Commun. Math. Phys. 324, 215–232 (2013)
Digital Object Identifier (DOI) 10.1007/s00220-013-1750-x

Communications in
**Mathematical
Physics**

Log-Gamma Polymer Free Energy Fluctuations via a Fredholm Determinant Identity

Alexei Borodin[1], Ivan Corwin[2], Daniel Remenik[3,4]

[1] Massachusetts Institute of Technology, Department of Mathematics, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA. E-mail: borodin@math.mit.edu
[2] Microsoft Research, New England, 1 Memorial Drive, Cambridge, MA 02142, USA. E-mail: ivan.corwin@gmail.com
[3] Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, ON M5S 2E4, Canada, E-mail: dremenik@math.toronto.edu
[4] Departamento de Ingeniería Matemática, Universidad de Chile, Av. Blanco Encalada 2120, Santiago, Chile.

Received: 15 November 2012 / Accepted: 16 November 2012
Published online: 3 July 2013 –

Abstract: We prove that under $n^{1/3}$ scaling, the limiting distribution as $n \to \infty$ of the free energy of Seppäläinen's log-Gamma discrete directed polymer is GUE Tracy-Widom. The main technical innovation we provide is a general identity between a class of n-fold contour integrals and a class of Fredholm determinants. Applying this identity to the integral formula proved in Corwin et al. (Tropical combinatorics and Whittaker functions. http://arxiv.org/abs/1110.3489v3 [math.PR], 2012) for the Laplace transform of the log-Gamma polymer partition function, we arrive at a Fredholm determinant which lends itself to asymptotic analysis (and thus yields the free energy limit theorem). The Fredholm determinant was anticipated in Borodin and Corwin (Macdonald processes. http://arxiv.org/abs/1111.4408v3 [math.PR], 2012) via the formalism of Macdonald processes yet its rigorous proof was so far lacking because of the nontriviality of certain decay estimates required by that approach.

1. Introduction and Main Results

The log-Gamma polymer was introduced and studied by Seppäläinen [16]. It is an example of a more general class of discrete directed random polymers in 1+1 dimensions, which have received considerable attention in the last decade due to their connections with random matrix theory, integrable systems and the Kardar-Parisi-Zhang universality class [13]. We will focus on the log-Gamma polymer, and refer the reader to the review [10] for more details on the general model.

Definition 1.1. *Let θ be a positive real. A random variable X has* **inverse-Gamma distribution with parameter** $\theta > 0$ *if it is supported on the positive reals where it has distribution*

$$\mathbb{P}(X \in dx) = \frac{1}{\Gamma(\theta)} x^{-\theta-1} \exp\left\{-\frac{1}{x}\right\} dx.$$

We abbreviate this $X \sim \Gamma^{-1}(\theta)$.

Definition 1.2. *The* **log-Gamma polymer partition function** *with parameter* $\gamma > 0$ *is given by*

$$Z(n,N) = \sum_{\pi:(1,1)\to(n,N)} \prod_{(i,j)\in\pi} d_{i,j},$$

where π *is an up/right directed lattice path from the Euclidean point* $(1,1)$ *to* (n,N) *and where the random variables* $d_{i,j}$ *are i.i.d,* $d_{i,j} \sim \Gamma^{-1}(\gamma)$.

In [16] it was proved that

$$\lim_{n\to\infty} \frac{\log Z(n,n)}{n} = \bar{f}_\gamma, \qquad \limsup_{n\to\infty} \frac{\operatorname{var}\log Z(n,n)}{n^{2/3}} \le C,$$

where $\bar{f}_\gamma = -2\Psi(\gamma/2)$ and C is a large constant. Here $\Psi(x) = [\log\Gamma]'(x)$ is the digamma function. The scale of the variance upper-bound is believed to be tight, since directed polymers at positive temperature should have KPZ universality class scalings (see e.g. the review [10]). Moreover, it is believed that, when centered by $n\bar{f}_\gamma$ and scaled by $n^{1/3}$, the distribution of the free energy $\log Z(n,n)$ should limit to the GUE Tracy-Widom distribution [17].

We presently prove this form of KPZ universality for the log-Gamma polymer for $\gamma < \gamma^*$ for some $\gamma^* > 0$. This assumption is purely technical and comes from the asymptotic analysis. It is likely that this assumption can be removed following the approach of [8], where a similar assumption was removed in the case of the semi-discrete polymer. For this model γ plays a role akin to temperature.

Theorem 1. *There exists* $\gamma^* > 0$ *such that the log-Gamma polymer free energy with parameter* $\gamma \in (0,\gamma^*)$ *has limiting fluctuation distribution given by*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log Z(n,n) - n\bar{f}_\gamma}{n^{1/3}} \le r\right) = F_{\mathrm{GUE}}\left(\left(\frac{\bar{g}_\gamma}{2}\right)^{-1/3} r\right),$$

where $\bar{f}_\gamma = -2\Psi(\gamma/2)$, $\bar{g}_\gamma = -2\Psi''(\gamma/2)$ *and* F_{GUE} *is the GUE Tracy-Widom distribution function.*

We give the proof of this theorem in Sect. 2. There are two ingredients in the proof, and then some asymptotic analysis. The first ingredient is the n-fold integral formula given in [11] for the Laplace transform of the polymer partition function. This is given below as Proposition 1.4. The second ingredient in the proof is a general identity between a class of n-fold contour integrals and a class of Fredholm determinants. This is given below as Theorem 2. Applying this identity to Proposition 1.4 yields Corollary 1.8 which is a new Fredholm determinant expression for the Laplace transform of the log-Gamma polymer partition function. This formula lends itself to straightforward asymptotic analysis, as is done in Sect. 2.

The log-Gamma polymer may be generalized, as done in [11], so that the distributions of the $\gamma_{i,j}$ depend on two collections of parameters.

Discrete directed polymer partition functions, under *intermediate disorder* scaling [2,14], converge to the solution of the multiplicative stochastic heat equation (whose logarithm is the KPZ equation). If the two collections of parameters determining the distributions of the $\gamma_{i,j}$ are tuned correctly, then the initial data for the limiting stochastic heat equation is determined by two collections of parameters as well. The Fredholm determinant formula of Corollary 1.8 should limit to an analogous formula for the Laplace transform of the stochastic heat equation with this general class of initial data which would be a finite temperature analog of the results of [9]. When only one of the collections of parameters is tuned, this formula was computed in [8] via a similar limit of the Fredholm determinant formula for the Laplace transform of the semi-discrete polymer partition function (see also [12] in the case where only a single parameter is tuned), and this is a finite temperature analog of the results of [6].

We turn now to the n-fold integral formula for the Laplace transform of the log-Gamma polymer partition function.

Definition 1.3. *The* **Sklyanin measure** s_N *on* $\mathbb{C}^N$ *is given by*

$$s_N(dw_1,\dots,dw_N) = \frac{1}{(2\pi \mathrm{i})^N N!} \prod_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{\Gamma(w_i - w_j)} \prod_{i=1}^{N} dw_i. \tag{1.1}$$

The following result is taken from [11], Thm. 3.8.ii.

Proposition 1.4. *Fix $n \geq N$, and choose parameters $\alpha_i > 0$ for $1 \leq i \leq n$ and $a_j > 0$ for $1 \leq j \leq N$ such that $\gamma_{i,j} = \alpha_i - a_j > 0$. Consider the log-Gamma polymer partition function, where $d_{i,j} \sim \Gamma^{-1}(\gamma_{i,j})$. Then for all u with $\mathrm{Re}(u) > 0$,*

$$\mathbb{E}\left[e^{-uZ(n,N)}\right] = \int_{(\mathrm{i}\mathbb{R})^N} s_N(dw_1,\dots,dw_N) \prod_{i,j=1}^{N} \Gamma(a_j - w_i) \prod_{j=1}^{N} \frac{F(w_j)}{F(a_j)},$$

where

$$F(w) = u^w \prod_{m=1}^{n} \Gamma(\alpha_m - w). \tag{1.2}$$

Until now, there has not been progress in extracting asymptotics from this formula. The following theorem, however, transforms this integral formula into a Fredholm determinant for which we can readily perform asymptotic analysis. This identity should be considered the main technical contribution of this paper, of which Theorem 1 is essentially a corollary (after some asymptotic analysis).

