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Characterizing the efficient points without closedness or free-disposability

Fabián Flores-Bazán · Sigifredo Laengle · Gino Loyola

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Abstract Many multicriteria problems in economics and finance require that efficient solutions be found. A recent contribution to production theory established a characterization of efficient points under closedness and free-disposability (Bonnisseau and Crettez in Econ Theory 31(2):213–223, 2007, Theorem 1). However, as will be shown using a number of examples, these results cannot be applied to simple and plausible production sets, nor can they be extended to other classic multicriteria problems such as those arising in optimal portfolio theory and bargaining theory. To address these limitations, a reformulation of the above theorem without closedness or free-disposability is proposed. This enables efficient solutions for a wider range of multicriteria problems to be identified.

Keywords Production efficiency · Portfolio theory · Bargaining theory · Nash demand game · Free-disposability

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F. Flores-Bazán
Departamento de Ingeniería Matemática, CI2MA, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile
e-mail: fflores@ing-mat.udec.cl

S. Laengle (✉) · G. Loyola
Departamento Control de Gestión, Universidad de Chile, Diagonal Paraguay 257, Santiago de Chile, Chile
e-mail: slaengle@fen.uchile.cl

G. Loyola
e-mail: gloyola@fen.uchile.cl

1 Introduction

Many classic multicriteria problems arising in economic and financial theory involve the identification of efficient solutions. Typical examples include production theory (Debreu 1954), portfolio selection (Markowitz 1952) and Pareto-optimal solution choice in Nash demand games (Nash 1950). In the case of production theory, it is known that the points on the boundary of a closed production set satisfying certain conditions such as free disposability are always weakly efficient. The conditions under which boundary points are efficient have been recently established by Bonnisseau and Crettez, see Theorem 1 below (Bonnisseau and Crettez 2007, Theorem 1, p. 216).

It would be desirable to apply this theorem to simple and plausible sets in production models as well as portfolio selection and Pareto-optimal Nash demand game solution choices, the other two examples mentioned above. As we will see, however, this is not possible with the above-cited theorem in its original form. We therefore propose a reformulation that will allow it to be extended to these and potentially other multicriteria problems.

Our modified version of the theorem does not in fact replace the original one, hereafter referred to also (in keeping with Bonnisseau and Crettez) as Theorem 1. To identify the efficient solutions a method is needed that will eliminate those points preventing this theorem from fulfilling this task. Thus, Theorem 1 must be applied to identify the points that do not satisfy either of its assumptions. Given a set of feasible solutions, we propose an approach that consists of the following three steps. First, apply Theorem 1 to the original solution set if possible. Second, identify those boundary points that do not satisfy some of theorem's assumptions and eliminate them from the feasible solution set. And third, apply our main theorem to the (modified) feasible set whose efficient points coincide with those of the original set.

In the next section we present Bonnisseau- Crettez's theorem (Theorem 1), our main result (Theorem 2), whereas in Sect. 3, a method (Theorem 3) showing the construction of the modified set satisfying the assumptions of Theorem 2 is presented. We then apply this method to determine the efficient points of simple and plausible solution sets for production, portfolio and Nash demand game models. Some concluding remarks end the paper.

2 Basic definitions and the main results

Let Y be a (not necessarily closed or convex) non-empty set in $\mathbb{R}^m$, that will play the role of the production set, the set of possible portfolios or the set of feasible strategy profiles of Nash demand game in the three respective theoretical models already cited. Let $K \subseteq \mathbb{R}^m$ be a proper closed convex cone with nonempty interior, i.e., $\operatorname{int} K \neq \emptyset$, where by proper we mean $\{0\} \neq K \neq \mathbb{R}^m$. This cone induces the following preference relations:

$$y \geq_K y' \quad \text{if and only if} \quad y - y' \in K;$$
$$y >_K y' \quad \text{if and only if} \quad y - y' \in \operatorname{int} K.$$

