Concentrating standing waves for the fractional nonlinear Schrödinger equation

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Abstract
We consider the semilinear equation

$$\epsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N)$$

where $0 < s < 1$, $1 < p < \frac{N+2s}{N-2s}$, $V(x)$ is a sufficiently smooth potential with $\inf_{\mathbb{R}} V(x) > 0$, and $\epsilon > 0$ is a small number. Letting $w_\lambda^s$ be the radial ground state of $(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0$ in $H^{2s}(\mathbb{R}^N)$, we build solutions of the form

$$u_\epsilon(x) \sim \sum_{i=1}^{k} w_{\lambda_i} \left( (x - \xi_i^\epsilon) / \epsilon \right),$$

where $\lambda_i = V(\xi_i^\epsilon)$ and the $\xi_i^\epsilon$ approach suitable critical points of $V$. Via a Lyapunov–Schmidt variational reduction, we recover various existence results already known for the case $s = 1$. In particular such a solution exists around $k$ nondegenerate critical points of $V$. For $s = 1$ this corresponds to the classical results by Floer and Weinstein [13] and Oh [24,25].

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1. Introduction and main results

We consider the fractional nonlinear Schrödinger equation

\[ i\hbar \psi_t = \hbar^{2s}(-\Delta)^s \psi + W(x)\psi - |\psi|^{p-1}\psi \]  

(1.1)

where \((-\Delta)^s\), \(0 < s < 1\), denotes the usual fractional Laplace operator, \(W(x)\) is a bounded potential, \(p > 1\) and \(\hbar\) designates the usual Planck constant. Eq. (1.1) was introduced by Laskin [19] as an extension of the classical nonlinear Schrödinger equation \(s = 1\) in which the Brownian motion of the quantum paths is replaced by a Lévy flight. Here \(\psi = \psi(x,t)\) represents the quantum mechanical probability amplitude for a given unit-mass particle to have position \(x\) at time \(t\) (the corresponding probability density is \(|\psi|^2\)), under a confinement due to the potential \(W\). We refer to [19–21] for detailed physical discussions and motivation of Eq. (1.1).

We are interested in the semi-classical limit regime, \(0 < \varepsilon := \hbar \ll 1\). For small values of \(\varepsilon\) the wave function tends to concentrate as a material particle.

Our purpose is to find standing-wave solutions of (1.1), which are those of the form \(\psi(x,t) = u(x)e^{iEt/\varepsilon}\) with \(u(x)\) a real-valued function. Letting \(V(x) = W(x) + E\), Eq. (1.1) becomes

\[ \varepsilon^{2s}(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N. \]  

(1.2)

We assume in what follows that \(V\) satisfies

\[ V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \inf_{\mathbb{R}^N} V(x) > 0. \]  

(1.3)

We are interested in finding solutions with a spike pattern concentrating around a finite number of points in space as \(\varepsilon \to 0\). This has been the topic of many works in the standard case \(s = 1\), relating the concentration points with critical points of the potential, starting in 1986 with the pioneering work by Floer and Weinstein [13], then continued by Oh [24,25]. The natural place to look for solutions to (1.2) that decay at infinity is the space \(H^{2s}(\mathbb{R}^N)\), of all functions \(u \in L^2(\mathbb{R}^N)\) such that

\[ \int_{\mathbb{R}^N} \left(1 + |\xi|^{4s}\right)\left|\hat{u}(\xi)\right|^2d\xi < +\infty, \]

where \(\hat{\cdot}\) denotes Fourier transform. The fractional Laplacian \((-\Delta)^s u\) of a function \(u \in H^{2s}(\mathbb{R}^N)\) is defined in terms of its Fourier transform by the relation

\[ (-\Delta)^s u = |\xi|^{2s}\hat{u} \in L^2(\mathbb{R}^N). \]

We will explain next what we mean by a spike pattern solution of Eq. (1.2). Let us consider the basic problem

\[ (-\Delta)^s v + v - |v|^{p-1}v = 0, \quad v \in H^{2s}(\mathbb{R}^N). \]  