Definition 1.5. *We introduce the following contours: C_δ is a positively oriented circle around the origin with radius δ; ℓ_δ is a line parallel to the imaginary axis from $-\mathrm{i}\infty + \delta$ to $\mathrm{i}\infty + \delta$; $\ell_{-\delta}$ is similarly the contour from $-\mathrm{i}\infty - \delta$ to $\mathrm{i}\infty - \delta$; and $\ell'_{\delta_1,\delta_2,M}$ is the horizontal line segment going from $-\delta_1 + \mathrm{i}M$ to $\delta_2 + \mathrm{i}M$. For any simple smooth contour γ in $\mathbb{C}$ we will write $L^2(\gamma)$ to mean the space $L^2(\gamma,\mu)$, where μ is the path measure along γ divided by $2\pi\mathrm{i}$.*

Theorem 2. *Fix $0 < \delta_2 < 1$, $0 < \delta_1 < \min\{\delta_2, 1-\delta_2\}$ and $a_1,\dots,a_N \in \mathbb{C}$ such that $|a_i| < \delta_1$. Suppose F is a meromorphic function such that all its poles have real part strictly larger than δ_2, F is non-zero along and inside C_{δ_1}, and for all $\kappa > 0$,*

$$\int_{\ell_{\pm\delta_2}} dw\, e^{\pi(\frac{N}{2}-1)|\mathrm{Im}(w)|} |\mathrm{Im}(w)|^\kappa |F(w)| < \infty,$$

$$\int_{\ell'_{\delta_1,\delta_2,M}} dw\, e^{\pi(\frac{N}{2}-1)|\mathrm{Im}(w)|} |\mathrm{Im}(w)|^\kappa |F(w)| \xrightarrow[|M|\to\infty]{} 0. \tag{1.3}$$

Then

$$\int_{\ell_{-\delta_1}} \cdots \int_{\ell_{-\delta_1}} s_N(dw_1, \ldots, dw_N) \prod_{i,j=1}^{N} \Gamma(a_j - w_i) \prod_{j=1}^{N} \frac{F(w_j)}{F(a_j)} = \det(I+K)_{L^2(C_{\delta_1})}, \tag{1.4}$$

where $\det(I+K)_{L^2(C_{\delta_1})}$ *is the Fredholm determinant of* $K : L^2(C_{\delta_1}) \longrightarrow L^2(C_{\delta_1})$ *with*

$$K(v, v') = \frac{1}{2\pi \mathrm{i}} \int_{\ell_{\delta_2}} dw \, \frac{\pi}{\sin(\pi(v-w))} \frac{F(w)}{F(v)} \frac{1}{w - v'} \prod_{m=1}^{N} \frac{\Gamma(v - a_m)}{\Gamma(w - a_m)}. \tag{1.5}$$

The proof of this theorem is given in Sect. 3. The argument uses only the Andréief identity [4] (sometimes referred to as the generalized Cauchy-Binet identity), the Cauchy determinant, the fact that $\det(I + AB) = \det(I + BA)$ and some simple contour shifts and residue computations to go from the Fredholm determinant formula on the right-hand side of (1.4) to the integral formula on the left-hand side. Going from the integral formula to the Fredholm determinant formula (even knowing that it should be true) is a much more challenging path, since one has to undo certain cancelations to discover determinants.

Remark 1.6. In the context of the related semi-discrete polymer (see the end of the Introduction) O'Connell describes (after Corollary 4.2 of [15]) another Fredholm determinant expression for the Laplace transform of the partition function. That expression and the one which arises from the application of Theorem 2 are different.

Remark 1.7. The condition in Theorem 2 on the $|a_i| < \delta_1$ can be relaxed at the cost of more complicated choices of contours. Instead of taking the w contour to be a vertical line ℓ_{δ_2} one could, in order to accommodate larger values of the a_i, shift the vertical line horizontally to the right by some positive integer K, and then augment the w contour with a collection of sufficiently small contours around the positive integers up to and including K. See Sect. 5 of [8] for an example of this sort of procedure.

We may apply Theorem 2 to show the following.

Corollary 1.8. *Fix* $n \geq N$ *and any* δ_1, δ_2 *such that* $0 < \delta_2 < 1$ *and* $0 < \delta_1 < \min\{\delta_2, 1 - \delta_2\}$*. Given a collection of* $\alpha_i > \delta_2$ *for* $1 \leq i \leq n$*, and* $0 \leq a_j < \delta_1$ *for* $1 \leq j \leq N$*, set* $\gamma_{i,j} = \alpha_i - a_j$ *and consider the log-Gamma polymer partition function with weights* $d_{i,j} \sim \Gamma^{-1}(\gamma_{i,j})$*. Then for all* u *with* $\mathrm{Re}(u) > 0$*,*

$$\mathbb{E}\Big[e^{-uZ(n,N)}\Big] = \det(I + K_u)_{L^2(C_{\delta_1})},$$

where $\det(I + K_u)_{L^2(C_{\delta_1})}$ *is the Fredholm determinant of* $K_u : L^2(C_{\delta_1}) \longrightarrow L^2(C_{\delta_1})$ *with*

$$K_u(v, v') = \frac{1}{2\pi \mathrm{i}} \int_{\ell_{\delta_2}} dw \, \frac{\pi}{\sin(\pi(v-w))} \frac{F(w)}{F(v)} \frac{1}{w - v'} \prod_{m=1}^{N} \frac{\Gamma(v - a_m)}{\Gamma(w - a_m)}$$

and $F(w)$ *as given in* (1.2)*.*

Proof. We start with the Laplace transform formula given in Proposition 1.4 (if some $a_j = 0$ we simply shift the $\mathrm{i}\mathbb{R}$ integration contour slightly to the left). In order to apply Theorem 2 we must check that all the conditions are satisfied. The function $F(w)$ has no poles with real part less than $\min\{\alpha_i\}$ and it is non-zero in the entire complex plane. This implies that given δ_1 and δ_2 as specified in the hypothesis, the conditions on the poles and zeros of F are satisfied. It remains to check the decay condition (1.3). This, however, is immediate from the estimate on the Gamma function as given in (3.2). Note that the condition $n \geq N$ becomes important in this case (actually $n \geq N - 1$ would do). The conditions having been satisfied, we may apply Theorem 2 to arrive at the corollary. □

The asymptotic analysis of this Fredholm determinant is performed in Sect. 2 and yields the proof of Theorem 1.

The existence of such an identity as in Theorem 2 did not arise out of the blue. Let us briefly explain the two results which suggested this identity (though only in the special case of $F(w)$ as in (1.6)).

Definition 1.9. *An* **up/right path** *in $\mathbb{R} \times \mathbb{Z}$ is an increasing path which either proceeds to the right or jumps up by one unit. For each sequence $0 < s_1 < \cdots < s_{N-1} < t$ we can associate an up/right path ϕ from $(0, 1)$ to (t, N) which jumps between the points (s_i, i) and $(s_i, i + 1)$, for $i = 1, \ldots, N - 1$, and is continuous otherwise. Fix a real vector $a = (a_1, \ldots, a_N)$ and let $B(s) = (B_1(s), \ldots, B_N(s))$ for $s \geq 0$ be independent standard Brownian motions such that B_i has drift a_i.*

Define the energy of a path ϕ to be

$$E(\phi) = B_1(s_1) + (B_2(s_2) - B_2(s_1)) + \cdots + (B_N(t) - B_N(s_{N-1})).$$

Then the O'Connell-Yor semi-discrete directed polymer partition function $Z^N(t)$ is given by

$$Z^N(t) = \int d\phi\, e^{E(\phi)},$$

where the integral is with respect to Lebesgue measure on the Euclidean set of all up/right paths ϕ (i.e., the simplex of jumping times $0 < s_1 < \cdots < s_{N-1} < t$).

