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On uniform H^2-estimates in periodic homogenization

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We introduce what are called regular materials for which, by definition, the corresponding solution of the classical periodic homogenization problem remains bounded in H^2_{loc}. We give examples of two types of such materials depending on whether the coefficients representing them belong to $W^{1,\infty}$ or not. A complete characterization is obtained in the former case.

1. Introduction

We consider the operator

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial y_k}\biggl(a_{k\ell}(y)\frac{\partial}{\partial y_\ell}\biggr), \quad y \in \mathbb{R}^N,$$

where the matrix $a(y) = [a_{k\ell}(y)]$ is symmetric and positive definite $a(y) \geqslant \alpha I$ with $\alpha > 0$. Its entries belong to $L^\infty_\#(Y)$ (Y is the cube $]0, 2\pi[^N$ and subscript '#' means that the space consists of Y-periodic functions). In the sequel, we will make various further length-scale regularity assumptions on the coefficients $a_{k\ell}$ which play a central role throughout the paper. Some of them are as follows:

$$a_{k\ell} \in W^{1,\infty}_\#(Y) \quad \forall k, \ell = 1, \dots, N, \tag{1.1}$$

$$\frac{\partial a_{k\ell}}{\partial y_k} \in L^\infty_\#(Y) \qquad \forall \ell = 1, \dots, N. \tag{1.2}$$

In (1.2) above, and throughout this paper, usual summation convention with respect to repeated indices is followed, unless stated otherwise explicitly. For each $\varepsilon > 0$, we introduce the operator A^ε, where

$$A^\varepsilon \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\biggl(a^\varepsilon_{k\ell}(x)\frac{\partial}{\partial x_\ell}\biggr) \quad \text{with } a^\varepsilon_{k\ell}(x) = a_{k\ell}\biggl(\frac{x}{\varepsilon}\biggr), \quad x \in \mathbb{R}^N.$$

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary domain and let $f \in H^{-1}(\Omega)$. We consider a sequence u^ε in $H^1(\Omega)$ such that

$$\left.\begin{aligned} A^\varepsilon u^\varepsilon = f \qquad & \text{in } \Omega, \\ u^\varepsilon \rightharpoonup u^* \qquad & \text{in } H^1(\Omega)\text{-weak.} \end{aligned}\right\} \tag{1.3}$$

Our aim in this work is to seek necessary and sufficient conditions on $a(y)$ for u^ε to be bounded in $H^2_{\text{loc}}(\Omega)$ under the hypothesis that

$$f \in L^2_{\text{loc}}(\Omega) \tag{1.4}$$

(see theorem 4.3 below). Even when the hypothesis (1.1) holds good, classical results (see the book by Gilberg and Trudinger [6, p. 175]) when applied to (1.3) yield the estimate

$$\|u^\varepsilon\|_{H^2_{\text{loc}}(\Omega)} \leqslant c\varepsilon^{-1}\|f\|_{L^2_{\text{loc}}(\Omega)},$$

which is clearly not uniform in ε. On the other hand, in the classical book by Bensoussan *et al.* [1], the authors obtain an asymptotic expansion (with $y = x/\varepsilon$) of the form

$$\begin{aligned} u^\varepsilon(x) = u^*(x) + \varepsilon\left\{\chi_\ell(y)\frac{\partial u^*}{\partial x_\ell}(x) + \tilde{u}_1(x)\right\} \\ + \varepsilon^2\left\{\chi_{\ell m}(y)\frac{\partial^2 u^*}{\partial x_\ell \partial x_m}(x) + \chi_\ell(y)\frac{\partial \tilde{u}_1}{\partial x_\ell}(x) + \tilde{u}_2(x)\right\} + \cdots. \end{aligned} \tag{1.5}$$

Here, χ_ℓ is the unique solution of the cell problem

$$\left.\begin{aligned} A\chi_\ell = \frac{\partial a_{k\ell}}{\partial y_k} \qquad & \text{in } Y, \\ \chi_\ell \in H^1_\#(Y), \qquad & \mathcal{M}_Y(\chi_\ell) \stackrel{\text{def}}{=} \frac{1}{|Y|}\int_Y \chi_\ell \, \mathrm{d}y = 0. \end{aligned}\right\} \tag{1.6}$$

The function $\chi_{\ell m}$ is characterized as the unique solution of

$$A\chi_{\ell m} = a_{\ell m} + a_{\ell k}\frac{\partial \chi_m}{\partial y_k} - \frac{\partial}{\partial y_k}(a_{k\ell}\chi_m) - \mathcal{M}_Y(a_{\ell m}) - \mathcal{M}_Y\left(a_{\ell k}\frac{\partial \chi_m}{\partial y_k}\right) \quad \text{in } Y,$$
$$\chi_{\ell m} \in H^1_\#(Y), \quad \mathcal{M}_Y(\chi_{\ell m}) = 0.$$

The first term in (1.5) satisfies the homogenized equation

$$A^* u^* \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(q_{k\ell}\frac{\partial u^*}{\partial x_\ell}\right) = f \quad \text{in } \Omega,$$

where the homogenized coefficients $q_{k\ell}$ are given by

$$q_{k\ell} = \mathcal{M}_Y\left(a_{k\ell} + a_{km}\frac{\partial \chi_\ell}{\partial y_m}\right) \quad \forall k, \ell = 1, \dots, N.$$

The above method also proves that $\tilde{u}_1(x), \tilde{u}_2(x), \dots$ are independent of ε and satisfy equations of the type $A^*\tilde{u}_j = \tilde{g}_j$ in Ω, where, for instance,

$$\tilde{g}_1(x) = b_{jk\ell}\frac{\partial^3 u^*}{\partial x_j \partial x_k \partial x_\ell}(x),$$

where $b_{jk\ell}$ are constants,

$$b_{jk\ell} = \mathcal{M}_Y\left(a_{jm}\frac{\partial \chi_{k\ell}}{\partial y_m} + a_{k\ell}\chi_j \right) \quad \forall j,k,\ell = 1,\dots,N.$$

Some comments on the expansion (1.5) are now in order. Because it contains infinitely many terms, it is not very useful to establish H^2_{loc}-estimates on u^ε, even though the effects of f and the medium are separated at various powers of ε. However, it offers important insight into the difficulties in the offing. For instance, it shows that the second-order derivatives of the ε-term involve

$$\varepsilon^{-1}\frac{\partial^2 \chi_\ell}{\partial y_j \partial y_k}(y)\frac{\partial u^*}{\partial x_\ell}(x), \qquad \frac{\partial \chi_\ell}{\partial y_k}\frac{\partial^2 u^*}{\partial x_j \partial x_\ell}(x), \qquad \varepsilon\chi_\ell(y)\frac{\partial^3 u^*}{\partial x_j \partial x_k \partial x_\ell}(x).$$

Because of the presence of the negative power of ε in the first of these terms, an easy way out of the difficulty is to annihilate it by requiring that $\chi_\ell = 0 \ \forall \ell = 1,\dots,N$. At one stroke, this eliminates other terms as well. However, it is not at all clear whether this condition is going to be sufficient to overcome the difficulties coming from higher powers of ε. This is due to the following reasons.

(i) The second-order derivatives of the terms containing ε^2 involve derivatives of u^* of higher order on which we have no control with our hypothesis (1.4). More and more higher-order derivatives of u^* appear, and thus this difficulty is amplified when we go up in powers of ε.

(ii) The second-order derivatives of u^* are multiplied by the second-order derivatives of $\chi_{k\ell}$ and so it is natural to require that $\chi_{k\ell} \in W^{2,\infty}_{\#}(Y)$. It is classically known that such a regularity result involving $L^\infty(Y)$-space is hard to come by if it is not impossible.

