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Rate of convergence estimates for the spectral approximation of a generalized eigenvalue problem

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Summary. The aim of this work is to derive rate of convergence estimates for the spectral approximation of a mathematical model which describes the vibrations of a solid-fluid type structure. First, we summarize the main theoretical results and the discretization of this variational eigenvalue problem. Then, we state some well known abstract theorems on spectral approximation and apply them to our specific problem, which allow us to obtain the desired spectral convergence. By using classical regularity results, we are able to establish estimates for the rate of convergence of the approximated eigenvalues and for the gap between generalized eigenspaces.

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1 Mathematical knowledge of the eigenvalue problem

This section is devoted to the presentation of the mathematical model which describes the physical problem and a summary of the most important results of theoretical study. We also deduce its approximate problem. Firstly, the physical hypotheses assumed for the system are established, and the periodical mathematical model which this represents is set out. The differential problem is then rigorously formulated and we summarize the theorem of existence and location of eigenvalues and the theorem on the optimal bound for the number of nonreal solutions. Finally, the problem is approximated by the discretization of the Hilbert spaces involved. To this end, certain simplifications are made in the geometry of the problem, which allow a more appropriate numerical approach. More details on the physical problem, deduction of the mathematical model, proofs of the theoretical results and numerical analysis are available in the references given below.

1.1 The physical problem

The physical problem which interests us is the study of the vibrations of a solid-fluid type structure. To be more exact, our purpose consists in determining the vibratory eigenfrequencies and eigenmotions of a bundle of metallic tubes immersed in an incompressible viscous fluid.

The fluid is assumed to be contained in a three-dimensional cavity with rigid walls. It is assumed that they are parallel to each other, that they are perfectly rigid (they do not allow deformations) and that they are elastically mounted in such a way that they can only vibrate on a transverse plane, perpendicular to the bundle. Furthermore, axial effects are not taken into account, and it is assumed that the tubes are of infinite length. The problem is then studied in two dimensions, restricting it to any of the sections of the cavity which are perpendicular to the tubes. With respect to dynamics, it is assumed that the solid-fluid system undergoes small vibrations around a state of equilibrium.

1.2 Formulation of the eigenvalue problem

Let Ω_0 be an open bounded subset of $\mathbb{R}^2$, with a locally Lipschitz continuous boundary Γ_0 (see [16] Chap. I) and let $\{\Theta_i\}_{i=1,K}$ be a family of K open subsets of Ω_0 which satisfy the following properties:

(1.1a) $\forall i = 1, \ldots, K, \quad \Theta_i$ is a non-empty connected open subset of Ω_0 .

(1.1b) $\forall i = 1, \ldots, K, \quad \bar{\Theta}_i \subset \Omega_0$.

(1.1c) $\forall i \neq j, \quad \bar{\Theta}_i \cap \bar{\Theta}_j = \phi$.

(1.1d) Each Θ_i has a locally Lipschitz boundary Γ_i .

Using the above notation we define Ω as follows

$$\Omega = \Omega_0 \setminus \bigcup_{i=1}^{K} \bar{\Theta}_i \, .$$

It should be observed that the boundary of Ω has $(K+1)$ connected components which are $\Gamma_0, \Gamma_1, \ldots, \Gamma_K$.

The periodical mathematical model which describes the solid-fluid interaction is a differential eigenvalue problem with non-local boundary conditions on the velocity. In this model, the tube sections are represented by the perforations $\{\Theta_i\}_{i=1,K}$, and the domain Ω represents the area occupied by the fluid. If the velocity of the fluid is denoted by $\mathbf{u} = \mathbf{u}(x)$ and the pressure by $p = p(x)$ then $(\mathbf{u}, p)$ satisfies:

(1.2a) $-2\nu \operatorname{div} e(\mathbf{u}) + \nabla p + \omega \mathbf{u} = \mathbf{0}$ in Ω ,

(1.2b) $\operatorname{div} \mathbf{u} = 0$ in Ω ,

(1.2c) $\mathbf{u} = \mathbf{0}$ on Γ_0 ,

(1.2d) $\mathbf{u} = \dfrac{-\omega}{k_i + m_i\omega^2} \left(\int\limits_{\Gamma_i} \sigma(\mathbf{u}, p)\mathbf{n}\, ds \right)$ on $\Gamma_i, \quad \forall i = 1, \ldots, K$,

where $\omega \in \mathbb{C}$ is the frequency, $\nu > 0$ represents the kinematic viscosity of the fluid, k_i and m_i, $i = 1, \ldots, K$, are strictly positive given constants associated with this system. They represent the mass per length unit of tube i and the stiffness constant of the spring system supporting the i^{th} tube (see [17]), $e(\mathbf{u})$ is the linear strain tensor, defined by

$$2e(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^{\text{t}} ,$$

and the term $\sigma(\mathbf{u}, p)$ represents the stress tensor of the system given by Stokes's law

$$\sigma(\mathbf{u}, p) = -p\, I + 2\nu\, e(\mathbf{u}) . \tag{1.3}$$

To obtain the mixed variational formulation of problem (1.2), on which we develop the numerical analysis, the following Sobolev spaces are introduced

$$\begin{aligned} H = \{\mathbf{v} \in H^1(\Omega)^2 \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0 \text{ and} \\ \mathbf{v} \text{ is a constant vector on } \Gamma_i,\ \forall i = 1, \ldots, K\} , \\ L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int\limits_{\Omega} q\, dx = 0\} . \end{aligned}$$

Clearly, H is a closed vector subspace of $H^1(\Omega)^2$ and therefore a Hilbert space with the induced norm.

Considering H and $L_0^2(\Omega)$ as complex Hilbert spaces, and multiplying (1.2a) by $\bar{\mathbf{v}}$ in H and (1.2b) by $\bar{q}$ in $L_0^2(\Omega)$ ($\bar{\mathbf{v}}, \bar{q}$ denote the complex conjugate of $\mathbf{v}, q$), integrating by parts in Ω, and using (1.2c,d) and (1.3), it follows that if the triplet $(\omega, \mathbf{u}, p)$ is a solution of (1.2), then $(\omega, \mathbf{u}, p)$ is a solution of the following variational eigenvalue problem:

$$\text{Find } \omega \in \mathbb{C}\,, \tag{1.4a}$$

$$(\mathbf{u}, p) \in H \times L^2_0(\Omega)\,,\ (\mathbf{u}, p) \neq (\mathbf{0}, 0) \tag{1.4b}$$
$$\text{such that } \forall\ (\mathbf{v}, q) \in H \times L^2_0(\Omega)\,,$$

$$2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\mathbf{v}})\ dx - \int_\Omega p \operatorname{div} \bar{\mathbf{v}}\, dx \tag{1.4c}$$
$$+\omega \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} dx + \sum_{i=1}^{K} \left(\frac{k_i + m_i\omega^2}{\omega}\right) \gamma_i(\mathbf{u}) \cdot \gamma_i(\bar{\mathbf{v}}) = 0\,,$$

$$\int_\Omega \operatorname{div}\ \mathbf{u}\, \bar{q}\, dx = 0\,, \tag{1.4d}$$

where, in (1.4c), $\gamma_i(\mathbf{u})$ denotes the trace of $\mathbf{u}$ on Γ_i; it is a complex constant vector. Conversely, it is straightforward to see that if $(\omega, \mathbf{u}, p)$ is a solution of (1.4), then $(\omega, \mathbf{u}, p)$ is a solution of (1.2) in a weak sense.

