

On SDE associated with continuous-state branching processes conditioned to never be extinct

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Abstract

We study the pathwise description of a (sub-)critical continuous-state branching process (CSBP) conditioned to be never extinct, as the solution to a stochastic differential equation driven by Brownian motion and Poisson point measures. The interest of our approach, which relies on applying Girsanov theorem on the SDE that describes the unconditioned CSBP, is that it points out an explicit mechanism to build the immigration term appearing in the conditioned process, by randomly selecting jumps of the original one. These techniques should also be useful to represent more general h -transforms of diffusion-jump processes.

Key words Stochastic Differential Equations; Continuous-state branching processes; Non-extinction; Immigration.

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1 Introduction and preliminaries

Stochastic differential equations (SDE) representing continuous-state branching processes (CSBP) or CSBP with immigration (CBI) have attracted increasing attention in the last years, as powerful tools for studying pathwise and distributional properties of these processes as well as some scaling limits, see e.g. Dawson and Li [5], [6], Lambert [17], Fu and Li [10] and Caballero *et al.* [4].

In this note, we are interested in SDE representations for (sub-)critical CSBP conditioned to never be extinct. It is well known that such conditioned CSBP correspond to CBIs with particular immigration mechanisms (see [25]). Thus, it is possible to obtain SDE representations for them by using general results and techniques developed in some of the aforementioned works, see [5] and [10]. However, our goal is to directly obtain such representation by rather using the fact that the law of the conditioned CSBP is obtained from the one of the non conditioned process, by means of an explicit h -transform. In accordance with that relation between the laws and to the “spine” or immortal particle picture of the conditioned process ([25], [9]), one should expect to identify, after measure change, copies of the original driving random processes and an independent subordinator accounting for immigration. Our proof will show how to obtain these processes by using Girsanov theorem and an enlargement of the probability space in order to select by a suitable marking procedure those jumps of the original (non conditioned) process that will constitute (or will not) the immigrants. The enlargement of the probability space and the marking procedure are both inspired in a construction of Lambert [17] on stable Lévy processes.

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They are also reminiscent of the sized biased tree representation of measure changes for Galton-Watson trees (Lyons *et. al* [22]) or for branching Brownian motions (see e.g. Kyprianou [15] and Englanger and Kyprianou [8]), but we do not aim at fully developing those ideas in the present framework. In a related direction, in a recently posted article [12] Henard obtains the same SDE description of the conditioned CSBP, using the look-down particle representation of CSBP of Donnelly and Kurtz [7].

We start by recalling some definitions and classic results about CSBPs and Levy processes along the lines of [16, Chap. 1,2 and 10], in particular the relationship between them through the Lamperti transform. (We also refer the reader to Le Gall [20] and Li [21] for further background on CSBP).

1.1 Continuous-state branching processes

Continuous-state branching processes (CSBP) were introduced by Jirina [13] in 1958. Later, Lamperti [19] showed that they can be obtained as scaling limits of a sequence of Galton-Watson processes. A CSBP with probability laws given the initial state $\{\mathbb{P}_x : x \geq 0\}$ is a cadlag $[0, \infty)$ -valued strong Markov processes $Z = \{Z_t : t \geq 0\}$ satisfying the branching property. That is, for any $t \geq 0$ and $z_1, z_2 \in [0, \infty)$, Z_t under $\mathbb{P}_{z_1+z_2}$ has the same law as the independent sum $Z_t^{(1)} + Z_t^{(2)}$, where the distribution of $Z_t^{(i)}$ is equal to that of Z_t under \mathbb{P}_{z_i} for $i = 1, 2$. Usually, Z_t represents the population at time t descending from an initial population x . The law of Z is completely characterized by its Laplace transform

$$\mathbb{E}_x(e^{-\theta Z_t}) = e^{-xu_t(\theta)}, \quad \forall x > 0, t \geq 0,$$

where u is a differentiable function in t satisfying

$$\begin{cases} \frac{\partial u_t}{\partial t}(\theta) + \psi(u_t(\theta)) = 0 \\ u_0(\theta) = \theta, \end{cases} \quad (1)$$

and ψ is called the *branching mechanism* of Z , which has the form

$$\psi(\lambda) = -q - a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{(x < 1)}) \Pi(dx) \quad \lambda \geq 0, \quad (2)$$

for some $q \geq 0$, $a \in \mathbb{R}$, $\sigma \geq 0$ and Π a measure supported in $(0, \infty)$ such that $\int_{(0,\infty)} (1 \wedge x^2) \Pi(dx) < \infty$. In particular, ψ is the characteristic exponent of a *spectrally positive* Levy process, i.e. one with no negative jumps. Since clearly, $\mathbb{E}_x(Z_t) = xe^{-\psi'(0+)t}$, defining $\rho := \psi'(0+)$ one has the following classification of CSBPs :

- (i) subcritical, if $\rho > 0$,
- (ii) critical, if $\rho = 0$ and
- (iii) supercritical, if $\rho < 0$,

according to whether the process will, on average, decrease, remain constant or increase.

In the following, we will assume that Z is *conservative*, i.e. $\forall t > 0, \mathbb{P}_x(Z_t < \infty) = 1$. By Grey (1974), this is true if and only if $\int_{0+} \frac{d\xi}{|\psi(\xi)|} = \infty$, so it is sufficient to assume $\psi(0) = 0$ and $|\psi'(0+)| < \infty$.

