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Rates of Decay and *h*-Processes for One Dimensional Diffusions Conditioned on Non-Absorption

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Let (X_t) be a one dimensional diffusion corresponding to the operator $\mathscr{L} = \frac{1}{2}\partial_{xx} - \alpha\partial_x$, starting from $x > 0$ and T_0 be the hitting time of 0. Consider the family of positive solutions of the equation $\mathscr{L}\psi = -\lambda\psi$ with $\lambda \in (0, \eta)$, where $\eta = -\lim_{t\to\infty}(1/t) \log \mathbb{P}_x(T_0 > t)$. We show that the distribution of the h-process induced by any such ψ is $\lim_{M\to\infty} \mathbb{P}_x(X \in A \mid S_M < T_0)$, for a suitable sequence of stopping times $(S_M : M \geqslant 0)$ related to ψ which converges to ∞ with M. We also give analytical conditions for $\eta = \underline{\lambda}$, where $\underline{\lambda}$ is the smallest point of increase of the spectral measure associated to $\mathscr{L}^*$.

KEY WORDS: One-dimensional diffusions; *h*-processes; absorption.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

We denote by $\mathbb{P}_x$ the probability law of a Brownian motion (B_t) starting at x. Consider the diffusion (X_t) given by

$$X_t = B_t - \int_0^t \alpha(X_s)\, dx$$

where we assume α to be C^1 and denote by T_a, the hitting time of a,

$$T_a = \inf\{t > 0 : X_t = a\}$$

We consider the sub Markovian semigroup given by $P_t f(x) = \mathbb{E}_x(f(X_t), T_0 > t)$, with density kernel denoted by $p(t, x, y)$.

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Let $\gamma(x) = 2\int_0^x \alpha(z)\,dz$. We will assume the following condition always holds

Hypothesis H: $\int_0^\infty e^{\gamma(x)}(\int_0^x e^{-\gamma(z)}\,dz)\,dx = \int_0^\infty e^{-\gamma(x)}(\int_0^x e^{\gamma(z)}\,dz)\,dx = \infty.$

Hypothesis H says that ∞ is a natural boundary for the process (see [4, p. 487]).

It is known (see for instance Ref. 1) that this implies

$$\lim_{x\to\infty} \mathbb{P}_x(T_0 < s) = 0 \qquad \text{and} \qquad \lim_{M\to\infty} P_x(T_M < s) = 0 \qquad \text{for any} \quad s > 0 \tag{1}$$

A relevant function in our study is the *scale function* $\Lambda(x) = \int_0^x e^{\gamma(z)}\,dz$. It satisfies $\mathscr{L}\Lambda \equiv 0$, $\Lambda(0) = 0$, $\Lambda'(0) = 1$ where $\mathscr{L} = \frac{1}{2}\partial_{xx} - \alpha\partial_x$. Hence $\Lambda(X_s)\,1_{T_0 > s}$ is a local martingale and

$$\text{for } x \in (0, M), \qquad \mathbb{P}_x(T_M < T_0) = \frac{\Lambda(x)}{\Lambda(M)}$$

Note that

$$\mathbb{P}_x(T_0 = \infty) > 0 \qquad \text{if and only if} \quad \Lambda(\infty) < \infty$$

Let $\mathscr{L}^* = \frac{1}{2}\partial_{xx} + \partial_x(\alpha\,\cdot)$ be the formal adjoint of $\mathscr{L}$. Denote by φ_λ the solution of

$$\mathscr{L}^*\varphi_\lambda = -\lambda\varphi_\lambda, \qquad \varphi_\lambda(0) = 0, \qquad \varphi'_\lambda(0) = 1 \tag{2}$$

and by ψ_λ the solution of

$$\mathscr{L}\psi_\lambda = -\lambda\psi_\lambda, \qquad \psi_\lambda(0) = 0, \qquad \psi'_\lambda(0) = 1 \tag{3}$$

It can be checked that

$$\varphi_\lambda = e^{-\gamma}\psi_\lambda \tag{4}$$

Let $\rho(\lambda)$ be the spectral measure of the operator $\mathscr{L}^*$. We will assume ρ is left-continuous (see [2, Chapter 9]). Let $\underline{\lambda}$ the smallest point of increase of $\rho(\lambda)$. In [5, Lemma 2] it was shown that

$$\underline{\lambda} = \sup\{\lambda : \varphi_\lambda(\cdot) \text{ does not change sign}\}$$

Since $\varphi_0 = e^{-\gamma}\Lambda$ does not change sign, we have $\underline{\lambda} \geqslant 0$. Also notice that continuity in λ implies that $\varphi_{\underline{\lambda}} \geqslant 0$. Notice that ρ is concentrated in $[\underline{\lambda}, \infty)$.