1.3 Main results

Here we deal with an existence and a location result for the eigenfrequencies of problem (1.4) and with an optimal bound for the number of nonreal solutions. Even though the physical problem behind (1.4) is truely two-dimensional, from a mathematical viewpoint there is no conceptual difficulty in higher dimensions. Nevertheless, in view of the numerical approximation of (1.4), we shall state all our theoretical results in the two-dimensional case and we shall therefore continue in the framework of Sect. 1.2.

The proof of the following result can be found in [8]:

Theorem 1.1 *The spectrum of* (1.4) *consists of a countable infinite quantity of complex numbers* $\omega_1, \ldots, \omega_\ell, \ldots$*, which converge in modulus to infinity, i.e.,*

$$\lim_{\ell \to \infty} \frac{1}{|\omega_\ell|} = 0. \tag{1.5a}$$

Moreover, the eigenvalues have the following properties:
(i) $\operatorname{Re}(\omega_i) < 0 \quad \forall i \geq 1.$
(ii) *If* $\operatorname{Im}(\omega_i) \neq 0$ *then*

$$|\omega_i|^2 < k/m\,, \tag{1.5b}$$

where k/m denotes the quantity $\max\limits_{1 \leq j \leq K} \{k_j/m_j\}$.

It is interesting to observe in (ii) that the spectrum of (1.4) can only contain a finite quantity of eigenvalues whose imaginary part is different from zero. It should be noted that Theorem 1.1 does not provide an estimate of the number of imaginary (non-real) eigenvalues. This is provided by the following result, the proof of which can be referred to in [9].

Theorem 1.2 *The spectrum of* (1.4) *admits a maximum of* $4K$ *non-real eigenvalues.*

1.4 Discretization of the problem

We begin by setting out some simplifications on the geometry of the domain. The first geometrical hypothesis adopted consists in assuming that Ω is a bounded polygonal domain in $\mathbb{R}^2$. That is, Γ_0 (the exterior boundary) and $\Gamma_i\ \forall i = 1, \ldots, K$ (the boundary of the holes) are polygonals. We associate to Ω a regular family of triangulations $\{\tau_h\}_{h>0}$ (in the sense of [6], Chap. 2, Sect. 2.1), which satisfies the classical condition

$$\bar{\Omega} = \bigcup_{T \in \tau_h} T \quad \forall\, h > 0\,,$$

where h is defined as the $\max\{h_T,\ T \in \tau_h\}$; h_T being the diameter of triangle T. We also assume, by simplicity and without loosing generality, that the constants k_i and m_i, $i = 1, \ldots, K$, have a single value, say k and m, respectively (the tubes are identical).

To approximate H and $L^2_0(\Omega)$ by means of Lagrange type finite elements on triangles, we introduce the finite dimensional spaces

$$\text{(1.6a)}\quad H_h \equiv \{\mathbf{v}_h \in C^\circ(\bar{\Omega})^2 \mid \mathbf{v}_h|_T \in P_2(T)^2 \forall T \in \tau_h,\ \mathbf{v}_h = \mathbf{0} \text{ on } \Gamma_0 \text{ and } \mathbf{v}_h \text{ is a constant vector on } \Gamma_i\,, \quad \forall i = 1, \ldots, K\}\,,$$

$$\text{(1.6b)}\quad L^2_{0h} \equiv \{q_h \in C^\circ(\bar{\Omega}) \mid q_h|_T \in P_1(T)\ \forall T \in \tau_h \text{ and } \int_\Omega q_h\, dx = 0\}\,,$$

where, in the above definitions $P_n(T)$, $n \geq 0$, denotes the space of polynomials of degree less than or equal to n. The positive integers $2N$ and M are defined as the dimensions of spaces H_h and L^2_{0h}, respectively.

In what follows we shall assume the following hypotheses of non-degeneracy and regularity concerning the cardinality $|\tau_h|$ of the triangulations, the integer K (number of tubes) and the degrees of freedom associated with the velocity and pressure:

$$\text{(1.7a)}\qquad 2N - M > 0 \quad \forall h > 0\,,$$

$$\text{(1.7b)}\qquad 2|\tau_h| + K \leq 2N \quad \forall h > 0\,.$$

In these conditions, we approximate problem (1.4) by

(1.8a) Find $\omega_h \in \mathbb{C}$

(1.8b) $(\mathbf{u}_h, p_h) \in H_h \times L^2_{0h},\ (\mathbf{u}_h, p_h) \neq (\mathbf{0}, 0)$
such that $\forall (\mathbf{v}_h, q_h) \in H_h \times L^2_{0h}$,

(1.8c) $$a(\mathbf{u}_h, \mathbf{v}_h) + \overline{b(\mathbf{v}_h, p_h)} + \frac{k}{\omega_h} c(\gamma(\mathbf{u}_h), \gamma(\mathbf{v}_h)) = -\omega_h (d(\mathbf{u}_h, \mathbf{v}_h) + m\, c(\gamma(\mathbf{u}_h), \gamma(\mathbf{v}_h))),$$

(1.8d) $$b(\mathbf{u}_h, q_h) = 0,$$

where, in (1.8c,d), $\gamma(\mathbf{u}) \equiv (\gamma_1(\mathbf{u}), \ldots, \gamma_K(\mathbf{u})) \in \mathbb{C}^{2K}$ and the sesquilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$, $d(\cdot,\cdot)$ are defined by

(1.9a) $$a(\mathbf{u}, \mathbf{v}) = 2\nu \int_{\Omega} e(\mathbf{u}) : e(\bar{\mathbf{v}})\, dx \quad \forall (\mathbf{u}, \mathbf{v}) \in H^1(\Omega)^2 \,,$$

(1.9b) $$b(\mathbf{u}, p) = -\int_{\Omega} \bar{p} \operatorname{div} \mathbf{u}\, dx \quad \forall (\mathbf{u}, p) \in H^1(\Omega) \times L^2(\Omega) \,,$$

(1.9c) $$c(\mathbf{s}, \mathbf{t}) = \sum_{i=1}^{K} \mathbf{s}_i \cdot \bar{\mathbf{t}}_i \quad \forall (\mathbf{s}, \mathbf{t}) \in \mathbb{C}^{2K} \times \mathbb{C}^{2K} \,,$$

(1.9d) $$d(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u} \cdot \bar{\mathbf{v}}\, dx \quad \forall (\mathbf{u}, \mathbf{v}) \in L^2(\Omega)^2 \,.$$

As is shown in [7], under conditions (1.7), problem (1.8) is well posed. Furthermore, several numerical experiments which were carried out in recent years are consistent with all the abstract results of Sect. 1.3.

2 Rate of convergence estimates

Our aim in this section is to estimate the rate of convergence of the approximated eigenvalues and eigenvectors of problems (1.4). To this end, we briefly present a well known general theoretical result on approximation of eigenvalue problems in variational form and its application to the particular case of a saddlepoint form eigenvalue problem. We will use some notations of [15], where a general discussion of these theorems can be found. We also refer to [3] for a more complete approach on numerical approximation of eigenvalue problems. Then, we prove that problem (1.4) and that the discretized problem (1.8) can be put in the standard form of a saddlepoint eigenvalue problem. We show that these equivalent formulations satisfy the hypotheses which allow the use of the results summarized previously, and the desired estimations are thereby obtained.

