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COMMUTATION PRINCIPLE FOR VARIATIONAL PROBLEMS ON EUCLIDEAN JORDAN ALGEBRAS*

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Abstract. This paper establishes a commutation result for variational problems involving spectral sets and spectral functions. The discussion takes places in the context of a general Euclidean Jordan algebra.

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1. Introduction.

Let the space $\mathcal{S}_n$ of real symmetric matrices of order n be equipped with the trace inner product $\langle X, Y\rangle = \operatorname{tr}(XY)$. The notation $\lambda(X)$ refers to the vector of eigenvalues of $X \in \mathcal{S}_n$ arranged in nonincreasing order. Recall that a spectral set in $\mathcal{S}_n$ is a set of the form

$$\Omega = \lambda^{-1}(Q) := \{X \in \mathcal{S}_n : \lambda(X) \in Q\},$$

where Q is a permutation invariant set in $\mathbb{R}^n$. A spectral function on $\mathcal{S}_n$ is a function $\Phi : \mathcal{S}_n \to \mathbb{R}$ admitting the representation $\Phi(X) = g(\lambda(X))$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a real-valued permutation invariant function. Spectrality is a property that some authors refer to as orthogonal invariance. General information on the theory of spectral sets and spectral functions can be found in [3, 9, 11] and the references therein. Iusem and Seeger [7, Lemma 4] recently established the following commutativity result for variational problems involving spectral data. By a local extremum of a function one understands a local minimum or a local maximum.

LEMMA 1. *Let $A, B \in \mathcal{S}_n$. Suppose that $\Omega \subseteq \mathcal{S}_n$ is a spectral set and that $\Phi : \mathcal{S}_n \to \mathbb{R}$ is a spectral function. Under these assumptions, if B is a local extremum of*

$$X \in \Omega \mapsto F(X) = \langle A, X\rangle + \Phi(X),$$

then A and B commute, i.e., $AB = BA$.

It is worthwhile to keep in mind that if two symmetric matrices commute, then it is possible to diagonalize them by means of a common orthogonal matrix. The possibility of simultaneous diagonalization opens the way to significant simplifications in the proofs of various linear algebra results. The commutation principle stated in Lemma 1 has applications in various fields: Fenchel conjugate and subdifferential of

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a convex spectral function (cf. [9]), distance to a spectral set (cf. [3, Proposition 2.3]), inradius and incenter of a spectral convex cone (cf. [5, Theorem 3.3]), distance between a pair of spectral convex cones (cf. [8, Proposition 6.6]), and antipodal pairs in spectral convex cones (cf. [7, Theorem 4]).

It turns out that Lemma 1 is a particular instance of a more general and deep commutation principle for variational problems on Euclidean Jordan algebras. The main result of this paper reads as follows.

Theorem 2. *Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be an Euclidean Jordan algebra and let $a, b \in \mathbb{V}$. Suppose that $\Omega \subseteq \mathbb{V}$ is a spectral set and that $\Phi : \mathbb{V} \to \mathbb{R}$ is a spectral function. Under these assumptions, if b is a local extremum of*

$$x \in \Omega \mapsto F(x) = \langle a, x\rangle + \Phi(x), \tag{1}$$

then a and b operator commute, i.e.,

$$a \circ (b \circ z) = b \circ (a \circ z) \quad \text{for all } z \in \mathbb{V}. \tag{2}$$

For simplicity in the exposition we consider Φ as a spectral function on the whole space $\mathbb{V}$, but one could restrict Φ to the spectral subset Ω. Section 2 reviews some basic material on Euclidean Jordan algebras and prepares the ground for proving Theorem 2. The proof itself is given in section 3. Some applications are mentioned in section 4.

