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EXISTENCE OF LOCAL AND GLOBAL SOLUTIONS OF FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS

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We study the existence of local and global mild solutions of the fractional-order differential equations in an arbitrary Banach space by using the semigroup theory and the Schauder fixed-point theorem. We also give some examples to illustrate the applications of abstract results.

1. Introduction

We consider the following fractional-order differential equation in a Banach space $(H, \|.\|)$:

$$\frac{d^\beta u(t)}{dt^\beta} + Au(t) = f(t, u(t)), \quad t \in (0, T],$$
$$u(0) = u_0, \tag{1.1}$$

where A is a closed linear operator defined on a dense set, $0 < \beta \le 1$, $0 < T < \infty$, and $\frac{d^\beta u(t)}{dt^\beta}$ denotes the derivative of u in the Caputo sense. We assume that $-A$ is the infinitesimal generator of a compact analytic semigroup $\{S(t) \colon t \ge 0\}$ in H, and the nonlinear map f is defined from $[0, T] \times H$ into H satisfying certain conditions to be specified later.

For the initial works on the existence and uniqueness of solutions of differential equations of different types, see [1–9] and references therein.

Jardat et al. [3] considered the following fractional-order differential equation in a Banach space:

$$\frac{d^\beta u(t)}{dt^\beta} = Au(t) + f(t, u(t), Gu(t), Su(t)), \quad t > t_0, \quad \beta \in (0, 1],$$
$$u(t_0) = u_0, \tag{1.2}$$

where A generates a strongly continuous semigroup. They have used the semigroup and fixed-point methods to prove the existence and uniqueness of solutions.

In the present paper, we use the Schauder fixed-point theorem and semigroup theory to prove the existence of local and global mild solutions of problem (1.1). With some extra assumptions, we can apply all the results of this paper to the problem considered by Jardat in [3].

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The plan of the paper is as follows: The introduction and preliminaries are given, respectively, in the first two sections. In Sec. 3, we prove the existence of local mild solutions, and the existence of global mild solutions for problem (1.1) is given in Sec. 4. In the last section, we give some examples

2. Preliminaries

We note that if $-A$ is the infinitesimal generator of an analytic semigroup, then, for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case where the semigroup is bounded and the generator is invertible. Hence, without loss of generality, we suppose that

$$\|S(t)\| \leq M \quad \text{for} \quad t \geq 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that, for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ dense in H. We have $H_\kappa \hookrightarrow H_\alpha$ for $0 < \alpha < \kappa$, and the imbedding is continuous. For more details on the fractional powers of closed linear operators, see [10]. It can easily be proved that $H_\alpha := D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ equivalent to the graph norm of A^α.

Note that the set $\mathcal{C}_T = C([0,T], H)$ of all continuous functions from $[0,T]$ into H is a Banach space under the supremum norm given by

$$\|\psi\|_T := \sup_{0 \leq \eta \leq T} \|\psi(\eta)\|, \quad \psi \in \mathcal{C}_T.$$

We consider the following assumptions:

(H$_1$) $-A$ is the infinitesimal generator of a compact analytic semigroup $S(t)$;

(H$_2$) the nonlinear map $f\colon [0,T] \times H \to H$ is continuous in the first variable and satisfies the following condition:

$$\|f(t,x) - f(s,y)\| \leq L_f(r)[|t-s| + \|x-y\|],$$

for all $x, y \in B_r(H, u_0)$ and $t, s \in [0,T]$. Here, $L_f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function and, for $r > 0$,

$$B_r(Z, z_1) = \{z \in Z\colon \|z - z_1\|_Z \leq r\},$$

where $(Z, \|.\|_Z)$ is a Banach space.

We need some basic definitions and properties from the fractional-calculus theory.

Definition 2.1. *A real function $g(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number p $(> \mu)$ such that $g(x) = x^p g_1(x)$, where $g_1 \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $g^{(m)} \in C_\mu$, $m \in \mathbb{N}$.*

Definition 2.2. *The Riemann–Liouville fractional integral operator of order* $\beta \ge 0$ *of a function* $g \in C_\mu$, $\mu \ge -1$, *is defined as*

$$I^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} g(\theta) d\theta, \quad t > 0.$$

Definition 2.3. *If* $g \in C_{-1}^m$ *and* m *is a positive integer, then we can define the fractional derivative of* $g(t)$ *in the Caputo sense as follows:*

$$\frac{d^\beta g(t)}{dt^\beta} = \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\theta)^{m-\beta-1} g^m(\theta) d\theta, \quad m-1 < \beta \le m, \quad t > 0.$$

Definition 2.4. *By a mild solution of the differential equation (1.1), we mean a continuous solution* u *of the following integral equation:*

$$u(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} S(t-s) f(s, u(s))\, ds.$$

For more details on mild solutions, see [3].