2.1 A general result on spectral approximation

Firstly, we expound the fundamental theorem on error estimates of variationally posed eigenvalue problems. Let H_1 and H_2 be two complex Hilbert spaces provided with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Let A and B be two continuous sesquilinear forms defined on $H_1 \times H_2$. We are interested in studying the spectral approximation of

(2.1a) Find $(\lambda, U) \in \mathbb{C} \times H_1\,,\ U \neq 0$ such that

(2.1b) $$A(U,V) = \lambda B(U,V) \quad \forall V \in H_2\,.$$

In order to approximate problem (2.1), we consider the following properties on $A(\cdot,\cdot)$

(2.2) $$\inf_{\substack{U\in H_1\\ \|U\|_1=1}} \sup_{\substack{V\in H_2\\ \|V\|_2=1}} |A(U,V)| > 0\,,$$

(2.3) $$\sup_{\substack{U\in H_1\\ \|U\|_1=1}} |A(U,V)| > 0 \quad \forall V \in H_2\,,\ V \neq 0\,,$$

and the linear operator T defined by

(2.4a) $$T : H_1 \to H_1$$

(2.4b) $$T\,U = W \quad \forall U \in H_1\,,$$

where W is the unique solution of the following problem

(2.5a) Find $W \in H_1$ such that

(2.5b) $$A(W,V) = B(U,V) \quad \forall V \in H_2\,.$$

It is clear that properties (2.2) and (2.3) imply that T is well defined. Also, it is easy to see that if (λ, U) is a solution of (2.1), therefore $(1/\lambda, U)$ is an eigenpair of T, and reciprocally.

It is worthwhile to note that the existence of solutions to (2.1), eigenvalues of the operator T, is not sure from the hypotheses stated above. Henceforth, for the approximation, we will assume their existence. Let us introduce finite dimensional subspaces $H_{1h} \subset H_1$ and $H_{2h} \subset H_2$, $h > 0$, such that dim $H_{1h} =$ dim H_{2h}, then we can approximate problem (2.1) as follows

$$\text{Find } (\lambda_h, U_h) \in \mathbb{C} \times H_{1h}\,, U_h \neq 0 \ \text{ such that} \tag{2.6a}$$

$$A(U_h, V_h) = \lambda_h\, B(U_h, V_h) \quad \forall V_h \in H_{2h}\,, \tag{2.6b}$$

and we consider the corresponding inf-sup condition

$$\inf_{\substack{U_h \in H_{1h} \\ \|U_h\|_1 = 1}} \sup_{\substack{V_h \in H_{2h} \\ \|V_h\|_2 = 1}} |A(U_h, V_h)| \, \geq \, \alpha\,, \tag{2.7}$$

where α is a strictly positive real constant independent of h.

With respect to the discretization of the space H_1, we state the following property of approximation

$$\lim_{h \to 0} \inf_{U_h \in H_{1h}} \|U - U_h\|_1 \, = \, 0 \quad \forall\, U \in H_1\,. \tag{2.8}$$

As in the continuous case, by using (2.7) and the fact that dim $H_{1h} =$ dim H_{2h}, we can define the family of linear operators

$$T_h : H_1 \to H_1 \tag{2.9a}$$

$$T_h\, U = W_h \quad \forall U \in H_1\,, \tag{2.9b}$$

where W_h is given like the unique solution of the following problem

$$\text{Find } W_h \in H_{1h} \text{ such that} \tag{2.10a}$$

$$A(W_h, V_h) = B(U, V_h) \quad \forall V_h \in H_{2h}\,. \tag{2.10b}$$

It is clear that if (λ_h, U_h) is a solution of (2.6), then $(1/\lambda_h, U_h)$ is an eigenpair of T_h. Conversely, if $(\mu_h, U_h) \in \mathbb{C} \times H_1$ is an eigenpair of T_h and if $\mu_h \neq 0$, then U_h belongs to H_{1h} and $(1/\mu_h, U_h)$ is a solution of (2.6).

It is well known that properties (2.7) and (2.8) imply that

$$\lim_{h \to 0} \|\Pi_h U - U\|_{H_1} \, = \, 0 \quad \forall\, U \in H_1\,,$$

where $\Pi_h : \ H_1 \to H_{1h} \subset H_1$ is the usual projector operator defined by

$$A(U - \Pi_h U\,, V_h) \, = \, 0 \quad \forall\, U \in H_1, \; \forall\, V_h \in H_{2h}\,.$$

Because $T_h = \Pi_h\, T$, it is well known that if T is assumed to be compact, then

$$\lim_{h \to 0} \|T - T_h\|_{\mathcal{L}(H_1, H_1)} \, = \, 0\,.$$

It follows that if λ is an eigenvalue of (2.1) of algebraic multiplicity $m > 0$, thus as h tends to zero, exactly m eigenvalues of (2.6), denoted $\lambda_{1h}, \ldots, \lambda_{mh}$ (including their algebraic multiplicity), converge to λ (see [13]). The next result, whose proof can be found in [1], [10] or [14], provides us the desired estimates.

In the following, $\hat{\lambda}_h$ will denote the arithmetic mean of the approximated eigenvalues $\{\lambda_{jh}\}_{j=1}^{m}$; the quantities ϵ_h and ϵ_h^* are defined by

$$\epsilon_h = \sup_{\substack{U\in E\\ \|U\|_1=1}} \inf_{V\in H_{1h}} \|U - V\|_1 , \tag{2.11a}$$

$$\epsilon_h^* = \sup_{\substack{U\in E^*\\ \|U\|_2=1}} \inf_{V\in H_{2h}} \|U - V\|_2 , \tag{2.11b}$$

where E (and E^* respectively) is the kernel of $(1/\lambda - T)^\alpha$ $((1/\bar{\lambda} - T_*)^\alpha$ respectively; where T_* is the formal adjoint of T with respect to $A(\cdot,\cdot)$); the real constant α is the ascent of $(1/\lambda - T)$; E_h denotes the direct sum of the generalized eigenspaces corresponding to $\lambda_{jh},\ \forall\, j = 1, \ldots, m$; and $\hat{\delta}(E, E_h)$ represents the gap between the subspaces E and E_h.