(1.4)
We assume the following constraint in $p$,\[1 < p < \begin{cases} \frac{N+2s}{N-2s} & \text{if } 2s < N, \\ +\infty & \text{if } 2s \geq N. \end{cases}\text{(1.5)}\]Under this condition it is known the existence of a positive, radial least energy solution $v = w(x)$, which gives the lowest possible value for the energy\[J(v) = \frac{1}{2} \int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{1}{2} \int_{\mathbb{R}^N} v^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |v|^{p+1},\]among all nontrivial solutions of (1.4). An important property, which has only been proven recently by Frank, Lenzmann and Silvestre [15] (see also [2,14]), is that there exists a radial least energy solution which is nondegenerate, in the sense that the space of solutions of the equation\[(-\Delta)^s \phi + \phi - pw^{p-1} \phi = 0, \quad \phi \in H^{2s}(\mathbb{R}^n)\text{(1.6)}\]consists of exactly of the linear combinations of the translation-generators, $\partial w/\partial x_j$, $j = 1, \ldots, N$. It is easy to see that the function\[w_\lambda(x) := \frac{1}{\lambda^{p-1}} w\left(\lambda^{\frac{1}{p}} x\right)\]satisfies the equation\[(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0 \quad \text{in } \mathbb{R}^N.\]Therefore for any point $\xi \in \mathbb{R}^N$, taking $\lambda = V(\xi)$, the spike-shape function\[u(x) = w_{V(\xi)}\left(\frac{x - \xi}{\varepsilon}\right)\text{(1.7)}\]satisfies\[\varepsilon^{2s} (-\Delta)^s u + V(\xi) u - u^p = 0.\]Since the $\varepsilon$-scaling makes it concentrate around $\xi$, this function constitutes a good positive approximate solution to Eq. (1.2), namely of\[\varepsilon^{2s} (-\Delta)^s u + V(x) u - u^p = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N).\text{(1.8)}\]We call a $k$-spike pattern solution of (1.8) one that looks approximately like a superposition of $k$ spikes like (1.7), namely a solution $u_\varepsilon$ of the form\[u_\varepsilon(x) = \sum_{i=1}^{k} w_{V(\xi_i^\varepsilon)}\left(\frac{x - \xi_i^\varepsilon}{\varepsilon}\right) + o(1)\text{(1.9)}\]for points $\xi_1^\varepsilon, \ldots, \xi_k^\varepsilon$, where $o(1) \to 0$ in $H^{2s}(\mathbb{R}^N)$ as $\varepsilon \to 0$. 
In what follows we assume that $p$ satisfies condition (1.5) and $V$ condition (1.3).

Our first result concerns the existence of multiple spike solution at separate places in the case of stable critical points.

**Theorem 1.** Let $\Lambda_i \subset \mathbb{R}^N$, $i = 1, \ldots, k \geq 1$ be disjoint bounded open sets in $\mathbb{R}^N$. Assume that

$$\deg(\nabla V, \Lambda_i, 0) \neq 0 \text{ for all } i = 1, \ldots, k.$$ 

Then for all sufficiently small $\epsilon$, problem (1.8) has a solution of the form (1.9) where $\xi^\epsilon_i \in \Lambda_i$ and

$$\nabla V(\xi^\epsilon_i) \to 0 \text{ as } \epsilon \to 0.$$ 

An immediate consequence of Theorem 1 is the following.

**Corollary 1.1.** Assume that $V$ is of class $C^2$. Let $\xi^0_1, \ldots, \xi^0_k$ be $k$ nondegenerate critical points of $V$, namely

$$\nabla V(\xi^0_i) = 0, \quad D^2 V(\xi^0_i) \text{ is invertible for all } i = 1, \ldots, k.$$ 

Then, a $k$-spike solution of (1.8) of the form (1.9) with $\xi^\epsilon_i \to \xi^0_i$ exists.

When $s = 1$, the result of Corollary 1.1 is due to Floer and Weinstein [13] for $N = 1$ and $k = 1$ and to Oh [24,25] when $N \geq 1, k \geq 1$. Theorem 1 for $s = 1$ was proven by Yanyan Li [22].

**Remark 1.1.** As the proof will yield, Theorem 1 for $0 < s < 1$ holds true under the following, more general condition introduced in [22]. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_k$ and assume that the function

$$\varphi(\xi_1, \ldots, \xi_k) = \sum_{i=1}^{k} V(\xi_i)^\theta, \quad \theta = \frac{p + 1}{p - 1} - \frac{N}{2s} > 0$$

has a stable critical point situation in $\Lambda$: there is a number $\delta_0 > 0$ such that for each $g \in C^1(\bar{\Lambda})$ with $\|g\|_{L^\infty(\Lambda)} + \|
abla g\|_{L^\infty(\Lambda)} < \delta_0$, there is a $\xi_g \in \Lambda$ such that $\nabla \varphi(\xi_g) + \nabla g(\xi_g) = 0$. Then for all sufficiently small $\epsilon$, problem (1.8) has a solution of the form (1.9) where $\xi^\epsilon = (\xi^\epsilon_1, \ldots, \xi^\epsilon_k) \in \Lambda$ and $\nabla \varphi(\xi^\epsilon) \to 0$ as $\epsilon \to 0$.