3. Existence of Local Solutions

To prove the existence of a mild solution of the evolution problem (1.1), we need the following lemma:

Lemma 3.1. *The differential equation (1.1) is equivalent to the following integral equation:*

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (-Au(s)) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, u(s)) ds,$$

where $0 < t \le T$.

Proof. For details, we refer to Lemma 1.1 in [3].

We now state the following theorem:

Theorem 3.1. *Assume that conditions (H*$_1$*) and (H*$_2$*) are satisfied and* $u_0 \in D(A)$. *Then there exists* t_0, $0 < t_0 < T$, *such that Eq. (1.1) has a local mild solution on* $[0, t_0]$.

Proof. Let $R > 0$ be such that

$$M\|u_0\| \le \frac{R}{2}$$

and let $A_1 = \|A^{-\alpha}\|$.

We choose t_0, $0 < t_0 \le T$, such that

$$t_0 < \left[\frac{R}{2} \left\{ \frac{M}{\beta\Gamma(\beta)} \{L_f(R)[T + R] + \|f(0, u_0)\|\} \right\}^{-1} \right]^{\frac{1}{\beta}}.$$

We set

$$Y = \{u \in \mathcal{C}_{t_0} : u(0) = u_0, \ \ \|u(t) - u_0\| \le R \ \text{ for } \ 0 \le t \le t_0\}.$$

Clearly, Y is a bounded, closed, convex subset of $\mathcal{C}_{t_0}$.

For any $0 < \tilde{T} \le T$, we define a mapping F from $\mathcal{C}_{\tilde{T}}$ into $\mathcal{C}_{\tilde{T}}$ as follows:

$$(Fu)(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} S(t-s) f(s, u(s))\, ds.$$

Clearly, F is well defined.

We need to show that $F: Y \to Y$. For any $u \in Y$, we have $(Fu)(0) = u_0$. If $t \in [0, t_0]$, then

$$\|(Fu)(t) - u_0\| \le \|S(t)u_0 - u_0\| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|S(t-s)\| \|f(s, u(s))\| ds$$

$$\le \frac{R}{2} + \frac{M}{\beta\Gamma(\beta)} \{L_f(R)[T + R] + \|f(0, u_0)\|\} t_0^{\beta} \le R.$$

Hence, $F: Y \to Y$.

We now show that F maps Y into a precompact subset $F(Y)$ of Y. For this purpose, we show that, for fixed $t \in [0, t_0]$, $Y(t) = \{(Fu)(t) : u \in Y\}$ is precompact in H and $F(Y)$ is a uniformly equicontinuous family of functions. Here, for $t = 0$, $Y(0) = \{u_0\}$ is precompact in H.

Let $t > 0$ be fixed. For an arbitrary $\epsilon \in (0, t)$, we define a mapping F_ϵ on Y by the formula

$$(F_\epsilon u)(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)} \int_0^{t-\epsilon} (t-s)^{\beta-1} S(t-s) f(s, u(s)) ds$$

$$= S(t)u_0 + \frac{S(\epsilon)}{\Gamma(\beta)} \int_0^{t-\epsilon} (t-s)^{\beta-1} S(t-s-\epsilon) f(s, u(s))\, ds.$$

Since $S(\epsilon)$ is compact for every $\epsilon > 0$, the set $Y_\epsilon(t) = \{(F_\epsilon u)(t) : u \in Y\}$ is precompact in H for every $\epsilon \in (0, t)$, where $t \in (0, t_0]$.

We also have

$$\|(Fu)(t) - (F_\epsilon u)(t)\| = \left\| \frac{1}{\Gamma(\beta)} \int_{t-\epsilon}^t (t-s)^{\beta-1} S(t-s) f(s, u(s))\, ds \right\| \le \epsilon^{\beta} R_1$$

for all $t \in (0, t_0]$, $u \in Y$, and

$$R_1 = \frac{M}{\beta\Gamma(\beta)}\{L_f(R)[T + R] + \|f(0, u_0)\|\}.$$

Consequently, the set $Y(t)$, where $t \geq 0$, is precompact in H.