We define the **set of weakly efficient points** of Y (with respect to K) as

$$\mathrm{WE}(Y;K) \doteq \{y \in Y : \nexists\, y' \in Y\ (y' >_K y)\},$$

and the **set of efficient points** of Y (with respect to K) as

$$\mathrm{E}(Y;K) \doteq \{y \in Y : \nexists\, y' \in Y\ (y' \geq_K y \quad \text{and} \quad y \ngeq_K y')\},$$

If K is pointed, i.e., $K \cap (-K) = \{0\}$, this reduces to

$$\mathrm{E}(Y;K) \doteq \{y \in Y : \nexists\, y' \in Y\ (y' \geq_K y \quad \text{and} \quad y' \neq y)\}.$$

Recall also that the *boundary* of Y is defined by $\partial Y \doteq \overline{Y} \setminus \mathrm{int}\, Y$, where $\overline{Y}$ is the closure of Y.

The classic *free-disposal* property [see Bonnisseau and Crettez (2007), p. 215] of $Y \neq \mathbb{R}^m$ is defined by $Y - K \subseteq Y$, or equivalently, $Y - K = Y$. It is expressed in **Assumption (P)**, which is the first assumption of Theorem 1. We will use the following weaker assumption:

Assumption (**P**′) $Y - \mathrm{int}\, K \subseteq Y$, or equivalently, $Y - \mathrm{int}\, K = \mathrm{int}\, Y$.

The latter equivalence follows from the equality $\mathrm{int}(Y - K) = Y - \mathrm{int}\, K$ provided $\mathrm{int}\, K \neq \emptyset$.
The second assumption of Theorem 1 is **Assumption (B)**, which states that for all $y, y' \in \partial Y$ such that $y' \geq_K y$ and $y \ngeq_K y'$, $ty + (1-t)y' \in \mathrm{int}\, Y$ for all $t \in]0,1[$. We propose instead the following weaker assumption:

Assumption (**B**′) For all $y, y' \in Y \setminus \mathrm{int}\, Y$ such that $y' \geq_K y$, $y \ngeq_K y'$, $ty + (1-t)y' \in \mathrm{int}\, Y$ for all $t \in\ ]0,1[$.

It is easily shown that $\mathrm{WE}(Y;K) = Y \setminus (Y - \mathrm{int}\, K)$. Hence, under Assumption (**P**′), we get

$$\mathrm{WE}(Y;K) = Y \setminus \mathrm{int}\, Y. \tag{1}$$

Now let us recall Theorem 1 in Bonnisseau and Crettez (2007), which specifies the condition under which the boundary of Y is the set of its efficient points.

Theorem 1 [Bonnisseau and Crettez (2007), Theorem 1] *Assume Y is closed and satisfies Assumption* **(P)**. *Then*

$$\mathrm{E}(Y;K) = \partial Y \quad \textit{if and only if } Y \textit{ satisfies Assumption } \mathbf{(B)}.$$

Observe the previous result remains true even if $Y = \mathbb{R}^m$, since in this case, Assumption **(B)** holds vacuously and $\mathrm{E}(\mathbb{R}^m;K) = \emptyset$.

By replacing Assumptions **(P)** and **(B)** with Assumptions (**P**′) and (**B**′) respectively, we obtain the following theorem:

Theorem 2 *Assume Y satisfies Assumption* (**P**$'$). *Then*

$$\mathrm{E}(Y;K) = Y \setminus \operatorname{int} Y \quad \textit{if and only if } Y \textit{ satisfies Assumption}(\mathbf{B}'). \tag{2}$$

Proof ($\Rightarrow$) Take any $y, y' \in Y \setminus \operatorname{int} Y$ satisfying $y' \geq_K y$ and $y \not\geq_K y'$. This yields a contradiction if $y \in \mathrm{E}(Y;K)$.
($\Leftarrow$) Let $y \in Y \setminus \operatorname{int} Y$. If $y \notin \mathrm{E}(Y;K)$ then there exists a $y' \in Y$ such that $y' \geq_K y$ and $y \not\geq_K y'$. We distinguish two cases:

(a) $y' \in \operatorname{int} Y$: By Assumption (**P**$'$), $\operatorname{int} Y = Y - \operatorname{int} K$. Thus, one obtains

$$y \in y' + (-K) \setminus K \subseteq Y - \operatorname{int} K + (-K) \setminus K \subseteq \operatorname{int} Y$$

which is a contradiction.