Due to the invariance principle, the semi-discrete polymer is a universal scaling limit for discrete directed polymers when N is fixed and n goes to infinity (and temperature is suitably scaled – see for instance [5]). As such, the semi-discrete polymer inherits the solvability of the log-Gamma polymer.

In fact, before the work of [11] on the log-Gamma polymer, O'Connell [15] showed (see Corollary 4.2) that $\mathbb{E}\left[e^{-uZ^N(t)}\right]$ was given by the left hand side of (1.4) with

$$F(w) = u^w e^{w^2 t/2}. \tag{1.6}$$

On the other hand, the theory of Macdonald processes was developed in [7]. The expectations of certain observables of the Macdonald processes were computed in [7] using Macdonald difference operators and the Cauchy identity for Macdonald polynomials. When formed into generating functions, these expectations lead to Fredholm determinants. After a particular limit transition, the Macdonald processes become the Whittaker processes introduced in [15] and the generating functions become a Fredholm determinant expression for $\mathbb{E}\left[e^{-uZ^N(t)}\right]$ – exactly in the form of the one on the right-hand side of (1.4). In particular the following result appeared in [7] as Thm. 5.2.10 (a change of variables $s + v = w$ brings it exactly to the form of (1.5)).

Theorem 3. *Fix* $N \geq 1$ *and a drift vector* $a = (a_1, \ldots, a_N)$. *Fix* $0 < \delta_2 < 1$, *and* $\delta_1 < \delta_2/2$ *such that* $|a_i| < \delta_1$. *Then for* $t \geq 0$,

$$\mathbb{E}\left[e^{-uZ^N(t)}\right] = \det(I + K_u),$$

where $\det(I + K_u)$ *is the Fredholm determinant of*

$$K_u : L^2(C_a) \to L^2(C_a)$$

for C_a *a positively oriented contour containing* $a_1, \ldots, a_N$ *and such that for all* $v, v' \in C_a$, *we have* $|v - v'| < \delta_2$. *The operator* K_u *is defined in terms of its integral kernel*

$$K_u(v, v') = \frac{1}{2\pi \mathrm{i}} \int_{\ell_{\delta_2}} ds\, \Gamma(-s)\Gamma(1+s) \prod_{m=1}^{N} \frac{\Gamma(v - a_m)}{\Gamma(s + v - a_m)} \frac{u^s e^{vts + ts^2/2}}{v + s - v'}.$$

This provides a very indirect proof of the identity given in (1.4), in the very particular case of $F(w)$ as in (1.6). An obvious question this development raised was to provide a direct proof of this identity and to understand how general it is. This is what Theorem 2 accomplishes.

It is worth noting that Macdonald processes also have a limit transition to the Whittaker processes described in [11], which are connected to the log-Gamma polymer. The Fredholm determinant given in Corollary 1.8 was anticipated in [7] via the formalism of Macdonald processes yet its rigorous proof was so far lacking because of the nontriviality of certain decay estimates required by that approach. If one only cares about the GUE Tracy-Widom asymptotics then the approach given here provides a direct, though non-obvious and rather ad hoc, route from the integral formula of [11] to the Fredholm determinant.

2. Fredholm Determinant Asymptotic Analysis: Proof of Theorem 1

Let us first recall two useful lemmas. In performing steepest descent analysis on Fredholm determinants, the following allows us to deform contours to descent curves.

Lemma 2.1 (Prop. 1 of [18]). *Suppose* $s \to \Gamma_s$ *is a deformation of closed curves and a kernel* $L(\eta, \eta')$ *is analytic in a neighborhood of* $\Gamma_s \times \Gamma_s \subset \mathbb{C}^2$ *for each* s. *Then the Fredholm determinant of* L *acting on* Γ_s *is independent of* s.

Lemma 2.2 (Lem. 4.1.38 of [7]). *Consider a sequence of functions* $\{f_n\}_{n\geq 1}$ *mapping* $\mathbb{R} \to [0, 1]$ *such that for each* n, $f_n(x)$ *is strictly decreasing in* x *with a limit of 1 at* $x = -\infty$ *and 0 at* $x = \infty$, *and for each* $\delta > 0$, *on* $\mathbb{R} \setminus [-\delta, \delta]$ f_n *converges uniformly to* $\mathbf{1}(x \leq 0)$. *Define the* r*-shift of* f_n *as* $f_n^r(x) = f_n(x - r)$. *Consider a sequence of random variables* X_n *such that for each* $r \in \mathbb{R}$,

$$\mathbb{E}[f_n^r(X_n)] \to p(r),$$

and assume that $p(r)$ *is a continuous probability distribution function. Then* X_n *converges weakly in distribution to a random variable* X *which is distributed according to* $\mathbb{P}(X \leq r) = p(r)$.

Proof of Theorem 1. Consider the function $f_n(x) = e^{-e^{n^{1/3}x}}$ and define $f_n^r(x) = f_n(x-r)$. Observe that this sequence of functions meets the criteria of Lemma 2.2. Setting

$$u = u(n, r, \gamma) = e^{-n\bar{f}_\gamma - rn^{1/3}}$$

observe that

$$e^{-uZ(n,n)} = f_n^r\left(\frac{\log Z(n,n) - n\bar{f}_\gamma}{n^{1/3}}\right).$$

By Lemma 2.2, if for each $r \in \mathbb{R}$ we can prove that

$$\lim_{n\to\infty} \mathbb{E}\left[f_n^r\left(\frac{\log Z(n,n) - n\bar{f}_\gamma}{n^{1/3}}\right)\right] = p_\gamma(r)$$

for $p_\gamma(r)$ a continuous probability distribution function, then it will follow that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log Z(n,n) - n\bar{f}_\gamma}{n^{1/3}} \le r\right) = p_\gamma(r)$$

as well.

Now, the starting point of our asymptotics is the Fredholm determinant formula given in Corollary 1.8 for $\mathbb{E}\big[e^{-uZ(n,n)}\big]$. Observe that by setting $a_j \equiv 0$ and $\alpha_i \equiv \gamma$, we can choose δ_1 and δ_2 as necessary for the corollary to hold. In particular, that implies that

$$\mathbb{E}\left[f_n^r\left(\frac{\log Z(n,n) - n\bar{f}_\gamma}{n^{1/3}}\right)\right] = \det(I + K_{u(n,r,\gamma)}),$$

where $\det(I + K_{u(n,r,\gamma)})$ is the Fredholm determinant of $K : L^2(C_{\delta_1}) \longrightarrow L^2(C_{\delta_1})$ with

$$K_{u(n,r,\gamma)}(v, v') = \frac{1}{2\pi \mathrm{i}} \int_{\ell_{\delta_2}} dw \, \frac{\pi}{\sin(\pi(v-w))} \exp\left(n\,[G(v) - G(w)] + rn^{1/3}(v-w)\right) \frac{dw}{w - v'}, \tag{2.1}$$

and

$$G(z) = \log\Gamma(z) - \log\Gamma(\gamma - z) + \bar{f}_\gamma z.$$

We have rewritten the kernel in the form necessary to perform steepest descent analysis. Let us first provide a critical point derivation of the asymptotics. We will then provide the rigorous proof. Besides standard issues of estimation, we must be careful about manipulating contours due to a mine-field of poles (this ultimately leads to our present technical limitation that $\gamma < \gamma^*$).

The idea of steepest descent is to find critical points for the argument of the function in the exponential, and then to deform contours so as to go close to the critical point. The contours should be engineered such that away from the critical point, the real part of the function in the exponential becomes large negative and hence as n gets large, has negligible contribution. This then justifies localizing and rescaling the integration around the critical point. The order of the first non-zero derivative (here third order) determines the rescaling in n (here $n^{1/3}$) which in turn corresponds with the scale of the fluctuations in the problem we are solving. It is exactly this third order nature that accounts for the emergence of Airy functions and hence the GUE Tracy-Widom distribution.

Before proceeding with the critical point derivation (and then rigorous proof) let us remark that when applying a change of variables to the variables in the kernel of a Fredholm determinant, in order that the determinant of the kernel remain unchanged it is necessary to include a single factor of the Jacobian of the transform. In the calculation below we just consider the pointwise limit of the kernel (taking into account the Jacobian factor) and then we strength the convergence to the whole determinant.