1.2 Levy Processes and their connection with CSBP

Let $X = \{X_t : t \geq 0\}$ be a spectrally positive Levy process with characteristic exponent ψ given by (2) with $q = 0$, and initial state $x \geq 0$. By the Levy-Ito decomposition it is well known that it can be

written as the following sum of independent processes

$$X_t = x + at + \sigma B_t^X + \int_0^t \int_1^\infty r N^X(ds, dr) + \int_0^t \int_0^1 r \tilde{N}^X(ds, dr),$$

where a is a real number, $\sigma \geq 0$, B^X is a Brownian motion, N^X is an independent Poisson measure on $[0, \infty) \times (0, \infty)$ with intensity measure $dt \times \Pi(dr)$ and $\tilde{N}^X(dt, dr) := N^X(dt, dr) - dt\Pi(dr)$ denotes the compensated measure associated to N^X (the last integral thus being a square integrable martingale of compensated jumps of magnitude less than unity).

Lamperti [18] established a one-to-one correspondence between CSBPs and spectrally positive Lévy processes via a random time change. More precisely, for a Lévy process X as above the process

$$Z := \{Z_t = X_{\theta_t \wedge T_0} : t \geq 0\},$$

where $T_0 = \inf\{t > 0 : X_t = 0\}$ and $\theta_t = \inf\left\{s > 0 : \int_0^s \frac{du}{X_u} > t\right\}$, is a continuous-state branching process with branching mechanism ψ and initial value $Z_0 = x$. Conversely, given $Z = \{Z_t : t \geq 0\}$ a CSBP with branching mechanism ψ , such that $Z_0 = x > 0$, we have that

$$X := \{X_t = Z_{\varphi_t \wedge T} : t \geq 0\},$$

where $T = \inf\{t > 0 : Z_t = 0\}$ and $\varphi_t = \inf\left\{s > 0 : \int_0^s Z_u du > t\right\}$, is a Lévy process with no negative jumps, stopped at T_0 and satisfying $\psi(\lambda) = \log \mathbb{E}(e^{-\lambda X_1})$, with initial position $X_0 = x$.

Relying on this relationship, Caballero *et al.* [4, Prop 4] provide a pathwise description of the dynamics of a CSBP: for $(Z_t, t \geq 0)$ there exist a standard Brownian motion B^Z , and an independent Poisson measure N^Z on $[0, \infty) \times (0, \infty) \times (0, \infty)$ with intensity measure $dt \times d\nu \times \Pi(dr)$ in an enlarged probability space such that

$$\begin{aligned} Z_t = & x + a \int_0^t Z_s ds + \sigma \int_0^t \sqrt{Z_s} dB_s^Z + \int_0^t \int_0^{Z_{s^-}} \int_1^\infty r N^Z(ds, d\nu, dr) \\ & + \int_0^t \int_0^{Z_{s^-}} \int_0^1 r \tilde{N}^Z(ds, d\nu, dr), \end{aligned} \quad (3)$$

where \tilde{N}^Z is the compensated Poisson measure associated with N^Z . Pathwise properties of stochastic differential equations driven by Brownian motion and Poisson point processes have been studied in more general settings in [5], [10] and [6]. In particular, strong existence and pathwise uniqueness for (3) is established [10]. Related SDE have also been considered in Bertoin and Le Gall [2], [3].

2 CSBPs conditioned to be never extinct as solutions of SDEs

2.1 CSBP conditioned to be never extinct

We assume from now on that Z is a (sub-)critical CSBP such that $\psi(\infty) = \infty$ and $\int^\infty \frac{d\xi}{\psi(\xi)} < \infty$. Under these and the previous conditions, the process does not explode and there is almost surely extinction in finite time. Branching processes conditioned to stay positive were first studied in the continuous-state framework by Roelly and Rouault [25], who proved that for Z as before,

$$\mathbb{P}_x^\uparrow(A) := \lim_{s \uparrow \infty} \mathbb{P}_x(A|T > t + s), \quad A \in \sigma(Z_s : s \leq t) \quad (4)$$

is a well defined probability measure which satisfies

$$\mathbb{P}_x^\uparrow(A) = \mathbb{E}(\mathbf{1}_A e^{\rho t} \frac{Z_t}{x}).$$

In particular, $\mathbb{P}_x^\uparrow(T < \infty) = 0$, and $\{e^{\theta t} Z_t : t \geq 0\}$ is a martingale under \mathbb{P}_x . Note that \mathbb{P}_x^\uparrow is the law of the so-called *Q-process* (for in-depth looks at this type of processes, we refer the reader to [17], [23] and references therein). They also proved that (Z, \mathbb{P}^\uparrow) has the same law as a CBI with branching mechanism ψ and immigration mechanism $\phi(\theta) = \psi'(\theta) - \rho$, $\theta \geq 0$. This means that (Z, \mathbb{P}^\uparrow) is a càdlàg $[0, \infty)$ -valued process, and for all $x, t > 0$ and $\theta \geq 0$

$$\mathbb{E}_x^\uparrow(e^{-\theta Z_t}) = \exp\{-xu_t(\theta) - \int_0^t \phi(u_{t-s}(\theta)) ds\},$$

where $u_t(\theta)$ is the unique solution to (1). Note also that ϕ is the Laplace exponent of a subordinator.

2.2 Main Result

The above result is the key for the study of CSBP conditioned on non-extinction, but we seek a more explicit description for the paths of Z under \mathbb{P}^\uparrow . To this end, we shall prove that (Z, \mathbb{P}^\uparrow) has a SDE representation, which agrees with the interpretation of a CSBP conditioned on non-extinction as a CBI, but also gives us a pathwise description for the conditioned process. In particular, this result extends Lambert's results for the stable case [17, Theorem 5.2] (see below for details) as well as equation (3).