(b) $y' \notin \operatorname{int} Y$, i.e., $y' \in Y \setminus \operatorname{int} Y$. By Assumptions (**B**$'$) and (**P**$'$), we get

$$y'' = \frac{1}{2}y + \frac{1}{2}y' \in \operatorname{int} Y = Y - \operatorname{int} K.$$

Thus $y - y'' = \frac{1}{2}(y - y') \in (-K) \setminus K$. Hence

$$y \in y'' + (-K) \setminus K \subseteq Y - \operatorname{int} K + (-K) \setminus K \subseteq Y - \operatorname{int} K - K \subseteq \operatorname{int} Y,$$

reaching a contradiction again.

This completes the proof of $Y \setminus \operatorname{int} Y \subseteq \mathrm{E}(Y;K)$. The other inclusion is straightforward. Consequently, $Y \setminus \operatorname{int} Y = \mathrm{E}(Y;K)$.

A result similar to Theorem 2 when Y is closed may be found in [Tammer and Zălinescu (2010), Proposition 6.1].
Finally, taking into account (1), we observe that (2) in Theorem 2 can be written as

$$\mathrm{E}(Y;K) = \mathrm{WE}(Y;K) \quad \text{if and only if } Y \text{ satisfies Assumption } (\mathbf{B}').$$

3 How to reach assumptions (P$'$) and (B$'$)

Given a set Y, we provide a general procedure allowing to construct a set $\tilde{Y}$ satisfying Assumptions (**P**$'$) and (**B**$'$) such that $\mathrm{E}(\tilde{Y};K) = \mathrm{E}(Y;K)$.

Let $\emptyset \neq Y \subsetneq \mathbb{R}^m$ and $K \subseteq \mathbb{R}^m$ be a proper pointed closed convex cone. Set $\hat{Y} \doteq Y - K$ and

$$Z \doteq \mathrm{WE}(\hat{Y};K) \setminus \mathrm{E}(\hat{Y};K), \quad \tilde{Y} \doteq \hat{Y} \setminus Z.$$

Obviously $\hat{Y} - K = \hat{Y}$, and therefore $\hat{Y} - \operatorname{int} K = \operatorname{int} \hat{Y}$. On the other hand, $\tilde{Y} - \operatorname{int} K \subseteq \hat{Y} - \operatorname{int} K = \operatorname{int}(\hat{Y} - K) = \operatorname{int} \hat{Y}$, which ensures that $y \in \tilde{Y} - \operatorname{int} K$

implies $y \in \hat{Y} \backslash Z = \tilde{Y}$. Whence $\hat{Y}$ and $\tilde{Y}$ satisfy Assumptions **(P)** and **(P′)** respectively. From equality (1), we get

$$\mathrm{WE}(\hat{Y}; K) = \hat{Y} \setminus \mathrm{int}\ \hat{Y}; \quad \mathrm{WE}(\tilde{Y}; K) = \tilde{Y} \setminus \mathrm{int}\ \tilde{Y}.$$

Thus,

$$Z = \left\{ y \in \hat{Y} \setminus \mathrm{int}\ \hat{Y} : \exists\, y' \in \hat{Y},\ y' \geq_K y \ \text{ and } y \neq y' \right\} \tag{3}$$

$$= \left\{ y \in \hat{Y} \setminus \mathrm{int}\ \hat{Y} : \hat{Y} \cap (y + K) \neq \{y\} \right\}. \tag{4}$$

