left[\frac{\partial A(\eta)}{\partial \eta_k} - \frac{\partial \lambda_1}{\partial \eta_k}(\eta)\right]\phi_1(\cdot;\eta) = 0,$$

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_\ell}(\eta) = (a_{k\ell}\phi_1(\cdot;\eta), \phi_1(\cdot;\eta)) &+ \frac{1}{2}\left(\left[\frac{\partial A(\eta)}{\partial \eta_k} - \frac{\partial \lambda_1}{\partial \eta_k}(\eta)\right]\frac{\partial \phi_1}{\partial \eta_\ell}(\cdot;\eta), \phi_1(\cdot;\eta)\right) \\ &+ \frac{1}{2}\left(\left[\frac{\partial A(\eta)}{\partial \eta_\ell} - \frac{\partial \lambda_1}{\partial \eta_\ell}(\eta)\right]\frac{\partial \phi_1}{\partial \eta_k}(\cdot;\eta), \phi_1(\cdot;\eta)\right).\end{aligned}$$

We know already that $\lambda_1(0) = 0$ and by our choice $\phi_1(y;0) = |Y|^{-\frac{1}{2}}$. If we use this information in the above relations and evaluate them at $\eta = 0$, we get successively (3.6), (3.8), and (3.7). $\square$

Before taking up the rigorous proof of Theorem 3.1, we pass to the limit in relation (3.2) in a heuristic manner to see the homogenized equation obtained via the Bloch-wave method. Let us take $m = 1$ in (3.2) and multiply both sides by $\varepsilon^{\frac{N}{2}}$:

$$\varepsilon^{-2}\lambda_1(\varepsilon\xi)\varepsilon^{\frac{N}{2}}\hat{u}_1^\varepsilon(\xi) = \varepsilon^{\frac{N}{2}}\hat{f}_1^\varepsilon(\xi), \tag{3.9}$$

where, we recall, $\hat{u}_1^\varepsilon$ and $\hat{f}_1^\varepsilon$ denote the first Bloch transform of u^ε and f, respectively. Expanding $\lambda_1(\varepsilon\xi)$ by Taylor's formula around $\xi = 0$ and using the results of Proposition 3.7, we get

$$\left[\frac{1}{2}\frac{\partial^2\lambda_1}{\partial\eta_k\partial\eta_\ell}(0)\xi_k\xi_\ell + O(\varepsilon\xi^3)\right]\varepsilon^{\frac{N}{2}}\hat{u}_1^\varepsilon(\xi) = \varepsilon^{\frac{N}{2}}\hat{f}_1^\varepsilon(\xi). \tag{3.10}$$

A simple passage to the limit yields

$$\frac{1}{2}\frac{\partial^2\lambda_1}{\partial\eta_k\partial\eta_\ell}(0)\xi_k\xi_\ell\ \hat{u}^*(\xi) = \hat{f}(\xi), \tag{3.11}$$

where, we recall, u^* is the L^2-weak limit of u^ε.

Thanks to (3.7), the above equation is nothing but the homogenized equation in the Fourier space; i.e., it is just the Fourier transform of the usual homogenized equation. It can be remarked that the passage to the limit is more direct because no derivatives are involved in (3.9). However, there is one flaw in our argument of letting $\varepsilon \to 0$ in (3.9). Strictly speaking we cannot apply Proposition 3.6 here since u^ε does not need to have uniform compact support. A natural way to overcome this difficulty is to use the cut-off function technique to localize the equation. This is what we carry out in the next paragraph.

3.3. Proof of Theorem 3.1. Let $\phi \in \mathcal{D}(\Omega)$ be arbitrary. If u^ε satisfies $A^\varepsilon u^\varepsilon = f$ in Ω then its localization ϕu^ε satisfies

$$A^\varepsilon(\phi u^\varepsilon) = \phi f + g^\varepsilon + h^\varepsilon \quad \text{in} \quad \mathbb{R}^N, \tag{3.12}$$

where

$$\begin{aligned} g^\varepsilon &= -2a_{k\ell}^\varepsilon\frac{\partial u^\varepsilon}{\partial x_\ell}\frac{\partial\phi}{\partial x_k} - a_{k\ell}^\varepsilon\frac{\partial^2\phi}{\partial x_k\partial x_\ell}u^\varepsilon = -2\sigma_k^\varepsilon\frac{\partial\phi}{\partial x_k} - a_{k\ell}^\varepsilon\frac{\partial^2\phi}{\partial x_k\partial x_\ell}u^\varepsilon, \\ h^\varepsilon &= -\frac{\partial a_{k\ell}^\varepsilon}{\partial x_k}\frac{\partial\phi}{\partial x_\ell}u^\varepsilon. \end{aligned}$$