Theorem 2.1 *Under \mathbb{P}^\uparrow , the process Z is the unique strong solution of the following stochastic differential equation:*

$$\begin{aligned} Z_t = & x + a \int_0^t Z_s ds + \sigma \int_0^t \sqrt{Z_s} dB_s^\uparrow + \int_0^t \int_0^{Z_{s-}} \int_1^\infty r N^\uparrow(ds, d\nu, dr) + \int_0^t \int_0^{Z_{s-}} \int_0^1 r \tilde{N}^\uparrow(ds, d\nu, dr) \\ & + \int_0^t \int_0^\infty r N^*(ds, dr) + \sigma^2 t \end{aligned} \quad (5)$$

where $\{B_t^\uparrow : t \geq 0\}$ is a Brownian motion, N^\uparrow and N^* are Poisson measures on $[0, \infty) \times (0, \infty)^2$ and $[0, \infty) \times (0, \infty)$ with intensities measures $ds \times d\nu \times \Pi(dr)$ and $ds \times r\Pi(dr)$, respectively, and these objects are mutually independent (as usual, \tilde{N}^\uparrow stands for the compensated measure associated with N^\uparrow). Moreover, the point processes N^\uparrow and N^* can be constructed by change of measure and a marking procedure on an enlargement of the probability space where B^Z and N^Z in (3) are defined, which supports and independent i.i.d. sequence of uniform random variables in the unit interval.

This result implies that we can recover Z conditioned on non-extinction as the solution of a SDE driven by a copy of B^Z , a copy of N^Z , and a Poisson random measure with intensity $ds \times r\Pi(dr)$, plus a drift. (Notice that taking out the last line, corresponding to a subordinator with drift, one again obtains equation (3).)

3 Relations to previous results

3.1 Stable processes

We will show that, as pointed out before, Lambert's SDE representation of stable branching processes given in [17, Theorem 5.2] can be seen as a special case of Theorem 2.1.

Let X be a spectrally positive α -stable process with characteristic exponent ψ and characteristic measure $\Pi(dr) = kr^{-(\alpha+1)}dr$, where k is some positive constant and $1 < \alpha \leq 2$. Let Z be the branching process with branching mechanism ψ . Thanks to Theorem 2.1 we know that, under \mathbb{P}^\uparrow , Z satisfies the following stochastic differential equation:

$$Z_t = \int_0^t \int_0^{Z_{s-}} \int_1^\infty r N^\uparrow(ds, d\nu, dr) + \int_0^t \int_0^{Z_{s-}} \int_0^1 r \tilde{N}^\uparrow(ds, d\nu, dr) + \int_0^t \int_0^\infty r N^*(ds, dr), \quad (6)$$

where N^\uparrow is a Poisson random measure with intensity $ds \times d\nu \times \Pi(dr)$ and N^\star is an independent Poisson random measure with intensity $ds \times r\Pi(dr)$. Now, we define

$$\theta_n = \frac{r_n^\uparrow \mathbf{1}_{(\nu_n^\uparrow \leq Z_{t_n-})}}{Z_{t_n-}^{1/\alpha}},$$

where $\{(t_n, r_n^\uparrow, \nu_n^\uparrow) : n \in \mathbb{N}\}$ are the atoms of N^\uparrow . We claim that, under \mathbb{P}^\uparrow , $\{(t_n, \theta_n) : n \in \mathbb{N}\}$ are atoms of a Poisson random measure N' with intensity $ds \times \Pi(du)$. Indeed, for any bounded non-negative predictable process H , and any positive bounded function f vanishing at zero,

$$M_t := \sum_{t_n \leq t} H_{t_n} f(\theta_n) - \int_0^t H_s ds \int_0^\infty \int_0^\infty f\left(\frac{r}{Z_s^{1/\alpha}}\right) \mathbf{1}_{(\nu \leq Z_s)} d\nu \Pi(dr)$$

is a martingale. If we change variables, the particular form of Π implies that

$$M_t = \sum_{t_n \leq t} H_{t_n} f(\theta_n) - \int_0^t H_s ds \int_0^\infty f(u) \Pi(du).$$

Taking expectations, our claim follows thanks to Lemma 4.2 below. Since $\sum_{t_n \leq t} r_n^\uparrow \mathbf{1}_{(\nu_n^\uparrow \leq Z_{t_n-})} = \sum_{t_n \leq t} Z_{t_n-}^{1/\alpha} \theta_n$, we can rewrite (6) as

$$Z_t = \int_0^t \int_1^\infty Z_{s-}^{1/\alpha} u N'(ds, du) + \int_0^t \int_0^1 Z_{s-}^{1/\alpha} u \tilde{N}'(ds, du) + \int_0^t \int_0^\infty r N^\star(ds, dr).$$

Defining

$$X_t := \int_0^t \int_1^\infty u N'(ds, du) + \int_0^t \int_0^1 u \tilde{N}'(ds, du),$$

by the Lévy-Ito decomposition it is easy to see that X is an α -stable Lévy process with characteristic exponent ψ . Similarly,

$$S_t := \int_0^t \int_0^\infty r N^\star(ds, dr)$$

is seen to be an $(\alpha-1)$ -stable subordinator. Independence of X and S is granted by construction, because the two processes do not have simultaneous jumps. Thus, we have

$$dZ_t = Z_t^{1/\alpha} dX_t + dS_t,$$

which corresponds to Lambert's result.