2. Preliminary material on Euclidean Jordan algebras.

Throughout this work one assumes that $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra (EJA) with unit element $e \in \mathbb{V}$. This means that $\mathbb{V}$ is a finite dimensional real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$ and a bilinear function $\circ : \mathbb{V}\times\mathbb{V} \to \mathbb{V}$ satisfying the axioms

$$\begin{cases} x \circ y = y \circ x & \text{for all } x, y \in \mathbb{V}, \\ x \circ (x^2 \circ y) = x^2 \circ (x \circ y) & \text{for all } x, y \in \mathbb{V}, \\ \langle x \circ y, z\rangle = \langle y, x \circ z\rangle & \text{for all } x, y, z \in \mathbb{V}, \\ e \circ x = x & \text{for all } x \in \mathbb{V}. \end{cases}$$

Here $x^2 = x \circ x$. Higher-order powers are defined recursively by $x^{k+1} = x \circ x^k$. The rank of $\mathbb{V}$ is $r = \max\{\deg(x) : x \in \mathbb{V}\}$, where $\deg(x)$ is the smallest positive integer k such that $\{e, x, x^2, \ldots, x^k\}$ is linearly dependent.

The Lyapunov operator associated with $x \in \mathbb{V}$ is the linear map $L_x : \mathbb{V} \to \mathbb{V}$ given by $L_x y = x \circ y$. The operator commutation property (2) amounts to saying that the bracket

$$[L_a, L_b] := L_a L_b - L_b L_a$$

is equal to the zero map on $\mathbb{V}$.

An element $c \in \mathbb{V}$ is an idempotent if $c^2 = c$. An idempotent c is primitive if it is nonzero and cannot be written as a sum of two nonzero idempotents. A Jordan frame is a collection $\{c_1, \ldots, c_r\}$ of primitive idempotents satisfying $\sum_{i=1}^r c_i = e$ and $c_i \circ c_j = 0$ when $i \neq j$. We recall below a spectral decomposition theorem taken from [4, Theorem III.1.2].

Theorem 3. *Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be an EJA with rank r. Then, for every $x \in \mathbb{V}$, there exist a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that $x = \lambda_1 c_1 + \cdots + \lambda_r c_r$. The λ_i's are uniquely determined by x.*

We write $\lambda_i(x)$ to underline the dependence with respect to x. Renumbering the c_i's if necessary, one may suppose that the $\lambda_i(x)$'s are arranged in nonincreasing order. By analogy with the case of symmetric matrices, one sees $\lambda(x) \in \mathbb{R}^r$ as the vector of "eigenvalues" of $x \in \mathbb{V}$. A spectral set in $\mathbb{V}$ is then a set of the form

$$\Omega = \lambda^{-1}(Q) := \{x \in \mathbb{V} : \lambda(x) \in Q\} \tag{3}$$

with $Q \subseteq \mathbb{R}^r$ permutation invariant. A spectral function on $\mathbb{V}$ is a function $\Phi : \mathbb{V} \to \mathbb{R}$ admitting the representation $\Phi(x) = g(\lambda(x))$ with $g : \mathbb{R}^r \to \mathbb{R}$ permutation invariant. The formulation of Theorem 2 is now perfectly clear.

Remark 4. Note that Lemma 1 can be derived from Theorem 2. It suffices to consider $\mathbb{V} = \mathcal{S}_n$ equipped with $\langle X, Y\rangle = \mathrm{tr}(XY)$ and $X \circ Y = (XY + YX)/2$.

The following result, borrowed from [1, Theorem 27], shows the importance of the concept of operator commutation. We mention in passing that this concept also admits other equivalent characterizations; see, for instance, [10, Theorem 1].

THEOREM 5. *Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be an EJA with rank r. Then $a \in \mathbb{V}$ and $b \in \mathbb{V}$ operator commute if and only if a and b admit a common Jordan frame, i.e., there exist a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_r$ such that*

$$a = \lambda_1 c_1 + \cdots + \lambda_r c_r \quad \textit{and} \quad b = \mu_1 c_1 + \cdots + \mu_r c_r.$$