Also observe that Hypotheses H implies $\int \varphi_0(x)\, dx = \int e^{-\gamma(x)} \Lambda(x)\, dx = \infty$. Therefore, we have

$$\int_0^\infty \varphi_{\underline{\lambda}}(x)\, dx < \infty \Rightarrow \underline{\lambda} > 0 \tag{5}$$

The analysis of one dimensional diffusions with absorption presented in Ref. 5, uses spectral tools. Let us introduce the concepts needed to present those results. Consider the following Hilbert spaces $\mathscr{H} = L^2(\mathbb{R}_+, e^{\gamma(x)}\, dx)$ and $\mathscr{K} = L^2(\mathbb{R}, d\rho(\lambda))$. The spectral decomposition is studied through the unitary transformation $\mathscr{U}\colon \mathscr{H} \to \mathscr{K}$ defined as

$$\mathscr{U}f(\lambda) = \int_0^\infty f(z)\, \varphi_\lambda(z)\, e^{\gamma(z)}\, dz = \int_0^\infty f(z)\, \psi_\lambda(z)\, dz$$

An important role is played by the subset $\mathscr{H}_0 \subset \mathscr{H}$, which consists of all nonnegative functions not equivalent to zero for which $\mathscr{U}f$ is bounded from below in some right neighborhood of $\underline{\lambda}$.

Theorem 2 of Ref. 5 asserts that if A and B are bounded measurable sets and $f \in \mathscr{H}_0$ then

$$\lim_{t \to \infty} \frac{\int_A \int f(z)\, p(t, z, y)\, dz\, dy}{\int_B \int f(z)\, p(t, z, y)\, dz\, dy} = \frac{\int_A \varphi_{\underline{\lambda}}(z)\, dz}{\int_B \varphi_{\underline{\lambda}}(z)\, dz} \tag{6}$$

A relevant quantity in our study is the exponential decay for the absorption probability

$$\eta = -\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_x(T_0 > t)$$

which exists and is independent of $x > 0$. In [3, Theorem C] it was shown that

$$\eta > 0 \Rightarrow \eta = \underline{\lambda}, \qquad \int_0^\infty \varphi_{\underline{\lambda}}(z)\, dz < \infty, \qquad \Lambda(\infty) = \infty \tag{7}$$

On the other hand, by using [5, Theorem 5] one can deduce that

$$\int_0^\infty \varphi_{\underline{\lambda}}(x)\, dx < \infty \qquad \text{and} \qquad \int_0^\infty e^{-\gamma(x)}\, dx < \infty \Rightarrow \eta = \underline{\lambda} > 0 \tag{8}$$

Our first result proves that these two conditions are equivalent and also gives necessary and sufficient conditions, in terms of the eigenvectors of $\mathscr{L}$ and $\mathscr{L}^*$, in order to have $\eta > 0$.

Theorem 1. The following properties are equivalent

(i) $\eta > 0$;

(ii) $\int_0^\infty \varphi_{\underline{\lambda}}(x)\,dx < \infty$ and $\int_0^\infty e^{-\gamma(y)}\,dy < \infty$;

(iii) $\int_0^\infty \varphi_{\underline{\lambda}}(x)\,dx < \infty$ and $\Lambda(\infty) = \infty$;

(iv) $\underline{\lambda} > 0$ and $\Lambda(\infty) = \infty$;

(v) $\exists \lambda > 0$ such that ψ_λ is increasing.

In the case when $\underline{\lambda} = 0$ we have from (7) that necessarily $\eta = 0$. On the other hand if $\underline{\lambda} > 0$ and $\Lambda(\infty) = \infty$ we conclude that $\eta > 0$ and therefore by (7) $\underline{\lambda} = \eta$. This means that the following result holds.

Theorem 2. If $\Lambda(\infty) = \infty$ then $\eta = \underline{\lambda}$.

In [3, Theorem B and Proposition D] it was proved that for any $s > 0$ and for any $A \in \mathscr{F}_s$:

$$\lim_{t \to \infty} \mathbb{P}_x(X \in A \mid T_0 > t) = \mathbb{P}_x\left(X \in A, \frac{\psi_\eta(X_s)}{\psi_\eta(x)}\, e^{\eta s}, T_0 > s\right) \tag{9}$$

and if $\Lambda(X_s)\, 1_{T_0 > s}$ is a martingale it was also shown that

$$\lim_{M \to \infty} \mathbb{P}_x(X \in A \mid T_M < T_0) = \mathbb{P}_x\left(X \in A, \frac{\Lambda(X_s)}{\Lambda(x)}, T_0 > s\right) \tag{10}$$

Our aim is to generalize (9) and (10) for $\lambda \in (0, \eta)$. With this goal in mind we will show in Lemma 4 that when $\eta > 0$ any of the solutions, ψ_λ, $\lambda \in (0, \eta)$, satisfies the semigroup property: $P_s \psi_\lambda(x) = e^{-\lambda s} \psi_\lambda(x)\ \forall x \geqslant 0$, $s > 0$. Hence

$$P_s^\lambda f(x) = \mathbb{E}_x\left(f(X_s), e^{\lambda s}\frac{\psi_\lambda(X_s)}{\psi_\lambda(x)}, T_0 > s\right)$$

defines a Markov process.

Let us consider an increasing sequence of stopping times (S_M^λ) associated to ψ_λ, defined as follows

$$S_M^\lambda = \inf\{s > 0 : F_\lambda(X_s, s) \geqslant M\}$$

where

$$F_\lambda(x, s) = e^{\lambda s} \psi_\lambda(x) \qquad \text{for} \quad x \geqslant 0, \quad s > 0$$

We will show the following result.