Theorem 2.1 *Assume that* (2.2), (2.3), (2.7) *and* (2.8) *hold true. Also, suppose that the operator* T, *defined by* (2.4), *is compact and that* λ^{-1} *is an eigenvalue of* T *with algebraic multiplicity* m. *Then there exist two strictly positive real constants* C *and* h_0, *such that for all* $h \in\,]0, h_0[$ *there exist exactly* m *eigenvalues* $\lambda_{1,h}^{-1}, \lambda_{2,h}^{-1}, \ldots, \lambda_{m,h}^{-1}$ *of* T_h *converging to* λ^{-1} *and*

$$|\lambda - \hat{\lambda}_h| \le C\, \epsilon_h \epsilon_h^* , \tag{2.12a}$$

$$|\lambda - \lambda_{jh}| \le C\, (\epsilon_h \epsilon_h^*)^{\frac{1}{\alpha}} \quad \forall\, j = 1, \ldots, m , \tag{2.12b}$$

$$\hat{\delta}(E, E_h) \le C\, \epsilon_h . \tag{2.12c}$$

Now we turn to the eigenvalue problems of saddlepoint form, that is, problems of the following type

(2.13a) Find $(\lambda, u, p) \in\ \mathbb{C} \times H \times F\, , (u, p) \ne (0, 0)$ such that

$$a(u, v) + \overline{b(v, p)} = \lambda\, r(u, v) \quad \forall\, v \in H , \tag{2.13b}$$

$$b(u, q) = \lambda\, s(p, q) \quad \forall\, q \in F , \tag{2.13c}$$

where H and F are two complex Hilbert spaces provide with norms $\|\cdot\|_H$ and $\|\cdot\|_F$, respectively; $a : H \times H \to \mathbb{C}$, $b : H \times F \to \mathbb{C}$, $r : H \times H \to \mathbb{C}$ and $s : F \times F \to \mathbb{C}$ are continuous sesquilinear forms. If we have the finite dimensional spaces $H_h \subset H$ and $F_h \subset F$, so we consider the approximation of (2.13)

(2.14a) Find $(\lambda_h, u_h, p_h) \in \mathbb{C} \times H_h \times F_h$,
$(u_h, p_h) \neq (0,0)$ such that

(2.14b) $a(u_h, v_h) + \overline{b(v_h, p_h)} = \lambda_h \, r(u_h, v_h) \quad \forall v_h \in H_h$,

(2.14c) $b(u_h, q_h) = \lambda_h \, s(p_h, q_h) \quad \forall q_h \in F_h$.

In the sequel, we show that Theorem 2.1 can be applied to estimate the rate of spectral convergence of the approximated problem (2.14). Below, we state some properties which allow us to use this result. To this end, let us consider the following coercivity hypothesis on $a(\cdot,\cdot)$

$$\text{(2.15a)} \qquad \operatorname{Re} a(u,u) \;\geq\; \alpha_1 \, \|u\|_H^2 \quad \forall\, u \in H_0 \,,$$

where α_1 is a strictly positive constant and the space H_0 is defined by

$$\text{(2.15b)} \qquad H_0 \;=\; \{ v \in H \mid b(v,q) = 0 \quad \forall\, q \in \, F \} \,,$$

and its discretized version

$$\text{(2.16a)} \qquad \operatorname{Re} a(u_h,u_h) \;\geq\; \alpha_2 \, \|u_h\|_H^2 \quad \forall\, u_h \in H_{0h} \,,$$

where α_2 is a strictly positive constant independent of h. The space H_{0h} is defined by

$$\text{(2.16b)} \qquad H_{0h} \;=\; \{ v_h \in H_h \mid b(v_h,q_h) = 0 \quad \forall\, q_h \in \, F_h \} \,.$$

Respect to the sesquilinear form $b(\cdot,\cdot)$, we take into account the conditions

$$\text{(2.17)} \qquad \inf_{\substack{p \in F \\ p \neq 0}} \; \sup_{\substack{u \in H \\ u \neq 0}} \; \frac{|b(u,p)|}{\|p\|_F \, \|u\|_H} \;\geq\; \beta_1 > 0 \,,$$

and

$$\text{(2.18)} \qquad \inf_{\substack{p_h \in F_h \\ p_h \neq 0}} \; \sup_{\substack{u_h \in H_h \\ u_h \neq 0}} \; \frac{|b(u_h,p_h)|}{\|p_h\|_F \, \|u_h\|_H} \;\geq\; \beta_2 > 0 \,,$$

where β_2 is independent of h.

Also, we state the following property of approximation

(2.19)
$$\lim_{h \to 0} \; \inf_{(u_h,p_h) \in H_h \times F_h} \; \|(u,p) - (u_h,p_h)\|_{H \times F} \;=\; 0 \quad \forall\, (u,p) \in H \times F \,.$$

If we identify

(2.20a) $H_1 = H_2 = H \times F\,,$

(2.20b) $U = (u,p)\,, V = (v,q)\,,$

(2.20c) $A(U,V) = a(u,v) + \overline{b(v,p)} + b(u,q)\,,$

(2.20d) $B(U,V) = r(u,v) + s(p,q)\,,$

and the discretized spaces

(2.20e) $H_{1h} = H_{2h} = H_h \times F_h\,,$

we can easily see that (2.13) and (2.14) respectively, can be formulated in the variational form (2.1) and (2.6), respectively. It is well known (see [2] or [4]) that conditions (2.15) to (2.19) imply, with the identification (2.20), the validity of (2.2),(2.3),(2.7) and (2.8). It follows that Theorem 2.1 can be applied. To this end, we define the continuous linear operator

(2.21a) $T : H \times F \to H \times F$

(2.21b) $T\,(f,g) = (u,p) \quad \forall\,(f,g) \in H \times F\,,$

by

(2.22a) $a(u,v) + \overline{b(v,p)} \;=\; r(f,v) \quad \forall\, v \in H\,,$

(2.22b) $b(u,q) \;=\; s(g,q) \quad \forall\, q \in F\,,$

and we can conclude that if (λ,u,p) is a solution of (2.13), then (u,p) is an eigenvector of T corresponding to the eigenvalue λ^{-1}. In a same way we define the continuous linear operators

(2.23a) $T_h : H \times F \to H_h \times F_h \subset H \times F$

(2.23b) $T_h\,(f,g) = (u_h,p_h) \quad \forall\,(f,g) \in H \times F\,,$

by

(2.24a) $a(u_h,v_h) + \overline{b(v_h,p_h)} \;=\; r(f,v_h) \quad \forall\, v_h \in H_h\,,$

(2.24b) $b(u_h,q_h) \;=\; s(g,q_h) \quad \forall\, q_h \in F_h\,,$

and if (λ_h,u_h,p_h) is a solution of (2.14), then (u_h,p_h) is an eigenvector of T_h corresponding to the eigenvalue λ_h^{-1}. A direct consequence of Theorem 2.1, using the identification (2.18), is given in (see [15]):

Theorem 2.2 *Assume that hypotheses* (2.15) *to* (2.19) *hold. In addition, we assume that* T *is compact and that* λ^{-1} *is an eigenvalue of* T *with algebraic multiplicity* m*. Then there exist two positive constants* C *and* h_0 *such that if* $h \in]0, h_0[$*, then there exist exactly* m *eigenvalues* $\lambda_{1,h}^{-1}, \lambda_{2,h}^{-1}, \ldots, \lambda_{m,h}^{-1}$ *(counted according to algebraic multiplicity) of* T_h *converging to* λ^{-1} *and satisfying*

$$|\lambda - \hat{\lambda}_h| \leq C\, \epsilon_h \epsilon_h^*\,, \tag{2.25a}$$

$$|\lambda - \lambda_{jh}| \leq C\, (\epsilon_h \epsilon_h^*)^{\frac{1}{\alpha}} \quad \forall\, j = 1, \ldots, m\,, \tag{2.25b}$$

where $\epsilon_h \epsilon_h^*$ *are defined as above,* α *being the ascent of* $(\lambda^{-1} - T)$*. Moreover, for the gap between generalized eigenspaces we have*

$$\hat{\delta}(E, E_h) \leq C\, \epsilon_h\,. \tag{2.25c}$$

Remark that hypotheses neither in Theorem 2.1, nor in Theorem 2.2, are enough to provide actually the existence of at least one non-zero eigenvalue of operator T and then the assumption of existence is needed.