**Theorem 2.** Let $\Lambda$ be a bounded, open set with smooth boundary such that $V$ is such that either

$$c = \inf_{\Lambda} V < \inf_{\partial \Lambda} V$$

or

$$c = \sup_{\Lambda} V > \sup_{\partial \Lambda} V$$

or, there exist closed sets $B_0 \subset B \subset \Lambda$ such that

$$c = \inf_{\Phi \in \Gamma} \sup_{x \in B} V(\Phi(x)) > \sup_{B_0} V.$$
where \( \Gamma = \{ \Phi \in C(\bar{B}, \Lambda)/ \Phi|_{B_0} = 1d \} \) and \( \nabla V(x) \cdot \tau \neq 0 \) for all \( x \in \partial \Lambda \) with \( V(x) = c \) and some tangent vector \( \tau \) to \( \partial \Lambda \) at \( x \).

Then, there exists a 1-spike solution of (1.8) with \( \xi^\varepsilon \in \Lambda \) with \( \nabla V(\xi^\varepsilon) \to 0 \) and \( V(\xi^\varepsilon_i) \to c \).

In the case \( s = 1 \), the above results were found by del Pino and Felmer [7,8]. The case of a (possibly degenerate) global minimizer was previously considered by Rabinowitz [26] and X. Wang [28]. An isolated maximum with a power type degeneracy appears in Ambrosetti, Badiale and Cingolani [1]. Condition (1.12) is called a nontrivial linking situation for \( V \). The cases of \( k \) disjoint sets where (1.11) holds was treated in [9,17]. Multiple spikes for disjoint nontrivial linking regions were first considered in [10], see also [5,16] for other multiplicity results.

Our last result concerns the existence of multiple spikes at the same point.

**Theorem 3.** Let \( \Lambda \) be a bounded, open set with smooth boundary such that \( V \) is such that

\[
\sup_{\Lambda} V > \sup_{\partial \Lambda} V.
\]

Then for any positive integer \( k \) there exists a \( k \)-spike solution of (1.8) with spikes \( \xi^\varepsilon_j \in \Lambda \) satisfying \( V(\xi^\varepsilon_j) \to \max_{\Lambda} V \).

In the case \( s = 1 \), Theorem 3 was proved by Kang and Wei [18]. D’Aprile and Ruiz [6] have found a phenomenon of this type at a saddle point of \( V \).

The rest of this paper will be devoted to the proofs of Theorems 1–3. The method of construction of a \( k \)-spike solution consists of a Lyapunov–Schmidt reduction in which the full problem is reduced to that of finding a critical point \( \xi^\varepsilon \) of a functional which is a small \( C^1 \)-perturbation of \( \varphi \) in (1.10). In this reduction the nondegeneracy result in [15] is a key ingredient.

After this has been done, the results follow directly from standard degree theoretical or variational arguments. The Lyapunov–Schmidt reduction is a method widely used in elliptic singular perturbation problems. Some results of variational type for \( 0 < s < 1 \) have been obtained for instance in [12] and [27]. We believe that the scheme of this paper may be generalized to concentration on higher dimensional regions, while that could be much more challenging. See [11, 23] for concentration along a curve in the plane and \( s = 1 \).

2. Generalities

Let \( 0 < s < 1 \). Various definitions of the fractional Laplacian \( (-\Delta)^s \phi \) of a function \( \phi \) defined in \( \mathbb{R}^N \) are available, depending on its regularity and growth properties.