Theorem 3. Assume $\eta > 0$. For any $\lambda \in (0, \eta)$ we have:

$$\forall s > 0, \quad \forall A \in \mathscr{F}_s, \quad \lim_{M \to \infty} \mathbb{P}_x(X \in A \mid S_M^\lambda < T_0)$$

$$= \mathbb{E}_x \left(X \in A, \frac{\psi_\lambda(X_s)}{\psi_\lambda(x)} e^{\lambda s}, T_0 > s \right)$$

Now, in Theorem C of Ref. 3 the quasi limiting distribution of the diffusion was entirely characterized as

$$\lim_{s \to \infty} \mathbb{P}_x(X_s \geqslant a \mid T_0 > s) = \begin{cases} 1 & \text{if} \quad \eta = 0 \\ c^{-1} \int_a^\infty \varphi_{\underline{\lambda}}(x)\, dx & \text{if} \quad \eta > 0 \end{cases} \tag{11}$$

where $c = \int_0^\infty \varphi_{\underline{\lambda}}(x)\, dx$. We shall prove later that $c^{-1} = 2\underline{\lambda}$. There are situations where $\eta < \underline{\lambda}$. In this case always $\eta = 0$ and $\Lambda(\infty) < \infty$. From (11) $\mathbb{P}_x(X_t \leqslant a \mid T_0 > t)$ converges to 0. An example of $\eta < \underline{\lambda}$ is when the drift is a positive constant (in our notation $\alpha(x) \equiv \alpha < 0$), in this case $\eta = 0 < \underline{\lambda} = \alpha^2/2$. We point out that when the drift is constant η depends on the sign of the drift whereas $\underline{\lambda}$ does not. We shall prove that under the condition $\eta < \underline{\lambda}$ the latter quantity governs the speed at which $\mathbb{P}_x(X_t \leqslant a \mid T_0 > t)$ converges to 0.

Theorem 4. If $\eta < \underline{\lambda}$ then for any $a > 0$

$$\limsup_{t \to \infty} e^{\underline{\lambda} t} \mathbb{P}_x(X_t \leqslant a \mid T_0 > t) < \infty$$

We point out that the study of general one dimensional diffusions can be reduced, under suitable conditions on the diffusion coefficient, to the previous setting.

2. PROOF OF THE MAIN RESULTS

A direct computation shows that ψ_λ and φ_λ introduced in (2) and (3) satisfy:

$$\psi_\lambda(x) = \int_0^x e^{\gamma(y)} \left(1 - 2\lambda \int_0^y \varphi_\lambda(z)\, dz \right) dy$$

$$= \Lambda(x) - 2\lambda \int_0^x e^{\gamma(y)} \int_0^y \psi_\lambda(z)\, e^{-\gamma(z)}\, dz\, dy \tag{12}$$

To simplify notation in the proofs of the results we shall denote $\psi = \psi_\lambda$, $\varphi = \varphi_\lambda$, $F = F_\lambda$, $S_M = S_M^\lambda$.

Lemma 1. Assume $\Lambda(\infty) = \infty$. The following statements are equivalent for $\lambda > 0$:

(i) ψ_λ (or equivalently φ_λ) is positive;

(ii) ψ_λ is strictly increasing;

(iii) φ_λ is strictly positive and integrable.

Moreover, if any of these conditions holds then

$$\lim_{M \to \infty} \frac{\psi_\lambda(M)}{\Lambda(M)} = 0 \qquad \text{and} \qquad \int_0^\infty \varphi_\lambda(x)\, dx = \frac{1}{2\lambda} \tag{13}$$

Proof. (i) $\Rightarrow$ (ii) From (12),

$$\psi'(x) = e^{\gamma(x)} \left(1 - 2\lambda \int_0^x \varphi(y)\, dy\right) \tag{14}$$

Assume $\psi'(x_0) < 0$. Then $\forall x \geqslant x_0$ we have:

$$\psi'(x) = e^{\gamma(x) - \gamma(x_0)} e^{\gamma(x_0)} \left(1 - 2\lambda \int_0^x \varphi(y)\, dy\right) \leqslant e^{\gamma(x) - \gamma(x_0)} \psi'(x_0)$$

Then $\psi(x) \leqslant \psi(x_0) + e^{-\gamma(x_0)} \psi'(x_0)(\Lambda(x) - \Lambda(x_0)) \to -\infty$, because $\Lambda(\infty) = \infty$. We deduce that ψ is increasing. Let us show that ψ is strictly increasing.

Since $\psi(0) = 0$ and $\psi'(0) = 1$ we deduce that $\psi(z) > 0$ for any $z > 0$. Now, assume ψ constant on some interval $[x, y]$. Hence for $z \in (x, y)$:

$$\tfrac{1}{2} \psi''(z) - \alpha(z)\, \psi'(z) = 0 \neq -\lambda \psi(z)$$

which is a contradiction. Thus (ii) holds.

(ii) $\Rightarrow$ (iii). From (14) we get $1 - 2\lambda \int_0^x \varphi(y)\, dy \geqslant 0$ for any $x \geqslant 0$ so $\int_0^\infty \varphi(y)\, dy \leqslant 1/2\lambda$. The function φ is strictly positive because of the relation (4) between φ and ψ.

(iii) $\Rightarrow$ (i). It is trivial.