2.1. The tangent space to the set of ordered Jordan frames. The proof of Theorem 2 relies on the analysis of an optimization problem of the form

$$\min\{f(\mathbf{c}) : \ \mathbf{c} \in \mathcal{O}_{\mathbb{V}}\}, \tag{4}$$

where $f : \mathbb{V}^r \to \mathbb{R}$ is a continuously differentiable function and

$$\mathcal{O}_{\mathbb{V}} := \{\mathbf{c} = (c_1, \ldots, c_r) \in \mathbb{V}^r : \{c_1, \ldots, c_r\} \text{ is a Jordan frame}\}. \tag{5}$$

Each element of (5) is called an ordered Jordan frame. A local solution $\bar{\mathbf{c}}$ to the problem (4) satisfies the first-order optimality condition

$$f'(\bar{\mathbf{c}})\mathbf{h} \geq 0 \quad \text{for all } \mathbf{h} \in T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}], \tag{6}$$

where $f'(\bar{\mathbf{c}}) : \mathbb{V}^r \to \mathbb{R}$ is the differential map of f at $\bar{\mathbf{c}}$ and $T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]$ is the Bouligand tangent set to $\mathcal{O}_{\mathbb{V}}$ at $\bar{\mathbf{c}}$ (cf. [2, Definition 4.1.1]).

The next lemma shows that $T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]$ is a linear subspace and provides an explicit formula for computing this set. It also characterizes the orthogonal complement

$$(T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}])^{\perp} := \left\{\mathbf{q} \in \mathbb{V}^r : \sum_{i=1}^{r} \langle q_i, h_i\rangle = 0 \text{ for all } \mathbf{h} \in T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]\right\}.$$

For convenience we introduce the index sets $N_r := \{1, \ldots, r\}$ and $M_r := \{(i, j) \in N_r \times N_r : i < j\}$.

LEMMA 6. *Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be an EJA with rank r and let $\bar{\mathbf{c}} \in \mathcal{O}_{\mathbb{V}}$. Then the following hold:*

(a) *$\mathbf{h} = (h_1, \ldots, h_r) \in \mathbb{V}^r$ belongs to $T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]$ if and only if*

$$2\bar{c}_i \circ h_i - h_i = 0 \quad \textit{for all } \ i \in N_r, \tag{7}$$

$$\bar{c}_i \circ h_j + \bar{c}_j \circ h_i = 0 \quad \textit{for all } \ (i, j) \in M_r. \tag{8}$$

In particular, $T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]$ is a linear subspace.

(b) $\mathbf{q} = (q_1, \ldots, q_r) \in \mathbb{V}^r$ *belongs to* $(T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}])^{\perp}$ *if and only if there are vectors*

$$\{\alpha_i : i \in N_r\} \subseteq \mathbb{V} \quad \text{and} \quad \{\beta_{i,j} : (i,j) \in M_r\} \subseteq \mathbb{V} \tag{9}$$

such that

$$q_i = 2\alpha_i \circ \bar{c}_i - \alpha_i + \sum_{j=1}^{i-1} \beta_{j,i} \circ \bar{c}_j + \sum_{j=i+1}^{r} \beta_{i,j} \circ \bar{c}_j \quad \text{for all } i \in N_r. \tag{10}$$

Proof. Let $\mathcal{R}_{\mathbb{V}}$ be the set of all $\mathbf{c} \in \mathbb{V}^r$ satisfying the nonlinear system

$$\left\{ \begin{array}{rcll} c_i^2 - c_i & = & 0 & \text{for all } \ i \in N_r, \\ c_i \circ c_j & = & 0 & \text{for all } \ (i,j) \in M_r. \end{array} \right. \tag{11}$$