As we have recalled in the introduction, for \( \phi \in H^{2s}(\mathbb{R}^N) \) the standard definition is given via Fourier transform \( \hat{\cdot} \). \( (-\Delta)^s \phi \in L^2(\mathbb{R}^N) \) is defined by the formula

\[
|\xi|^{2s} \hat{\phi}(\xi) = (-\Delta)^s \phi.
\]

When \( \phi \) is assumed in addition sufficiently regular, we obtain the direct representation

\[
(-\Delta)^s \phi(x) = d_{s,N} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^{N + 2s}} \, dy
\]

(2.2)
for a suitable constant $d_{s,N}$ and the integral is understood in a principal value sense. This integral makes sense directly when $s < \frac{1}{2}$ and $\phi \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha > 2s$, or if $\phi \in C^{1,\alpha}(\mathbb{R}^N)$, $1 + \alpha > 2s$. In the latter case, we can desingularize the integral representing it in the form

$$(-\Delta)^s \phi(x) = d_{s,N} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y) - \nabla \phi(x)(x - y)}{|x - y|^{N+2s}} dy.$$  

Another useful (local) representation, found by Caffarelli and Silvestre [3], is via the following boundary value problem in the half space $\mathbb{R}^{N+1}_+ = \{(x, y) / x \in \mathbb{R}^N, y > 0\}$:

$$\begin{cases}
\nabla \cdot (y^{1-2s} \nabla \tilde{\phi}) = 0 & \text{in } \mathbb{R}^{N+1}_+,

\tilde{\phi}(x, 0) = \phi(x) & \text{on } \mathbb{R}^N.
\end{cases}$$

Here $\tilde{\phi}$ is the $s$-harmonic extension of $\phi$, explicitly given as a convolution integral with the $s$-Poisson kernel $p_s(x, y)$,

$$\tilde{\phi}(x, y) = \int_{\mathbb{R}^N} p_s(x - z, y) \phi(z) dz,$$

where

$$p_s(x, y) = c_{N,s} \frac{y^{4s-1}}{(|x|^2 + |y|^2) \frac{N-1+4s}{2}}$$

and $c_{N,s}$ achieves $\int_{\mathbb{R}^N} p(x, y) dx = 1$. Then under suitable regularity, $(-\Delta)^s \phi$ is the Dirichlet-to-Neumann map for this problem, namely

$$(-\Delta)^s \phi(x) = \lim_{y \to 0^+} y^{1-2s} \partial_y \tilde{\phi}(x, y).$$  

(2.3)

Characterizations (2.1), (2.2), (2.3) are all equivalent for instance in Schwartz’s space of rapidly decreasing smooth functions.

Let us consider now for a number $m > 0$ and $g \in L^2(\mathbb{R}^N)$ the equation

$$(-\Delta)^s \phi + m \phi = g \quad \text{in } \mathbb{R}^N.$$  

Then in terms of Fourier transform, this problem, for $\phi \in L^2$, reads

$$(|\xi|^{2s} + m) \hat{\phi} = \hat{g}$$

and has a unique solution $\phi \in H^{2s}(\mathbb{R}^N)$ given by the convolution

$$\phi(x) = T_m[g] := \int_{\mathbb{R}^N} k(x - z) g(z) dz,$$  

(2.4)
where
\[ \hat{k}(\xi) = \frac{1}{|\xi|^{2s} + m}. \]

Using the characterization (2.3) written in weak form, \( \phi \) can then be characterized by \( \phi(x) = \tilde{\phi}(x, 0) \) in trace sense, where \( \tilde{\phi} \in H \) is the unique solution of
\[ \int \int_{\mathbb{R}^N + 1} \nabla \tilde{\phi} \nabla \varphi \ y^{1 - 2s} + m \int_{\mathbb{R}^N} \varphi \phi = \int_{\mathbb{R}^N} g \varphi, \quad \text{for all } \varphi \in H, \tag{2.5} \]

where \( H \) is the Hilbert space of functions \( \varphi \in H_{loc}^1(\mathbb{R}^N + 1) \) such that
\[ \| \varphi \|_H := \int \int_{\mathbb{R}^N + 1} |\nabla \varphi|^2 \ y^{1 - 2s} + m \int_{\mathbb{R}^N} |\varphi|^2 < +\infty, \]
or equivalently the closure of the set of all functions in \( C^\infty_c(\mathbb{R}^N + 1) \) under this norm.

A useful fact for our purposes is the equivalence of the representations (2.4) and (2.5) for \( g \in L^2(\mathbb{R}^N) \).

**Lemma 2.1.** Let \( g \in L^2(\mathbb{R}^N) \). Then the unique solution \( \tilde{\phi} \in H \) of problem (2.5) is given by the \( s \)-harmonic extension of the function \( \phi = T_m[g] = k * g \).