Let us show that $\lim_{M \to \infty} (\psi(M)/\Lambda(M))$ exists. From Itô's formula we get,

$$\begin{aligned} &\psi(x) - \lambda \mathbb{E}_x \int_0^{s \wedge T_M \wedge T_0} \psi(X_t)\, dt \\ &\quad = \psi(M)\, \mathbb{P}_x(T_M < T_0 \wedge s) + \mathbb{E}_x(\psi(X_s), s < T_M \wedge T_0) \end{aligned}$$

On the other hand, $|\mathbb{E}_x(\psi(X_s), s < T_M \wedge T_0)| \leqslant \psi(M)\, \mathbb{P}_x(s < T_0) \xrightarrow[s \to \infty]{} 0$.

Hence $\psi(x) - \lambda \mathbb{E}_x \int_0^{T_M \wedge T_0} \psi(X_t)\, dt = \psi(M)\, \mathbb{P}_x(T_M < T_0) = \psi(M)(\Lambda(x)/\Lambda(M))$. Therefore the following limit exists

$$\lim_{M \to \infty} \frac{\psi(M)}{\Lambda(M)} = \frac{\psi(x) - \lambda \mathbb{E}_x \int_0^{T_0} \psi(X_t)\, dt}{\Lambda(x)}$$

Now, if $\lim_{M \to \infty}(\psi(M)/\Lambda(M)) > 0$ we obtain from H that $\int_0^\infty \varphi(x)\, dx = \int_0^\infty \psi(x)\, e^{-\gamma(x)}\, dx = \infty$, which contradicts (iii). Thus, $\lim_{M \to \infty}(\psi(M)/\Lambda(M)) = 0$. If $\int_0^\infty \varphi(z)\, dz < 1/2\lambda$ we would obtain from (12), $\liminf_{x \to \infty}(\psi(x)/\Lambda(x)) > 0$. Hence the result holds. □

Lemma 2. Let $\lambda < \eta$. Then ψ_λ is a strictly increasing positive function on $\mathbb{R}_+$ and

$$\lim_{M \to \infty} \psi_\lambda(M) = \infty$$

Proof. If $\lambda = 0 < \eta$ then necessarily $\Lambda(\infty) = \infty$. Since $\psi_0 = \Lambda$ the result is verified. Hence we restrict ourselves to the case $\lambda \neq 0$.

First, let us show ψ is not bounded. On the contrary assume $\forall x \in \mathbb{R}_+$, $|\psi(x)| \leqslant K$. Consider $F(x, t) = \psi(x)\, e^{\lambda t}$. Then by Itô's formula we have

$$\psi(x) = \mathbb{E}_x(F(X_s, s), s < T_M \wedge T_0) + \psi(M)\, \mathbb{E}_x(e^{\lambda T_M}, T_M < s \wedge T_0)$$

Since $e^{\lambda T_M} 1_{T_M < s \wedge T_0} \leqslant (e^{\lambda s} \vee 1)\, 1_{T_M < s}$ we obtain

$$|\psi(x)| \leqslant K e^{\lambda s}\, \mathbb{P}_x(s < T_0) + K(e^{\lambda s} \vee 1)\, \mathbb{P}_x(T_M < s)$$

Letting $M \to \infty$ and using (1) we get $|\psi(x)| \leqslant K e^{\lambda s}\, \mathbb{P}_x(s < T_0)$. From the definition of η, $e^{\lambda s}\, \mathbb{P}_x(s < T_0) \xrightarrow[s \to \infty]{} 0$, for $\lambda < \eta$. We obtain $\psi \equiv 0$ which is a contradiction, so ψ is not bounded.

Now let us show ψ only vanishes at $x = 0$. On the contrary let $x_0 > 0$ be such that $\psi(x_0) = 0$ and $\psi(x) > 0$ for $x \in (0, x_0)$.

By Itô's formula we obtain,

$$\begin{aligned} |\psi(x)| &= |\mathbb{E}_x(\psi(X_s)\, e^{\lambda s}, s < T_{x_0} \wedge T_0)| \\ &\leqslant \max_{y \in [0, x_0]} |\psi(y)|\, e^{\lambda s}\, \mathbb{P}_x(s < T_0) \xrightarrow[s \to \infty]{} 0 \end{aligned}$$

Therefore, we get that $\psi(x)$ is strictly positive for $x > 0$.

Now, we prove that ψ is increasing. Assume there exist $x<y$ for which $\psi(x)>\psi(y)$. In the case $\lambda<0$ take $z<x$ such that $\psi(z)=\psi(y)$. Denote by $\bar{x}\in(z,y)$ a point verifying $\psi(\bar{x})=\max_{r\in[z,y]}\psi(r)$. Then

$$\tfrac{1}{2}\psi''(\bar{x})=\tfrac{1}{2}\psi''(\bar{x})-\alpha(\bar{x})\,\psi'(\bar{x})=-\lambda\psi(\bar{x})>0$$

which is a contradiction. Assume now $\lambda>0$. Since ψ is not bounded there exists $z>y$ such that $\psi(x)=\psi(z)$. Consider $\bar{x}\in(x,z)$ such that $\psi(\bar{x})=\min_{r\in[x,z]}\psi(r)>0$, then

$$\tfrac{1}{2}\psi''(\bar{x})=\tfrac{1}{2}\psi''(\bar{x})-\alpha(\bar{x})=-\lambda\psi(\bar{x})<0$$

which is again a contradiction. Hence ψ is increasing.