These fundamental issues and difficulties may be the reason why H^2-regularity questions have not been tackled in the literature. Thus the results obtained in this work seem to be new and are not easily obtainable using other classical methods. Given the above picture of difficulties, our results may be interpreted as follows: individually considered, the above troublesome terms are not in H^2_{loc} under the hypothesis (1.4); however, taken together, they seem to behave nicely.

The plan of this short paper is as follows. Bloch waves (which are our tool to understand the issues involved) are introduced in § 2. In § 3, we rapidly introduce a condition on the coefficients which seems necessary to eliminate the effects of the boundary. Main results are stated in the next section in the form of several theorems. Their proofs are presented in §§ 5 and 6. Notion of regular materials is introduced in § 7, wherein examples of such materials are also furnished.

Finally, a word about the notation adopted in this work. The constants appearing in various estimates independent of ε are generically denoted by c, c_1, c_2, etc. Apart from the usual norms in Sobolev spaces $H^1(Y)$, $H^2(Y)$, we will also use the following semi-norms,

$$|v|_{H^1(Y)} = \left\{ \sum_{j=1}^{N} \left\| \frac{\partial v}{\partial y_j} \right\|^2_{L^2(Y)} \right\}^{1/2}, \qquad |v|_{H^2(Y)} = \left\{ \sum_{k,\ell=1}^{N} \|D^2_{k,\ell} v\|^2_{L^2(Y)} \right\}^{1/2},$$

where $D^2_{k,\ell} v = \partial^2 v / \partial y_k \partial y_\ell$.

2. Bloch waves

To overcome the difficulties mentioned in § 1, we will adapt a different strategy, which involves the use of the Bloch waves associated with A. We have used them in the homogenization of eigenvalue problems and boundary-value problems associated with elliptic operators (see [3, 4]). For an application of Bloch waves in the case of Schrödinger equation with periodic potential, we refer to [5].

The basic idea of the method consists in representing the solution in terms of Bloch waves. We will thereby be able to transfer the questions of estimates on u^ε or its derivatives in an equivalent way to that of Bloch waves via Parseval's identity. Thus the method yields optimal results. Bloch waves are defined as eigenvectors of the problem

$$\left.\begin{aligned} A(\eta)\phi(\cdot;\eta) = \lambda(\eta)\phi(\cdot;\eta) \quad \text{in } \mathbb{R}^N,\\ \phi(\cdot;\eta) \text{ is } Y\text{-periodic,} \end{aligned}\right\} \tag{2.1}$$

where $A(\eta)$ is the following operator,

$$A(\eta) \stackrel{\text{def}}{=} -\left(\frac{\partial}{\partial y_k} + \mathrm{i}\eta_k\right)\left[a_{k\ell}(y)\left(\frac{\partial}{\partial y_\ell} + \mathrm{i}\eta_\ell\right)\right],$$

and η is the Bloch parameter confined to $Y' = [-\frac{1}{2}, \frac{1}{2}[^N$. This operator is referred to as the *shifted operator* in the literature. As is well known, for each η, these eigenvectors form a countable orthonormal basis in $L^2_\#(Y)$, and they are denoted by $\{\phi_m(\cdot;\eta)\}_{m=1}^\infty$:

$$\int_Y \phi_m(y;\eta)\bar{\phi}_{m'}(y;\eta)\,\mathrm{d}y = \delta_{mm'}.$$

The corresponding eigenvalues form a countable sequence with the following properties:

$$\left.\begin{aligned} 0 \leqslant \lambda_1(\eta) \leqslant \cdots \leqslant \lambda_m(\eta) \leqslant \cdots \to \infty,\\ \forall m \geqslant 1,\ \lambda_m(\eta) \text{ defines a Lipschitz continuous function of } \eta \text{ in } Y'. \end{aligned}\right\} \tag{2.2}$$

With the help of the above parametrized eigenvalues and eigenfunctions, one can describe the spectral resolution of A as an unbounded self-adjoint operator in $L^2(\mathbb{R}^N)$. Roughly, the results are as follows,

$$\{\mathrm{e}^{\mathrm{i}y\cdot\eta}\phi_m(y;\eta) \mid m \geqslant 1,\ \eta \in Y'\} \text{ forms a basis of } L^2(\mathbb{R}^N) \text{ in a generalized sense,}$$

and $L^2(\mathbb{R}^N)$ can be identified with $L^2(Y';\ell^2(\mathbb{N}))$ via Parseval's identity. The operator A corresponds to an operator with multipliers $\lambda_m(\eta)$,

$$A(\mathrm{e}^{\mathrm{i}y\cdot\eta}\phi_m(y;\eta)) = \lambda_m(\eta)\mathrm{e}^{\mathrm{i}y\cdot\eta}\phi_m(y;\eta).$$

What we need below are Bloch waves $\{\phi^\varepsilon_m(x;\xi)\}$ in the ε-scale and the corresponding eigenvalues $\{\lambda^\varepsilon_m(\xi)\}$. By homothecy, the following relations hold,

$$\lambda^\varepsilon_m(\xi) = \varepsilon^{-2}\lambda_m(\eta), \qquad \phi^\varepsilon_m(x;\xi) = \phi_m(y;\eta),$$

where (x,ξ) and (y,η) are related by

$$y = \frac{x}{\varepsilon}, \qquad \eta = \varepsilon\xi.$$

It is clear that ξ varies in $\varepsilon^{-1}Y'$. The spectral decomposition of A^ε is described in terms of these waves in the following result, a proof of which can be found, for example, in [1] or [4].

THEOREM 2.1. *Let $g \in L^2(\mathbb{R}^N)$. The mth Bloch coefficient of g is defined as follows:*

$$B_m^\varepsilon g(\xi) = \int_{\mathbb{R}^N} g(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\bar{\phi}_m\left(\frac{x}{\varepsilon};\varepsilon\xi\right)\mathrm{d}x \quad \forall\xi \in \varepsilon^{-1}Y', \quad m \geqslant 1. \tag{2.3}$$

Then the following inverse formula holds:

$$g(x) = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} B_m^\varepsilon g(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\phi_m\left(\frac{x}{\varepsilon};\varepsilon\xi\right)\mathrm{d}\xi \quad \forall x \in \mathbb{R}^N. \tag{2.4}$$

Further, we have Parseval's identity,

$$\int_{\mathbb{R}^N} |g(x)|^2\,\mathrm{d}x = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} |B_m^\varepsilon g(\xi)|^2\,\mathrm{d}\xi. \tag{2.5}$$

The above result, which has been exploited by us in homogenization (see [3]), does not seem to be fully adequate for the purposes of the present paper. As will be pointed out later, some intermediate steps leading to the above-mentioned result are necessary.

We present these now. For full details, the reader is referred to the literature already cited (see, for example, [4, paragraph III.2]).

THEOREM 2.2. *Any $g \in L^2(\mathbb{R}^N)$ can be decomposed as*

$$g(x) = \varepsilon^N \int_{\varepsilon^{-1}Y'} g_\#^\varepsilon(x;\xi)\,\mathrm{d}\xi, \tag{2.6}$$

where $g_\#^\varepsilon$ is $(\varepsilon\xi, \varepsilon Y)$-periodic with respect to x (see [4, p. 187]), i.e.

$$g_\#^\varepsilon(x + 2\pi\varepsilon q;\xi) = \mathrm{e}^{2\pi\varepsilon q\cdot\xi} g_\#^\varepsilon(x;\xi) \quad \forall q \in \mathbb{Z}^N.$$

Further, we have Parseval's relation,

$$\int_{\mathbb{R}^N} |g(x)|^2\,\mathrm{d}x = \varepsilon^N \int_{\varepsilon Y}\int_{\varepsilon^{-1}Y'} |g_\#^\varepsilon(x;\xi)|^2\,\mathrm{d}\xi\mathrm{d}x. \tag{2.7}$$

REMARK 2.3. The element $g_\#^\varepsilon$ associated to g in theorem 2.2 is, in fact, explicitly given by

$$g_\#^\varepsilon(x;\xi) = \sum_{p\in\mathbb{Z}^N} g(x + 2\pi\varepsilon p)\mathrm{e}^{-2\pi\mathrm{i}\varepsilon p\cdot\xi}. \tag{2.8}$$

For fixed ξ, expanding this in terms of the basis functions $\{\mathrm{e}^{\mathrm{i}x\cdot\xi}\phi_m^\varepsilon(x,\xi) \mid m \geqslant 1\}$, we arrive at theorem 2.1.