Since $\mathcal{O}_{\mathbb{V}} \subseteq \mathcal{R}_{\mathbb{V}}$, one has $T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}] \subseteq T_{\bar{\mathbf{c}}}[\mathcal{R}_{\mathbb{V}}]$. Let $\mathcal{B}_{\bar{\mathbf{c}}}$ be the set of all $\mathbf{h} \in \mathbb{V}^r$ satisfying the linear system (7)–(8). Observe that (7)–(8) is obtained by linearizing (11) around the reference point $\bar{\mathbf{c}}$. This observation shows that $T_{\bar{\mathbf{c}}}[\mathcal{R}_{\mathbb{V}}] \subseteq \mathcal{B}_{\bar{\mathbf{c}}}$. For completing the proof of (a) one needs to check that $\mathcal{B}_{\bar{\mathbf{c}}} \subseteq T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]$. Let $\mathbf{h} \in \mathcal{B}_{\bar{\mathbf{c}}}$. For proving that $\mathbf{h} \in T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]$, we construct a continuously differentiable function $\gamma : \mathbb{R} \to \mathbb{R}^r$ such that

$$\gamma(0) = \bar{\mathbf{c}}, \ \gamma'(0) = \mathbf{h}, \ \text{ and } \ \gamma(t) \in \mathcal{O}_{\mathbb{V}} \quad \text{for all } \ t \in \mathbb{R}. \tag{12}$$

By adapting a technique used in [6, section 3], we take as the ith component of γ a function of the form $\gamma_i(t) = \exp(tD)\bar{c}_i$, where $D : \mathbb{V} \to \mathbb{V}$ is given by

$$D := 2 \sum_{k=1}^{r} [L_{h_k}, L_{\bar{c}_k}]$$

and $\{\exp(tD)\}_{t \in \mathbb{R}}$ is the associated semigroup. A long and tedious computation shows that such a particular γ satisfies all the conditions listed in (12). We now prove (b). The linear system (7)–(8) is formed by $s_r := r + (1/2)r(r-1)$ equations. So, the linear subspace $\mathcal{B}_{\bar{\mathbf{c}}}$ corresponds to the kernel of a linear map $\mathcal{M} : \mathbb{V}^r \to \mathbb{V}^{s_r}$ whose definition is clear. Hence, the orthogonal complement of $\mathcal{B}_{\bar{\mathbf{c}}}$ is equal to the range of the adjoint map $\mathcal{M}^* : \mathbb{V}^{s_r} \to \mathbb{V}^r$. The explicit computation of this adjoint leads to the announced formula (10). The details are omitted. ∎

3. Proof of the general commutation principle. This section takes care of the proof of Theorem 2. In fact, without extra effort one can demonstrate a generalized version of Theorem 2 in which the linear functional $\langle a, \cdot \rangle$ is changed by a nonlinear differentiable function $\mathcal{E} : \mathbb{V} \to \mathbb{R}$.

THEOREM 7. *Let* $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ *be an EJA and let* $b \in \mathbb{V}$ *be a point at which* $\mathcal{E} : \mathbb{V} \to \mathbb{R}$ *is continuously differentiable. Suppose that* $\Omega \subseteq \mathbb{V}$ *is a spectral set and that* $\Phi : \mathbb{V} \to \mathbb{R}$ *is a spectral function. Under these assumptions, if* b *is a local extremum of*

$$x \in \Omega \mapsto F(x) = \mathcal{E}(x) + \Phi(x), \tag{13}$$

then b *and* $\nabla \mathcal{E}(b)$ *operator commute.*

Proof. Suppose that b is a local minimum of (13); the case of a local maximum can be treated in a similar way. One has $b \in \Omega$ and

$$\mathcal{E}(b) + \Phi(b) \leq \mathcal{E}(x) + \Phi(x) \quad \text{for all } x \in \Omega \cap \mathcal{N}_b,$$

where $\mathcal{N}_b$ is some neighborhood of b. Let $\bar{\mathbf{c}} \in \mathcal{O}_{\mathbb{V}}$ be an ordered Jordan frame such that $b = \sum_{i=1}^r \bar{\lambda}_i \bar{c}_i$ with $\bar{\lambda}_i = \lambda_i(b)$. Consider the linear function $\Gamma : \mathbb{V}^r \to \mathbb{V}$ defined by $\Gamma(\mathbf{c}) = \sum_{i=1}^r \bar{\lambda}_i c_i$. Let $x := \Gamma(\mathbf{c})$ with $\mathbf{c} \in \mathbb{V}^r$. If $\mathbf{c}$ is taken in a small neighborhood $\mathcal{N}_{\bar{\mathbf{c}}}$ of $\bar{\mathbf{c}}$, then $x \in \mathcal{N}_b$ by the continuity of Γ. On the other hand, if $\mathbf{c}$ belongs to $\mathcal{O}_{\mathbb{V}}$, then $\lambda(x) = \lambda(b)$ and, a posteriori, $x \in \Omega$. Hence,