Finally if $\psi(x)=\psi(y)$ for $x<y$ we see that ψ is constant on $[x,y]$ and therefore: $0\neq-\lambda\psi(z)=\frac{1}{2}\psi''(z)-\alpha(z)\,\psi'(z)=0$ for $z\in(x,y)$. The result follows from this. □

Since $\psi_\lambda(x)$ is continuous in λ, we deduce from the last Lemma that ψ_η is also positive and increasing (in fact the above arguments show it is strictly increasing). Hence if $\lambda\in(0,\eta]$ we deduce from (12) $\psi_\lambda(x)<\Lambda(x)\ \forall x>0$.

Lemma 3. If ψ_λ is increasing for some $\lambda>0$ then $\lambda\leqslant\eta$.

Proof. Let $F(x,t)=\psi(x)\,e^{\lambda t}$. According to Itô's formula we obtain,

$$\psi(x)=\psi(M)\,\mathbb{E}_x(e^{\lambda T_M},T_M<T_0\wedge s)+\mathbb{E}_x(\psi(X_s)\,e^{\lambda s},s<T_0\wedge T_M)$$

Therefore

$$\psi(x)\geqslant\mathbb{E}_x(\psi(X_s)\,e^{\lambda s},s<T_0)$$

Since ψ is an increasing function we obtain for any $a>0$,

$$\psi(x)\geqslant\psi(a)\,\mathbb{E}_x(1_{X_s\geqslant a}e^{\lambda s},s<T_0)=\psi(a)\,e^{\lambda s}\,\mathbb{P}_x(X_s\geqslant a\mid T_0>s)\,\mathbb{P}_x(T_0>s)$$

From (11) we deduce

$$\limsup_{s\to\infty}e^{\lambda s}\,\mathbb{P}_x(T_0>s)\leqslant\frac{\psi(x)}{\psi(a)}\Big(\lim_{s\to\infty}(\mathbb{P}_x(X_s\geqslant a\mid T_0>s))^{-1}<\infty$$

Then, for fixed $x>0$, there exists $K>0$ such that $\mathbb{P}_x(T_0>s)\leqslant Ke^{-\lambda s}$. Therefore $\eta\geqslant\lambda>0$. □

We remark that if $\eta > 0$ we have $\eta = \underline{\lambda}$. From relations (11) and (13) we obtain $\lim_{s \to \infty} \mathbb{P}_x(X_s \geqslant a \mid T_0 > s) = 2\underline{\lambda} \int_0^\infty \varphi_{\underline{\lambda}}(z)\, dz$. From Lemma 1, $\psi_{\underline{\lambda}}$ is strictly increasing and therefore we obtain as in the previous proof the following estimate

$$\limsup_{s \to \infty} e^{\underline{\lambda} s} \mathbb{P}_x(T_0 > s) \leqslant \frac{\psi_{\underline{\lambda}}(x)}{2\underline{\lambda}} \left(\sup_{a > 0} \left(\psi_{\underline{\lambda}}(a) \int_a^\infty \varphi_{\underline{\lambda}}(y)\, dy \right) \right)^{-1}$$

Proof of Theorem 1. (i) $\Leftrightarrow$ (v). If follows directly from Lemmas 2 and 3.

(i) $\Rightarrow$ (ii). Since $\eta > 0$ we get from (7), $\int_0^\infty \varphi_{\underline{\lambda}}(x)\, dx < \infty$. On the other hand there exists $\lambda > 0$ (given by (v)) such that ψ_λ is increasing. From (13), $\int_0^\infty \psi_\lambda(y)\, e^{-\gamma(y)}\, dy = (2\lambda)^{-1} < \infty$. Then the relation $\int_0^\infty e^{-\gamma(y)}\, dy < \infty$ follows from the fact that ψ_λ is increasing.

(ii) $\Rightarrow$ (i). This follows from (8).

(i) $\Rightarrow$ (iii). This follows from (7).

(iii) $\Rightarrow$ (i). From Lemma 1, $\psi_{\underline{\lambda}}$ is increasing and from (5) $\underline{\lambda} > 0$. From Lemma 3 we get $\eta \geqslant \underline{\lambda} > 0$.

(i) $\Rightarrow$ (iv). This is a direct consequence of (7).

(iv) $\Rightarrow$ (iii). This follows from Lemma 1, since $\underline{\lambda} > 0$ and $\varphi_{\underline{\lambda}} \geqslant 0$ we get $\varphi_{\underline{\lambda}}$ is integrable. □

Let us now turn to the proof of Theorem 3. First we will show some technical lemmas.

Lemma 4. Assume $\eta > 0$. Then for any $\lambda \in (0, \eta]$ we have:

(i) $\psi_\lambda(x) = \lambda \mathbb{E}_x(\int_0^{T_0} \psi_\lambda(X_s)\, ds)\ \forall x \geqslant 0$.

(ii) $\mathbb{E}_x(\psi_\lambda(X_s), s < T_0) = e^{-\lambda s} \psi_\lambda(x)\ \forall x \geqslant 0, \forall s > 0$.