3. Localization

Since we are dealing with estimates in $H^2_{\mathrm{loc}}(\Omega)$, the first step in the method is to reduce to the case $\Omega = \mathbb{R}^N$ by means of a cut-off function $\theta \in \mathcal{D}(\Omega)$. Indeed, a simple calculation shows that

$$A^\varepsilon(\theta u^\varepsilon) = \theta f + g^\varepsilon + h^\varepsilon \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

where

$$g^\varepsilon = -2a^\varepsilon_{k\ell}\frac{\partial u^\varepsilon}{\partial x_\ell}\frac{\partial \theta}{\partial x_k} - a^\varepsilon_{k\ell}\frac{\partial^2\theta}{\partial x_k \partial x_\ell}u^\varepsilon,$$

and

$$h^\varepsilon = -\frac{\partial a^\varepsilon_{k\ell}}{\partial x_k}\frac{\partial \theta}{\partial x_\ell}u^\varepsilon.$$

It is easily seen that the right-hand side of (3.1) is bounded in $L^2(\mathbb{R}^N)$, provided

$$\frac{\partial a_{k\ell}}{\partial y_k} = 0 \quad \text{in } \mathbb{R}^N \quad \forall \ell = 1, \dots, N. \tag{3.2}$$

This hypothesis is obviously equivalent to saying that $\chi_\ell = 0$ for all ℓ (see (1.6)). It is somewhat surprising that the above condition, which we came across in § 1 for obtaining H^2_{loc}-estimates, appears in the localization process, too.

The above reduction procedure leads us to the following question. Consider $u^\varepsilon \rightharpoonup u^*$ in $H^1(\mathbb{R}^N)$ weak such that $A^\varepsilon u^\varepsilon = f$ in $\mathbb{R}^N$. Under what conditions on $a(\cdot)$ does there exist a constant $c > 0$ independent of ε such that

$$|u^\varepsilon|_{H^2(\mathbb{R}^N)} \leqslant c\|f\|_{L^2(\mathbb{R}^N)}? \tag{3.3}$$

We give an answer to this question in theorem 4.3 below.

4. Main results

The aim of this section is to state the principal results of this paper. Proof of some of these results are indicated, while others are deferred to later sections. We start with a consequence of theorem 2.1, which established that the operator A^ε corresponds, under Bloch transform, to an operator on $L^2(\varepsilon^{-1}Y'; \ell^2(\mathbb{N}))$ given by multipliers $\{\lambda^\varepsilon_m(\xi) \mid m \geqslant 1,\ \xi \in \varepsilon^{-1}Y'\}$. This can be considered as a *diagonal integral operator* on $\ell^2(\mathbb{N})$ parametrized by $\xi \in \varepsilon^{-1}Y'$. The following result shows that the operator $\partial^2/\partial x_k \partial x_\ell$, for fixed $k, \ell = 1, \dots, N$, corresponds to a more general integral operator on $\ell^2(\mathbb{N})$.

THEOREM 4.1. *For $k, \ell = 1, \dots, N$, we have*

$$\frac{\partial^2 g}{\partial x_k \partial x_\ell}(x) = \varepsilon^{-2}\int_{\varepsilon^{-1}Y'}\sum_{n=1}^\infty\left\{\sum_{m=1}^\infty B^\varepsilon_m g(\xi)\alpha_{mn}(\varepsilon\xi)\right\}\mathrm{e}^{\mathrm{i}x\cdot\xi}\phi^\varepsilon_n(x;\xi)\,\mathrm{d}\xi,$$

for some coefficients $\{\alpha_{mn}(\varepsilon\xi) \mid m \geqslant 1,\ n \geqslant 1,\ \xi \in \varepsilon^{-1}Y'\}$. Therefore, by Parseval's relation, we get

$$\left\|\frac{\partial^2 g}{\partial x_k \partial x_\ell}\right\|^2_{L^2(\mathbb{R}^N)} = \varepsilon^{-4}\int_{\varepsilon^{-1}Y'}\sum_{n=1}^\infty\left|\sum_{m=1}^\infty B^\varepsilon_m g(\xi)\alpha_{mn}(\varepsilon\xi)\right|^2\mathrm{d}\xi.$$

Proof. Differentiating (2.4) with respect to x_k and x_ℓ, we get

$$\frac{\partial^2 g}{\partial x_k \partial x_\ell}(x) = \varepsilon^{-2} \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} B_m^\varepsilon g(\xi) \mathrm{e}^{\mathrm{i}x\cdot\xi} \rho_m\left(\frac{x}{\varepsilon}; \varepsilon\xi\right) \mathrm{d}\xi,$$

where we have set

$$\rho_m(y;\eta) = -\eta_k \eta_\ell \phi_m(y;\eta) + \mathrm{i}\left\{ \eta_\ell \frac{\partial \phi_m}{\partial y_k}(y;\eta) + \eta_k \frac{\partial \phi_m}{\partial y_\ell}(y;\eta) \right\} + \frac{\partial^2 \phi_m}{\partial y_k \partial y_\ell}(y;\eta).$$

Since $\rho_m(\cdot;\eta)$ is Y-periodic, we can expand it in terms of the Bloch basis $\{\phi_n(\cdot,\eta) \mid n \geqslant 1\}$,

$$\rho_m(y;\eta) = \sum_{n=1}^{\infty} \alpha_{mn}(\eta) \phi_n(y;\eta) \quad \forall m \geqslant 1,$$

with the coefficients α_{mn} defined by

$$\alpha_{mn}(\eta) = \int_Y \rho_m(y;\eta) \bar{\phi}_n(y;\eta) \, \mathrm{d}y.$$

The proof in now complete. □

Though we will need the above result in the sequel for other purposes, it does not seem to be useful in estimating the second-order derivatives $D^2_{k,\ell} u^\varepsilon$ in $L^2(\mathbb{R}^N)$ because it has the drawback of mixing the indices m, n. In other words, the operator $D^2_{k,\ell}$ is not diagonal in the decomposition of theorem 2.1. We prefer a decomposition invariant under the above operator. The decomposition provided by theorem 2.2 seems to be more suitable for the above purpose, since it is invariant under $D^2_{k,\ell}$ and A^ε. Indeed, it follows directly that

$$D^2_{k,\ell} g^\varepsilon_\#(x;\xi) = (D^2_{k,\ell} g)^\varepsilon_\#(x;\xi)$$

and, as a consequence, it follows that

$$D^2_{k,\ell} g(x) = \varepsilon^N \int_{\varepsilon^{-1}Y'} (D^2_{k,\ell} g)^\varepsilon_\#(x;\xi) \, \mathrm{d}\xi, \tag{4.1}$$

$$\int_{\mathbb{R}^N} |D^2_{k,\ell} g(x)|^2 \, \mathrm{d}x = \varepsilon^N \int_{\varepsilon Y} \int_{\varepsilon^{-1}Y'} |(D^2_{k,\ell} g)^\varepsilon_\#(x;\xi)|^2 \, \mathrm{d}\xi \, \mathrm{d}x. \tag{4.2}$$

Applying the above general decomposition results, we see that our equation $A^\varepsilon u^\varepsilon = f$ in $\mathbb{R}^N$ is equivalent to the following parametrized problems:

$$\left.\begin{array}{c} A^\varepsilon u^\varepsilon_\#(x;\xi) = f^\varepsilon_\#(x;\xi) \quad \text{for } x \in \mathbb{R}^N, \\ u^\varepsilon_\#(\cdot;\xi) \text{ is } (\varepsilon\xi, \varepsilon Y)\text{-periodic.} \end{array}\right\} \tag{4.3}$$

Our main result is the following estimate on the above problem. There exists a constant $c_2 > 0$ independent of $\varepsilon > 0$ and $\xi \in \varepsilon^{-1}Y'$, but depending on $\|a\|_{W^{1,\infty}(Y)}$, such that

$$\|D^2_{k,\ell} u^\varepsilon_\#(\cdot;\xi)\|_{L^2(\varepsilon Y)} \leqslant c_2 \|f^\varepsilon_\#(\cdot;\xi)\|_{L^2(\varepsilon Y)}, \tag{4.4}$$

provided (3.2) is satisfied.