It is not difficult to check that

$$\tilde{Y} \setminus \mathrm{int}\ \tilde{Y} = [(\hat{Y} \setminus \mathrm{int}\ \hat{Y}) \setminus Z] \cup [((\mathrm{int}\ \hat{Y}) \cap \overline{Z}) \setminus Z] \cup [(\hat{Y} \setminus \mathrm{int}\ \hat{Y}) \cap \overline{Z} \setminus Z].$$

Since $(\mathrm{int}\ \hat{Y}) \cap \overline{Z} = \emptyset$, the preceding equality reduces to

$$\tilde{Y} \setminus \mathrm{int}\ \tilde{Y} = [(\hat{Y} \setminus \mathrm{int}\ \hat{Y}) \setminus Z] \cup [(\hat{Y} \setminus \mathrm{int}\ \hat{Y}) \cap \overline{Z} \setminus Z], \tag{5}$$

or equivalently,

$$\tilde{Y} \setminus \mathrm{int}\ \tilde{Y} = \mathrm{E}(\hat{Y}; K).$$

One can prove, due to pointedness of K, that $\mathrm{E}(\hat{Y}; K) = \mathrm{E}(Y; K)$. Hence,

$$\mathrm{WE}(\tilde{Y}; K) = \mathrm{E}(\hat{Y}; K) = \mathrm{E}(Y; K).$$

We have the following theorem.

Theorem 3 *Assume that K and Y are as above. Then*

(a) *$\tilde{Y}$ satisfies Assumptions* **(P′)** *and* **(B′)***;*
(b) $\mathrm{E}(\tilde{Y}; K) = \mathrm{E}(Y; K) = \tilde{Y} \setminus \mathrm{int}\ \tilde{Y}$.

Proof (a): The fulfilment of Assumption **(P′)** was showed above. We will check that $\tilde{Y}$ verifies Assumption **(B′)** vacuously. Indeed, let y, y' be points in $\tilde{Y} \setminus \mathrm{int}\ \tilde{Y}$ such that $y - y' \notin K$ and $y' - y \in K$. By (5), $y', y \in \hat{Y} \setminus \mathrm{int}\ \hat{Y}$ and $y \notin Z$. Then, $\hat{Y} \cap (y + K) = \{y\}$ leading to a contradiction.
(b): We first show $\mathrm{E}(Y; K) \subseteq \tilde{Y}$ and $\mathrm{E}(\tilde{Y}; K) \subseteq Y$. Indeed, for the first inclusion, we obviously get $\mathrm{E}(Y; K) \subseteq Y \subseteq \hat{Y}$. Take any $y \in \mathrm{E}(Y; K)$: if $y \in Z$ then $y \in \hat{Y} \setminus \mathrm{int}\ \hat{Y}$ and there exists $y' \in \hat{Y}$ such that $y' \geq_K y$ and $y' \neq y$. This gives a contradiction, and the first claim is proved.

For the second inclusion, let us take any $y \in \mathrm{E}(\tilde{Y}; K) \subseteq \mathrm{WE}(\tilde{Y}; K) = \tilde{Y} \setminus \mathrm{int}\ \tilde{Y}$. By (5), $y \in \hat{Y} \setminus \mathrm{int}\ \hat{Y}$, $y \notin Z$. Thus, $\hat{Y} \cap (y + K) = \{y\}$, which is absurd, if we take into account that $y \in \hat{Y} = Y - K$ and assume that $y \notin Y$. This proves the second claim.
We now prove that $\mathrm{E}(Y; K) \subseteq \mathrm{E}(\tilde{Y}; K)$. Let $\bar{y} \in \mathrm{E}(Y; K)$ and $y \in \tilde{Y}$ such that