Proof. (i) Using Itô's formula we get

$$\psi(x) = \psi(M)\, \mathbb{P}_x(T_M < T_0) + \lambda \mathbb{E}_x \int_0^{T_M \wedge T_0} \psi(X_s)\, ds$$

$$= \psi(M) \frac{\Lambda(x)}{\Lambda(M)} + \lambda \mathbb{E}_x \int_0^{T_M \wedge T_0} \psi(X_s)\, ds$$

From (13) we obtain the result.

(ii) From Itô's formula,

$$\psi(x)=\mathbb{E}_x(\psi(X_s)\,e^{\lambda s},\ s<T_M\wedge T_0)+\psi(M)\,\mathbb{E}_x(e^{\lambda T_M},\ T_M<s\wedge T_0)$$

Since

$$\begin{aligned}\psi(M)\,\mathbb{E}_x(e^{\lambda T_M},\ T_M<s<T_0)&\leqslant \psi(M)\,e^{\lambda s}\,\mathbb{P}_x(T_M<T_0)\\ &=\frac{\psi(M)}{\Lambda(M)}\,\Lambda(x)\,e^{\lambda s}\xrightarrow[M\to\infty]{}0\end{aligned}$$

the result follows. □

Lemma 5. For $0<\lambda<\eta$ we have:

(i) $\mathbb{P}_x(S_M^\lambda<T_0)=\psi_\lambda(x)/M$ if $\psi_\lambda(x)\in(0,M)$.

(ii) $\lim_{M\to\infty}(\mathbb{P}_x(S_M^\lambda<T_0\wedge s)/\mathbb{P}_x(S_M^\lambda<T_0))=0$

Proof. (i) We will write S_M instead of S_M^λ. Take x such that $\psi(x)\in(0,M)$. If $t\leqslant S_M\wedge T_0$ then $X_t\in[0,\psi^{-1}(M)]$. Therefore, from Itô's formula we get,

$$\begin{aligned}\psi(x)&=\mathbb{E}_x(F(X_{S_M\wedge T_0\wedge s},\ S_M\wedge T_0\wedge s))\\ &=M\mathbb{P}_x(S_M<T_0\wedge s)+\mathbb{E}_x(F(X_s,s),\ s<S_M\wedge T_0)\end{aligned}$$

Now $\mathbb{E}_x(F(X_s,s),\ s<S_M\wedge T_0)\leqslant M\mathbb{P}_x(s<T_0)\xrightarrow[s\to\infty]{}0$. Therefore (i) holds.

(ii) We can assume $\psi(x)<M$. On the set $\{S_M<T_0\wedge s\}$ we have $\psi(X_{S_M})=Me^{-\lambda S_M}\geqslant Me^{-\lambda s}$. Therefore $T_{\psi^{-1}(Me^{-\lambda s})}\leqslant S_M$.

Hence:

$$\frac{\mathbb{P}_x(S_M<T_0\wedge s)}{\mathbb{P}_x(S_M<T_0)}\leqslant\frac{\mathbb{P}_x(T_{\psi^{-1}(Me^{-\lambda s})}<T_0)}{\mathbb{P}_x(S_M<T_0)}=\frac{M}{\psi(x)}\,\frac{\Lambda(x)}{\Lambda(\psi^{-1}(Me^{-\lambda s}))}$$

Put $N=\psi^{-1}(Me^{-\lambda s})$, then $\psi(N)=Me^{-\lambda s}$ and N converges to ∞ with M.

Thus, $(\mathbb{P}_x(S_M<T_0\wedge s)/\mathbb{P}_x(S_M<T_0))\leqslant e^{\lambda s}(\Lambda(x)/\psi(x))(\psi(N)/\Lambda(N))$ $\xrightarrow[M\to\infty]{}0$ by using Lemma 1. □

Proof of Theorem 3. Let θ_s be the shift operator in s units of time. It can be checked that $S_{Me^{-\lambda s}}\circ\theta_s=S_M-s$ on the set $\{s\leqslant S_M<\infty\}$. Observe that on this set $\psi(X_s)\leqslant Me^{-\lambda s}$, therefore from Lemma 5 (i),

$\mathbb{P}_{X_s}(S_{Me^{-\lambda s}} < T_0) = (\psi(X_s)/M)\, e^{\lambda s}$. Now, by using the Markov property we get,

$$\frac{\mathbb{P}_x(X\in A,\, s\leqslant S^{\lambda}_M < T_0)}{\mathbb{P}_x(S_M < T_0)}$$

$$= \frac{\mathbb{E}_x(X\in A,\, \mathbb{P}_{X_s}(S^{\lambda}_{Me^{-\lambda s}} < T_0),\, T_0 > s,\, S^{\lambda}_M \geqslant s)}{\mathbb{P}_x(S_M < T_0)}$$

$$= \mathbb{E}_x\left(X\in A,\, \frac{\psi_\lambda(X_s)}{\psi_\lambda(x)}\, e^{\lambda s},\, T_0 > s,\, S_M \geqslant s\right)$$

$$\xrightarrow[M\to\infty]{} \mathbb{E}_x\left(X\in A,\, \frac{\psi_\lambda(X_s)}{\psi_\lambda(x)}\, e^{\lambda s},\, T_0 > s\right)$$

because $\{S_M \geqslant s\} \supseteq \{T_{\psi_\lambda^{-1}(Me^{-\lambda s})} \geqslant s\} \nearrow \{T_\infty \geqslant s\} = \Omega$, $\mathbb{P}_x$ a.e.