3.2 CSBP flows as SDE solutions

A family of CSBP processes $Z = \{Z_t(a) : t \geq 0, a \geq 0\}$ allowing the initial population size $Z_0(a) = a$ to vary, can be constructed simultaneously as a two parameter process or *stochastic flow* satisfying the branching property. This was done by Bertoin and Le-Gall [1] by using families of subordinators. In [2], [3] they later used Poisson measure driven SDE to formulate such type of flows in related contexts, including equations close to (3). In the same line, Dawson and Li [6] proved the existence of strong solutions for stochastic flows of continuous-state branching processes with immigration, as SDE families driven by white noise processes and Poisson random measures with joint regularity properties. The stochastic equations they study (in particular equation (1.5)) are close to equation (5), the main difference being the immigration behavior which in their case only covers linear drifts. For simplicity reasons Theorem 2.1 is presented in the case of a Brownian motion and Poisson measure driven SDE, but our arguments can be extended to the white-noise and Poisson measure driven stochastic flow considered in [6] (in absence of immigration).

4 Proof of the main theorem

In [17], a suitable marking of Poisson point processes was used to firstly construct a stable Lévy process, conditioned to stay positive, out of the realization of the unconditioned one. After time-changing the author takes advantage of the scaling property of α -stable processes to derive an SDE for the branching process. Our proof is inspired in his marking argument but in turn it is carried out directly in the time scale of the CSBP. We will need the following version of Girsanov's theorem (c.f. Theorem 37 in Chapter III.8 of [24]):

Theorem 4.1 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, and let M be a \mathbb{P} -local martingale with $M_0 = 0$. Let \mathbb{P}^* be another probability measure absolutely continuous with respect to \mathbb{P} , and let $D_t = \mathbb{E}(\frac{d\mathbb{P}^*}{d\mathbb{P}} | \mathcal{F}_t)$. Assume that $\langle M, D \rangle$ exists for \mathbb{P} . Then $A_t = \int_0^t \frac{1}{Z_{s-}} d\langle M, Z \rangle_s$ exists a.s. for the probability \mathbb{P}^* , and $M_t - A_t$ is a \mathbb{P}^* -local martingale.*

The following well-known characterization of Poisson point processes will also be useful:

Lemma 4.2 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, (S, \mathcal{S}, η) an arbitrary σ -finite measure space, and $\{(t_n, \delta_n) \in \mathbb{R}_+ \times S\}$ a countable family of random variables such that $\{t_n \leq t, \delta_n \in A\} \in \mathcal{F}_t$ for all $n \in \mathbb{N}$, $t \geq 0$ and $A \in \mathcal{S}$, and moreover*

$$\mathbb{E} \sum_{n:t_n \leq t} F_{t_n} g(\delta_n) = \mathbb{E} \int_0^t F_s ds \int_S g(x) m(dx) \quad (7)$$

for any nonnegative predictable process F_s and any nonnegative function $g : S \rightarrow \mathbb{R}$. Then, $(t_n, \delta_n)_{n \in \mathbb{N}}$ are the atoms of a Poisson random measure N on $\mathbb{R}_+ \times S$ with intensity $dt \times m(dx)$.

Proof. Writing

$$e^{\left\{ \sum_{t_n \leq t} f(\delta_n) \right\}} = \sum_{n:t_n \leq t} \left[\prod_{k:t_k < t_n} e^{f(\delta_k)} \right] (e^{f(\delta_n)} - 1) = \sum_{n:t_n \leq t} \left[e^{\sum_{k:t_k \leq s} f(\delta_k)} \right] (e^{f(\delta_n)} - 1)$$

we get from (7) that

$$\mathbb{E} \left[e^{\sum_{t_n \leq t} f(\delta_n)} \right] = \int_0^t \mathbb{E} \left[e^{\sum_{k:t_k \leq s} f(\delta_k)} \right] ds \int_S (e^{f(x)} - 1) m(dx)$$

since $F_s := \prod_{t_k < s} e^{f(\delta_k)}$ is a predictable process. Solving this differential equation yields

$$\mathbb{E} \left[e^{\sum_{t_n \leq t} f(\delta_n)} \right] = e^{-t \int_S (1 - e^{f(x)}) m(dx)},$$

and the statement follows by Campbell's formula (see e.g. [14]) ■

Proof of Theorem 3.1. We will prove that under the laws \mathbb{P}_x^\uparrow the process Z in equation (3) is a weak solution of (5). Pathwise uniqueness, which then classically implies also strong existence, can be shown as in [10].

We write $B = B^Z$ and $N = N^Z$, and we denote by $\{\mathcal{F}_t\}$ the filtration

$$\mathcal{F}_t := \sigma(B_s, (r_n, \nu_n) \mathbf{1}_{t_n \leq s}; n \in \mathbb{N}, s \leq t),$$

where $\{(t_n, r_n, \nu_n) \in [0, \infty) \times (0, \infty) \times (0, \infty)\}_{n \in \mathbb{N}}$ are the atoms of the Poisson point process N . We will use the absolute continuity of \mathbb{P}^\uparrow w.r.t. \mathbb{P} and the Radon-Nikodym density $D_t = \frac{e^{\rho t} Z_t}{x}$ applying the previous theorem to the process $\{B_t : t \geq 0\}$ and, indirectly, to the Poisson random measure N and its compensated measure.

Dealing with the diffusion part is standard since $d\langle D, B \rangle_t = \frac{e^{\rho t}}{x} \sigma \sqrt{Z_t} dt$, so that

$$B_t^\uparrow := B_t - \int_0^t \frac{d\langle D, B \rangle_s}{D_s} = B_t - \sigma \int_0^t Z_s^{-\frac{1}{2}} ds$$

is a Brownian motion under \mathbb{P}^\uparrow by Girsanov theorem.