$y \geq_K \bar{y}$, i.e., $y - \bar{y} \in K$. Since $y \in \hat{Y}$, being $y \in \tilde{Y}$ by the above remark, $y = y_0 - k$, for some $y_0 \in Y$ and $k_0 \in K$. Thus, $y_0 - \bar{y} \in K$, therefore, $\bar{y} - y_0 \in K$ by the choice of $\bar{y}$. Moreover, $\bar{y} - y = \bar{y} - y_0 + k \in K$, showing that $\bar{y} \geq_K y$, which proves $\bar{y} = y$ due to pointedness of K, and therefore $\bar{y} \in \mathrm{E}(\tilde{Y}; K)$.
We now prove that $\mathrm{E}(\tilde{Y}; K) \subseteq \mathrm{E}(Y; K)$. Let $\bar{y} \in \mathrm{E}(\tilde{Y}; K)$ and $y \in Y$ such that $y \geq_K \bar{y}$, i.e., $y - \bar{y} \in K$. Since $y \in Y$ by the above remark. From (5) it follows that $\bar{y} \in \hat{Y} \setminus \mathrm{int}\, \hat{Y}$ and $\bar{y} \notin Z$. Hence $\hat{Y} \cap (\bar{y} + K) = \{\bar{y}\}$. By the choice of y, we obtain $y = \bar{y}$, concluding that $\bar{y} \in \mathrm{E}(Y; K)$.
The last equality of (b) is a consequence of Theorem 2. □

Remark 1 Theorems 2 and 3 are valid even in any real normed vector space.

4 Some illustrative examples

In this section we apply the preceding theorems to some simple and plausible example models from production, portfolio selection and Nash bargaining theory. Certainly we first check whether the original feasible set Y satisfies Assumptions **(P)** and **(B)** of Theorem 1. Second, if it does not (which will always be the case in our examples) we modify the set by using Theorem 3. This leaves us with a modified set $\tilde{Y}$. Third and finally, we apply Theorem 2 to this modified set, thus identifying the efficient points of the original set Y.

Before proceeding, however, we briefly review the theoretical concepts behind the three examples.

4.1 Production theory example

The standard Arrow-Debreu theory of production [see Debreu (1954)], posits an economy with m commodities modeled as an m-dimensional space denoted $\mathbb{R}^m$. A production plan in this economy is a specification of inputs and outputs constrained by the existing technology in which the outputs are represented by non-negative numbers and the inputs by non-positive ones. This production space is traditionally denoted as the subset Y of the commodities space $\mathbb{R}^m$ and describes the production possibilities of the entire economy.

For simplicity the example studied here assumes a single input, single output production plan. As depicted in Fig. 1a, the feasibility set, Y, has two goods: an input resource and a output obtained from it. The resource lies on the negative x-axis and the output on the y-axis. A quantity $|x_0|$ of the resource is needed to initiate production (at less than this minimum amount, no production is possible given that the economy must pay a *setup cost*). Output beyond this quantity exhibits *constant returns to scale* up to y_1. A *saturation point* is reached at (x_1, y_1), no further output increases being possible even though more of the resource is available. This model is described in formal terms below.

The original set Y. Let $(x_0, 0)$ and (x_1, y_1) be pairs in $\mathbb{R}^2$, where $x_1 < x_0 < 0$ and $0 < y_1$. The closed set $Y \subset \mathbb{R}^2$ is given by all pairs $(x, y) \in \mathbb{R}^2$ such that

(i) $y \leq 0$ for all $x_0 \leq x \leq 0$;

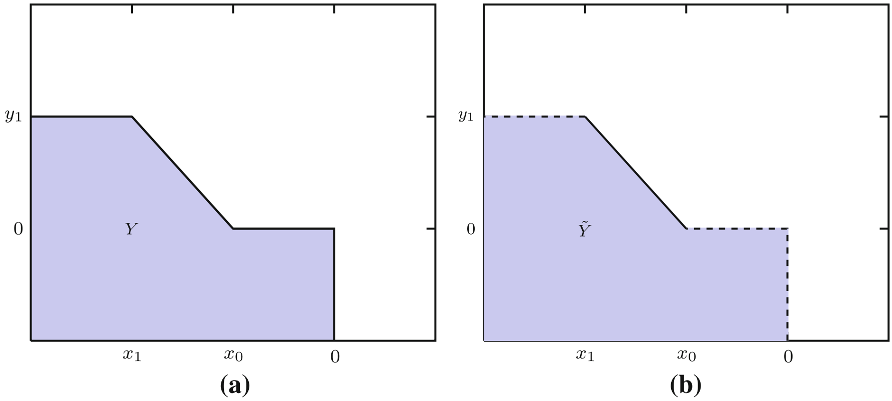

Fig. 1 A typical production theory example. In **a**, Y is a closed set satisfying Assumption **(P)** (*free-disposal*) but not Assumption **(B)**. In **b**, the modified set $\tilde{Y}$ is depicted, which satisfies both Assumptions **(P′)** and **(B′)**