Let us start by recording the first three derivatives of $G(z)$:

$$G'(z) = \Psi(z) + \Psi(\gamma - z) + \bar{f}_\gamma,$$
$$G''(z) = \Psi'(z) - \Psi'(\gamma - z),$$
$$G'''(z) = \Psi''(z) + \Psi''(\gamma - z).$$

Define $\bar{t}_\gamma$ such that $G''(\bar{t}_\gamma) = 0$; that is, $\bar{t}_\gamma = \gamma/2$. Notice that $\bar{f}_\gamma$ was defined such that $G'(\bar{t}_\gamma) = 0$ as well. Around $z = \bar{t}_\gamma$, the first non-vanishing derivative of G is the third derivative. Notice that $\bar{g}_\gamma = -G'''(\bar{t}_\gamma)$. This indicates that near $z = \bar{t}_\gamma$,

$$G(v) - G(w) = -\frac{\bar{g}_\gamma (v - \bar{t}_\gamma)^3}{6} + \frac{\bar{g}_\gamma (w - \bar{t}_\gamma)^3}{6} + \text{h.o.t.},$$

where h.o.t. denotes higher order terms in $(v - \bar{t}_\gamma)$. This cubic behavior suggests rescaling around $\bar{t}_\gamma$ by the change of variables

$$\tilde{v} = n^{1/3}(v - \bar{t}_\gamma), \qquad \tilde{w} = n^{1/3}(w - \bar{t}_\gamma).$$

Clearly the steepest descent contour for v from $\bar{t}_\gamma$ departs at an angle of $\pm 2\pi/3$ whereas the contour for w departs at angle $\pm\pi/3$. The w contour must lie to the right of the v contour (so as to avoid the pole from $1/(w - v')$). As n goes to infinity, neglecting the contribution away from the critical point, the point-wise limit of the kernel becomes

$$K_{r,\gamma}(\tilde{v}, \tilde{v}') = \frac{1}{2\pi \mathrm{i}} \int \frac{1}{\tilde{v} - \tilde{w}} \frac{\exp\left\{\frac{-\bar{g}_\gamma}{6}\tilde{v}^3 + r\tilde{v}\right\}}{\exp\left\{\frac{-\bar{g}_\gamma}{6}\tilde{w}^3 + r\tilde{w}\right\}} \frac{d\tilde{w}}{\tilde{w} - \tilde{v}'}, \tag{2.5}$$

where the kernel acts on the contour $e^{-2\pi \mathrm{i}/3}\mathbb{R}_+ \cup e^{2\pi \mathrm{i}/3}\mathbb{R}_+$ (oriented from negative imaginary part to positive imaginary part) and the integral in $\tilde{w}$ is on the (likewise oriented) contour $\{e^{-\pi \mathrm{i}/3}\mathbb{R}_+ + \delta\} \cup \{e^{\pi \mathrm{i}/3}\mathbb{R}_+ + \delta\}$ for any horizontal shift $\delta > 0$. This is owing to the fact that

$$n^{-1/3} \frac{\pi}{\sin(\pi(v - w))} \to \frac{1}{\tilde{v} - \tilde{w}}, \qquad \frac{dw}{w - v'} \to \frac{d\tilde{w}}{\tilde{w} - \tilde{v}'}.$$

The factor of $n^{-1/3}$ came from the Jacobian associated with the change of variables in v and v'.

Another change of variables to rescale by $(\tilde{g}_\gamma/2)^{1/3}$ results in a kernel

$$K'_{r,\gamma}(\tilde{v}, \tilde{v}') = \frac{1}{2\pi \mathrm{i}} \int \frac{1}{\tilde{v} - \tilde{w}} \frac{\exp\left\{-\tilde{v}^3/3 + (\bar{g}_\gamma/2)^{-1/3} r\tilde{v}\right\}}{\exp\left\{-\tilde{w}^3/3 + (\bar{g}_\gamma/2)^{-1/3} r\tilde{w}\right\}} \frac{d\tilde{w}}{\tilde{w} - \tilde{v}'}.$$

One now recognizes that the Fredholm determinant of this kernel is one way to define the GUE Tracy-Widom distribution (see for instance Lemma 8.6 in [8]). Assuming that the determinants converge (in addition to the pointwise convergence of their kernels) this shows that the limiting expectation $p_\gamma(r) = F_{\mathrm{GUE}}\left((\bar{g}_\gamma/2)^{-1/3} r\right)$ which shows that it is a continuous probability distribution function and thus Lemma 2.2 applies. This completes the critical point derivation.

The challenge now is to rigorously prove that $\det(I + K_{u(n,r,\gamma)})$ converges to $\det(I + K_{r,\gamma})$, where the two operators act on their respective L^2 spaces and are defined with respect to the kernels above in (2.1) and (2.5).

We will consider the case where $\gamma < \gamma^*$ for γ^* small enough and perform certain estimates given that assumption. Let us record some useful estimates: for z close to zero we have (g is being used to represent the Euler Mascheroni constant $g = 0.577...$)

$$\begin{aligned}
\log \Gamma(z) &= -\log z - gz + O(z^2),\\
\Psi(z) &= -\frac{1}{z} - g + O(z),\\
\Psi'(z) &= \frac{1}{z^2} + O(1),\\
\Psi''(z) &= -\frac{2}{z^3} + O(1),\\
\Psi'''(z) &= \frac{6}{z^4} + O(1).
\end{aligned}$$

From this we can immediately observe that for γ small,

$$\begin{aligned}
\bar{t}_\gamma &= \gamma/2,\\
\bar{f}_\gamma &= 4\gamma^{-1} + 2g + O(\gamma),\\
\bar{g}_\gamma &= 32\gamma^{-3} + O(1).
\end{aligned}$$

With the change of variables $z = \gamma\tilde{z}$ we may estimate

$$\begin{aligned}
G(z) - G(\bar{t}_\gamma) &= \log\Gamma(z) - \log\Gamma(\gamma - z) + \bar{f}_\gamma(z - \gamma/2) + \kappa\bar{t}_\kappa^2/2 - \bar{f}_\kappa\bar{t}_\kappa\\
&= f(\tilde{z}) + O(\gamma^2),
\end{aligned} \tag{2.9}$$

where the error is uniform for $\tilde{z}$ in any compact domain and

$$f(\tilde{z}) = \log(1 - \tilde{z}) - \log(\tilde{z}) + 4\tilde{z} - 2.$$

Our approach will be as follows: Step 1: We will deform the contour C_{δ_1} on which v and v' are integrated as well as the contour on which w is integrated so that they both locally follow the steepest descent curve for $G(z)$ coming from the critical point $\bar{t}_\gamma$ and so that along them there is sufficient global decay to ensure that the integral localizes to the vicinity of $\bar{t}_\gamma$. Step 2: In order to show this desired localization we will use a Taylor series with remainder estimate in a small ball of radius approximately γ around $\bar{t}_\gamma$, and outside that ball we will use the estimate (2.9) for the v and v' contour, and a similarly straightforward estimate for the w contour. Step 3: Given these estimates we can show convergence of the Fredholm determinants as desired.

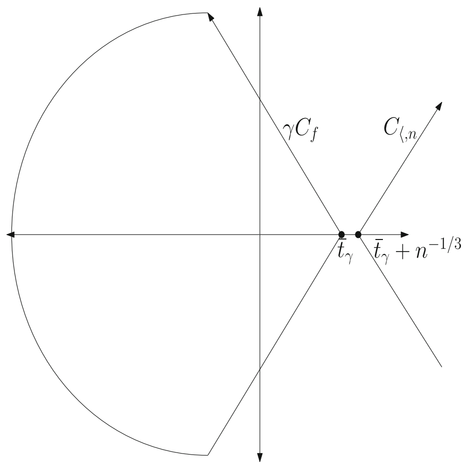

Fig. 1. Steep descent contours

Step 1. Define the contour C_f which corresponds to a steep descent contour[1] of the function $f(\tilde{z})$ leaving $\tilde{z} = 1/2$ at an angle of $2\pi/3$ and returning to $\tilde{z} = 1/2$ at an angle of $-2\pi/3$, given a positive orientation. In particular we will take C_f to be composed of a line segment from $\tilde{z} = 1/2$ to $\tilde{z} = 1/2 + e^{2\pi \mathrm{i}/3}$, then a circular arc (centered at 0) going counter-clockwise until $\tilde{z} = 1/2 + e^{-2\pi \mathrm{i}/3}$ and then a line segment back to $\tilde{z} = 1/2$. The contour γC_f will serve as our steep descent contour for $G(v)$ (see Fig. 1).