Our strategy of proving (4.4) consists of transforming the problem (4.3) into a Y-periodic problem by means of the change of variables

$$u^\varepsilon_\#(x;\xi) = \mathrm{e}^{\mathrm{i}x\cdot\xi} U^\varepsilon(y;\eta), \qquad f^\varepsilon_\#(x;\xi) = \mathrm{e}^{\mathrm{i}x\cdot\xi} F^\varepsilon(y;\eta). \tag{4.5}$$

This idea is standard in Bloch analysis. It is well known that U^ε and F^ε satisfy

$$\left.\begin{aligned} A(\eta)(\varepsilon^{-2}U^\varepsilon(y;\eta)) = F^\varepsilon(y;\eta) \quad \text{for } y \in \mathbb{R}^N,\\ U^\varepsilon(\cdot;\eta) \text{ is } Y\text{-periodic.} \end{aligned}\right\} \tag{4.6}$$

Our following theorem is concerned with establishing estimates on problems of the type (4.6).

THEOREM 4.2. *Consider the problem where $a \in W^{1,\infty}(Y)$,*

$$\left.\begin{aligned} A(\eta)U = F \quad \text{in } \mathbb{R}^N,\\ U \text{ is } Y\text{-periodic.} \end{aligned}\right\} \tag{4.7}$$

Then there exists a constant $c_2 > 0$ depending on $\|a\|_{W^{1,\infty}(Y)}$, but independent of $\eta \in Y'$, such that

$$|\eta|^2\|U\|_{L^2(Y)} + |\eta||U|_{H^1(Y)} + |\eta||U|_{H^2(Y)} \leqslant c_2\|F\|_{L^2(Y)}.$$

If, in addition, we suppose (3.2), then

$$|U|_{H^2(Y)} \leqslant c_2\|F\|_{L^2(Y)}.$$

A proof of this result will be presented in the next section. For the moment, we observe the singular behaviour of the second-order derivatives of U as $\eta \to 0$. The assertion is that it disappears if we suppose (3.2). A simple application of the above theorem to problem (4.6) yields the desired estimate (4.4). Indeed,

$$\begin{aligned} \|D^2_{k,\ell}u^\varepsilon_\#(\cdot;\xi)\|^2_{L^2(\varepsilon Y)} &\leqslant c\varepsilon^N\{|\xi|^4\|U^\varepsilon(\cdot;\eta)\|^2_{L^2(Y)} \\ &\qquad + \varepsilon^{-2}|\xi|^2|U^\varepsilon(\cdot;\eta)|^2_{H^1(Y)} + \varepsilon^{-4}|U^\varepsilon(\cdot;\eta)|^2_{H^2(Y)}\} \\ &\leqslant c_2\varepsilon^N\|F^\varepsilon(\cdot;\eta)\|^2_{L^2(Y)} \qquad \text{(by theorem 4.2)} \\ &= c_2\|f^\varepsilon_\#(\cdot;\xi)\|^2_{L^2(\varepsilon Y)}. \end{aligned}$$

Thanks to (2.7), it is an easy matter to deduce the required estimate from (4.4). More precisely, we have the following result.

THEOREM 4.3. *We assume that the coefficients satisfy (1.1) and (3.2). Then there exists a constant $c_2 > 0$ independent of $\varepsilon > 0$, but depending on $\|a\|_{W^{1,\infty}(Y)}$, such that, for all $f \in L^2(\mathbb{R}^N)$, we have*

$$|u^\varepsilon|_{H^2(\mathbb{R}^N)} \leqslant c_2\|f\|_{L^2(\mathbb{R}^N)}.$$

Our next result presents estimates on the Bloch eigenvectors $\phi_m(\cdot;\eta)$. We highlight the singular behaviour of the first Bloch mode $\phi_1(\cdot;\eta)$ as $\eta \to 0$. Though the Bloch eigenvector problem (2.1) is of the form (4.7), the following estimate does not immediately follow from the one stated in theorem 4.2. This is one of the reasons why we prefer to state this result separately.

THEOREM 4.4.

(i) *There is a constant c_1 depending on $\|a\|_{L^\infty(Y)}$, and independent of m and η, such that*

$$|\phi_m(\cdot;\eta)|_{H^1(Y)} \leqslant c_1\lambda_m(\eta)^{1/2} \quad \forall\eta \in Y', \quad m \geqslant 1.$$

(ii) *There is a constant c_2 depending on $\|a\|_{W^{1,\infty}(Y)}$, but independent of m and η, such that*

$$|\phi_1(\cdot;\eta)|_{H^2(Y)} \leqslant c_2\lambda_1(\eta)^{1/2} \quad \forall\eta \in Y',$$
$$|\phi_m(\cdot;\eta)|_{H^2(Y)} \leqslant c_2\lambda_m(\eta) \quad \forall\eta \in Y', \quad m \geqslant 2.$$

(iii) *With the additional hypotheses (3.2), we have*

$$|\phi_1(\cdot;\eta)|_{H^2(Y)} \leqslant c_2\lambda_1(\eta) \quad \forall\eta \in Y'. \tag{4.8}$$

Theorem 4.3 above establishes the sufficiency of the condition (3.2) to have H^2-estimates uniform with respect to ε. We now proceed to show the necessity of the condition. It is worthwhile to recall that the heuristic arguments advanced in § 1 already indicate this fact. Our aim here is to show how this property can be rigorously deduced using Bloch analysis. The first result in this direction is the following converse of theorem 4.4.

THEOREM 4.5. *Assume that the coefficients $a_{k\ell}$ have the regularity (1.1). If the first Bloch mode satisfies the estimate (4.8), then condition (3.2) is true.*

Our next result gives an equivalent formulation of the uniform H^2-estimates. In doing so, we find theorem 4.1 very useful.

THEOREM 4.6. *We assume (1.1). Then the following statements are equivalent.*

(A) *The estimate (3.3) holds for all $f \in L^2(\mathbb{R}^N)$.*

(B) *There exists a constant $c > 0$ independent of $\eta \in Y'$ such that*

$$\sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty}\beta_m\frac{\alpha_{mn}(\eta)}{\lambda_m(\eta)}\right|^2 \leqslant c\sum_{m=1}^{\infty}|\beta_m|^2 \quad \forall\beta = (\beta_m) \in \ell^2(\mathbb{N}),$$

where the coefficients $\alpha_{mn}(\eta)$ are the ones introduced in theorem 4.1.

As a corollary of the previous two results, we will deduce our final conclusion.