**Proof.** Let us assume first that \( \hat{g} \in C^\infty_c(\mathbb{R}^N) \). Then \( \phi \) given by (2.4) belongs to \( H^{2s}(\mathbb{R}^N) \). Take a test function \( \psi \in C^\infty_c(\mathbb{R}^N + 1) \). Then the well-known computation by Caffarelli and Silvestre shows that
\[ \int \int_{\mathbb{R}^N + 1} \nabla \tilde{\phi} \nabla \psi \ y^{1 - 2s} \ dx \ dy = \int_{\mathbb{R}^N} \lim_{y \to 0} \ y^{1 - 2s} \partial_y \tilde{\phi}(y, \cdot) \ \psi \ dx \]
\[ = \int_{\mathbb{R}^N} \psi (-\Delta)^s \phi \ dx = \int_{\mathbb{R}^N} (g - m \phi) \ dx. \]

By taking \( \psi = \tilde{\phi} \eta_R \) for a suitable sequence of smooth cut-off functions equal to one on expanding balls \( B_R(0) \) in \( \mathbb{R}^N + 1 \), and using the behavior at infinity of \( \tilde{\phi} \) which resembles the Poisson kernel \( p_*(x, y) \), we obtain
\[ \int \int_{\mathbb{R}^N + 1} |\nabla \tilde{\phi}|^2 \ y^{1 - 2s} \ dx \ dy + m \int_{\mathbb{R}^N} |\phi|^2 = \int_{\mathbb{R}^N} g \phi \]
and hence \( \| \tilde{\phi} \|_H \leq C \| g \|_{L^2} \) and satisfies (2.5). By density, this fact extends to all \( g \in L^2(\mathbb{R}^N) \). The result follows since the solution of problem (2.5) in \( H \) is unique. \( \square \)
Let us recall the main properties of the fundamental solution $k(x)$ in the representation (2.4), which are stated for instance in [15] or in [12].

We have that $k$ is radially symmetric and positive, $k \in C^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying

- $|k(x)| + |x| |\nabla k(x)| \leq \frac{C}{|x|^{N-2s}}$ for all $|x| \leq 1$,
- $\lim_{|x| \to \infty} k(x)|x|^{N+2s} = \gamma > 0$, and
- $|x| |\nabla k(x)| \leq \frac{C}{|x|^{N+2s}}$ for all $|x| \geq 1$.

The operator $T_m$ is not just defined on functions in $L^2$. For instance it acts nicely on bounded functions. The positive kernel $k$ satisfies $\int_{\mathbb{R}^N} k = \frac{1}{m}$. We see that if $g \in L^\infty(\mathbb{R}^N)$ then

$$\| T_m[g] \|_\infty \leq \frac{1}{m} \| g \|_\infty.$$  

We have indeed the validity of an estimate like this for $L^\infty$ weighted norms as follows.

**Lemma 2.2.** Let $0 \leq \mu < N + 2s$. Then there exists a $C > 0$ such that

$$\| (1 + |x|)^\mu T_m[g] \|_{L^\infty(\mathbb{R}^N)} \leq C \| (1 + |x|)^\mu g \|_{L^\infty(\mathbb{R}^N)}.$$  

**Proof.** Let us assume that $0 \leq \mu < N + 2s$ and let $\tilde{g}(x) = \frac{1}{(1+|x|)^\mu}$. Then

$$T[\tilde{g}](x) = \int_{|y-x| < \frac{1}{2}|x|} \frac{k(y)}{(1 + |y-x|)^\mu} dy + \int_{|y-x| > \frac{1}{2}|x|} \frac{k(y)}{(1 + |y-x|)^\mu} dy.$$  

Then, as $|x| \to \infty$ we find

$$|x|^\mu \int_{|y-x| < \frac{1}{2}|x|} \frac{k(y)}{(1 + |y-x|)^\mu} dy \sim |x|^{-2s} \to 0,$$

and since $k \in L^1(\mathbb{R}^N)$, by dominated convergence we find that as $|x| \to \infty$

$$\int_{|x-y| > \frac{1}{2}|x|} \frac{k(y)|x|^\mu}{(1 + |y-x|)^\mu} dy \to \int_{\mathbb{R}^N} k(z) dz = \frac{1}{m}.$$  

We conclude in particular that for a suitable constant $C > 0$, we have

$$T_m[(1 + |x|)^{-\mu}] \leq C (1 + |x|)^{-\mu}.$$  

Now, we have that
\[ \pm T_m[g] \leq \left\| (1 + |x|)^\mu g \right\|_{L^\infty(\mathbb{R}^N)} T_m[(1 + |x|)^{-\mu}], \]
and then
\[ \left\| (1 + |x|)^\mu T[g] \right\|_{L^\infty(\mathbb{R}^N)} \leq C \left\| (1 + |x|)^\mu g \right\|_{L^\infty(\mathbb{R}^N)} \]
as desired. \qed

We also have the validity of the following useful estimate.