2.2 Application of Theorem 2.2 to problem (1.4)

Let us henceforth establish the desired rate of convergence estimate. We begin by showing that problem (1.4) and (1.8) respectively, can be put in the standard form of a saddlepoint eigenvalue problem. Then, we will prove that all the hypotheses of Theorem 2.2 are satisfied and that the corresponding operator T is compact. This will allow us to apply Theorem 2.2.

Let $(\omega, \mathbf{u}, p) \in \mathbb{C} \times H \times L_0^2(\Omega)$ be a solution of problem (1.4), where here the spaces H and $L_0^2(\Omega)$ are defined in Sect. 1.2. Let $\mathbf{s} \in \mathbb{C}^{2K}$ be defined by

$$\mathbf{s} = \frac{\sqrt{k}}{\omega}\, \gamma(\mathbf{u})\,, \tag{2.26}$$

where $\gamma(\mathbf{u}) = (\gamma_1(\mathbf{u}), \ldots, \gamma_K(\mathbf{u})) \in \mathbb{C}^{2K}$ is the vector trace on $\Gamma_i,\ i = 1, \ldots, K$. It is straightforward to prove that $(\omega, \mathbf{u}, p, \mathbf{s})$ is a solution of the following variational eigenvalue problem

$$\text{Find } (\omega, \mathbf{u}, p, \mathbf{s}) \in \mathbb{C} \times H \times L_0^2(\Omega) \times \mathbb{C}^{2K}, \quad (\mathbf{u}, p, \mathbf{s}) \neq (\mathbf{0}, 0, \mathbf{0}) \text{ such that} \tag{2.27a}$$

$$a(\mathbf{u}, \mathbf{v}) + \overline{b(\mathbf{v}, p)} + \sqrt{k}\; c(\mathbf{s}, \gamma(\mathbf{v})) = -\omega(d(\mathbf{u}, \mathbf{v}) + m\, c(\gamma(\mathbf{u}), \gamma(\mathbf{v}))) \quad \forall \mathbf{v} \in H\,, \tag{2.27b}$$

$$b(\mathbf{u}, q) = 0 \qquad \forall q \in L_0^2(\Omega)\,, \tag{2.27c}$$

$$\sqrt{k}\; c(\gamma(\mathbf{u}), \mathbf{t}) = \omega\, c(\mathbf{s}, \mathbf{t}) \quad \forall\, \mathbf{t} \in \mathbb{C}^{2K}\,. \tag{2.27d}$$

We use the notation for the sesquilinear forms defined in (1.9). Conversely, it is direct to see that if $(\omega, \mathbf{u}, p, \mathbf{s})$ is a solution of (2.27) then $(\omega, \mathbf{u}, p)$ is a solution of (1.4) and we have (2.26). Thus problems (1.4) and (2.27) are equivalent. In order to obtain the rate of spectral convergence, this formulation presents an improvement on the previous one, since problem (2.27) can be posed like an eigenvalue problem of saddlepoint form. In fact, if we consider the space

$$F \;=\; L^2_0(\Omega) \times \mathbb{C}^{2K}\,, \tag{2.28a}$$

provided with the induced norm, that is

$$\|\mathbf{y}\|_F = (\|q\|^2_{0,\Omega} \;+\; |\mathbf{t}|^2)^{\frac{1}{2}} \quad \forall\, \mathbf{y} = (q, \mathbf{t}) \in \; F\,, \tag{2.28b}$$

and if $\tilde{b}(\cdot,\cdot)$ denotes the sesquilinear form given by

$$\tilde{b} : H \times F \to \mathbb{C} \tag{2.29a}$$

$$\tilde{b}(\mathbf{v}, \mathbf{y}) = b(\mathbf{v}, q) + \sqrt{k}\, c(\gamma(\mathbf{v}), \mathbf{t}) \qquad \forall \mathbf{v} \in H,\ \forall \mathbf{y} = (q, \mathbf{t}) \in F\,, \tag{2.29b}$$

then we can rewrite problem (2.27) equivalently as follows

(2.30a) Find $(\omega, \mathbf{u}, \mathbf{x}) \in \; \mathbb{C} \times H \times F\,, (\mathbf{u}, \mathbf{x}) \neq (\mathbf{0}, \mathbf{0})$ such that

$$a(\mathbf{u}, \mathbf{v}) + \overline{\tilde{b}(\mathbf{v}, \mathbf{x})} \;=\; \omega\, r(\mathbf{u}, \mathbf{v}) \quad \forall\, \mathbf{v} \in H\,, \tag{2.30b}$$

$$\tilde{b}(\mathbf{u}, \mathbf{y}) \;=\; \omega\, s(\mathbf{x}, \mathbf{y}) \quad \forall\, \mathbf{y} \in F\,, \tag{2.30c}$$

where the sesquilinear forms $r(\cdot,\cdot)$ and $s(\cdot,\cdot)$ are defined by

$$r(\mathbf{u}, \mathbf{v}) = -d(\mathbf{u}, \mathbf{v}) - m\, c(\gamma(\mathbf{u}), \gamma(\mathbf{v})) \quad \forall\, \mathbf{u}, \mathbf{v} \in H\,,$$
$$s(\mathbf{x}, \mathbf{y}) = c(\mathbf{s}, \mathbf{t}) \quad \forall \mathbf{x} = (p, \mathbf{s}),\ \mathbf{y} = (q, \mathbf{t}) \in F\,.$$

Now, we shall prove that problem (2.30) satisfies (2.15) to (2.19). Let us begin by characterizing the subspace H_0 defined in (2.15b), when we replace the sesquilinear form $b(\cdot,\cdot)$ by $\tilde{b}(\cdot,\cdot)$.

Lemma 2.3 *The space H_0 is given by*

$$H_0 \;= \{\mathbf{v} \in \; H^1_0(\Omega)^2 \mid \text{div}\ \mathbf{v} = 0 \textit{ in } \Omega\ \}\,.$$

Its proof is left to the reader. We deduce

Corollary 2.4 *The sesquilinear form $a(\cdot,\cdot)$ is H_0-elliptic.*

Proof. It is straigtforward by virtue of Poincaré and Korn's first inequalities. □

The discretized coercivity condition of $a(\cdot,\cdot)$ can be proved in the same way. In fact, taking the finite dimensional spaces H_h and L^2_{0h}, as defined by (1.6) in Sect. 1.4, and approximating the space F by

$$F_h \,=\, L^2_{0h} \times \mathbb{C}^{2K}\,,$$

we deduce that H_{0h} is nothing but

$$H_{0h} = \{\mathbf{v}_h \in H_h \cap H^1_0(\varOmega)^2 \mid \int_\varOmega q_h \operatorname{div} \mathbf{v}_h \, dx \,=\, 0 \quad \forall\, q_h \in L^2_{0h}\}\,,$$

from which we can directly establish the uniform coercivity condition of $a(\cdot,\cdot)$ on H_{0h}. Then (2.15) and (2.16) hold.

Let us next prove the inf-sup condition on $\tilde{b}(\cdot,\cdot)$. To this effect, we state the two previous results that we will use after.