Using the arguments outlined above leading to (3.11), we can pass to the limit in (3.12): since $\{\phi u^\varepsilon\}$ is bounded in $H^1(\mathbb{R}^N)$, we can neglect all the harmonics corresponding to $m \geq 2$. The component corresponding to $m = 1$ yields at the limit

$$\frac{1}{2}\frac{\partial^2\lambda_1}{\partial\eta_k\partial\eta_\ell}(0)\xi_k\xi_\ell\widehat{(\phi u^*)}(\xi) = \widehat{(\phi f)}(\xi) + \lim_{\varepsilon\to 0}\varepsilon^{\frac{N}{2}}\hat{g}_1^\varepsilon(\xi) + \lim_{\varepsilon\to 0}\varepsilon^{\frac{N}{2}}\hat{h}_1^\varepsilon(\xi), \tag{3.13}$$

where $\hat{g}_1^\varepsilon$, $\hat{h}_1^\varepsilon$ are the first Bloch transform of g^ε and h^ε, respectively. The sequence $\{\sigma_k^\varepsilon\}$ is bounded in $L^2(\Omega)$; we can therefore extract a subsequence (still denoted by ε)

which is weakly convergent in $L^2(\Omega)$. Let σ_k^* denote its limit as well as its extension by zero outside Ω. Using this convergence and the definition of g^ε we see that

$$g^\varepsilon \rightharpoonup g^* \stackrel{\text{def}}{=} -2\sigma_k^* \frac{\partial \phi}{\partial x_k} - \mathcal{M}(a_{k\ell}) \frac{\partial^2 \phi}{\partial x_k \partial x_\ell} u^* \quad \text{in} \quad L^2(\mathbb{R}^N) \text{ weak},$$

where $\mathcal{M}(a_{k\ell})$ is the average of $a_{k\ell}$ on Y. From Proposition 3.6, it follows that

$$\varepsilon^{\frac{N}{2}} \hat{g}_1^\varepsilon(\xi) \rightharpoonup \hat{g}^*(\xi) \quad \text{in} \quad L^2_{loc}(\mathbb{R}^N_\xi) \text{ weak}.$$

Concerning the sequence $\{\hat{h}_1^\varepsilon\}$, we cannot apply Proposition 3.6 directly because $\{h^\varepsilon\}$ is not bounded in $L^2(\mathbb{R}^N)$. However, following the idea of Proposition 3.6, we decompose

$$(3.14)\quad \begin{cases} \varepsilon^{\frac{N}{2}} \hat{h}_1^\varepsilon(\xi) = \displaystyle\int_{\mathbb{R}^N} h^\varepsilon(x) e^{-ix\cdot\xi} \bar{\phi}_1\left(\frac{x}{\varepsilon}, 0\right) dx \\ \qquad\qquad\qquad + \displaystyle\int_{\mathbb{R}^N} h^\varepsilon(x) e^{-ix\cdot\xi} \left(\bar{\phi}_1\left(\frac{x}{\varepsilon}; \varepsilon\xi\right) - \bar{\phi}_1\left(\frac{x}{\varepsilon}; 0\right)\right) dx. \end{cases}$$

The main point is that since h^ε is not bounded in $L^2(\mathbb{R}^N)$, the second term will also contribute. (The proof of Proposition 3.6 shows that the second term tends to zero if the sequence is bounded in $L^2(\mathbb{R}^N)$.) In fact, using the Taylor expansion of $\phi_1(y;\eta)$ at $\eta = 0$ (which is valid), we see that the second term is equal to

$$-\varepsilon^{-1} \int_{\mathbb{R}^N} \frac{\partial a_{k\ell}}{\partial y_k}\left(\frac{x}{\varepsilon}\right) \frac{\partial \phi}{\partial x_\ell}(x) u^\varepsilon(x) e^{-ix\cdot\xi} \left[\varepsilon \frac{\partial \bar{\phi}_1}{\partial \eta_j}\left(\frac{x}{\varepsilon}; 0\right) \xi_j + O(\varepsilon^2 \xi^2)\right] dx,$$

which evidently converges to

$$-\mathcal{M}\left(\frac{\partial a_{k\ell}}{\partial y_k} \frac{\partial \bar{\phi}_1}{\partial \eta_j}(y; 0)\right) \xi_j \int_{\mathbb{R}^N} \frac{\partial \phi}{\partial x_\ell} u^* e^{-ix\cdot\xi} dx$$

in $L^\infty_{loc}(\mathbb{R}^N_\xi)$ strong.