For any $t_1, t_2 \in (0, t_0]$ with $t_1 < t_2$ and $u \in Y$, we have

$$\begin{aligned}(Fu)(t_2) - (Fu)(t_1) = [S(t_2) - S(t_1)]u_0 + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2 - s)^{\beta-1}S(t_2 - s)f(s, u(s))\,ds \\ + \frac{-1}{\Gamma(\beta)}\int_0^{t_1}[(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}]S(t_2 - s)f(s, u(s))\,ds \\ + \frac{1}{\Gamma(\beta)}\int_0^{t_1}(t_1 - s)^{\beta-1}[S(t_2 - s) - S(t_1 - s)]\,f(s, u(s))\,ds \\ = I_1 + I_2 + I_3 + I_4. \qquad (3.1)\end{aligned}$$

Hence,

$$\|(Fu)(t_2) - (Fu)(t_1)\| \leq \|I_1\| + \|I_2\| + \|I_3\| + \mathcal{I}_4\|. \qquad (3.2)$$

We get

$$I_1 = [S(t_2) - S(t_1)]u_0.$$

It follows from Theorem 2.6.13 in [10] that, for every $0 < \eta < 1 - \alpha$, $t_2 > t_1 > 0$, we have

$$\|I_1\| \leq A_1\|(S(t_2) - S(t_1))A^\alpha u_0\| \leq A_1C_\eta C_{\alpha+\eta}t_1^{-(\alpha+\eta)}(t_2 - t_1)^\eta\|u_0\| \leq M_1(t_2 - t_1)^\eta,$$

where C_η is some positive constant such that $\|A^\eta S(t)\| \leq C_\eta t^{-\eta}$ for all $t > 0$. Furthermore, M_1 depends on t_1 and blows up as t_1 decreases to zero.

Using Eq. (3.1), we get

$$\|I_2\| \leq \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2 - s)^{\beta-1}\|S(t_2 - s)\|\|f(s, u(s))\|ds \leq \frac{MA_2}{\beta\Gamma(\beta)}(t_2 - t_1)^\beta,$$

where $A_2 = \{L_f(R)[T + R] + \|f(0, u_0)\|\}$. We have

$$I_3 = \frac{-1}{\Gamma(\beta)}\int_0^{t_1}[(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}]S(t_2 - s)f(s, u(s))\,ds.$$

Hence,

$$\|I_3\| \leq \frac{A_2\, A_1\, C_\alpha}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\lambda - 1}[(t_1 - s)^{-\lambda\mu} - (t_2 - s)^{-\lambda\mu}]\, ds,$$

where

$$\lambda = 1 - \alpha, \quad \mu = \frac{1-\beta}{1-\alpha}, \quad \text{and} \quad \alpha \neq 1.$$

Hence, after some calculation, we get

$$\|I_3\| \leq \frac{A_2\, A_1\, C_\alpha}{\Gamma(\beta)} \mu {\delta_1}^{\mu-1}(1-c)^{-\lambda(1-\mu)-1}(t_2 - t_1)^{\lambda(1-\mu)},$$

where

$$c = \left(1 - \left(\frac{\mu}{\lambda}\right)^1 \lambda\mu\right) \quad \text{and} \quad 0 < \delta_1 \leq 1.$$

Similarly, we obtain

$$\|I_4\| \leq \frac{A_1\, A_2\, C_{1+\alpha}}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1}[(t_1 - s)^{-1} - (t_2 - s)^{-1}]\, ds$$

$$\leq \frac{A_1\, A_2\, C_{1+\alpha}}{\alpha\Gamma(\beta)} {\delta_2}^{(\frac{1}{\beta}-1)}(1 - c_1)^{-\beta}(t_2 - t_1)^{\beta(1-\frac{1}{\beta})},$$

where

$$C_{1+\alpha} = \left(1 - \frac{1}{\beta^2}\right), \quad 0 < \delta_2 \leq 1,$$

and $C_{1+\alpha}$ is some positive constant such that $\|A^{1+\alpha}S(t)\| \leq C_{1+\alpha}t^{-1-\alpha}$ for all $t > 0$.

Thus, it follows from the above calculations that the right-hand side of inequality (3.2) tends to zero as $t_2 - t_1 \to 0$. Hence, $F(Y)$ is a family of equicontinuous functions. Furthermore, $F(Y)$ is bounded. Thus, according to the Arzelà–Ascoli theorem (see [11]), $F(Y)$ is precompact. The existence of a fixed point of F in Y is a consequence of the Schauder fixed-point theorem.

Hence, there exists $u \in Y$ such that, for all $t \in [0, t_0]$, we have

$$u(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} S(t-s) f(s, u(s)) ds, \tag{3.3}$$

where $u(0) = u_0$.