(ii) the pairs are on or below the path between the points $(x_0, 0)$ and (x_1, y_1); and
(iii) $y \leq y_1$ for all $x \leq x_1$.

These conditions are illustrated in Fig. 1a. Here, $K = \mathbb{R}^2_+$. As can be seen, Y satisfies Assumption **(P)** but not Assumption **(B)**. Therefore, we must modify the original set Y.

The modified set $\tilde{Y}$. It is obtained following Theorem 3 and is given by all pairs $(x, y) \in \mathbb{R}^2$ such that

(i) $y < 0$ for all $x_0 < x < 0$;
(ii) the pairs are on or below the path between the points $(x_0, 0)$ and (x_1, y_1); and
(iii) $y < y_1$ for all $x < x_1$.

These conditions are illustrated in Fig. 1b. As can be seen, the modified set satisfies Assumptions **(P′)** and **(B′)**.

4.2 Portfolio theory example

Portfolio theory is a mathematical formulation of the concept of investment diversification. It seeks to identify the proportions of various assets that maximize expected return given a given amount of portfolio risk, or equivalently, to minimize risk subject to a given level of return. The theory was first stated by Markowitz (1952) and is considered a significant advance in mathematical modeling of financial theory.

Consider now a simple example of an investment portfolio with two assets denoted Asset 2 and Asset 3. The returns on both are normally distributed, with expected values of y_2 for the first asset and y_3 for the second, and corresponding standard deviations of x_2 and x_3. Suppose in this example that the coefficient of correlation ρ between the two random variables belongs to the interval $]0, 1[$. If $t \in [0, 1]$ is the proportion of Asset 2 in the portfolio, $1 - t$ thus being the proportion of Asset 3, it is easily demonstrated that the portfolio's expected return is given by $y(t) \doteq ty_2 + (1-t)y_3$ and the standard deviation by $x(t) \doteq \sqrt{t^2x_2^2 + (1-t)^2x_3^2 + 2t(1-t)\rho x_2 x_3}$, as represented by the

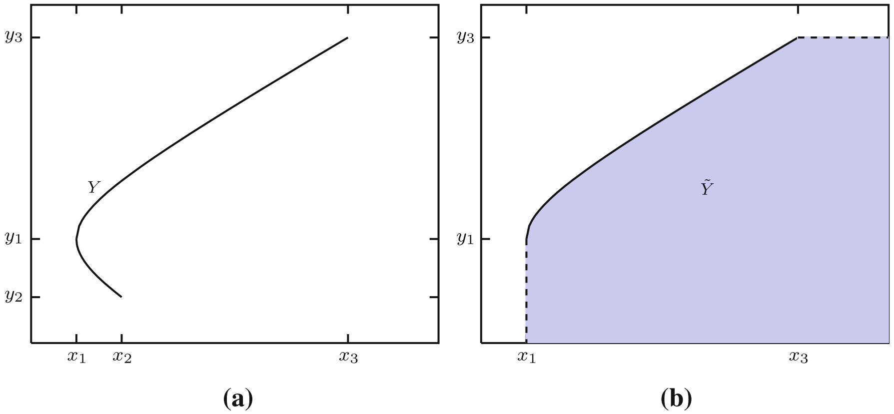

Fig. 2 A typical portfolio theory example. In **a**, Y is a closed set but does not satisfy either Assumption **(P)** (*free-disposal*) or Assumption **(B)**. In **b** the modified set $\tilde{Y}$ is shown, which satisfies Assumptions **(P′)** and **(B′)**

parametric curve in Fig. 2a. Observe that $0 < x_1 < x_2 < x_3$ and $0 < y_2 < y_1 < y_3$. In this case it is customary to considerer that the cone defining the partial order on Y is represented by the second quadrant of the plane (rather than the first quadrant as in the other examples studied in this article).