We next study the way the Poisson random measure N is affected by the change of probability, which is the main part of the proof. Enlarging the probability space and filtration if needed, we may and shall assume that there is a sequence $(u_n)_{n \geq 1}$ of independent random variables uniformly distributed on $[0, 1]$, independent of B and N and such that $u_n \mathbf{1}_{t_n \leq t}$ is \mathcal{F}_t -measurable. Define random variables $(\Delta_n, \delta_n) \in [0, \infty)^2 \times [0, \infty)$ by

$$(\Delta_n, \delta_n) := \begin{cases} ((0, 0), r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n^-}\}}) & \text{if } u_n > \frac{D_{t_n^-}}{D_{t_n}} = \frac{Z_{t_n^-}}{Z_{t_n}} \text{ and } Z_{t_n} > 0, \\ ((r_n, \nu_n), 0) & \text{if } u_n \leq \frac{D_{t_n^-}}{D_{t_n}} \text{ and } Z_{t_n} > 0, \\ ((0, 0), 0) & \text{if } Z_{t_n} = 0. \end{cases}$$

Let $f_{R,\epsilon}$ be a nonnegative function such that for all (r, ν, s)

- $f_{R,\epsilon}((r, \nu), s) = 0$ when $\nu \geq R$, for some fixed $R \geq 0$,
- $f_{R,\epsilon}((r, \nu), s) = 0$ when $r < \epsilon$, for some fixed $0 < \epsilon \leq 1$, and
- $f_{R,\epsilon}((0, 0), 0) = 0$.

For any non-negative predictable process F , we have

$$\begin{aligned} \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}(\Delta_n, \delta_n) &= \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}((0, 0), r_n \mathbf{1}_{\{\nu \leq Z_{t_n^-}\}}) \mathbf{1}_{\{u_n > \frac{Z_{t_n^-}}{Z_{t_n}}\}} \\ &\quad + \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}((r_n, \nu_n), 0) \mathbf{1}_{\{u_n \leq \frac{Z_{t_n^-}}{Z_{t_n}}\}}. \end{aligned}$$

Therefore, since $1 - \frac{Z_{t_n^-}}{Z_{t_n}} = \frac{r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n^-}\}}}{Z_{t_n}}$, the process

$$\begin{aligned} S_t &:= \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}(\Delta_n, \delta_n) - \int_0^t ds F_s \int_0^\infty \int_0^\infty f_{R,\epsilon}((0, 0), r \mathbf{1}_{\{\nu \leq Z_s\}}) \frac{r \mathbf{1}_{\{\nu \leq Z_s\}}}{Z_s + r \mathbf{1}_{\{\nu \leq Z_s\}}} \Pi(dr) d\nu \\ &\quad - \int_0^t ds F_s \int_0^\infty \int_0^\infty f_{R,\epsilon}((r, \nu), 0) \frac{Z_s}{Z_s + r \mathbf{1}_{\{\nu \leq Z_s\}}} \Pi(dr) d\nu \end{aligned}$$

is a martingale under \mathbb{P} . The quadratic covariation of S and D is given by

$$\begin{aligned} [S, D]_t &= \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}(\Delta_n, \delta_n) \frac{e^{\rho t_n}}{x} r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n^-}\}} \\ &= \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}((0, 0), r_n \mathbf{1}_{\{\nu \leq Z_{t_n^-}\}}) \frac{e^{\rho t_n}}{x} r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n^-}\}} \mathbf{1}_{\{u_n > \frac{Z_{t_n^-}}{Z_{t_n}}\}} \\ &\quad + \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}((r_n, \nu_n), 0) \frac{e^{\rho t_n}}{x} r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n^-}\}} \mathbf{1}_{\{u_n \leq \frac{Z_{t_n^-}}{Z_{t_n}}\}}, \end{aligned}$$

since S is a jump process. Thus, the conditional quadratic covariation is

$$\begin{aligned}\langle D, S \rangle_t &= \int_0^t \frac{e^{\rho s}}{x} F_s ds \int_0^\infty \int_0^\infty f_{R,\epsilon}((0,0), r \mathbf{1}_{\{\nu \leq Z_s\}}) \frac{r \mathbf{1}_{\{\nu \leq Z_s\}}}{Z_s + r \mathbf{1}_{\{\nu \leq Z_s\}}} r d\Pi(dr) d\nu \\ &\quad + \int_0^t \frac{e^{\rho s}}{x} F_s ds \int_0^\infty \int_0^\infty f_{R,\epsilon}((r,\nu), 0) \frac{Z_s}{Z_s + r \mathbf{1}_{\{\nu \leq Z_s\}}} r \mathbf{1}_{\{\nu \leq Z_s\}} d\Pi(dr) d\nu.\end{aligned}$$