THEOREM 4.7. *We assume (1.1). Then the following statements are equivalent.*

(i) *The condition (3.2) holds.*

(ii) *There exists a constant $c_2 > 0$ depending on $\|a\|_{W^{1,\infty}(Y)}$, but independent of ε, such that, for all $f \in L^2(\mathbb{R}^N)$, we have*

$$|u^\varepsilon|_{H^2(\mathbb{R}^N)} \leqslant c_2\|f\|_{L^2(\mathbb{R}^N)}.$$

Classically, it is well known that under the hypothesis (3.2), one can pass to the limit in the product $\sigma_k^\varepsilon = a_{k\ell}^\varepsilon(\partial u^\varepsilon/\partial x_\ell)$ in $L^2(\mathbb{R}^N)$-weak by using the compensated-compactness theory of Murat [8] and Tartar [9]. The new aspect of the above theorem is the H^2-estimate, which allows the use of Rellich's lemma to pass to the limit and obtain

$$a_{k\ell}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_\ell} \rightharpoonup \mathcal{M}_Y(a_{k\ell}) \frac{\partial u^*}{\partial x_\ell} \quad \text{in } L^2(\mathbb{R}^N)\text{-weak.}$$

Since (3.2) implies $\chi_\ell = 0$ for all ℓ, we have $\mathcal{M}_Y(a_{k\ell}) = q_{k\ell}$, and the classical homogenization result is therefore recovered from the above convergence result. Of course, it should be mentioned that compensated-compactness method is very general and goes beyond the case of periodically oscillating coefficients.

As an immediate application of our theorem and the Hardy space regularity result of Coifman *et al.* [2, paragraph III.2, p. 258], we can deduce that

$$\sigma_k^\varepsilon \text{ is bounded in } \mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}^N),$$

where $\mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}^N)$ is the localized Hardy space. The above bound on σ_k^ε is better than the classical L^2-estimate on σ_k^ε, especially at points where σ_k^ε vanishes.

5. Estimates on cell problems

This section in devoted to the proof of theorems 4.2, 4.4, 4.5. We introduce the bilinear form associated with the operator $A(\eta)$

$$b(\eta;\phi,\psi) = \int_Y a_{k\ell}(y)\left(\frac{\partial\phi}{\partial y_\ell} + \mathrm{i}\eta_\ell\phi\right)\overline{\left(\frac{\partial\psi}{\partial y_k} + \mathrm{i}\eta_k\psi\right)}\,\mathrm{d}y$$

for all $\phi,\psi \in H^1_\#(Y)$. The basic estimate on this bilinear form is as follows (cf. [4, p. 190]. For all $\phi \in H^1_\#(Y)$ and $\eta \in Y'$, we have

$$d_1(|\phi|^2_{H^1(Y)} + |\eta|^2\|\phi\|^2_{L^2(Y)}) \leqslant b(\eta;\phi,\phi) \leqslant d_2(|\phi|^2_{H^1(Y)} + |\eta|^2\|\phi\|^2_{L^2(Y)}), \tag{5.1}$$

where d_1 and d_2 depend only on α and $\|a\|_{L^\infty(Y)}$. As a consequence, we have the Poincaré inequality

$$|\phi|_{H^1(Y)} + |\eta|\|\phi\|_{L^2(Y)} \leqslant c\|\nabla\phi + \mathrm{i}\eta\phi\|_{L^2(Y)} \quad \forall\phi \in H^1_\#(Y). \tag{5.2}$$

Here, c is a constant depending only on α and $\|a\|_{L^\infty(Y)}$.

Proof of theorem 4.2. Multiplying (4.7) by U and using the above estimates, we deduce

$$\begin{aligned}\|\nabla U + \mathrm{i}\eta U\|^2_{L^2(Y)} &\leqslant \frac{1}{\alpha}\|F\|_{L^2(Y)}\|U\|_{L^2(Y)} \\ &\leqslant \frac{c}{\alpha|\eta|}\|F\|_{L^2(Y)}\|\nabla U + \mathrm{i}\eta U\|_{L^2(Y)}.\end{aligned}$$

Thus

$$\|\nabla U + \mathrm{i}\eta U\|_{L^2(Y)} \leqslant \frac{c}{|\eta|}\|F\|_{L^2(Y)}.$$

The above inequality immediately implies

$$|\eta|^2\|U\|_{L^2(Y)} + |\eta||U|_{H^1(Y)} \leqslant c\|F\|_{L^2(Y)}. \tag{5.3}$$

So far, we have not used our hypothesis that $a \in W^{1,\infty}(Y)$. We will use it now to estimate $|U|_{H^2(Y)}$. To this end, we rewrite (4.7) as

$$\left.\begin{aligned} AU &= \tilde{F}, \\ U &\text{ is } Y\text{-periodic,} \end{aligned}\right\} \tag{5.4}$$

where

$$\tilde{F} = F + \mathrm{i}\eta_k a_{k\ell}\frac{\partial U}{\partial y_\ell} + \mathrm{i}\eta_\ell \frac{\partial}{\partial y_k}(a_{k\ell}U) - a_{k\ell}\eta_k\eta_\ell U.$$

Thanks to (5.3), we see that

$$\|\tilde{F}\|_{L^2(Y)} \leqslant c\left\{\|F\|_{L^2(Y)} + |\eta|\left\|\frac{\partial a_{k\ell}}{\partial y_k}\right\|_{L^\infty(Y)}\|U\|_{L^2(Y)}\right\}. \tag{5.5}$$

Multiplying the above inequality by $|\eta|$ and using again (5.3), we deduce

$$|\eta|\|\tilde{F}\|_{L^2(Y)} \leqslant c\left(1 + \|\frac{\partial a_{k\ell}}{\partial y_k}\|_{L^\infty(Y)}\right)\|F\|_{L^2(Y)}. \tag{5.6}$$

On the other hand, regarding the problem (5.4), it is well known (see, for instance, [6, p. 173]) that under the hypothesis (1.1) there exists a constant $c_2 = c_2(\|a\|_{W^{1,\infty}(Y)})$ such that

$$|U|_{H^2(Y)} \leqslant c_2\|\tilde{F}\|_{L^2(Y)}. \tag{5.7}$$

Combination of the estimates (5.5)–(5.7) completes the proof of theorem 4.2. □

Proof of theorem 4.4. The technique of the proof is the same as the one followed for the proof of theorem 4.2. The only difference is that we need to use the additional information on the behaviour of Bloch eigenvalues,

$$d_1|\eta|^2 \leqslant \lambda_m(\eta) \quad \forall m \geqslant 1, \quad \eta \in Y', \tag{5.8}$$

$$0 < \lambda \leqslant \lambda_m(\eta) \quad \forall m \geqslant 2, \quad \eta \in Y'. \tag{5.9}$$

While (5.8) is a direct consequence of (5.1), the inequality (5.9) is a consequence of min–max principle of eigenvalues. It is proved in [3, p. 1653] that (5.9) holds, where λ is the second eigenvalue for A in the cell Y with Neumann boundary condition on ∂Y. It is also proved in [3] that

$$\left.\begin{aligned} \lambda_1(0) = \lambda_1'(0) &= 0, \\ \lambda_1(\eta) &\leqslant c|\eta|^2. \end{aligned}\right\} \tag{5.10}$$

The above inequalities clearly illustrate the singular behaviour of the first eigenvalue which is distinct from the rest of the eigenvalues. This accounts for the singular behaviour of the first Bloch mode $\phi_1(\cdot;\eta)$ as $\eta \to 0$ stated in theorem 4.4.