Lemma 2.3. *Along C_f,* $\mathrm{Re}(f)$ *achieves its maximal value of 0 at the unique point $\tilde{z} = 1/2$ and is negative everywhere else.*

Proof. We need only consider z in the upper half plane by symmetry considerations. For the line segment, parameterize $z = 1/2 + re^{2\pi \mathrm{i}/3}$ and observe that

$$\mathrm{Re}(f(z)) = \tfrac{1}{2}\log\left(r^2 - \tfrac{r}{2} + \tfrac{1}{4}\right) - \tfrac{1}{2}\log\left(r^2 + \tfrac{r}{2} + \tfrac{1}{4}\right) - r.$$

It suffices to prove that the derivative in r of this function is negative for $r > 0$, since that will imply the function is decreasing. The derivative is readily calculated to equal

$$\frac{-3 + 4r^2 - 16r^4}{1 + 4r^2 + 16r^4}.$$

For $r > 0$ the denominator is clearly positive. The numerator can be rewritten as $-(4r^2 - \frac{1}{2})^2 - \frac{1}{4} - 3$ which is negative for all r. Hence the derivative is negative for all $r > 0$.

Turning now to the circular arc, we parameterize $z = Re^{\theta \mathrm{i}}$ with $R = \sqrt{3}/2$ and observe that

$$\mathrm{Re}(f(z)) = \tfrac{1}{2}\log\left(R^2 - 2R\cos(\theta) + 1\right) - R + 4R\cos(\theta) - 2.$$

[1] For an integral $I = \int_\gamma dz\, e^{tf(z)}$, we say that γ is a steep descent path if (1) $\mathrm{Re}(f(z))$ reaches the maximum at some $z_0 \in \gamma$: $\mathrm{Re}(f(z)) < \mathrm{Re}(f(z_0))$ for $z \in \gamma \setminus \{z_0\}$, and (2) $\mathrm{Re}(f(z))$ is monotone along γ except at its maximum point z_0 and, if γ is closed, at a point z_1 where the minimum of $\mathrm{Re}(f)$ is reached.

It suffices to prove (by symmetry considerations) that the derivative in θ for θ going from $\pi/2$ to π is negative, since that will imply the function is decreasing. The derivative is readily calculated to equal

$$-\frac{R(2+4R^2-8R\cos(\theta))\sin(\theta)}{1+R^2-2R\cos(\theta)}.$$

The denominator equals $(1-R\cos(\theta))^2+(R\sin(\theta))^2$ and hence is positive. For $\theta \in [\pi/2, \pi]$, $\sin(\theta) \geq 0$ and $\cos(\theta) < 0$ which means that the denominator is positive. Taking into account the negative sign in front, we find that the derivative is negative on this interval of θ. $\square$

Returning to Step 1, one should note that for any $r \in (0, 1)$, for γ small enough, the contour γC_f is such that for v and v' along it, $|v - v'| < r$. By virtue of this fact, we may employ Lemma 2.1 to deform our initial contour C_{δ_1} to the contour γC_f without changing the value of the Fredholm determinant.

Likewise, we must deform the contour along which the w integration is performed. Given that $v, v' \in \gamma C_f$ now, we may use Cauchy's theorem to deform the w integration to a contour $C_{\langle,n}$ which is defined symmetrically over the real axis by a ray from $\bar{t}_\gamma + n^{-1/3}$ leaving at an angle of $\pi/3$ (again, see Fig. 1). It is easy to see that this deformation does not pass any poles. Since the contour is infinite, we must also provide decay of the integrand (in particular $\exp\{-nG(w)\}$) for w near infinity in the angular region between the original and terminal contour in this deformation (i.e., between angles $\pm\pi/2$ and $\pm\pi/3$). Due to the fact that $\bar{f}_\kappa > 0$, if we neglect everything but the linear part of $G(w)$, it is clear that there is exponential decay in w in this region. Finally, recall from (6.1.41) in [1]) that for w with argument bounded from π,

$$\log\Gamma(z) = (z - \tfrac{1}{2})\log(z) - z + \tfrac{1}{2}\log(2\pi) + o(1).$$

The nonlinear terms in $G(w)$ are $\log\Gamma(w) - \log\Gamma(\gamma - w)$ and the above bound shows that this difference grows sublinearly in $|w|$. Therefore $\exp\{-nG(w)\}$ has exponential decay in $|w|$ uniformly in the angular region, and Cauchy's theorem is justifiable.

Thus, the outcome of this first step is that the v, v' contour is now given by γC_f and the w contour is now given by $C_{\langle,n}$ which is independent of v.

Step 2. We will presently provide two types of estimates for our function G along the specified contours: those valid in a small ball around the critical point and those valid outside the small ball. Let us first focus on the small ball estimate.

Lemma 2.4. *There exists $\gamma^* > 0$ such that for all $\gamma < \gamma^*$ the following two facts hold:*

(1) There exists a constant $c_1 > 0$ such that for all v along the straight line segments of γC_f:

$$\mathrm{Re}\left[G(v) - G(\bar{t}_\gamma)\right] \leq \mathrm{Re}\left[-c_1\gamma^3(v-\bar{t}_\gamma)^3\right].$$

(2) There exists a constant $c_2 > 0$ such that for all w along the contour $C_{\langle,n}$ at distance less than γ from $\bar{t}_\gamma$:

$$\mathrm{Re}\left[G(w) - G(\bar{t}_\gamma)\right] \geq \mathrm{Re}\left[-c_2\gamma^3(w-\bar{t}_\gamma)^3\right].$$

Proof. Recall the Taylor expansion remainder estimate for a function $F(z)$ expanded around $\bar{z}$,

$$\Big|F(z)-\Big(F(\bar{z})+F'(\bar{z})(z-\bar{z})+\tfrac{1}{2}F''(\bar{z})(z-\bar{z})^2+\tfrac{1}{6}F'''(\bar{z})(z-\bar{z})^3\Big)\Big|$$
$$\leq \max_{\zeta\in B(\bar{z},|z-\bar{z}|)}\tfrac{1}{24}|F''''(\zeta)||z-\bar{z}|^4.$$

We may apply this to our function $G(z)$ around the point $\bar{t}_\gamma$ giving

$$\Big|G(z)-G(\bar{t}_\gamma)+\tfrac{1}{6}\bar{g}_\gamma(z-\bar{t}_\gamma)^3\Big|\leq \max_{\zeta\in B(\bar{t}_\gamma,|z-\bar{t}_\gamma|)}\tfrac{1}{24}|G''''(\zeta)||z-\bar{t}_\gamma|^4.$$

Let $z=\gamma\tilde{z}$ and also let $\zeta=\gamma\tilde{\zeta}$. Then for γ small, we have the following estimate:

$$\Big|G(z)-G(\bar{t}_\gamma)+\tfrac{16}{3}(\tilde{z}-1/2)^3\Big|\leq\frac{1}{4}\left|\frac{1}{\tilde{z}^3}-\frac{1}{(1-\tilde{z})^4}\right||\tilde{z}-1/2|^4+O(\gamma^2).$$

Both parts of the lemma follow readily from this estimate and the comparison along the contours of interest of $\frac{16}{3}(\tilde{z}-1/2)^3$ with the right-hand side above. □

We may now turn to the estimate outside the ball of size γ.