Then, using Girsanov's theorem, we see that the process

$$\begin{aligned}S_t^\uparrow &:= S_t - \int_0^t \int_0^\infty \int_0^\infty F_s f_{R,\epsilon}((0,0), r \mathbf{1}_{\{\nu \leq Z_s\}}) \frac{r \mathbf{1}_{\{\nu \leq Z_s\}}}{Z_s + r \mathbf{1}_{\{\nu \leq Z_s\}}} \frac{r}{Z_s} \Pi(dr) d\nu ds \\ &\quad - \int_0^t \int_0^\infty \int_0^\infty F_s f_{R,\epsilon}((r,\nu), 0) \frac{Z_s}{Z_s + r \mathbf{1}_{\{\nu \leq Z_s\}}} \frac{r \mathbf{1}_{\{\nu \leq Z_s\}}}{Z_s} \Pi(dr) d\nu ds\end{aligned}$$

is a (\mathcal{F}_t) -martingale under \mathbb{P}^\uparrow . By the definition of S ,

$$\begin{aligned}S_t^\uparrow &= \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}(\Delta_n, \delta_n) - \int_0^t F_s ds \int_0^\infty \int_0^\infty \left[f_{R,\epsilon}((0,0), r \mathbf{1}_{\{\nu \leq Z_s\}}) \frac{r \mathbf{1}_{\{\nu \leq Z_s\}}}{Z_s} + f_{R,\epsilon}((r,\nu), 0) \right] \Pi(dr) d\nu \\ &= \sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}(\Delta_n, \delta_n) - \int_0^t F_s ds \int_0^\infty \int_0^\infty \left[f_{R,\epsilon}((0,0), r) \frac{r}{Z_s} \mathbf{1}_{\{\nu \leq Z_s\}} + f_{R,\epsilon}((r,\nu), 0) \right] \Pi(dr) d\nu\end{aligned}$$

since $f_{R,\epsilon}((0,0), 0) = 0$. Recalling that S^\uparrow is a (\mathcal{F}_t) -martingale on \mathbb{P}^\uparrow starting from 0, we deduce that

$$\begin{aligned}\mathbb{E}^\uparrow \left[\sum_{t_n \leq t} F_{t_n} f_{R,\epsilon}(\Delta_n, \delta_n) \right] &= \mathbb{E}^\uparrow \left[\int_0^t F_s ds \int_0^\infty f_{R,\epsilon}((0,0), r) r \Pi(dr) \right] \\ &\quad + \mathbb{E}^\uparrow \left[\int_0^t F_s ds \int_0^\infty \int_0^\infty f_{R,\epsilon}((r,\nu), 0) \Pi(dr) d\nu \right].\end{aligned}$$

By standard arguments, this formula is also true for any nonnegative function f such that $f((0,0), 0) = 0$. By Lemma 4.2 we see that, under \mathbb{P}^\uparrow , $(t_n, \Delta_n)_{n \geq 0}$ and $(t_n, \delta_t)_{n \geq 0}$ are atoms of two Poisson point processes N^\uparrow and N^* with intensity measures $dt \times d\nu \times \Pi(dr)$ and $dt \times r \Pi(dr)$ on $[0, \infty) \times (0, \infty) \times (0, \infty)$ and $[0, \infty) \times (0, \infty)$ respectively. By construction, N^\uparrow and N^* are independent because they never jump simultaneously. Now set

$$J_t := \int_0^t \int_0^{Z_s^-} \int_1^\infty r N(ds, d\nu, dr) = \sum_{t_n \leq t} r_n \mathbf{1}_{\{\nu_n \leq Z_{t_n}^-\}} \mathbf{1}_{\{r_n \geq 1\}}.$$

From above, we have

$$J_t = \sum_{t_n \leq t} \Delta_n^{(1)} \mathbf{1}_{\{\Delta_n^{(2)} \leq Z_{t_n}^-\}} \mathbf{1}_{\{\Delta_n^{(1)} \geq 1\}} + \sum_{t_n \leq t} \delta_n \mathbf{1}_{\{\delta_n \geq 1\}},$$

where $\Delta_n^{(i)}$ is the i -th coordinate of Δ_n , $i = 1, 2$. Therefore

$$J(t) = \int_0^t \int_0^{Z_s^-} \int_1^\infty r N^\uparrow(ds, d\nu, dr) + \int_0^t \int_1^\infty r N^*(ds, dr).$$

Finally, given $0 < \varepsilon < 1$, let $\{\tilde{M}_t^{(\varepsilon)}, t \geq 0\}$ be the \mathbb{P} -martingale

$$\begin{aligned}\tilde{M}_t^{(\varepsilon)} &:= \int_0^t \int_0^{Z_{s^-}} \int_\varepsilon^1 r N^Z(ds, d\nu, dr) - \int_0^t \int_0^{Z_{s^-}} \int_\varepsilon^1 r ds d\nu \Pi(dr) \\ &= \sum_{t_n \leq t} r_n \mathbf{1}_{(\nu_n \leq Z_{t_n^-})} \mathbf{1}_{(\varepsilon < r_n < 1)} - \int_0^t \int_0^{Z_{s^-}} \int_\varepsilon^1 r ds d\nu \Pi(dr),\end{aligned}$$

which converges in the $L^2(\mathbb{P})$ sense when $\varepsilon \rightarrow 0$ to $\tilde{M}_t := \int_0^t \int_0^{Z_{s^-}} \int_0^1 r \tilde{N}^Z(ds, d\nu, dr)$. In terms of (Δ_n) and (δ_n) , we can write

$$\begin{aligned}\tilde{M}^{(\varepsilon)} &= \left(\sum_{t_n \leq t} \Delta_n^{(1)} \mathbf{1}_{(\Delta_n^{(2)} \leq Z_{t_n^-})} \mathbf{1}_{(\varepsilon < \Delta_n^{(1)} < 1)} - \int_0^t \int_0^{Z_{s^-}} \int_\varepsilon^1 r ds d\nu \Pi(dr) \right) \\ &\quad + \sum_{t_n \leq t} \delta_n \mathbf{1}_{(\varepsilon < \delta_n < 1)} \\ &= \left(\int_0^t \int_0^{Z_{s^-}} \int_\varepsilon^1 r N^\uparrow(ds, d\nu, dr) - \int_0^t \int_0^{Z_{s^-}} \int_\varepsilon^1 r ds d\nu \Pi(dr) \right) \\ &\quad + \int_0^t \int_\varepsilon^1 r N^*(ds, dr).\end{aligned}$$