Lemma 2.5 *There exists a strictly positive constant* η_1 *such that for all* $p \in L^2_0(\varOmega)$ *there is a function* $\mathbf{u}_1 \in H^1_0(\varOmega)^2$ *satisfying*

(2.31a) $\qquad \|\mathbf{u}^{\mathrm{r}}_1\|_{1,\varOmega} \,=\, 1$ *and* $\|\mathbf{u}^{\mathrm{i}}_1\|_{1,\varOmega} \,=\, 1\,,$

(2.31b) $\qquad \operatorname{Re} b(\mathbf{u}_1, p) \,\geq\, \eta_1 \, \|p\|_{0,\varOmega}\,,$

where $\mathbf{u}^{\mathrm{r}}_1$ *and* $\mathbf{u}^{\mathrm{i}}_1$ *are the real and imaginary parts of* $\mathbf{u}_1$*, respectively.*

Proof. If we take the real part in the definition of the sesquilinear form $b(\cdot,\cdot)$, we obtain

$$\begin{aligned}\operatorname{Re} b(\mathbf{v}, p) = &-\int_\varOmega p^{\mathrm{r}} \operatorname{div} \mathbf{v}^{\mathrm{r}} \, dx \\ &-\int_\varOmega p^{\mathrm{i}} \operatorname{div} \mathbf{v}^{\mathrm{i}} \, dx \quad \forall \mathbf{v} \in\, H^1_0(\varOmega)^2\,,\ \forall p \in L^2_0(\varOmega)\,.\end{aligned}$$

By virtue of the classical inf-sup condition for real-valued functions in the spaces $H^1_0(\varOmega)^2$ and $L^2_0(\varOmega)$ (see [11], Chap. 1, Sect. 5.1 or [5], Chap. 4, Sect. 2), we deduce the existence of $\eta_1 > 0$ such that for all real-valued functions $p^{\mathrm{r}}, p^{\mathrm{i}} \in\, L^2_0(\varOmega)$ there exist two real-valued vector functions $\mathbf{v}_1, \mathbf{v}_2 \in H^1_0(\varOmega)^2$ satisfying

(2.32a) $$-\int_\varOmega p^{\mathrm{r}} \operatorname{div} \mathbf{v}_1 \, dx \geq \eta_1 \, \|p^{\mathrm{r}}\|_{0,\varOmega} \, \|\mathbf{v}_1\|_{1,\varOmega}\,,$$

(2.32b) $$-\int_\varOmega p^{\mathrm{i}} \operatorname{div} \mathbf{v}_2 \, dx \geq \eta_1 \, \|p^{\mathrm{i}}\|_{0,\varOmega} \, \|\mathbf{v}_2\|_{1,\varOmega}\,.$$

Then, if we define

$$\mathbf{u}_1^{\mathrm{r}} = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|_{1,\Omega}} \quad \text{and} \quad \mathbf{u}_1^{\mathrm{i}} = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|_{1,\Omega}} ,$$

we see that: $\mathbf{u}_1 \equiv \mathbf{u}_1^{\mathrm{r}} + \mathrm{i}\, \mathbf{u}_1^{\mathrm{i}}$, fulfils (2.31a) and the inequality

$$\mathrm{Re}\, b(\mathbf{u}_1, p) \geq \eta_1 \|p\|_{0,\Omega} ,$$

which proves (2.31b) and completes the proof. □

Lemma 2.6 *There exists a strictly positive constant η_2 such that for all $\mathbf{s} \in \mathbb{C}^{2K}$ there is a function $\mathbf{u}_2 \in H^1(\Omega)^2$ satisfying*

(2.33a) $$\mathrm{div}\, \mathbf{u}_2 = 0 \quad \textit{and} \quad \|\mathbf{u}_2\|_{1,\Omega} = 1 ,$$

(2.33b) $$c(\gamma(\mathbf{u}_2), \mathbf{s}) \geq \frac{1}{\eta_2} |\mathbf{s}| .$$

Proof. Since

$$\int_{\Gamma_i} \mathbf{s}_i \cdot \mathbf{n}\, ds = 0 \quad \forall\, i = 1, \ldots, K,$$

we can conclude that there exists a unique pair $(\mathbf{v}, q) \in H^1(\Omega)^2 \times L_0^2(\Omega)$, solution of the following system of equations (see [11], Chap. 1, Sect. 5.1, Theorem 5.1)

(2.34a) $$-2\nu\, \mathrm{div}\, e(\mathbf{v}) + \nabla q = \mathbf{0} \quad \text{in } \Omega ,$$

(2.34b) $$\mathrm{div}\, \mathbf{v} = 0 \quad \text{in } \Omega ,$$

(2.34c) $$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_0 ,$$

(2.34d) $$\mathbf{v} = \mathbf{s}_i \quad \text{on } \Gamma_i,\ \forall i = 1, \ldots, K.$$

Moreover, we have

$$\|\mathbf{v}\|_{1,\Omega} + \|q\|_{0,\Omega} \leq \eta\, |\mathbf{s}| ,$$

where η is a constant independent of $\mathbf{s}, \mathbf{v}$ and q. If we define

$$\mathbf{u}_2 = \frac{\mathbf{v}}{\|\mathbf{v}\|_{1,\Omega}} ,$$

it is clear that $\mathbf{u}_2$ satisfies (2.33a) and

$$c(\gamma(\mathbf{u}_2), \mathbf{s}) = \frac{1}{\|\mathbf{v}\|_{1,\Omega}}\, c(\mathbf{s}, \mathbf{s}) \geq \eta_2\, |\mathbf{s}| ,$$

where $\eta_2 = 1/\eta$. This proves (2.33b). □

Once the previous Lemmata have been established, we are in a position to set out the inf-sup condition on $\tilde{b}(\cdot,\cdot)$.

Proposition 2.7 *There exists a strictly positive real constant β_1 such that*

$$\inf_{\substack{\mathbf{x}\in F\\ \mathbf{x}\neq\mathbf{0}}} \sup_{\substack{\mathbf{u}\in H\\ \mathbf{u}\neq\mathbf{0}}} \frac{|\tilde{b}(\mathbf{u},\mathbf{x})|}{\|\mathbf{x}\|_F\|\mathbf{u}\|_{1,\Omega}} \geq \beta_1 . \tag{2.35}$$

Proof. We will show that there exists $\beta_1 > 0$ such that

$$\sup_{\substack{\mathbf{u}\in H\\ \mathbf{u}\neq\mathbf{0}}} \frac{\operatorname{Re}\tilde{b}(\mathbf{u},\mathbf{x})}{\|\mathbf{u}\|_{1,\Omega}} \geq \beta_1 \|\mathbf{x}\|_F \quad \forall\, \mathbf{x}\in F , \tag{2.36}$$

which clearly implies (2.35).