The estimate on $|\phi_m(\cdot;\eta)|_{H^1(Y)}$ is a direct consequence of (5.1) and the fact that $b(\eta;\phi_m,\phi_m) = \lambda_m(\eta)$. To estimate $|\phi_m(\cdot;\eta)|_{H^2(Y)}$, we rewrite the eigenrelation (2.1) in the form $A\phi_m(\cdot;\eta) = \tilde{F}_m$, where

$$\tilde{F}_m = \lambda_m(\eta)\phi_m + \mathrm{i}\eta_k a_{k\ell}\frac{\partial \phi_m}{\partial y_\ell} + \mathrm{i}\eta_\ell \frac{\partial}{\partial y_k}(a_{k\ell}\phi_m) - a_{k\ell}\eta_k\eta_\ell\phi_m.$$

It is immediate that

$$\|\tilde{F}_m\|_{L^2(Y)} \leqslant c\lambda_m(\eta) \quad \text{if } m \geqslant 2$$

and, for $m = 1$, we have

$$\begin{aligned} \|\tilde{F}_1\|_{L^2(Y)} &\leqslant c\lambda_1(\eta)^{1/2} && \text{in general,} \\ \|\tilde{F}_1\|_{L^2(Y)} &\leqslant c\lambda_1(\eta) && \text{provided (3.2) holds.} \end{aligned}$$

The proof is complete if we apply the estimate (5.7) to the equation $A\phi_m = \tilde{F}_m$. □

Proof of theorem 4.5. We begin by rewriting $A(\eta)\phi_1 = \lambda_1(\eta)\phi_1$ as follows:

$$-\frac{\partial a_{k\ell}}{\partial y_k}\left(\frac{\partial \phi_1}{\partial y_\ell} + \mathrm{i}\eta_\ell\phi_1\right) = a_{k\ell}\frac{\partial^2 \phi_1}{\partial y_k \partial y_\ell} + \lambda_1(\eta)\phi_1 + \mathrm{i}\eta_k a_{k\ell}\frac{\partial \phi_1}{\partial y_\ell} + \mathrm{i}\eta_\ell a_{k\ell}\frac{\partial \phi_1}{\partial y_k} - a_{k\ell}\eta_k\eta_\ell\phi_1.$$

Thanks to (4.8) and previously stated inequalities, we see that the right-hand side of the above relation can be estimated in $L^2(Y)$, and hence we obtain

$$\left\| \frac{\partial a_{k\ell}}{\partial y_k}\left(\frac{\partial \phi_1}{\partial y_\ell} + \mathrm{i}\eta_\ell\phi_1\right)\right\|_{L^2(Y)} \leqslant c_2\lambda_1(\eta) \quad \forall \eta \in Y'.$$

We now confine η to a small neighbourhood V of the origin where, by results of [3], we know that $\phi_1(\cdot;\eta)$ is analytical, and can be developed as follows:

$$\phi_1(\cdot;\eta) = \frac{1}{|Y|^{1/2}}(1 + \mathrm{i}\eta_j\chi_j(\cdot)) + O(|\eta|^2).$$

Thus we get

$$|\eta_j|\left\|\frac{\partial a_{k\ell}}{\partial y_k}\left(\delta_{j\ell} + \frac{\partial \chi_j}{\partial y_\ell}\right)\right\|_{L^2(Y)} \leqslant c_2\lambda_1(\eta) \quad \text{for } \eta \in V.$$

Using (5.10) and letting $\eta \to 0$, we obtain

$$\frac{\partial a_{k\ell}}{\partial y_k}\left(\delta_{j\ell} + \frac{\partial \chi_j}{\partial y_\ell}\right) = 0 \quad \forall j = 1, \dots, N. \tag{5.11}$$

Thanks to the definition of χ_j (cf. (1.6)), the above condition is equivalent to

$$a_{k\ell}(y)\frac{\partial^2 \chi_j}{\partial y_k \partial y_\ell} = 0 \quad \text{in } Y \quad \forall j = 1, \dots, N.$$

This is an elliptic equation in non-divergence form for which the uniqueness result of [6, p. 170] applies, and gives $\chi_j \equiv 0$. Now (3.2) follows simply from (5.11). It is worthwhile to remark that the above uniqueness result requires only (1.2) and not (1.1). □

6. Proof of the main result

This section is devoted to the proof of theorem 4.7. The implication (i) $\Rightarrow$ (ii) was already established in theorem 4.3. For the reverse implication, we need theorem 4.6 and thus we start by proving it.

Proof of theorem 4.6. The proof is based on theorem 4.1. Since we have

$$\lambda_m^\varepsilon(\xi) B_m^\varepsilon u^\varepsilon(\xi) = B_m^\varepsilon f(\xi) \quad \forall \xi \in \varepsilon^{-1}Y', \quad m \geqslant 1, \tag{6.1}$$

we have

$$\|D_{k\ell}^2 u^\varepsilon\|_{L^2(\mathbb{R}^N)}^2 = \int_{\varepsilon^{-1}Y'} \sum_{n=1}^\infty \left| \sum_{m=1}^\infty B_m^\varepsilon f(\xi) \frac{\alpha_{mn}(\varepsilon\xi)}{\lambda_m(\varepsilon\xi)} \right|^2 \mathrm{d}\xi. \tag{6.2}$$

We show first (B) $\Rightarrow$ (A), which is easy. Since $\{B_m^\varepsilon f(\xi) \mid m \geqslant 1\} \in \ell^2(\mathbb{N})$ for almost all ξ, we get, by applying the inequality in (B),

$$\sum_{n=1}^\infty \left| \sum_{m=1}^\infty B_m^\varepsilon f(\xi) \frac{\alpha_{mn}(\varepsilon\xi)}{\lambda_m(\varepsilon\xi)} \right|^2 \leqslant c \sum_{m=1}^\infty |B_m^\varepsilon f(\xi)|^2$$

for almost all ξ. Integrating this inequality with respect to ξ over $\varepsilon^{-1}Y'$ and applying Parseval's identity (2.5), we deduce (A).

The other implication (A) $\Rightarrow$ (B) in theorem 4.6 involves a localization in ξ in the following inequality which results from (A):

$$\int_{\varepsilon^{-1}Y'} \sum_{n=1}^\infty \left| \sum_{m=1}^\infty B_m^\varepsilon f(\xi) \frac{\alpha_{mn}(\varepsilon\xi)}{\lambda_m(\varepsilon\xi)} \right|^2 \mathrm{d}\xi \leqslant c \int_{\varepsilon^{-1}Y'} \sum_{m=1}^\infty |B_m^\varepsilon f(\xi)|^2 \,\mathrm{d}\xi. \tag{6.3}$$

More precisely, for arbitrary $\beta = (\beta_m) \in \ell^2(\mathbb{N})$ and test function $\theta \in \mathcal{D}(\varepsilon^{-1}Y')$, we can choose $f \in L^2(\mathbb{R}^N)$ such that $B_m^\varepsilon f(\xi) = \beta_m \theta(\xi)$ $\forall \xi \in \varepsilon^{-1}Y'$, $m \geqslant 1$. This is possible because Bloch transform establishes a unitary isomorphism between $L^2(\mathbb{R}^N)$ and $L^2(\varepsilon^{-1}Y'; \ell^2(\mathbb{N}))$. A simple application of (6.3) to f yields

$$\int_{\varepsilon^{-1}Y'} \sum_{n=1}^\infty \left| \sum_{m=1}^\infty \beta_m \frac{\alpha_{mn}(\varepsilon\xi)}{\lambda_m(\varepsilon\xi)} |^2 |\theta(\xi) \right|^2 \mathrm{d}\xi \leqslant c \int_{\varepsilon^{-1}Y'} \left(\sum_{m=1}^\infty |\beta_m|^2 \right) |\theta(\xi)|^2 \,\mathrm{d}\xi.$$

Since θ is arbitrary, the inequality given in (B) follows. □

Completion of the proof of theorem 4.7. As pointed out earlier, it remains to show (ii) $\Rightarrow$ (i). According to theorem 4.6, part (ii) implies that there is a constant $c > 0$ such that

$$\sum_{n=1}^\infty \left| \sum_{m=1}^\infty \beta_m \frac{\alpha_{mn}(\eta)}{\lambda_m(\eta)} \right|^2 \leqslant c \sum_{m=1}^\infty |\beta_m|^2 \quad \forall \beta \in \ell^2(\mathbb{N}).$$

To extract information from the above inequality, we make the choice of

$$\beta = \{0, 0, \dots, 1^{m\text{th}}, 0, 0, \dots\}.$$

This yields

$$\sum_{n=1}^\infty |\alpha_{mn}(\eta)|^2 \leqslant c \lambda_m(\eta)^2 \quad \forall \eta \in Y', \quad m \geqslant 1. \tag{6.4}$$