Applying similar arguments as above, we see that the function u given by Eq. (3.3) is uniformly Hölder continuous on $[0, t_0]$. With the help of condition (H_2), we can show that the map $t \longmapsto f_1(t, u(t))$ is Hölder continuous on $[0, t_0]$. This completes the proof of the theorem.

4. Existence of Global Solutions

Theorem 4.1. *Suppose that* $0 \in \rho(-A)$ *and* $-A$ *generates a compact analytic semigroup* $S(t)$ *with* $\|S(t)\| \leq M$ *for* $t \geq 0$, $u_0 \in D(A)$, *and the function* $f_1 \colon [0, \infty) \times H \to H$ *satisfies condition* (H_2). *If there is a continuous, nondecreasing, real-valued function* $k(t)$ *such that*

$$\|f_1(t, \psi)\| \leq k(t)(1 + \|\psi\|) \quad \textit{for} \quad t \geq 0, \quad \psi \in H,$$

then Eq. (1.1) has a unique mild solution u, *which exists for all* $t \geq 0$.

Proof. According to Theorem 3.1, we can continue the solution of Eq. (1.1) as long as $\|u(t)\|$ stays bounded. Therefore, we need to show that if u exists on $[0, T)$, then $\|u(t)\|$ is bounded as $t \uparrow T$.

For $t \in [0, T)$, we have

$$u(t) = S(t)u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} S(t-s) f(s, u(s)) ds.$$

From the above equation, we get

$$\|u(t)\| \leq M\|u_0\| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|S(t-s)\| \|f(s, u(s))\| ds.$$

Hence,

$$\|u(t)\| \leq C_2 + C_3 \int_0^t (t-s)^{(\beta-1)} \|u(s)\| \, ds,$$

where

$$C_2 = M\|u_0\| + \frac{1}{\beta\Gamma(\beta)} M k(T) T^{\beta}$$

and

$$C_3 = \frac{1}{\Gamma(\beta)} M k(T).$$

Therefore, it follows from Lemma 6.7 ([10], Chap. 5) that u is a global solution.

To complete the proof of the theorem we only need to show that u is unique on the whole interval.

Let u_1 and u_2 be two solutions of the given fractional integral equation (1.1). Then, by a similar argument as above, we conclude that

$$\|u_1(t) - u_2(t)\| \leq \frac{1}{\Gamma(\beta)} M L_f(R) \int_0^t (t-s)^{(\beta-1)} \|u_1(s) - u_2(s)\| \, ds.$$

Hence, according to Lemma 6.7 ([10], Chap. 5), the solution u is unique. This completes the proof of the theorem.

5. Examples

Let $H = L^2((0,1);\mathbb{R})$. Consider the following fractional partial differential equations:

$$\frac{\partial^\beta}{\partial t^\beta} w(t,x) - \partial_x^2 w(t,x) = F(t, w(t,x)) \quad x \in (0,1), \quad t > 0,$$
$$w(0,x) = u_0, \quad w(t,0) = w(t,1) = 0, \quad t \in [0,T], \quad 0 < T < \infty, \tag{5.1}$$

where F is a given function and $0 < \beta < 1$.

We define an operator A as follows:

$$Au = -u'', \quad u \in D(A) = H_0^1(0,1).$$

Here, clearly, the operator A is self-adjoint, has a compact resolvent, and is the infinitesimal generator of a compact analytic semigroup $S(t)$. Let $\alpha = 1/2$ and let $D(A^{1/2})$ be a Banach space with the norm

$$\|x\|_{1/2} := \|A^{1/2}x\|, \quad x \in D(A^{1/2});$$

denote this space by $H_{1/2}$.

Equation (5.1) can be reformulated as the following abstract equation in $H = L^2((0,1);\mathbb{R})$:

$$\frac{d^\beta u(t)}{dt^\beta} + Au(t) = f(t, u(t)), \quad t > 0,$$

$$u(0) = u_0,$$

where $u(t) = w(t,.)$, i.e., $u(t)(x) = w(t,x)$, $t \in [0,T]$, $x \in (0,1)$, and the function $f\colon [0,T] \times H \to H$ is given by

$$f(t,u(t))(x) = F(t, w(t,x)).$$

We can take $f(t,u) = h(t)g(u^{'})$, where h is Lipschitz continuous and $g\colon H \to H$ is Lipschitz continuous on H. In particular, we can take $g(u) = \sin u$, $g(u) = \xi u$, and $g(u) = \arctan(u)$, where ξ is constant.

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