The original set Y. This set is given by the parametric curve $(x(t), y(t))$, where $t \in [0, 1]$. Observe that it does not satisfy Assumption **(P)** with $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0, x_2 \geq 0\}$, meaning that Theorem 1 cannot be applied. The set must therefore be modified.

The modified set $\tilde{Y}$. By Theorem 3, it is described by all pairs $(x, y) \in \mathbb{R}^2$ such that

(i) the pairs are on the parametric curve from point (x_3, y_3) to (x_1, y_1);
(ii) they are below the curve except directly underneath the point (x_1, y_1); and
(iii) $y < y_3$ for all $x_3 < x$;

These conditions are illustrated in Fig. 2b. The modified set satisfies both Assumption **(P′)** and Assumption **(B′)**.

4.3 Bargaining theory example

The Nash demand game is a non-cooperative game that models negotiation between two risk-neutral players. It was first posited and solved by John Nash (1950). The model assumes that a divisible good (usually of size 1) is available for distribution between two players in accordance with their simultaneous demands. If the two demands add up to no more than the total good, both players get what they demanded and the game ends. Otherwise, neither get anything.

More formally, assume that each strategy of each player is represented by x and y chosen from the interval $[0, 1]$ where 1 is the total good to be distributed. If $x + y$ is less than or equal to 1, the first player gets x and the second y. Otherwise, both get 0, which is what they get if they do not negotiate. It is proved that each pair (x, y) satisfying $x + y = 1$ is a Nash equilibrium and a Pareto-efficient solution too.

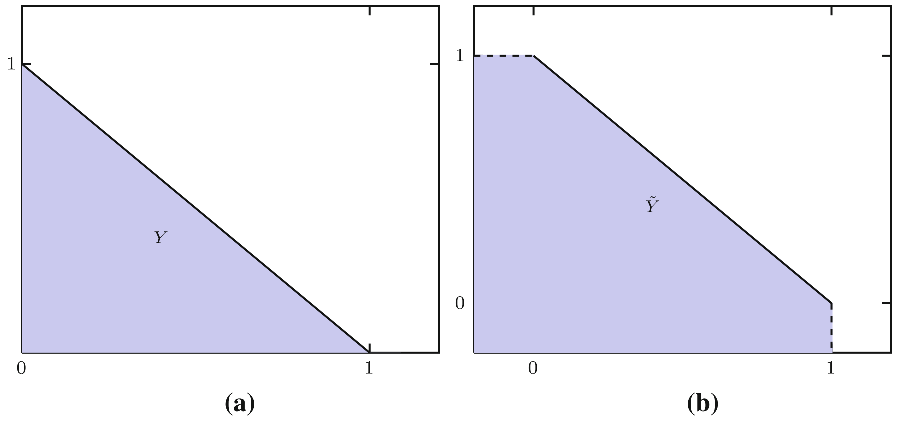

Fig. 3 Typical Nash demand game example. In **a**, Y is a closed set but does not satisfy either Assumption **(P)** or Assumption **(B)**. In **b** the modified set $\tilde{Y}$ is shown, which satisfies both Assumptions **(P′)** and **(B′)**

The original set Y. Consider the set Y given by all points $(x, y) \in \mathbb{R}^2$ such that $y \leq 1 - x$ for all $0 \leq x \leq 1$, as illustrated in Fig. 3a. By taking $K = \mathbb{R}^2_+$, the set Y does not satisfy either Assumption **(P)** or Assumption **(B)** and Theorem 1 cannot therefore be applied.