At this point, let us make the observation that $\{\alpha_{mn}(\eta)\}_{n=1}^{\infty}$ are nothing but Fourier coefficients of $\rho_m(\cdot;\eta)$ in the orthonormal basis $\{\phi_n(\cdot;\eta)\}_{n=1}^{\infty}$ (cf. theorem 4.1), and hence by Parseval's relation we get

$$\|\rho_m(\cdot;\eta)\|_{L^2(Y)}^2 = \sum_{n=1}^{\infty} |\alpha_{mn}(\eta)|^2 \quad \forall m \geqslant 1. \tag{6.5}$$

From the definition of ρ_m, it then follows that $\phi_m \in H^2(Y)$. To estimate its norm, we introduce

$$\Phi_m^\ell = \frac{\partial \phi_m}{\partial y_\ell} + \mathrm{i}\eta_\ell \phi_m \quad \forall m \geqslant 1, \quad \ell = 1, \dots, N.$$

It is then easy to express

$$\rho_m(y;\eta) = \frac{\partial \Phi_m^\ell}{\partial y_k} + \mathrm{i}\eta_k \Phi_m^\ell.$$

Combining all this information, we arrive at

$$\|\nabla \Phi_m^\ell + \mathrm{i}\eta \Phi_m^\ell\|_{L^2(Y)}^2 \leqslant c\lambda_m(\eta)^2 \quad \forall m \geqslant 1, \quad \ell = 1, \dots, N. \tag{6.6}$$

Thanks to (5.2), we can write the following chain of inequalities where k is fixed,

$$\begin{aligned}
\left|\frac{\partial \phi_m}{\partial y_k}\right|_{H^1(Y)}^2 &\leqslant c \left\| \nabla\left(\frac{\partial \phi_m}{\partial y_k}\right) + \mathrm{i}\eta \frac{\partial \phi_m}{\partial y_k} \right\|_{L^2(Y)}^2 \\
&= c \sum_{\ell=1}^{N} \left\| \frac{\partial^2 \phi_m}{\partial y_k \partial y_\ell} + \mathrm{i}\eta_\ell \frac{\partial \phi_m}{\partial y_k} \right\|_{L^2(Y)}^2 \\
&= c \sum_{\ell=1}^{N} \left\| \frac{\partial \Phi_m^\ell}{\partial y_k} \right\|_{L^2(Y)}^2 \\
&\leqslant c \sum_{\ell=1}^{N} \|\nabla \Phi_m^\ell\|_{L^2(Y)}^2 \\
&\leqslant c \sum_{\ell=1}^{N} \|\nabla \Phi_m^\ell + \mathrm{i}\eta \Phi_m^\ell\|_{L^2(Y)}^2 \\
&\leqslant c\lambda_m(\eta)^2;
\end{aligned}$$

the last inequality being a consequence of (6.6). The above inequality with $m = 1$ implies the estimate $|\phi_1(\cdot;\eta)|_{H^2(Y)} \leqslant c\lambda_1(\eta)$, which, according to theorem 4.5, implies (3.2). The proof is finished. □

7. Regular materials

We call the material represented by the matrix $a(y) = [a_{k\ell}(y)]$ *regular* if it admits H^2_{loc}-estimates uniform in ε, i.e. statement (ii) of theorem 4.7 is true. Results of the previous sections establish that materials satisfying (1.1), (3.2) are regular materials. More precisely, among the materials with regularity (1.1), the regular materials are precisely those which satisfy (3.2). Thus it seems natural to keep (3.2) in seeking other examples of regular materials. Another motivation for working with (3.2) is that it was already shown to be necessary in the localization process (cf. (3.1)). From the classical results of Murat and Tartar in homogenization theory, it is well known that L^∞-weak* limit provides an upper bound for all homogenized coefficients. As remarked at the end of § 4, the homogenized coefficients associated with regular materials are arithmetic averages of $a_{k\ell}$, which are nothing but the L^∞-weak* limit of $a^\varepsilon_{k\ell}$. Viewed in this manner, regular materials possess an optimal property, namely that their homogenization limit coincides with the upper bound.

Examples of regular materials satisfying (3.2) (but not (1.1)) are presented in this section. The proof of theorems 4.2, 4.3 show that we will have uniform H^2_{loc}-estimates if the following regularity property holds: the solution u of

$$Au = f \quad \text{in } Y, \quad u \in H^1_\#(Y) \tag{7.1}$$

(note that u is determined uniquely up to an additive constant) admits a bound

$$\sum_{k,\ell=1}^{N} \left\| \frac{\partial^2 u}{\partial y_k \partial y_\ell} \right\|_{L^2(Y)} \leqslant c\|f\|_{L^2(Y)} \tag{7.2}$$

for all $f \in L^2(Y)$ with $\mathcal{M}_Y(f) = 0$.

We will now exhibit situations where the above property holds without the coefficients $a_{k\ell}(Y)$ being Lipschitz. More precisely, apart form (3.2), let us make the following hypothesis on the coefficients:

$$\text{there exists } q \in [N, \infty] \text{ such that } a_{k\ell} \in W^{1,q}_\#(Y) \cap L^\infty_\#(Y) \text{ for all } k, \ell = 1, \dots, N. \tag{7.3}$$

This index q, which depends on Meyer's exponent p_0 (see theorem 7.2 below), can be quite large and will be suitably restricted (cf. (7.6) below). Nevertheless, it is somewhat surprising to know that uniform H^2_{loc}-estimates hold without coefficients $a_{k\ell}(y)$ being Lipschitz. In this direction, let us state and prove our first result.

Lemma 7.1. *Under the assumptions (3.2) and (7.3), we have the following identity for all u in $H^2_\#(Y)$:*

$$\int_Y a_{k\ell} a_{mn} \frac{\partial^2 u}{\partial y_k \partial y_n} \frac{\partial^2 u}{\partial y_\ell \partial y_m} \,\mathrm{d}y = \int_Y a_{k\ell} \frac{\partial^2 u}{\partial y_k \partial y_\ell} a_{mn} \frac{\partial^2 u}{\partial y_m \partial y_n} \,\mathrm{d}y + \int_Y \left(a_{m\ell} \frac{\partial a_{kn}}{\partial y_m} - a_{mn} \frac{\partial a_{k\ell}}{\partial y_m} \right) \frac{\partial^2 u}{\partial y_n \partial y_k} \frac{\partial u}{\partial y_\ell} \,\mathrm{d}y. \tag{7.4}$$

Proof. We have

$$\|Au\|^2_{L^2(Y)} = \int_Y a_{k\ell} \frac{\partial^2 u}{\partial y_k \partial y_\ell} a_{mn} \frac{\partial^2 u}{\partial y_m \partial y_n} \,\mathrm{d}y.$$

Integrating by parts several times and using (3.2) repeatedly, we arrive at

$$\|Au\|^2_{L^2(Y)} = \int_Y \frac{\partial a_{mn}}{\partial y_k}\frac{\partial u}{\partial y_n}\frac{\partial}{\partial y_m}\left(a_{k\ell}\frac{\partial u}{\partial y_\ell}\right)\mathrm{d}y + \int_Y a_{k\ell}a_{mn}\frac{\partial^2 u}{\partial y_k \partial y_n}\frac{\partial^2 u}{\partial y_\ell \partial y_m}\,\mathrm{d}y + \int_Y a_{mn}\frac{\partial a_{k\ell}}{\partial y_m}\frac{\partial u}{\partial y_\ell}\frac{\partial^2 u}{\partial y_n \partial y_k}\,\mathrm{d}y.$$

One more integration by parts and application of (3.2) in the first term of the right-hand side of the above relation obviously leads us to the required identity. Thanks to (7.3) and the Sobolev inclusion, we note that the last integral in the above identity is well defined and this completes the proof. □