$$\mathcal{E}\left(\sum_{i=1}^r \bar{\lambda}_i \bar{c}_i\right) + \Phi\left(\sum_{i=1}^r \bar{\lambda}_i \bar{c}_i\right) \le \mathcal{E}\left(\sum_{i=1}^r \bar{\lambda}_i c_i\right) + \Phi\left(\sum_{i=1}^r \bar{\lambda}_i c_i\right) \quad \text{for all } \mathbf{c} \in \mathcal{O}_{\mathbb{V}} \cap \mathcal{N}_{\bar{\mathbf{c}}}.$$

The spectrality of Φ leads to the simpler inequality

$$\mathcal{E}\left(\sum_{i=1}^r \bar{\lambda}_i \bar{c}_i\right) \le \mathcal{E}\left(\sum_{i=1}^r \bar{\lambda}_i c_i\right) \quad \text{for all } \mathbf{c} \in \mathcal{O}_{\mathbb{V}} \cap \mathcal{N}_{\bar{\mathbf{c}}}.$$

We have shown in this way that $\bar{\mathbf{c}}$ is a local minimum on $\mathcal{O}_{\mathbb{V}}$ of the function

$$\mathbf{c} \in \mathbb{V}^r \mapsto f(\mathbf{c}) = \mathcal{E}\left(\sum_{i=1}^r \bar{\lambda}_i c_i\right).$$

Note that f is differentiable at $\bar{\mathbf{c}}$ because $\mathcal{E}$ is differentiable at b. The optimality condition (6) takes the particular form

$$\underbrace{(\bar{\lambda}_1 a, \ldots, \bar{\lambda}_r a)}_{\nabla f(\bar{\mathbf{c}})} \in \left(T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]\right)^{\perp},$$

where $a := \nabla \mathcal{E}(b)$. In view of the characterization (10) of the subspace $\left(T_{\bar{\mathbf{c}}}[\mathcal{O}_{\mathbb{V}}]\right)^{\perp}$, one gets

$$\bar{\lambda}_i a = 2\alpha_i \circ \bar{c}_i - \alpha_i + \sum_{j=1}^{i-1} \beta_{j,i} \circ \bar{c}_j + \sum_{j=i+1}^{r} \beta_{i,j} \circ \bar{c}_j \quad \text{for all } i \in N_r \tag{14}$$

for suitable vectors α_i and $\beta_{i,j}$ as in (9). With this information at hand, we are now ready to show that a and b operator commute. One has

$$\begin{aligned}[L_a, L_b] &= L_a\left(\sum_{i=1}^r \bar{\lambda}_i L_{\bar{c}_i}\right) - \left(\sum_{i=1}^r \bar{\lambda}_i L_{\bar{c}_i}\right) L_a \\ &= \left(\sum_{i=1}^r L_{\bar{\lambda}_i a} L_{\bar{c}_i}\right) - \left(\sum_{i=1}^r L_{\bar{c}_i} L_{\bar{\lambda}_i a}\right) = \sum_{i=1}^r [L_{\bar{\lambda}_i a}, L_{\bar{c}_i}].\end{aligned}$$

But (14) implies that $L_{\bar{\lambda}_i a} = 2L_{\alpha_i \circ \bar{c}_i} - L_{\alpha_i} + L_{\nu_i}$ with

$$\nu_i := \sum_{j=1}^{i-1} \beta_{j,i} \circ \bar{c}_j + \sum_{j=i+1}^{r} \beta_{i,j} \circ \bar{c}_j.$$