**Lemma 2.3.** Assume that \( g \in L^2 \cap L^\infty \). Then the following holds: if \( \phi = T_m[g] \) then there is a \( C > 0 \) such that

\[ \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha} \leq C \|g\|_{L^\infty(\mathbb{R}^N)} \tag{2.6} \]

where \( \alpha = \min\{1, 2s\} \).

**Proof.** Since \( \|T_m[g]\|_\infty \leq C \|g\|_\infty \), it suffices to establish (2.6) for \( |x - y| < \frac{1}{3} \). We have
\[ |\phi(x) - \phi(y)| \leq \int_{\mathbb{R}^N} |k(z + y - x) - k(z)| \ dz \|g\|_\infty. \]

Now, we decompose
\[ \int_{\mathbb{R}^N} |k(z + y - x) - k(z)| \ dz \]
\[ = \int_{|z| > 3|y - x|} |k(z + y - x) - k(z)| \ dz + \int_{|z| < 3|y - x|} |k(z + y - x) - k(z)| \ dz. \]

We have
\[ \int_{|z| > 3|y - x|} |k(z + (y - x)) - k(z)| \leq \int_0^1 dt \int_{|z| > 3|y - x|} |\nabla k(z + t(y - x))| \ dz |y - x|, \]
and, since \( 3|y - x| < 1, \)
\[ \int_{|z| > 3|y - x|} |\nabla k(z + t(y - x))| \ dz \leq C \left( 1 + \int_{1 > |z| > 3|y - x|} \frac{dz}{|z|^N + 1 - 2s} \right) \leq C(1 + |y - x|^{2s - 1}). \]
On the other hand
\[
\int_{|z|<3|y-x|} |k(z + y - x) - k(z)| \, dz \leq 2 \int_{|z|<4|y-x|} |k(z)| \, dz \leq C|y - x|^{2s},
\]
and (2.6) readily follows. □

Next we consider the more general problem
\[
(-\Delta)^s \phi + W(x)\phi = g \quad \text{in } \mathbb{R}^N \tag{2.7}
\]
where \(W\) is a bounded potential.

We start with a form of the weak maximum principle.

**Lemma 2.4.** Let us assume that
\[
\inf_{x \in \mathbb{R}^N} W(x) =: m > 0
\]
and that \(\phi \in H^{2s}(\mathbb{R}^N)\) satisfies Eq. (2.7) with \(g \geq 0\). Then \(\phi \geq 0\) in \(\mathbb{R}^N\).

**Proof.** We use the representation for \(\phi\) as the trace of the unique solution \(\tilde{\phi} \in H\) to the problem
\[
\iint_{\mathbb{R}_+^{N+1}} \nabla \tilde{\phi} \nabla \varphi y^{1-2s} + \int_{\mathbb{R}^N} W \varphi = \int_{\mathbb{R}^N} g \varphi, \quad \text{for all } \varphi \in H.
\]
It is easy to check that the test function \(\varphi = \phi_- = \min\{\phi, 0\}\) does indeed belong to \(H\). We readily obtain
\[
\iint_{\mathbb{R}_+^{N+1}} |\nabla \tilde{\phi}|^2 y^{1-2s} + \int_{\mathbb{R}^N} W \phi_-^2 = \int_{\mathbb{R}^N} g \phi_-.
\]
Since \(g \geq 0\) and \(W \geq m\), we obtain that \(\phi_- \equiv 0\), which means precisely \(\phi \geq 0\), as desired. □

We want to obtain a priori estimates for problems of the type (2.7) when \(W\) is not necessarily positive. Let \(\mu > \frac{N}{2}\), and let us assume that
\[
\left\| (1 + |x|^\mu)g \right\|_{L^\infty(\mathbb{R}^N)} < +\infty.
\]
The assumption in \(\mu\) implies that \(g \in L^2(\mathbb{R}^N)\).