To finish the proof we use Lemma 5 (ii) to get the result:

$$\left|\mathbb{P}_x(X\in A \mid S_M < T_0) - \frac{\mathbb{P}_x(X\in A,\, s\leqslant S_M < T_0)}{\mathbb{P}_x(S_M < T_0)}\right|$$

$$\leqslant \frac{\mathbb{P}_x(S_M < T_0\wedge s)}{\mathbb{P}_x(S_M < T_0)} \xrightarrow[M\to\infty]{} 0 \qquad \square$$

Proof of Theorem 4. We prove first that $\psi_{\underline{\lambda}}$ is bounded and moreover ultimately decreasing (the proof also shows that the same property holds for any $\lambda\in(0,\underline{\lambda}]$). We have that $\psi_{\underline{\lambda}}$ is a positive function and from Lemma 3 it cannot be an increasing function in the whole domain. Therefore there exists $x^* > 0$ such that $\psi'_{\underline{\lambda}}(x) > 0$, $x\in[0, x^*)$ and $\psi'_{\underline{\lambda}}(x^*) = 0$. From $\psi''_{\underline{\lambda}}(x^*) - 2\alpha(x^*)\,\psi'_{\underline{\lambda}}(x^*) = -2\underline{\lambda}\psi_{\underline{\lambda}}(x^*)$, we obtain $\psi''_{\underline{\lambda}}(x^*) < 0$. In particular $\psi'_{\underline{\lambda}}(x) < 0$ in an interval $(x^*, x^*+\delta)$. Using (14) we find that

$$\psi'_{\underline{\lambda}}(x^*) = e^{\gamma(x^*)}\left(1 - 2\underline{\lambda}\int_0^{x^*} \varphi_{\underline{\lambda}}(y)\, dy\right) = 0$$

and for $x > x^*$

$$\psi'_{\underline{\lambda}}(x) = e^{\gamma(x)}\left(1 - 2\underline{\lambda}\int_0^{x} \varphi_{\underline{\lambda}}(y)\, dy\right) < 0$$

because $\varphi_{\underline{\lambda}}$ is a positive function. Hence, $\psi_{\underline{\lambda}}$ is bounded and ultimately decreasing.

We shall prove that for $\lambda \geqslant \underline{\lambda}$ the function ψ_λ is also bounded. The inequality $\psi_\lambda \leqslant \psi_{\underline{\lambda}}$ follows directly from an inequality of Caplygin type (see Theorem 1, Chap. XI of Ref. 6). Let us show now that ψ_λ is also bounded from below. It follows from (12) and the upper bound for ψ_λ that

$$\psi_\lambda(x) \geqslant \Lambda(x) - 2\lambda \int_0^x e^{\gamma(y)} \int_0^y \psi_{\underline{\lambda}}(z)\, e^{-\gamma(z)}\, dz\, dy \geqslant \Lambda(x) - \frac{\lambda}{\underline{\lambda}} (\Lambda(x) - \psi_{\underline{\lambda}}(x))$$

Since we are in the case $\Lambda(\infty) < \infty$ we deduce ψ_λ is bounded below.

Using the fact that ψ_λ is bounded we get that the process $\psi_\lambda(X_t)\, e^{\lambda t} 1_{T_0 > t}$ is a martingale and therefore we obtain

$$\psi_\lambda(x)\, e^{-\lambda t} = \mathbb{E}_x(\psi_\lambda(X_t), T_0 > t) = \int p(t, x, y)\, \psi_\lambda(y)\, dy \tag{15}$$

In particular for $\underline{\lambda}$ we obtain the following estimate when $b > x^*$

$$\begin{aligned}
\psi_{\underline{\lambda}}(x) &= \mathbb{E}_x(\psi_{\underline{\lambda}}(X_t)\, e^{\underline{\lambda} t}, t < T_0) \geqslant e^{\underline{\lambda} t}\, \mathbb{E}_x(\psi_{\underline{\lambda}}(X_t), X_t \in [x^*, b], t < T_0) \\
&\geqslant e^{\underline{\lambda} t} \psi_{\underline{\lambda}}(b)\, \mathbb{P}_x(X_t \in [x^*, b], t < T_0) \\
&= \psi_{\underline{\lambda}}(b)\, \mathbb{P}_x(t < T_0)\, e^{\underline{\lambda} t}\, \mathbb{P}_x(X_t \in [x^*, b] \mid t < T_0)
\end{aligned}$$

where we have used the observation that $\psi_{\underline{\lambda}}$ is nonnegative and decreasing on $[x^*, \infty)$. From the relation $\lim_{t \to \infty} \mathbb{P}_x(t < T_0) = (\Lambda(x)/\Lambda(\infty)) > 0$, we find

$$\limsup_{t \to \infty} e^{\underline{\lambda} t}\, \mathbb{P}_x(X_t \in [x^*, b] \mid T_0 > t) \leqslant \frac{\psi_{\underline{\lambda}}(x)\, \Lambda(\infty)}{\psi_{\underline{\lambda}}(b)\, \Lambda(x)} \tag{16}$$