Hence

$$[L_a, L_b] = \underbrace{\sum_{i=1}^r \left\{2[L_{\alpha_i \circ \bar{c}_i}, L_{\bar{c}_i}] - [L_{\alpha_i}, L_{\bar{c}_i}]\right\}}_{\Delta_1} + \underbrace{\sum_{i=1}^r [L_{\nu_i}, L_{\bar{c}_i}]}_{\Delta_2}.$$

We claim that Δ_1 is the zero map on $\mathbb{V}$. Indeed, $2[L_{\alpha_i \circ \bar{c}_i}, L_{\bar{c}_i}] = [L_{\alpha_i}, L_{\bar{c}_i}]$ for all $i \in N_r$, as one can see from the general identity (cf. [4, Proposition II.1.1])

$$2[L_{u \circ z}, L_z] = [L_u, L_{z^2}] \quad \text{for all } \ u, z \in \mathbb{V}$$

and the fact that $\bar{c}_i$ is idempotent. Also Δ_2 is the zero map on $\mathbb{V}$. To see this, we write

$$\begin{aligned}\Delta_2 &= \sum_{i=1}^{r} \left[\sum_{j=1}^{i-1} L_{\beta_{j,i} \circ \bar{c}_j} + \sum_{j=i+1}^{r} L_{\beta_{i,j} \circ \bar{c}_j}, L_{\bar{c}_i} \right] \\ &= \sum_{i=1}^{r} \left\{ \sum_{j=1}^{i-1} \left[L_{\beta_{j,i} \circ \bar{c}_j}, L_{\bar{c}_i} \right] + \sum_{j=i+1}^{r} \left[L_{\beta_{i,j} \circ \bar{c}_j}, L_{\bar{c}_i} \right] \right\} \\ &= \sum_{(i,j) \in M_r} \left\{ \left[L_{\beta_{i,j} \circ \bar{c}_i}, L_{\bar{c}_j} \right] + \left[L_{\beta_{i,j} \circ \bar{c}_j}, L_{\bar{c}_i} \right] \right\}\end{aligned}$$

and observe that $\left[L_{\beta_{i,j} \circ \bar{c}_i}, L_{\bar{c}_j} \right] + \left[L_{\beta_{i,j} \circ \bar{c}_j}, L_{\bar{c}_i} \right] = 0$. The above equality follows from the general identity (cf. [4, Proposition II.1.1])

$$[L_{u \circ y}, L_z] + [L_{u \circ z}, L_y] = [L_u, L_{zy}] \quad \text{for all } \ u, y, z \in \mathbb{V}$$

and the fact that $\bar{c}_i \circ \bar{c}_j = 0$ for all $(i, j) \in M_r$. This shows that a and b operator commute, finishing the proof. ∎

4. Applications. Two simple but illuminating examples suffice to illustrate how the commutation principle works in practice. The first example concerns a variational inequality of the form

$$\left\{ \begin{array}{l} \langle G(x), u - x \rangle + \Phi(u) - \Phi(x) \geq 0 \ \text{ for all } \ u \in \Omega, \\ x \in \Omega, \end{array} \right. \tag{15}$$

where $G : \mathbb{V} \to \mathbb{V}$ is an arbitrary function.

PROPOSITION 8. *Let* $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ *be an EJA. Suppose that* $\Omega \subseteq \mathbb{V}$ *is a spectral set and that* $\Phi : \mathbb{V} \to \mathbb{R}$ *is a spectral function. Under these assumptions, if* b *is a solution to the variational inequality* (15)*, then* b *and* $G(b)$ *operator commute.*

Proof. If b solves (15), then $b \in \Omega$ and

$$\langle G(b), u \rangle + \Phi(u) \geq \langle G(b), b \rangle + \Phi(b) \quad \text{for all } \ u \in \Omega.$$

Hence, b minimizes the function $\langle G(b), \cdot \rangle + \Phi(\cdot)$ on Ω. Theorem 2 leads to the announced conclusion. ∎