Let $\mathbf{x} = (p, \mathbf{s})$ be a fixed element of F. Let $\mathbf{u}_1, \mathbf{u}_2$ be the functions given by Lemma 2.5 and Lemma 2.6 associated with p and $\mathbf{s}$, respectively. If we define $\mathbf{u} \equiv \mathbf{u}_1 + \mathbf{u}_2$, then the pair $(\mathbf{u}, \mathbf{x})$ satisfies

$$\tilde{b}(\mathbf{u},\mathbf{x}) = b(\mathbf{u}_1, p) + \sqrt{k}\, c(\gamma(\mathbf{u}_2), \mathbf{s}) ,$$

and this yields

$$\operatorname{Re}\tilde{b}(\mathbf{u},\mathbf{x}) \geq \beta\|\mathbf{x}\|_F , \tag{2.37}$$

where β denotes the quantity $\min\{\eta_1, \sqrt{k}\,\eta_2\} > 0$ and $\mathbf{u}$ satisfies: $\|\mathbf{u}\|_{1,\Omega} \leq 3$.

Let us consider the identity

$$\sup_{\substack{\mathbf{u}\in H\\ \mathbf{u}\neq\mathbf{0}}} \frac{\operatorname{Re}\tilde{b}(\mathbf{u},\mathbf{x})}{\|\mathbf{u}\|_{1,\Omega}} = \frac{1}{3} \sup_{\substack{\mathbf{u}\in H\\ \|\mathbf{u}\|_{1,\Omega}\leq 3}} \mathbb{R}e\, \tilde{b}(\mathbf{u},\mathbf{x}) \quad \forall\, \mathbf{x}\in F . \tag{2.38}$$

Then, combining (2.37) and (2.38), we infer that

$$\sup_{\substack{\mathbf{u}\in H\\ \mathbf{u}\neq\mathbf{0}}} \frac{\operatorname{Re}\tilde{b}(\mathbf{u},\mathbf{x})}{\|\mathbf{u}\|_{1,\Omega}} \geq \frac{1}{3}\beta\|\mathbf{x}\|_F \quad \forall\, \mathbf{x}\in F ,$$

that is (2.36), with $\beta_1 = \beta/3$, which completes the proof. $\square$

It is worth remarking that all the results used to prove the inf-sup condition of $\tilde{b}(\cdot,\cdot)$ on the space $H \times F$ can also be established for the discrete eigenvalue problems, with all the constants involved independent of the parameter $h > 0$. In fact, one can show that (2.32) is still true in the discrete version when we choose the spaces H_h and $L^2_{0,h}$ as above. Below, we state the discretized version of Proposition 2.7, whose proof we shall omit.

Proposition 2.8 *There exists a strictly positive real constant β_2, independent of $h > 0$, such that*

$$\textit{inf}_{\substack{\mathbf{x}_h \in F_h \\ \mathbf{x}_h \neq \mathbf{0}}} \ \textit{sup}_{\substack{\mathbf{u}_h \in H_h \\ \mathbf{u}_h \neq \mathbf{0}}} \frac{|\tilde{b}(\mathbf{u}_h, \mathbf{x}_h)|}{\|\mathbf{x}_h\|_F \ \|\mathbf{u}_h\|_{1,\Omega}} \geq \beta_2 \quad \forall\, h > 0\,. \tag{2.39}$$

The last two propositions ensure the validity of the hypotheses (2.17) and (2.18). The corresponding approximation property (2.19) of the spaces H_h and L^2_{0h} is classical. We refer to [18] for the proof of the following convergence results

$$\lim_{h \to 0} \inf_{\mathbf{u}_h \in H_h} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} = 0 \quad \forall\, \mathbf{u} \in H\,, \tag{2.40a}$$

$$\lim_{h \to 0} \inf_{p_h \in L^2_{0h}} \|p - p_h\|_{0,\Omega} = 0 \quad \forall\, p \in L^2_0(\Omega)\,. \tag{2.40b}$$

It must be noted that hypotheses (2.15) to (2.19) hold for the problem (2.30). In what follows, we characterize the linear operator T (defined by (2.21)) corresponding to (2.30) and we prove its compactness. We define the following continuous linear operator

$$T : H \times F \to H \times F \tag{2.41a}$$

$$T\,(\mathbf{v}, \mathbf{y}) = (\mathbf{u}, \mathbf{x}) \quad \forall\, (\mathbf{v}, \mathbf{y}) \in H \times F\,, \tag{2.41b}$$

by

$$-2\nu \operatorname{div} e(\mathbf{u}) + \nabla p = -\mathbf{v} \quad \text{ in } \Omega\,, \tag{2.42a}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{ in } \Omega\,, \tag{2.42b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{ on } \Gamma_0\,, \tag{2.42c}$$

$$\mathbf{u} = \frac{1}{\sqrt{k}}\, \mathbf{t}_i \quad \text{ on } \Gamma_i,\ \forall i = 1, \ldots, K\,, \tag{2.42d}$$

and

$$\mathbf{s}_i = \frac{-1}{\sqrt{k}}\, \Big(m\, \gamma_i(\mathbf{v}) + \int_{\Gamma_i} \sigma(\mathbf{u}, p)\, \mathbf{n}\, ds\Big) \quad \forall i = 1, \ldots, K\,, \tag{2.42e}$$

where $\mathbf{x} = (p, \mathbf{s}),\ \mathbf{y} = (q, \mathbf{t})$. It is interesting to note that T is not a selfadjoint, injective operator. In fact, the spectrum of T admits non-real eigenvalues and the fonctions in $L^2_0(\Omega)$ are not used to define the operator, i.e., we have

$$T(\mathbf{v}, \mathbf{y}) = T(\mathbf{v}, \mathbf{y}') \quad \forall\, \mathbf{y} = (q, \mathbf{t}),\ \mathbf{y}' = (q', \mathbf{t}) \in F\,.$$

Proposition 2.9 *The operator T defined above, corresponds to the one associated with problem (2.30). In addition, it is compact.*

Proof. A direct computation shows that operator T is effectively the one associated to (2.30). It remains to prove the compactness of T. Let $\{(\mathbf{v}_n, \mathbf{y}_n)\}_{n\in\mathbb{N}}$ be a sequence of $H \times F$ such that

$$\|(\mathbf{v}_n, \mathbf{y}_n)\|_{H\times F} \le C \quad \mathbf{y}_n = (q, \mathbf{t}_n),\ \forall\, n \ge 1\,,$$

where $q \in L^2(\Omega)$ is any fixed element and C is a strictly positive constant independent of n. By using the compact imbedding, we deduce the existence of $\mathbf{v} \in L^2(\Omega)$ and $\mathbf{t} \in \mathbb{C}^{2K}$ such that

$$\begin{aligned} &\mathbf{v}_{n'} \to \mathbf{v} \quad \text{in } L^2(\Omega)\,,\\ &\mathbf{y}_{n'} \to \mathbf{y} = (q, \mathbf{t}) \quad \text{in } F\,. \end{aligned}$$

Hence, if we take the subsequence $\{(\mathbf{u}_{n'}, \mathbf{x}_{n'})\} \subset H \times F$ solutions of (2.42) with data $\{(\mathbf{v}_{n'}, \mathbf{y}_{n'})\}$, we have

$$\mathbf{u}_{n'} \to \mathbf{u} \quad \text{in } H\,, \tag{2.43a}$$

$$\mathbf{x}_{n'} = (p_{n'}, \mathbf{s}_{n'}) \to \mathbf{x} = (p, \mathbf{s}) \quad \text{in } F\,, \tag{2.43b}$$

where $(\mathbf{u}, \mathbf{x})$ is the solution of (2.42) lying to $(\mathbf{v}, \mathbf{y})$. Moreover, we can suppose

$$\sigma(\mathbf{u}_{n'}, p_{n'}) \to \sigma(\mathbf{u}, p) \quad \text{in } L^2(\Omega) \times L^2_0(\Omega)\,, \tag{2.43c}$$

which implies

$$T(\mathbf{v}_{n'}, \mathbf{y}_{n'}) \to T(\mathbf{v}, \mathbf{y}) \quad \text{in } H \times F\,,$$

and complete the proof. $\square$

Since the hypotheses concerning the spectral approximation are assumed to hold, we can apply Theorems 2.1 and 2.2 to our original problem (1.4). Before that, we set out the following regularity proposition about the generalized eigenspaces $\{E_\ell\}_{\ell=1}^{\infty}$, associated to problem (1.4). It will allow us to estimate the rate of convergence.