Below, and in all what follows, we will say that \(\phi \in L^2(\mathbb{R}^N)\) solves Eq. (2.7) if and only if \(\phi\) solves the linear problem
\[
\phi = T_m((m - W)\phi + g).
\]
Similarly, we will say that
\[ (-\Delta)^s \phi + W(x) \phi \geq g \quad \text{in} \quad \mathbb{R}^N \]
if for some \( \tilde{g} \in L^2(\mathbb{R}^N) \) with \( \tilde{g} \geq g \) we have
\[ \phi = T_m((m - W) \phi + \tilde{g}). \]

The next lemma provides an a priori estimate for a solution \( \phi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) of (2.7).

**Lemma 2.5.** Let \( W \) be a continuous function, such that for \( k \) points \( q_i \ i = 1, \ldots, k \) a number \( R > 0 \) and \( B = \bigcup_{i=1}^k B_R(q_i) \) we have
\[ \inf_{x \in \mathbb{R}^N \setminus B} W(x) =: m > 0. \]

Then, given any number \( \frac{N}{2} < \mu < N + 2s \) there exists a constant \( C = C(\mu, k, R) > 0 \) such that for any \( \phi \in H^{2s} \cap L^\infty(\mathbb{R}^N) \) and \( g \) with
\[ \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} < +\infty \]
that satisfy Eq. (2.7) we have the validity of the estimate
\[ \| \rho^{-1} \phi \|_{L^\infty(\mathbb{R}^N)} \leq C \left( \| \phi \|_{L^\infty(B)} + \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \right). \]

Here
\[ \rho(x) = \sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu}. \]

**Proof.** We start by noticing that \( \phi \) satisfies the equation
\[ (-\Delta)^s \phi + \hat{W} \phi = \hat{g} \]
where
\[ \hat{g} = (m - W)\chi_B \phi, \quad \hat{W} = m \chi_B + W(1 - \chi_B). \]

Observe that
\[ |\hat{g}(x)| \leq M \sum_{i=1}^k (1 + |x - q_i|)^{-\mu}, \quad M = C \left( \| \phi \|_{L^\infty(B)} + \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \right) \]
where \( C \) depends only on \( R, k \) and \( \mu \) and
\[ \inf_{x \in \mathbb{R}^N} \hat{W}(x) \geq m. \]
Now, from Lemma 2.2, since $0 < \mu < N + 2s$ we find a solution $\phi_0(x)$ to the problem

$$(-\Delta)^s \tilde{\phi} + m\tilde{\phi} = (1 + |x|)^{-\mu}$$

such that $\tilde{\phi} = O(|x|^{-\mu})$ as $|x| \to \infty$. Then we have that

$$((-\Delta)^s + \tilde{W})(\tilde{\phi}) \geq M \sum_{i=1}^{k} (1 + |x - q_i|)^{-\mu}$$

where

$$\tilde{\phi}(x) = M \sum_{i=1}^{k} \phi_0(x - q_i).$$

Setting $\psi = (\phi - \tilde{\phi})$ we get

$$(-\Delta)^s \psi + \tilde{W} \psi = \tilde{g} \leq 0$$

with $\tilde{g} \in L^2$. Using Lemma 2.4 we obtain $\phi \leq \tilde{\phi}$. Arguing similarly for $-\phi$, and using the form of $\tilde{\phi}$ and $M$, the desired estimate immediately follows. □

Examining the proof above, we obtain immediately the following.

**Corollary 2.1.** Let $\rho(x)$ be defined as in the previous lemma. Assume that $\phi \in H^{2s}(\mathbb{R}^N)$ satisfies Eq. (2.7) and that

$$\inf_{x \in \mathbb{R}^N} W(x) =: m > 0.$$ 

Then we have that $\phi \in L^\infty(\mathbb{R}^N)$ and it satisfies

$$\|\rho^{-1} \phi\|_{L^\infty(\mathbb{R}^N)} \leq C \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)}. \quad (2.8)$$

A last useful fact is that if $f, g \in L^2(\mathbb{R}^N)$ and $W = T(f)$, $Z = T(g)$ then the following holds:

$$\int_{\mathbb{R}^N} Z(-\Delta)^s W - \int_{\mathbb{R}^N} W(-\Delta)^s Z = \int_{\mathbb{R}^N} T_m[f]g - \int_{\mathbb{R}^N} T_m[g]f = 0,$$

the latter fact since the kernel $k$ is radially symmetric.
3. Formulation of the problem: the ansatz