The modified set $\tilde{Y}$. In this case, again by Theorem 3, the set is given by all pairs $(x, y) \in \mathbb{R}^2$ such that

(i) $y = 1 - x$ for all $0 \leq x \leq 1$;
(ii) the pairs are below the curve except directly underneath the point $(1, 0)$; and
(iii) $y < 1$ for all $x < 0$.

This new set satisfies both Assumptions **(P′)** and **(B′)**, as can be observed in Fig. 3b.

4.4 Application of Theorem 2

The characteristics of the sets for our three examples are summarized in Table 1. Observe that Assumption **(P)** is not satisfied by any of the original example sets, Y, with the exception of Prod-1, meaning that Theorem 1 cannot be applied. Even for Prod-1, Theorem 1 can only establish that the boundary does not coincide with the set of efficient solutions (i.e., $\mathrm{E}(Y) \neq \partial Y$).

On the other hand, in all the three modified examples (i.e., those ending in -2 in the table headings), the set $\tilde{Y}$ satisfies both Assumption **(P′)** and Assumption **(B′)**. Theorem 2 can therefore be applied to identify the efficient points through the equality $\mathrm{E}(Y; K) = \mathrm{E}(\tilde{Y}; K) = \tilde{Y} \setminus \operatorname{int} \tilde{Y}$ via Theorem 3.

5 Conclusions

A theorem originally presented in Bonnisseau and Crettez (2007) (Theorem 1) and a modified version developed in this article (Theorem 2) are applied to classic exam-

Table 1 Properties of Y and $\tilde{Y}$ for each example

Property	Prod-1 Y	Prod-2 $\tilde{Y}$	Port-1 Y	Port-2 $\tilde{Y}$	Nash-1 Y	Nash-2 $\tilde{Y}$
Closed	Yes	No	Yes	No	Yes	No
P	Yes	Yes	No	No	No	No
B	No	No	No	No	No	No
P′	Yes	Yes	No	Yes	No	Yes
B′	No	Yes	No	Yes	No	Yes

As all the modified examples (i.e., those ending in -2) satisfy both Assumptions (**P′**) and (**B′**), Theorem 2 can be applied to $\tilde{Y}$ providing the efficient points of Y

Table 2 Results of applying Theorem 1 and Theorem 2

Example	Prod-1	Prod-2	Port-1	Port-2	Nash-1	Nash-2
Theorem 1	$\mathrm{E} \neq \partial Y$	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable
Theorem 2	$\mathrm{E} \neq Y \backslash \mathrm{int}\, Y$	$\mathrm{E} = \tilde{Y} \backslash \mathrm{int}\, \tilde{Y}$	Not applicable	$\mathrm{E} = \tilde{Y} \backslash \mathrm{int}\, \tilde{Y}$	Not applicable	$\mathrm{E} = \tilde{Y} \backslash \mathrm{int}\, \tilde{Y}$

Observe that Theorem 1 either cannot be applied to any of the examples or asserts that all boundary points are not efficient. By contrast, Theorem 2, the modified version, determines the efficient solutions for all of the modified sets $\tilde{Y}$ (those ending in -2), and hence those of Y

ples of three economic and financial theories. The results are summarized in Table 2. The first two examples (Prod-1 and Prod-2) are from production theory, the second two (Port-1 and Port-2) from optimal portfolio theory and the third two (Nash-1 and Nash-2) from the feasible solutions set for the Nash demand game. In each case, the first example gives the set of feasible points typically characterized in the literature while the second example is a modified version that satisfies conditions imposed here for finding the points of the efficient frontier.

For all of the examples, application of Theorem 1 is either impossible or does not identify the efficient points (e.g., Prod-1 where $\mathrm{E}(Y) \neq \partial Y$). In contrast with this formulation, applying the modified version Theorem 2 (by means of Theorem 3) identifies the efficient solutions in all cases with the help of the equality $\mathrm{E}(Y; K) = \mathrm{E}(\tilde{Y}; K) = \tilde{Y} \setminus \mathrm{int}\, \tilde{Y}$, under Assumptions (**P′**) and (**B′**).

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