Proposition 2.10 *Let ω_ℓ be the eigenvalues of problem (1.4) given by Theorem 1.1 and let $(\mathbf{u}_\ell, p_\ell)$ be the corresponding eigenfunctions. We denote by E_ℓ the generalized eigenspace related to ω_ℓ. Then there exists $\eta \in\ (0,1]$, which depends only on the re-entrant corners of the domain, such that $E_\ell \subset\ H^{1+\eta}(\Omega)^2 \times H^\eta(\Omega),\ \forall\, \ell \in \mathbb{N}$.*

Proof. For the sake of simplicity, we restrict our proof to the eigenfunctions, because it is easy to verify that the general case have the same property. Given $\ell \in \mathbb{N}$ there exists a lifting function $\mathbf{u}_{\mathrm{R}} \in H^1(\Omega)^2$ (see [11], Chap. 1, Sect. 2, Lemma 2.2) satisfying

$$\text{(2.44a)} \quad \operatorname{div} \mathbf{u}_{\mathrm{R}} = 0 \quad \text{in } \Omega\,,$$

$$\text{(2.44b)} \quad \mathbf{u}_{\mathrm{R}} = \mathbf{0} \quad \text{on } \Gamma_0\,,$$

$$\text{(2.44c)} \quad \mathbf{u}_{\mathrm{R}} = \frac{\omega_\ell}{k + m\,\omega_\ell^2}\Big(\int\limits_{\Gamma_i} \sigma(\mathbf{u}_\ell, p_\ell)\mathbf{n}\,ds\Big) \text{on } \Gamma_i,\ \forall i = 1, \ldots, K\,.$$

Moreover, this function can be chosen in $C^\infty(\bar{\Omega})^2$ because $\mathbf{u}_{\mathrm{R}}$ can be constructed to be constant valued in a regular neighborhood of each Γ_i, and then a more precise result for smoother lifting functions holds (see [11], Chap. 1, Sect. 3, Lemma 3.2). Therefore, the function $\mathbf{u}_0 \equiv \mathbf{u}_\ell + \mathbf{u}_{\mathrm{R}}$ verifies the following classical Stokes system

$$\text{(2.45a)} \quad -2\nu \operatorname{div} e(\mathbf{u}_0) + \nabla p_\ell = \mathbf{f} \quad \text{in } \Omega\,,$$

$$\text{(2.45b)} \quad \operatorname{div} \mathbf{u}_0 = 0 \quad \text{in } \Omega\,,$$

$$\text{(2.45c)} \quad \mathbf{u}_0 = \mathbf{0} \quad \text{on } \Gamma_i,\ \forall i = 0, \ldots, K\,,$$

where $\mathbf{f} \in L^2(\Omega)^2$ is given by

$$\mathbf{f} = -\omega_\ell\,\mathbf{u}_\ell - \nu\,\Delta \mathbf{u}_{\mathrm{R}}\,.$$

It is well known that $\mathbf{u}_0 \in H^{1+\eta}(\Omega)^2$ for some $\eta \in (0, 1]$ (see for example [12], Chap. 6, Sect. 2, Theorem 6.2.3) and we conclude that $(\mathbf{u}_\ell, p_\ell) \in H^{1+\eta}(\Omega)^2 \times H^\eta(\Omega)$. $\square$

Proposition 2.11 *The spectral approximation of problem* (1.4) *is convergent, i.e.,*

$$\text{(2.46a)} \quad \lim_{h \to 0} |\omega_\ell - \omega_{\ell,h}| = 0 \quad \forall\,\ell \in \mathbb{N}\,.$$

Moreover, there exist two strictly positive constants h_0 *and* C *such that,* $\forall\,h \le h_0$ *we have*

$$\text{(2.46b)} \quad |\omega_\ell - \hat{\omega}_{\ell,h}| \le C\,h^{2\eta}\,,$$

$$\text{(2.46c)} \quad |\omega_\ell - \omega_{\ell,h}| \le C\,h^{\frac{2\eta}{\alpha}}\,,$$

$$\text{(2.46d)} \quad \hat{\delta}(E_\ell, E_{\ell,h}) \le C\,h^\eta\,,$$

where $\hat{\omega}_{\ell,h}$ *represents the arithmetic mean of the approximated eigenvalues of* ω_ℓ *and* η *is given by Proposition 2.10.*

Proof. The convergence (2.46a) ensues immediately by virtue of Theorem 2.2. On the other hand, using the Lagrange elements (1.6), to discretize the functional spaces, and a classical interpolation property between Sobolev spaces of positive order (see for example [19], Sect. 4) we deduce that there exist strictly positive constants C and h_0 such that $\forall h \leq h_0$, $\forall(\mathbf{u}, p) \in E_\ell$

$$\inf_{(\mathbf{v}_h, q_h) \in H_h \times L^2_{0h}} \|(\mathbf{u}, p) - (\mathbf{v}_h, q_h)\|_{H \times L^2_0(\Omega)}$$
$$\leq C\, h^\eta \,\|(\mathbf{u}, p)\|_{H^{1+\eta}(\Omega)^2 \times H^\eta(\Omega)}\,,$$

where the existence of at least one number $\eta \in (0, 1]$ is given by Proposition 2.10. From this last inequality, we may straightforward to conclude the desired estimates. $\square$

Remark that if, for all $\ell \in \mathbb{N}$, the spaces E_ℓ of generalized eigenvectors and E^*_ℓ of generalized adjoint eigenvectors associated with the eigenvalue ω_ℓ satisfy: $E_\ell \subset H^2(\Omega)^2 \times H^1(\Omega) \cap H \times L^2_0(\Omega)$ and $E^*_\ell \subset H^2(\Omega)^2 \times H^1(\Omega) \cap H \times L^2_0(\Omega)$, then the following inequality holds

$$\|(\mathbf{u}, p) - \Pi_h(\mathbf{u}, p)\|_{H \times L^2_0(\Omega)} \leq C\, h \,\|(\mathbf{u}, p)\|_{H^2(\Omega)^2 \times H^1(\Omega)}\,,$$

that is $\eta = 1$. Also, it is important to note that we have not proved optimality of the inequalities (2.46b,c,d).

Several open question are left. The operator T is not selfadjoint, nor normal, however numerical experiments suggest that the ascent α of T is one, which seems not obvious to prove. It is also interesting to study the asymptotic behaviour of the spectrum as the physical parameter K (the number of tubes in the structure) goes to infinity. This question is not trivial because we do not have the spectral decomposition associated to the operator T.

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