Lemma 2.5. *There exists $\gamma^*>0$ and $c>0$ such that for all v along the circular part of the contour γC_f, the following holds:*

$$\mathrm{Re}\big[G(v)-G(\bar{t}_\gamma)\big]\leq -c.$$

Proof. Writing $v=\gamma\tilde{v}$ we may appeal to the second line of the estimate (2.9) and the fact that along the circular part of the contour γC_f, the real part of function $f(\tilde{v})$ is strictly negative and the error in the estimate is $O(\gamma)$. □

The above bound suffices for the v contour since it is finite length. However, the w contour is infinite so our estimate must be sufficient to ensure that the contour's tails do not play a role.

Lemma 2.6. *There exists $\gamma^*>0$ and $c>0$ such that for all w along the contour $C_{\langle,n}$ at distance exceeding γ from $\bar{t}_\gamma$, the following holds:*

$$\mathrm{Re}\big[G(w)-G(\bar{t}_\gamma)\big]\geq \mathrm{Re}\Big[c\gamma^{-1}w\Big].$$

Proof. This estimate is best established in three parts. We first estimate for ζ between distance γ and distance $c_1\gamma$ from $\bar{t}_\gamma$ (c_1 large). Second we estimate for w between distance $c_1\gamma$ and distance c_2 from $\bar{t}_\gamma$. Finally we estimate for all w yet further. This third estimate is immediate from the first line of (2.9) in which the linear term in z clearly exceeds the other terms for γ small enough and $|z|>c_2$.

To make the first estimate we use the bottom line of (2.9). The function $f(\tilde{z})$ has sufficient growth along this contour to overwhelm the $O(\gamma^2)$ error as long as γ is small enough. The $O(\gamma^2)$ error in (2.9) is only valid for $\tilde{z}$ in a compact domain though. So, for the second estimate we must use the cruder bound that $G(z)-G(\bar{t}_\gamma)=f(\tilde{z})+O(1)$ for z along the contour and of size less than c_2 from $\bar{t}_\gamma$. Since in this regime of z, $f(\tilde{z})$ behaves like $(z-\gamma/2)\bar{f}_\gamma=4\gamma^{-1}z+O(1)$, one sees that for γ small enough, the $O(1)$ error is overwhelmed and the claimed estimate follows. □

Step 3. We now employ the estimates given above to conclude that $\det(I + K_{u(n,r,\gamma)})$ converges to $\det(I + K_{r,\gamma})$, where the two operators act on their respective L^2 spaces and are defined with respect to the kernels above in (2.1) and (2.5). The approach is standard (see for instance [8,3]) so we just briefly review what is done. Convergence can either be shown at the level of Fredholm series expansion or trace-class convergence of the operators. Focusing on the Fredholm series expansion, the estimates provided by Lemmas 2.4, 2.5 and 2.6, along with Hadamard's bound, show that for any ϵ there is a k large enough such that for all n, the contribution of the terms of the Fredholm series expansion past index k can be bounded by ϵ. This localizes the problem of asymptotics to a finite number of integrals involving the kernels. The same estimates then show that these integrals can be localized in a large window of size $n^{-1/3}$ around the critical point $\bar{t}_\gamma$. The cost of throwing away the portion of the integrals outside this window can be uniformly (in n) bounded by ϵ, assuming the window is large enough. Finally, after a change of variables to rescale this window by $n^{1/3}$ we can use the Taylor series with remainder to show that as n goes to infinity, the integrals coming from the kernel $K_{u(n,r,\gamma)}$ converge to those coming from $K_{r,\gamma}$. This last step is essentially the content of the critical point computation given earlier. □

3. Equivalence of Contour Integral and Fredholm Determinants

Let us first recall a version of the Andréief identity [4] (sometimes referred to as the generalized Cauchy-Binet identity): assuming all integrals exist,

$$\frac{1}{N!}\int\cdots\int d\mu(z_1)\cdots d\mu(z_N)\det\left[f_i(z_j)\right]_{i,j=1}^N \det\left[g_i(z_j)\right]_{i,j=1}^N$$
$$= \det\left[\int d\mu(z)\, f_i(z) g_j(z)\right]_{i,j=1}^N. \tag{3.1}$$

Proof of Theorem 2. We start working from the right-hand side of the formula. We regard the Fredhom determinant as given by its Fredholm series expansion (the convergence of the series is a consequence of the identity we will prove). Note that our assumptions on δ_1 and δ_2 imply that $\delta_1 < \frac{1}{2}$, so the factors $\Gamma(v - a_m)$ have no poles for $v \in C_{\delta_1}$. They also imply that the factor $\sin(\pi(v-w))(w-v')$ in the denominator in (1.5) is never zero on ℓ_{δ_2}. To see that the integral is convergent, recall the asymptotics $|\Gamma(x+\mathrm{i}y)|e^{\pi/2|y|}|y|^{1/2-x} \to \sqrt{2\pi}$ as $y \to \pm\infty$ for $x, y \in \mathbb{R}$ (see (6.1.45) in [1]), which implies that, for $\mathrm{Re}(z)$ in some finite interval,

$$c_1 e^{-\frac{\pi}{2}|\mathrm{Im}(z)|}|\mathrm{Im}(z)|^\eta \le |\Gamma(z)| \le c_2 e^{-\frac{\pi}{2}|\mathrm{Im}(z)|}|\mathrm{Im}(z)|^\eta, \tag{3.2}$$

as $|\mathrm{Im}(z)| \to \infty$ for some $c_1, c_2 > 0$ and $\eta \in \mathbb{R}$. We also have, under the same assumption on z, $|\sin(\pi z)| \ge c_3 e^{\pi|\mathrm{Im}(z)|}$ for some $c_3 > 0$. Then

$$\left|\frac{\pi}{\sin(\pi(v-w))}\frac{F(w)}{F(v)}\frac{1}{w-v'}\right|\prod_{m=1}^N \frac{\Gamma(v-a_m)}{\Gamma(w-a_m)}$$
$$\le c_3 e^{\pi(\frac{N}{2}-1)|\mathrm{Im}(w)|}|\mathrm{Im}(w)|^{-N\eta-1}|F(w)|, \tag{3.3}$$

as $|\mathrm{Im}(w)| \to \infty$ for some $c_4 > 0$, and hence the integral in (1.5) is convergent by (1.3).

Observe that $K = AB$ with $A\colon L^2(\ell_{\delta_2}) \longrightarrow L^2(C_{\delta_1})$ and $B\colon L^2(C_{\delta_1}) \longrightarrow L^2(\ell_{\delta_2})$ given by

$$A(v,w) = \frac{\pi}{\sin(\pi(v-w))}\frac{F(w)}{F(v)}\prod_{m=1}^{N}\frac{\Gamma(v-a_m)}{\Gamma(w-a_m)} \qquad \text{and} \qquad B(w,v') = \frac{1}{w-v'}$$

(checking that A and B map $L^2(\ell_{\delta_2})$ to $L^2(C_{\delta_1})$ and vice-versa is easy, and uses (1.3) in the case of A). Let $\widetilde{K} = BA\colon L^2(\ell_{\delta_2}) \longrightarrow L^2(\ell_{\delta_2})$, which is then given by

$$\widetilde{K}(w,w') = \frac{1}{2\pi\mathrm{i}}\oint_{C_{\delta_1}} dv\,\frac{1}{w-v}\frac{\pi}{\sin(\pi(v-w'))}\frac{F(w')}{F(v)}\prod_{m=1}^{N}\frac{\Gamma(v-a_m)}{\Gamma(w'-a_m)}.$$

Now recall in general that if A and B have integral kernels so that the integrals $\int dw\,A(v,w)B(w,v')$ and $\int dv\,B(w,v)A(v,w')$ both converge for v, v', w, w' in the right domains, then $\det(I+AB) = \det(I+BA)$ in the sense that the two convergent Fredholm expansions coincide termwise (this can be seen as an application of the Andréief identity (3.1)). Using this we deduce that

$$\det(I+K)_{L^2(C_{\delta_1})} = \det(I+\widetilde{K})_{L^2(\ell_{\delta_2})}. \tag{3.4}$$