On the other hand, the first term of the right side of (3.14), after one integration by parts, becomes

$$\int_{\mathbb{R}^N} a_{k\ell}^\varepsilon \left[\frac{\partial^2 \phi}{\partial x_k \partial x_\ell} u^\varepsilon + \frac{\partial \phi}{\partial x_\ell} \frac{\partial u^\varepsilon}{\partial x_k} - i\xi_k \frac{\partial \phi}{\partial x_\ell} u^\varepsilon\right] e^{-ix\cdot\xi} \bar{\phi}_1\left(\frac{x}{\varepsilon}; 0\right) dx.$$

Recalling that $\phi_1(y;0) = |Y|^{-\frac{1}{2}}$, it is easily seen that the above integral converges to

$$|Y|^{-\frac{1}{2}} \left[\mathcal{M}(a_{k\ell}) \int_{\mathbb{R}^N} \frac{\partial^2 \phi}{\partial x_k \partial x_\ell} u^* e^{-ix\cdot\xi} + \int_{\mathbb{R}^N} \sigma_\ell^* \frac{\partial \phi}{\partial x_\ell} e^{-ix\cdot\xi} dx - i\xi_k \mathcal{M}(a_{k\ell}) \int_{\mathbb{R}^N} \frac{\partial \phi}{\partial x_\ell} u^* e^{-ix\cdot\xi}\right]$$

in $L^2(\mathbb{R}^N)$ weak. Using this information in (3.13) and using Proposition 3.7, we arrive at

$$q_{k\ell} \xi_k \xi_\ell \widehat{(\phi u^*)}(\xi) = \widehat{(\phi f)}(\xi) - |Y|^{-\frac{1}{2}} \int_{\mathbb{R}^N} \sigma_k^* \frac{\partial \phi}{\partial x_k} e^{-ix\cdot\xi} dx - i\xi_k |Y|^{-\frac{1}{2}} q_{k\ell} \int_{\mathbb{R}^N} \frac{\partial \phi}{\partial x_\ell} u^* e^{-ix\cdot\xi} dx.$$

This can be rewritten as

$$(3.15)\quad \begin{cases} [\widehat{A^*(\phi u^*)}](\xi) = \widehat{(\phi f)}(\xi) \\ \qquad - |Y|^{-\frac{1}{2}} \displaystyle\int_{\mathbb{R}^N} \sigma_k^* \frac{\partial \phi}{\partial x_k} e^{-ix\cdot\xi} dx - i\xi_k |Y|^{-\frac{1}{2}} q_{k\ell} \int_{\mathbb{R}^N} \frac{\partial \phi}{\partial x_\ell} u^* e^{-ix\cdot\xi} dx. \end{cases}$$

We can call this *localized homogenized equation* in the Fourier space. The conclusions of Theorem 3.1 are easy consequences of this equation. In fact, taking the inverse Fourier transform of (3.15) we obtain

$$A^*(\phi u^*) = \phi f - \sigma_k^* \frac{\partial \phi}{\partial x_k} - q_{k\ell} \frac{\partial}{\partial x_k}\left(\frac{\partial \phi}{\partial x_\ell} u^*\right) \quad \text{in} \quad \mathbb{R}^N. \tag{3.16}$$

On the other hand, we can calculate $A^*(\phi u^*)$ directly:

$$A^*(\phi u^*) = -q_{k\ell} \frac{\partial^2 \phi}{\partial x_k \partial x_\ell} u^* - 2q_{k\ell} \frac{\partial \phi}{\partial x_k} \frac{\partial u^*}{\partial x_\ell} + \phi A^* u^* \quad \text{in} \quad \mathbb{R}^N. \tag{3.17}$$

A simple comparison between (3.16) and (3.17) yields

$$\phi(A^* u^* - f) = \left(q_{k\ell} \frac{\partial u^*}{\partial x_\ell} - \sigma_k^*\right) \frac{\partial \phi}{\partial x_k} \quad \text{in} \quad \mathbb{R}^N. \tag{3.18}$$

Since the above relation is true for all ϕ in $\mathcal{D}(\Omega)$, the desired conclusions follow. In fact, let us choose $\phi(x) = \phi_0(x) e^{inx\cdot\omega}$, where ω is a unit vector in $\mathbb{R}^N$ and $\phi_0(x) \in \mathcal{D}(\Omega)$ is fixed. Letting $n \to \infty$ in the resulting relation and varying the unit vector ω, we can easily deduce successively that $\sigma_k^* = q_{k\ell} \frac{\partial u^*}{\partial x_\ell}$ in Ω and $A^* u^* = f$ in Ω. This completes the proof of Theorem 3.1. ∎

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