The second example is bit more involved. It concerns a formula for computing the distance

$$\text{dist}[a, \Omega] := \inf_{x \in \Omega} \|a - x\| \tag{16}$$

from a point $a \in \mathbb{V}$ to a spectral set $\Omega \subseteq \mathbb{V}$. We need first to introduce some notation. The trace operator on an EJA $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ with rank r is the function

$$u \in \mathbb{V} \mapsto \text{Tr}(u) = \sum_{i=1}^{r} \lambda_i(u).$$

An EJA is said to be *scalarizable* if there exists a positive constant θ such that

$$\langle x, y\rangle = \theta \operatorname{Tr}(x \circ y) \quad \text{for all } x, y \in \mathbb{V}. \tag{17}$$

Such a constant is unique and given by $\theta = \langle e, e\rangle / \operatorname{Tr}(e)$. It is called the scaling factor of the EJA. As a consequence of the scalarization property (17) one gets

$$\|x\| = \sqrt{\theta}\, \|\lambda(x)\|_2 \quad \text{for all } x \in \mathbb{V}, \tag{18}$$

where $\|\cdot\|_2$ is the usual Euclidean norm on $\mathbb{R}^r$.

Remark 9. If an EJA is *simple* in the sense that it does not contain any nontrivial ideal, then it is scalarizable; see [4, Proposition III.4.1].

The next proposition is a generalization of [3, Proposition 2.3].

PROPOSITION 10. *Let* $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ *be an EJA with rank* r *and scaling factor* θ. *Let* $\Omega = \lambda^{-1}(Q)$ *with* $Q \subseteq \mathbb{R}^r$ *permutation invariant. Then for all* $a \in \mathbb{V}$ *one has*

$$\frac{1}{\sqrt{\theta}} \operatorname{dist}[a, \Omega] = \inf_{\mu \in Q} \|\lambda(a) - \mu\|_2. \tag{19}$$

In particular, $\operatorname{dist}[\,\cdot\,, \Omega]$ *is a spectral function.*

Proof. The topological closure of Q is permutation invariant and $\operatorname{cl}(\Omega) = \lambda^{-1}(\operatorname{cl}(Q))$. Hence, $\operatorname{cl}(\Omega)$ is a spectral set. Since a distance function is blind with respect to topological closure, there is no loss of generality in assuming that Q is already closed (in which case Ω is also closed). Let b be a solution to (16). Then b is a global maximum of

$$x \in \Omega \mapsto F(x) = \langle a, x\rangle - \frac{1}{2}\|x\|^2.$$

But (18) implies that $-(1/2)\|\cdot\|^2$ is a spectral function on $\mathbb{V}$. Hence, a and b operator commute by Theorem 2. Thanks to Theorem 5, there exist $\mathbf{c} \in \mathcal{O}_{\mathbb{V}}$ and $\bar{\mu} \in \mathbb{R}^r$ such that

$$a = \sum_{i=1}^{r} \lambda_i(a) c_i \quad \text{and} \quad b = \sum_{i=1}^{r} \bar{\mu}_i c_i.$$

Clearly,

$$\operatorname{dist}[a, \Omega] = \|a - b\| = \sqrt{\theta}\, \|\lambda(a) - \bar{\mu}\|_2.$$

We claim that $\bar{\mu}$ solves the minimization problem on the right-hand side of (19). Up to a permutation, the vector $\bar{\mu}$ is equal to $\lambda(b)$. Hence, $\bar{\mu} \in Q$. Suppose that there exists $\tilde{\mu} \in Q$ such that $\|\lambda(a) - \tilde{\mu}\|_2 < \|\lambda(a) - \bar{\mu}\|_2$. In such a case $\tilde{x} = \sum_{i=1}^{r} \tilde{\mu}_i c_i$ belongs to Ω and

$$\|a - \tilde{x}\| = \sqrt{\theta}\, \|\lambda(a) - \tilde{\mu}\|_2 < \sqrt{\theta}\, \|\lambda(a) - \bar{\mu}\|_2 = \operatorname{dist}[a, \Omega],$$

a clear contradiction. ∎

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