Thanks to [16, Theorem 2.10], the limit as $\varepsilon \rightarrow 0$ in the $L^2(\mathbb{P}^\uparrow)$ sense of the \mathbb{P}^\uparrow -martingale given by the first term on the right hand side exists, and it is equal to the martingale $\int_0^t \int_0^{Z_{s^-}} \int_0^1 r \tilde{N}^\uparrow(ds, d\nu, dr)$, where \tilde{N}^\uparrow is the compensated measure associated with N^\uparrow . Also, as $\int_0^\infty (1 \wedge x^2) \Pi(dx) < \infty$, by [16, Theorem 2.9] the second term on the right hand side converges \mathbb{P}^\uparrow -a.s., so we have

$$\tilde{M}_t = \int_0^t \int_0^{Z_{s^-}} \int_0^1 r \tilde{N}^\uparrow(ds, d\nu, dr) + \int_0^t \int_0^1 r N^*(ds, dr).$$

Bringing all parts together, we have shown that Z satisfies under \mathbb{P}^\uparrow the desired SDE, except for the independence of the processes B^\uparrow and (N^\uparrow, N^*) , which we shall establish in what follows.

Since N^\uparrow and N^* have σ -finite intensities and thanks to the Markov property of the three processes with respect to the filtration (\mathcal{F}_t) , it is enough to show that for every $t > s \geq 0$, $\zeta \in \mathbb{R}$, $\lambda_k, \gamma_k \in \mathbb{R}_+$, $k \in \{1, \dots, m\}$, and $m \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}^\uparrow \left[e^{-\zeta(B_t^\uparrow - B_s^\uparrow)} e^{-\sum_{k=1}^m \lambda_k N^\uparrow((s, t] \times W_k)} e^{-\sum_{k=1}^m \gamma_k N^*((s, t] \times V_k)} \middle| \mathcal{F}_s \right] \\ = e^{-\frac{\zeta^2}{2}(t-s)} e^{\sum_{k=1}^m \int_s^t \int_{W_k} (e^{-\lambda_k} - 1) \Pi(dr) d\nu du} e^{\sum_{k=1}^m \int_s^t \int_{V_k} (e^{-\gamma_k} - 1) r \Pi(dr) du},\end{aligned}$$

where $\{W_k\}_{k=1}^m$ and $\{V_k\}_{k=1}^m$ are disjoint sets of $(0, \infty) \times (0, \infty)$ and $(0, \infty)$ such that $\int_0^t \int_{W_k} \Pi(dr) d\nu du$ and $\int_0^t \int_{V_k} r \Pi(dr) du$ are finite. To that end, set

$$F(x, y_1, \dots, y_m, z_1, \dots, z_m) := e^{-\zeta x} e^{-\sum_{k=1}^m \lambda_k y_k} e^{-\sum_{k=1}^m \gamma_k z_k}.$$

Applying Itô's formula to the semimartingale

$$X(t) = \left(B^\uparrow(t), N^\uparrow((0, t] \times W_1), \dots, N^\uparrow((0, t] \times W_m), N^*((0, t] \times V_1), \dots, N^*((0, t] \times V_m) \right),$$

we obtain:

$$\begin{aligned}
F(X(t)) &= F(X(s)) - \int_s^t \zeta F(X(u)) dB_u^\uparrow - \sum_{j=1}^m \int_s^t \int_{W_j} \lambda_j F(X(u)) N^\uparrow(du, d\nu, dr) \\
&\quad - \sum_{j=1}^m \int_s^t \int_{V_j} \gamma_j F(X(u)) N^*(du, dr) + \frac{\zeta^2}{2} \int_s^t F(X(u)) du + \sum_{s < u \leq t} F(X(u)) - F(X(u^-)) \\
&\quad + \sum_{s < t_n \leq t} \sum_{j=1}^m \left[\lambda_j F(X(t_n)) \mathbf{1}_{\{\Delta_n \in W_j\}} + \gamma_j F(X(t_n)) \mathbf{1}_{\{\delta_n \in V_j\}} \right].
\end{aligned}$$

From above, we deduce that

$$\begin{aligned}
F(X(t)) - F(X(s)) &= \bar{M}_t - \bar{M}_s + \frac{\zeta^2}{2} \int_s^t F(X(u)) du \\
&\quad + \sum_{s < t_n \leq t} \left[F(X(t_n^-) + (0, \mathbf{1}_{\{\Delta_n \in W_1\}}, \dots, \mathbf{1}_{\{\delta_n \in V_m\}})) - F(X(t_n^-)) \right],
\end{aligned}$$

where (\bar{M}_t) is a (\mathcal{F}_t) -martingale. Defining $f(\Delta_n, \delta_n) := e^{-\sum_{k=1}^m \lambda_k \mathbf{1}_{\{\Delta_n \in W_k\}} - \sum_{k=1}^m \lambda_k \mathbf{1}_{\{\delta_n \in V_k\}}} - 1$, we have

$$F(X(t)) - F(X(s)) = \bar{M}_t - \bar{M}_s + \frac{\zeta^2}{2} \int_s^t F(X(u)) du + \sum_{s < t_n \leq t} F(X(t_n^-)) [f(\Delta_n, \delta_n)].$$