Recalling that $\Gamma(1+s) = s\Gamma(s)$, let

$$G(v) = F(v)\prod_{m=1}^{N}\frac{v-a_m}{\Gamma(v-a_m+1)} \tag{3.5}$$

(not to be confused with the function G introduced in Sect. 2) and rewrite $\widetilde{K}$ as

$$\widetilde{K}(w,w') = \frac{1}{2\pi\mathrm{i}}\oint_{C_{\delta_1}} dv\,\frac{1}{w-v}\frac{\pi}{\sin(\pi(v-w'))}\frac{G(w')}{G(v)}.$$

By the assumption $\delta_1 < \min\{\delta_2, 1-\delta_2\}$, $w-v$ and $\sin(\pi(v-w'))$ are never zero for $v \in C_{\delta_1}$ and $w, w' \in \ell_{\delta_2}$. Thus the only poles of the integrand inside C_{δ_1} are those coming from $G(v)^{-1}$, which can only come from the factors $(v-a_m)$ in the definition of $G(v)$. For simplicity we will assume that all the a_i's are pairwise distinct, the general case follows likewise by taking limits. Then the integrand has simple poles at $v = a_i$, $i = 1, \dots, N$, and thus

$$\begin{aligned}\widetilde{K}(w,w') &= \frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{N} G(w')\operatorname*{Res}_{v=a_i}\left[\frac{1}{w-v}\frac{\pi}{\sin(\pi(v-w'))}\frac{1}{G(v)}\right]\\ &= \frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{N} G(w')\frac{1}{w-a_i}\frac{\pi}{\sin(\pi(a_i-w'))}\frac{1}{F(a_i)}\prod_{m\neq i}\frac{\Gamma(a_i-a_m+1)}{a_i-a_m}.\end{aligned}$$

We rewrite this as

$$\widetilde{K}(w,w') = \frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{N} f_i(w)g_i(w')$$

with

$$f_i(w) = \frac{1}{w - a_i}, \quad g_i(w') = C_i G(w') \frac{\pi}{\sin(\pi(a_i - w'))} \quad \text{and} \quad C_i = \frac{1}{F(a_i)} \prod_{m \neq i} \Gamma(a_i - a_m). \tag{3.6}$$

In particular, this means that $\widetilde{K} = \widetilde{A}\widetilde{B}$, where $\widetilde{A} \colon \ell^2(\{1, \ldots, N\}) \longrightarrow L^2(\ell_{\delta_2})$ is given by $\widetilde{A}(w, i) = (2\pi \mathrm{i})^{-1} f_i(w)$ and $\widetilde{B} \colon L^2(\ell_{\delta_2}) \longrightarrow \ell^2(\{1, \ldots, N\})$ by $\widetilde{B}(i, w') = g_i(w')$, so that

$$\widetilde{B}\widetilde{A}(i, j) = \frac{1}{2\pi \mathrm{i}} \int_{\ell_{\delta_2}} dw\, f_i(w) g_i(w).$$

The integral is convergent for the same reason as the integral in (1.5), and hence using again the formula $\det(I + AB) = \det(I + BA)$ we deduce that

$$\det(I + \widetilde{K})_{L^2(\ell_{\delta_2})} = \det\left[\mathbf{1}_{i=j} + \frac{1}{2\pi \mathrm{i}} \int_{\ell_{\delta_2}} dw\, f_i(w) g_j(w)\right]_{i,j=1}^{N}. \tag{3.7}$$

Our next goal is to shift the contour ℓ_{δ_2} in the integral appearing on the right-hand side of (3.7) to $\ell_{-\delta_1}$. Note that, as we do this, we will cross all the a_i's. We have

$$f_i(w) g_j(w) = F(w) \frac{C_j \pi}{\sin(\pi(a_j - w))} \frac{1}{\Gamma(w - a_i + 1)} \prod_{m \neq i} \frac{1}{\Gamma(w - a_m)}.$$

Since all the poles of F lie to the right of ℓ_{δ_2}, the only singularities of $f_i(w)g_j(w)$ we encounter as we shift the contour are those coming from the zeros of the sine. Observe that for $i \neq j$ there is no pole at $w = a_j$ due to the factor $\Gamma(w - a_j)^{-1}$, and hence we only see simple poles at $w = a_i$ in the case $i = j$. On the other hand, the integral of $f_i(w)g_j(w)$ along the segments going from $-\delta_1 \pm \mathrm{i}M$ to $\delta_2 \pm \mathrm{i}M$ goes to 0 as $|M| \to \infty$ by (1.3) and (3.3) as before. The conclusion is that

$$\frac{1}{2\pi \mathrm{i}} \int_{\ell_{\delta_2}} dw\, f_i(w) g_j(w) = \mathbf{1}_{i=j} \operatorname*{Res}_{w=a_i} [f_i(w) g_i(w)] + \frac{1}{2\pi \mathrm{i}} \int_{\ell_{-\delta_1}} dw\, f_i(w) g_j(w).$$

Since

$$\operatorname*{Res}_{w=a_i} [f_i(w) g_i(w)] = -C_i F(a_i) \prod_{m \neq i} \frac{a_i - a_m}{\Gamma(a_i - a_m + 1)} = -1,$$

we deduce from (3.7) that

$$\begin{aligned} \det(I + \widetilde{K})_{L^2(\ell_{\delta_1})} &= \det\left[\frac{1}{2\pi \mathrm{i}} \int_{\ell_{-\delta_1}} dw\, f_i(w) g_j(w)\right]_{i,j=1}^{N} \\ &= \frac{1}{(2\pi \mathrm{i})^N N!} \int_{\ell_{-\delta_1}} \cdots \int_{\ell_{-\delta_1}} dw_1 \cdots dw_N\, \det[f_i(w_j)]_{i,j=1}^{N} \det[g_i(w_j)]_{i,j=1}^{N}, \end{aligned} \tag{3.8}$$

where the second equality follows from the Andréief identity (3.1).

Note that the contours in this last integral are exactly the ones appearing in the formula we seek to prove. Hence what is left is to compute the two determinants and show that the integrand coincides with the one appearing in (1.4). We start by observing that $\det[f_i(w_j)]_{i,j=1}^N$ is a Cauchy determinant, so

$$\det[f_i(w_j)]_{i,j=1}^N = \frac{\prod_{i<j}(a_j - a_i)(w_i - w_j)}{\prod_{i,j=1}^N (w_i - a_j)}. \tag{3.9}$$

On the other hand

$$\det[g_i(w_j)]_{i,j=1}^N = \prod_{i=1}^N G(w_i)C_i \det\left[\frac{\pi}{\sin(\pi(a_i - w_j))}\right]_{i,j=1}^N.$$

The determinant on the right-hand side can be turned into another Cauchy determinant by writing

$$\frac{\pi}{\sin(\pi(a_i - w_j))} = \frac{2\pi\mathrm{i}}{e^{-\mathrm{i}\pi(a_i+w_j)}} \frac{1}{e^{2\mathrm{i}\pi a_i} - e^{2\mathrm{i}\pi w_j}},$$

so that

$$\det\left[\frac{\pi}{\sin(\pi(a_i - w_j))}\right]_{i,j=1}^N = \prod_{i=1}^N \frac{2\pi\mathrm{i}}{e^{-\mathrm{i}\pi(a_i+w_i)}} \det\left[\frac{1}{e^{2\mathrm{i}\pi a_i} - e^{2\mathrm{i}\pi w_j}}\right]_{i,j=1}^N,$$

and therefore

$$\begin{aligned}&\det[g_i(w_j)]_{i,j=1}^N \\ &\quad = (2\pi\mathrm{i})^N e^{\mathrm{i}\pi\sum_i(a_i+w_i)} \prod_{i=1}^N G(w_i)C_i \frac{\prod_{i<j}(e^{2\mathrm{i}\pi a_j} - e^{2\mathrm{i}\pi a_i})(e^{2\mathrm{i}\pi w_i} - e^{2\mathrm{i}\pi w_j})}{\prod_{i,j=1}^N (e^{2\mathrm{i}\pi a_i} - e^{2\mathrm{i}\pi w_j})}.\end{aligned} \tag{3.10}$$