To conclude the result we will use a pointwise version of (6). In fact observe that if in (6) we take $f(z) = p(\Delta, x, z)$, for $\Delta > 0$, $x > 0$, then we will have

$$\lim_{t \to \infty} \frac{\mathbb{P}_x(X_t \in A, T_0 > t)}{\mathbb{P}_x(X_t \in A, T_0 > t)} = \frac{\int_A \varphi_{\underline{\lambda}}(z)\, dz}{\int_B \varphi_{\underline{\lambda}}(z)\, dz} \tag{17}$$

provided that $p(\Delta, x, \cdot) \in \mathscr{H}_0$, which amounts to proving that

(i) $\int p^2(\Delta, x, z)\, e^{\gamma(z)}\, dz < \infty$;

(ii) $\int_0^\infty p(\Delta, x, z)\, \psi_\lambda(z)\, dz$ is bounded in some right neigborhood of $\underline{\lambda}$.

Now, (i) follows from the fact that $p(\Delta, x, \cdot)$ is bounded and that $\Lambda(\infty) = \int e^{\gamma(z)}\, dz < \infty$. (ii) follows immediately from (15).

The result now follows from (16) and (17), and moreover

$$\limsup_{t\to\infty} e^{\underline{\lambda} t}\,\mathbb{P}_x(X_t\leqslant a \mid T_0>t)\leqslant \Gamma\,\frac{\psi_{\underline{\lambda}}(x)}{\Lambda(x)}\int_0^a \varphi_{\underline{\lambda}}(z)\,dz$$

where $\Gamma=\Lambda(\infty)(\sup_{b\geqslant x^*}\{\psi_{\underline{\lambda}}(b)\int_{x^*}^b \varphi_{\underline{\lambda}}(z)\,dz\})^{-1}$. □

3. SOME MONOTONICITY PROPERTIES

The fact that $\underline{\lambda}=\eta$, under the hypothesis of Theorem 2, allow us to get some information about the dependence of $\underline{\lambda}$ on α. For this purpose we denote Λ^α, $\underline{\lambda}(\alpha)$ and $\eta(\alpha)$ the quantities Λ, $\underline{\lambda}$ and η associated to α.

Corollary 1. Let $\alpha\geqslant\beta$ satisfy the hypothesis H and $\Lambda^\alpha(\infty)=\infty=\Lambda^\beta(\infty)$. Then

$$\underline{\lambda}(\alpha)\geqslant\underline{\lambda}(\beta)$$

Proof. From Theorem 3, $\underline{\lambda}(\alpha)=\eta(\alpha)$ and $\underline{\lambda}(\beta)=\eta(\beta)$. Let

$$dX_t=dB_t-\alpha(X_t)\,dt,\qquad X_0=x$$

$$dY_t=dB_t-\beta(Y_t)\,dt,\qquad Y_0=x$$

Since $\alpha\geqslant\beta$ we get $X_t\leqslant Y_t$ for every t, and so $T_0(X)\leqslant T_0(Y)$. Therefore

$$\mathbb{P}_x(T_0(X)>s)\leqslant\mathbb{P}_x(T_0(Y)>s)$$

which implies that $\underline{\lambda}(\alpha)\geqslant\underline{\lambda}(\beta)$. □

In particular if $\forall x\ \ \alpha(x)\geqslant k\geqslant 0$ then $\underline{\lambda}(\alpha)\geqslant\underline{\lambda}(k)=k^2/2$ (this last equality can be computed directly, also see Ref. 5).

Corollary 2. Let α be a non-negative C^1 function for which H holds, and $\Lambda^\alpha(\infty)=\infty$. Then $\underline{\lambda}(\alpha)\leqslant(\limsup_{x\to\infty}\alpha(x))^2/2$. In particular if α is also decreasing $\underline{\lambda}(\alpha)=(\lim_{x\to\infty}\alpha(x))^2/2$.

Proof. The result is obvious if $\limsup_{x\to\infty}\alpha(x)=\infty$. Let $dX_t=dB_t-\alpha(X_t)\,dt$, $X_0=x$. For $\beta>\limsup_{x\to\infty}\alpha(z)$ take x_0 large enough such that $\forall z\geqslant x_0$, $\alpha(z)\leqslant\beta$. Consider the process

$$dY_t=dB_t-\beta\,dt,\qquad Y_0=x$$

If $x > x_0$

$$Y_t \leqslant X_t \qquad \forall t \leqslant T_{x_0}(X)$$

Therefore, for such x

$$\mathbb{P}_x(T_0(X) > s) \geqslant \mathbb{P}_x(T_{x_0}(X) > s) \geqslant \mathbb{P}_x(T_{x_0}(T) > s)$$
$$= \mathbb{P}_{x-x_0}(T_0(Y) > s)$$

Since $\underline{\lambda}(\alpha) = -\lim_{s\to\infty}(1/s) \log \mathbb{P}_x(T_0(X) > s) \leqslant -\lim_{s\to\infty}(1/s) \log \mathbb{P}_{x-x_0} (T_0(Y) > s) = \beta^2/2$, the result follows. □

There are examples where $\Lambda(\infty) = \infty$, H is verified, the condition $\int_0^\infty e^{-\gamma(z)}\, dt < \infty$ but $\underline{\lambda} = \eta = 0$. It suffices to take $\alpha(x) = 1/(1+x)$.

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