Let now $A \in \mathcal{F}_s$. Multiplying both sides by $F(-(X(s))) \mathbf{1}_A$, yields:

$$\begin{aligned}
\mathbb{E}^\uparrow [F(X(t-s)) \mathbf{1}_A] - \mathbb{P}^\uparrow(A) &= \frac{\zeta^2}{2} \int_s^t \mathbb{E}^\uparrow [F(X(u-s)) \mathbf{1}_A] du \\
&\quad + \int_s^t \mathbb{E}^\uparrow [F(X(u-s)) \mathbf{1}_A] du \sum_{k=1}^m \int_{W_k} (e^{-\lambda_k} - 1) \Pi(dr) d\nu \\
&\quad + \int_s^t \mathbb{E}^\uparrow [F(X(u-s)) \mathbf{1}_A] du \sum_{k=1}^m \int_{V_k} (e^{-\gamma_k} - 1) r \Pi(dr).
\end{aligned}$$

Thus,

$$\mathbb{E}^\uparrow [F(X(t-s)) \mathbf{1}_A] = \mathbb{P}^\uparrow(A) e^{-\frac{\zeta^2}{2}(s-t)} e^{\sum_{k=1}^m \int_s^t \int_{W_k} (e^{-\lambda_k} - 1) \Pi(dr) d\nu du} e^{\sum_{k=1}^m \int_s^t \int_{V_k} (e^{-\gamma_k} - 1) r \Pi(dr) du}$$

which means that the three processes are mutually independent, which ends the proof of weak existence.

As concerns pathwise uniqueness, we just remark that the proof of Theorem 3.2 in [10] covers the case of equation (5). Indeed, if B^\uparrow , N^\uparrow and N^* are independent processes as before driving two solutions $\{Z_t^{(1)}\}$ and $\{Z_t^{(2)}\}$ of (5), setting $\zeta_t := Z_t^{(1)} - Z_t^{(2)}$ one gets that

$$\begin{aligned}
\zeta_t &= \zeta_0 + \int_0^t a \left(Z_s^{(1)} - Z_s^{(2)} \right) ds + \int_0^t \sigma \left(\sqrt{Z_s^{(1)}} - \sqrt{Z_s^{(2)}} \right) dB_s^\uparrow \\
&\quad + \int_0^t \int_{U_0} r \left(\mathbf{1}_{(\nu < Z_s^{(1)})} - \mathbf{1}_{(\nu < Z_s^{(2)})} \right) N^\uparrow(ds, d\nu, dr) \\
&\quad + \int_0^t \int_{U_1} r \left(\mathbf{1}_{(\nu < Z_s^{(1)})} - \mathbf{1}_{(\nu < Z_s^{(2)})} \right) \tilde{N}^\uparrow(ds, d\nu, dr),
\end{aligned} \tag{8}$$

where $U_0 = [0, \infty) \times [1, \infty)$ and $U_1 = [0, \infty) \times (0, 1)$. From this point on, the proof of Theorem 3.2 in [10] applies, since conditions (2.a,b) and (3.a,b) therein are satisfied. Indeed, in their notations, we

have the intensity measure $\mu(du) = \Pi(dr)d\nu$ for $N^0 = N^\uparrow|_{U_0}$ and $N^1 = N^\uparrow|_{U_1}$ (where $u = (r, \nu)$), continuous functions on \mathbb{R} given by $b(x) := ax\mathbf{1}_{0 \leq x}$ and $\sigma(x) := \sigma\sqrt{x}\mathbf{1}_{0 \leq x}$, and Borel functions on $\mathbb{R} \times U_i, i = \{0, 1\}$ given by $g(x, u) = g_0(x, u) = g_1(x, u) = r\mathbf{1}_{\nu < x}$ such that $g(x, u) + x \geq 0$ for $x > 0$ and $g(x, u) = 0$ for $x \leq 0$. Moreover,

1. there is a constant $K := |a| + M \geq 0$, where $\int_1^\infty r\Pi(dr) = M < \infty$, such that

$$|ax| + \int_0^\infty \int_1^\infty r\mathbf{1}_{\nu < x}\Pi(dr)d\nu \leq K(x + 1);$$

2. there is a non-negative and non-decreasing function $L(x) = (\sigma^2 + I)(x)$ on \mathbb{R}_+ , with $I = \int_0^1 r^2\Pi(dr)$, so that

$$\sigma^2x + \int_0^\infty \int_0^1 r^2\mathbf{1}_{\nu < x}\Pi(dr)d\nu \leq L(x);$$

3. there is a continuous non-decreasing function $x \rightarrow b_2(x) := x$ on \mathbb{R}_+ such that for $b_1(x) = b(x) + b_2(x)$, on has

$$|(a + 1)(b_1(x) - b_1(y))| + \int_0^\infty \int_1^\infty r\mathbf{1}_{y < \nu < x}\Pi(dr)d\nu \leq r(|x - y|);$$

where r is the non-decreasing and concave function $r(z) =: (|a + 1| + M)z$ on \mathbb{R}_+ satisfying $\int_{0+} r(z)^{-1}dz = \infty$; and

4. for every fixed $u \in U_0$ the function $x \rightarrow g(x, u)$ is non-decreasing, and there is a non-negative and non-decreasing function $\rho(z) := [\sigma^2 + I]\sqrt{z}$ on \mathbb{R}_+ so that $\int_{0+} \rho(z)^{-2}dz = \infty$ and

$$(\sigma\sqrt{x} - \sigma\sqrt{y})^2 + \int_0^\infty \int_0^1 r^2\mathbf{1}_{y < \nu < x}\Pi(dr)d\nu \leq \rho(|xy|)^2.$$

Conditions 1,2,3 and 4 respectively ensure that hypotheses (2.a,b) and (3.a,b) in [10] hold, and pathwise uniqueness follows. ■

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