Taking into account the structure of the left-hand side of (7.4), we now proceed to prove that there exists $\nu > 0$ such that

$$a_{k\ell}(y)a_{mn}(y)\eta_{kn}\eta_{\ell m} \geqslant \nu\|\eta\|^2 \tag{7.5}$$

for all symmetric matrices $\eta = [\eta_{k\ell}]$ and for $y \in Y$ almost everywhere. We will see that the above inequality is a consequence of our assumptions that the matrix $a(y) = [a_{k\ell}(y)]$ is symmetric and uniformly positive definite. First of all, a simple computation shows that

$$a_{k\ell}a_{mn}\eta_{kn}\eta_{\ell m} = \mathrm{Trace}(a\eta a\eta).$$

Next, the matrix a can be diagonalized in an orthonormal basis consisting of eigenvectors of a. There exists an orthogonal matrix Q such that

$$Q^* a Q = \Lambda,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, λ_k being eigenvalues of A. Exploiting the fact that Trace is invariant under change of variables, we see that

$$\mathrm{Trace}(a\eta a\eta) = \mathrm{Trace}(\Lambda\eta'\Lambda\eta'),$$

where $\eta' = Q^*\eta Q$. Note that η' is symmetric. Thus we are reduced to the case where the matrix a is diagonal. Another simple computation shows that

$$\mathrm{Trace}(\Lambda\eta'\Lambda\eta') = \lambda_k\lambda_\ell(\eta'_{k\ell})^2.$$

From this expression, we see that (7.5) holds with $\nu = \alpha^2$, provided we use $\lambda_k \geqslant \alpha$ for all k and the fact that $\|\eta'\| = \|\eta\|$.

One consequence of (7.5) is that the left-hand side of (7.4) is bounded below by the semi-norm $|u|^2_{H^2(Y)}$. We will now exploit this to prove the regularity result (7.2). To this end, let us denote by

$$\dot{D}(A) = \{v \in L^2(Y)/\mathbb{R} \mid Av \in L^2(Y)\}.$$

The Lax–Milgram lemma implies that $\|Av\|_{L^2(Y)}$ is a norm on $\dot{D}(A) \cap \dot{H}^1_\#(Y)$ equivalent to $|v|_{H^1(Y)} + \|Av\|_{L^2(Y)}$. (Here we have used $\dot{}$ to mean the space modulo $\mathbb{R}$.) Under additional assumptions on a, we are going to prove that this norm is equivalent to $\|v\|_{\dot{H}^2_\#(Y)}$. Before this, let us recall the following important result of Meyers [7], whose proof can be found, for example, in [1, p. 38].

THEOREM 7.2. *There is a $p_0 > 2$ (which depends on $\|a\|_{L^\infty(Y)}$) such that, for all $2 \leqslant p \leqslant p_0$, the solution u of (7.1) belongs to $W^{1,p}_{\#}(Y)$ whenever $f \in W^{-1,p}_{\#}(Y)$. Further, we have the estimate*

$$\|\nabla u\|_{L^p(Y)} \leqslant c\|f\|_{W^{-1,p}_{\#}(Y)},$$

where c depends on $\|a\|_{L^\infty(Y)}$.

THEOREM 7.3. *We assume (3.2). We suppose further that (7.3) holds with*

$$q \geqslant \max\left\{\frac{2p_0}{p_0 - 2}, N\right\}, \tag{7.6}$$

where p_0 is the exponent occurring in theorem 7.2. Then the solution u of (7.1) is in $H^2_{\#}(Y)$ whenever $f \in L^2(Y)$, and we have the estimate

$$|u|_{H^2(Y)} + |u|_{H^1(Y)} \leqslant c\|f\|_{L^2(Y)}.$$

Proof. First of all, we will prove the estimate

$$|u|_{H^2(Y)} \leqslant c\|Au\|_{L^2(Y)} \quad \forall u \in H^2_{\#}(Y). \tag{7.7}$$

Indeed, this follows from (7.4) because we can estimate the second integral on the right-hand side of (7.4) by

$$c\|a\|_{L^\infty(Y)}\|\nabla_y a\|_{L^q(Y)}|u|_{H^2(Y)}\|\nabla u\|_{L^r(Y)}$$

with $1/q + 1/r = \frac{1}{2}$. Because of (7.6), we have $r \leqslant p_0$ and so by theorem 7.2, it follows that

$$\|\nabla u\|_{L^r(Y)} \leqslant c\|Au\|_{W^{-1,r}_{\#}(Y)}.$$

Finally, we use the inclusion $L^2(Y) \hookrightarrow W^{-1,2^*}(Y)$ to deduce

$$\|Au\|_{W^{-1,r}_{\#}(Y)} \leqslant c\|Au\|_{L^2(Y)}.$$

Combination of all these estimates easily leads to (7.7).

Next, we assert that the inequality (7.7) is valid for $u \in \dot{D}(A) \cap \dot{H}^1_{\#}(Y)$. For this, it is sufficient to verify that $\dot{H}^2_{\#}(Y)$ is dense in $\dot{D}(A) \cap \dot{H}^1_{\#}(Y)$ with respect to the norm $\|Av\|_{L^2(Y)}$, which is stronger than $|v|_{H^1(Y)}$. Indeed, an element u is in the orthogonal complement of $\dot{H}^2_{\#}(Y)$ if and only if u satisfies

$$\int_Y Au \cdot Av \,\mathrm{d}y = 0 \quad \forall v \in H^2_{\#}(Y).$$

This is equivalent to saying that

$$u \in \dot{D}(A^2) \quad \text{and} \quad A^2 u = 0.$$

Taking the scalar product with u, we deduce that $Au = 0$, and hence $u \equiv \text{const}$. This completes the proof. □

We now conclude this section by giving two types of examples of materials for which the results established in this paper would apply and show that they are regular.

Example 7.4. Here we seek $a_{k\ell}$ of the form

$$a_{k\ell}(y) = a_k(y)\delta_{k\ell} \quad \forall k, \ell = 1, \dots N, \tag{7.8}$$

where the vector (a_k) is required to satisfy

$$a_k \in L^\infty_\#(Y) \cap W^{1,q}_\#(Y) \quad \text{and} \quad a_k(y) \geqslant \alpha > 0 \quad y \in Y \text{ a.e.}, \tag{7.9}$$

where q is chosen according to (7.6). Furthermore, in order to satisfy (3.2), we chose a_k such that

$$a_k(y) \text{ is independent of } y_k \text{ for each } k = 1, \dots, N.$$

Example 7.5. We take $N = 2$. In this case, it is well known that any symmetric matrix $a' = [a'_{k\ell}]$ satisfying (3.2) is of the form

$$a' = \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial y_2^2} & -\dfrac{\partial^2 \psi}{\partial y_1 \partial y_2} \\ -\dfrac{\partial^2 \psi}{\partial y_1 \partial y_2} & \dfrac{\partial^2 \psi}{\partial y_1^2} \end{pmatrix},$$

where ψ is the so-called *Airy potential.* We assume

$$\psi \in W^{2,\infty}_\#(Y) \cap W^{3,q}_\#(Y),$$

where q satisfies (7.6). What remains to be imposed is the positive definiteness condition. Obviously, this cannot be done directly on a' defined above. However, it is a simple matter to check that this can be achieved by adding a suitable constant matrix to a',

$$a = \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial y_2^2} + b_1 & -\dfrac{\partial^2 \psi}{\partial y_1 \partial y_2} + b_2 \\ -\dfrac{\partial^2 \psi}{\partial y_1 \partial y_2} + b_2 & \dfrac{\partial^2 \psi}{\partial y_1^2} + b_3 \end{pmatrix}.$$

There are constants b_1, b_2, b_3 such that a is positive definite. Thus the above matrix defines a regular material in two dimensions, and conversely all regular materials in two dimensions are obtained in this way.

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