Now we collect the factors involving only w_i's, only a_i's, and cross-terms: from (3.9), (3.10) and the definitions (3.5) and (3.6) of $G(w_i)$ and C_i we get

$$\det[f_i(w_j)]_{i,j=1}^N \det[g_i(w_j)]_{i,j=1}^N = (2\pi\mathrm{i})^N D_{a,a} D_{w,w} D_{a,w} \tag{3.11}$$

with

$$\begin{aligned}
D_{a,a} &= e^{\mathrm{i}\pi\sum_i a_i} \prod_{i<j}(a_j - a_i)(e^{2\mathrm{i}\pi a_j} - e^{2\mathrm{i}\pi a_i}) \prod_{i=1}^N \frac{\prod_{m\neq i}\Gamma(a_i - a_m)}{F(a_i)},\\
D_{w,w} &= e^{\mathrm{i}\pi\sum_i w_i} \prod_{i=1}^N F(w_i) \prod_{i<j}(w_i - w_j)(e^{2\mathrm{i}\pi w_i} - e^{2\mathrm{i}\pi w_j}),\\
D_{a,w} &= \prod_{i,j=1}^N \frac{1}{w_i - a_j} \frac{1}{\Gamma(w_i - a_j)} \frac{1}{e^{2\mathrm{i}\pi a_i} - e^{2\mathrm{i}\pi w_j}}.
\end{aligned}$$

$D_{a,a}$ can be rearranged in the following way:

$$D_{a,a} = e^{\mathrm{i}\pi \sum_i a_i} \prod_{i<j} (a_j - a_i) e^{\mathrm{i}\pi(a_i+a_j)} (e^{\mathrm{i}\pi(a_j-a_i)} - e^{\mathrm{i}\pi(a_i-a_j)}) \frac{\prod_{i\neq j} \Gamma(a_i - a_j)}{\prod_{i=1}^N F(a_i)}$$

$$= e^{\mathrm{i}\pi N \sum_i a_i} (2\mathrm{i})^{\frac{1}{2}N(N-1)} \prod_{i<j} \Gamma(a_j - a_i + 1)\Gamma(a_i - a_j) \sin(\pi(a_j - a_i)) \prod_{i=1}^N F(a_i)^{-1}$$

$$= e^{\mathrm{i}\pi N \sum_i a_i} (-2\pi\mathrm{i})^{\frac{1}{2}N(N-1)} \prod_{i=1}^N F(a_i)^{-1},$$

where have used

$$\Gamma(-s)\Gamma(1+s) = \frac{\pi}{\sin(-\pi s)}.$$

Similar computations give

$$D_{w,w} = e^{\mathrm{i}\pi N \sum_i w_i} (2\pi\mathrm{i})^{\frac{1}{2}N(N-1)} \prod_{i<j} \frac{\sin(\pi(w_i - w_j))}{\pi} (w_i - w_j) \prod_{i=1}^N F(w_i),$$

$$D_{a,w} = e^{-\mathrm{i}\pi N \sum_i (a_i + w_i)} (2\pi\mathrm{i})^{-N^2} \prod_{i,j=1}^N \Gamma(a_j - w_i).$$

Hence

$$D_{a,a} D_{w,w} D_{a,w}$$

$$= \frac{1}{(2\pi\mathrm{i})^N} (-1)^{\frac{1}{2}N(N-1)} \prod_{i<j} \frac{\sin(\pi(w_i - w_j))}{\pi} (w_i - w_j) \prod_{i,j=1}^N \Gamma(a_j - w_i) \prod_{i=1}^N \frac{F(w_i)}{F(a_i)}$$

$$= \frac{1}{(2\pi\mathrm{i})^N} \prod_{i\neq j} \frac{1}{\Gamma(w_i - w_j)} \prod_{i=1}^N \frac{F(w_i)}{F(a_i)},$$

and thus from (3.8) and (3.11) we deduce that

$$\det(I + \widetilde{K})_{L^2(\ell_{\delta_1})}$$

$$= \frac{1}{(2\pi\mathrm{i})^N N!} \int_{\ell_{-\delta_1}} \cdots \int_{\ell_{-\delta_1}} dw_1 \cdots dw_N \prod_{i\neq j} \frac{1}{\Gamma(w_i - w_j)} \prod_{i,j=1}^N \Gamma(a_j - w_i) \prod_{i=1}^N \frac{F(w_i)}{F(a_i)}.$$

Putting this together with (3.4) and the definition (1.1) of s_N yields (1.4). □

Acknowledgements. AB was partially supported by the NSF grant DMS-1056390. IC was partially supported by the NSF through DMS-1208998 as well as by the Clay Research Fellowship and by Microsoft Research through the Schramm Memorial Fellowship. DR was partially supported by the Natural Science and Engineering Research Council of Canada, by a Fields-Ontario Postdoctoral Fellowship and by Fondecyt Grant 1120309. DR is appreciative for MIT's hospitality during the visit in which this project was initiated.

References

1. Abramowitz M., Stegun I. A.: *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Volume 55. National Bureau of Standards Applied Mathematics Series, Washington, DC: Nat. Bur. of Standards, 1964
2. Alberts, T., Khanin, K., Quastel, J.: *Intermediate disorder regime for 1+1 dimensional directed polymers.* http://arxiv.org/abs/1202.4398v2 [math.PR], 2012
3. Amir, G., Corwin, I., Quastel, J.: Probability distribution of the free energy of the continuum directed random polymer in 1+1 dimensions. Comm. Pure Appl. Math. **64**, 466–537 (2011)
4. Andréief, C.: Note sur une relation entre les intégrales définies des produits des fonctions. Mém. de la Soc. Sci., Bordeaux **2**, 1–14 (1883)
5. Auffinger, A., Baik, J., Corwin, I.: *Universality for directed polymers in thin rectangles.* http://arxiv.org/abs/1204.4445v2 [math.PR], 2012
6. Baik, J., Ben Arous, G., Peché, S.: Phase transition of the largest eigenvalue for non-null complex sample covariance matrices. Ann. Probab. **33**, 1643–1697 (2006)
7. Borodin, A., Corwin, I.: Macdonald processes. Probab. Theor. Rel. Fields. http://arxiv.org/abs/1111.4408v3 [math.PR], (2012, to appear)
8. Borodin, A., Corwin, I., Ferrari, P.: Free energy fluctuations for directed polymers in random media in 1+1 dimension. Commun. Pure Appl. Math. http://arxiv.org/abs/1204.1024v1 [math.PR], (2012, to appear)
9. Borodin, A., Péché, S.: Airy kernel with two sets of parameters in directed percolation and random matrix theory. J. Stat. Phys. **132**, 275–290 (2008)
10. Corwin, I.: The Kardar-Parisi-Zhang equation and universality class. Random Matrices Theory Appl. **1**, 1130001 (2012)
11. Corwin, I., O'Connell, N., Seppäläinen, T., Zygouras, N.: *Tropical combinatorics and Whittaker functions.* http://arxiv.org/abs/1110.3489v3 [math.PR], 2012
12. Corwin, I., Quastel, J.: Universal distribution of fluctuations at the edge of the rarefaction fan. Ann. Probab. **41**(3), 1243–1314 (2013)
13. Kardar, K., Parisi, G., Zhang, Y.Z.: Dynamic scaling of growing interfaces. Phys. Rev. Lett. **56**, 889–892 (1986)
14. Moreno Flores, G., Quastel, J., Remenik, D.: Scaling to KPZ for discrete polymers and q-TASEP. In preparation
15. O'Connell, N.: Directed polymers and the quantum Toda lattice. Ann. Probab. **40**, 437–458 (2012)
16. Seppäläinen, T.: Scaling for a one-dimensional directed polymer with boundary. Ann. Probab. **40**, 19–73 (2012)
17. Tracy, C., Widom, H.: Level-spacing distributions and the Airy kernel. Commun. Math. Phys. **159**, 151–174 (1994)
18. Tracy, C., Widom, H.: Asymptotics in ASEP with step initial condition. Commun. Math. Phys. **290**, 129–154 (2009)

Communicated by H. Spohn