By a solution of the problem

\[ \varepsilon^{2s} (-\Delta)^s u + V(x) u - u^p = 0 \quad \text{in } \mathbb{R}^N \]

we mean a \( u \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) such that the above equation is satisfied. Let us observe that it suffices to solve

\[ \varepsilon^{2s} (-\Delta)^s u + V(x) u - u^p_+ = 0 \quad \text{in } \mathbb{R}^N \tag{3.1} \]

where \( u_+ = \max\{u, 0\} \). In fact, if \( u \) solves (3.1) then

\[ \varepsilon^{2s} (-\Delta)^s u + V(x) u \geq 0 \quad \text{in } \mathbb{R}^N \]

and, as a consequence to Lemma 2.4, \( u \geq 0 \).

After absorbing \( \varepsilon \) by scaling, the equation takes the form

\[ (-\Delta)^s v + V(\varepsilon x) v - v^p_+ = 0 \quad \text{in } \mathbb{R}^N. \tag{3.2} \]

Let us consider points \( \xi_1, \ldots, \xi_k \in \mathbb{R}^N \) and designate

\[ q_i = \varepsilon^{-1} \xi_i, \quad q = (q_1, \ldots, q_k). \]

Given numbers \( \delta > 0 \) small and \( R > 0 \) large, we define the configuration space \( \Gamma \) for the points \( q_i \) as

\[ \Gamma := \left\{ q = (q_1, \ldots, q_k) / R \leq \max_{i \neq j} |q_i - q_j|, \max_i |q_i| \leq \delta^{-1} \varepsilon^{-1} \right\}. \tag{3.3} \]

We look for a solution with concentration behavior near each \( \xi_j \). Letting \( \tilde{v}(x) = v(x + \xi_j) \) translating the origin to \( q_j \), Eq. (3.2) reads

\[ (-\Delta)^s \tilde{v} + V(\xi_j) v - \tilde{v}^p_+ = 0 \quad \text{in } \mathbb{R}^N. \]

Letting formally \( \varepsilon \to 0 \) we are left with the equation

\[ (-\Delta)^s \tilde{v} + \lambda_j \tilde{v} - \tilde{v}^p_+ = 0 \quad \text{in } \mathbb{R}^N, \quad \lambda_j = V(\xi_j). \]

So we ask that \( v(x) \approx w_{\lambda_j}(x - q_j) \) near \( q_j \). We consider the sum of these functions as a first approximation. Thus, we look for a solution \( v \) of (3.2) of the form

\[ v = W_q + \phi, \quad W_q(x) = \sum_{j=1}^k w_j(x), \quad w_j(x) = w_{\lambda_j}(x - q_j), \quad \lambda_j = V(\xi_j), \]
where \( \phi \) is a small function, disappearing as \( \epsilon \to 0 \). In terms of \( \phi \), Eq. (3.2) becomes
\[
(\Delta^s \phi + V(\epsilon x)\phi - pW_q^{p-1}\phi) = E + N(\phi) \quad \text{in } \mathbb{R}^N
\]
(3.4)

where
\[
N(\phi) := (W_q + \phi)_+^p - pW_q^{p-1}\phi - W_q^p,
\]
\[
E := \sum_{j=1}^k (\lambda_j - V(\epsilon x))w_j + \left( \sum_{j=1}^k w_j \right)^p - \sum_{j=1}^k w_j^p.
\]
(3.5)

Rather than solving problem (3.4) directly, we consider first a projected version of it. Let us consider the functions
\[
Z_{ij}(x) := \partial_j w_i(x)
\]
and the problem of finding \( \phi \in H^2_s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) such that for certain constants \( c_{ij} \)
\[
(\Delta^s \phi + V(\epsilon x)\phi - pW_q^{p-1}\phi + g(x) = E + N(\phi) + \sum_{i=1}^k \sum_{j=1}^N c_{ij}Z_{ij},
\]
(3.6)
\[
\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j.
\]
(3.7)

Let \( Z \) be the linear space spanned by the functions \( Z_{ij} \), so that Eq. (3.6) is equivalent to
\[
(\Delta^s \phi + V(\epsilon x)\phi - pW_q^{p-1}\phi - E - N(\phi) \in Z.
\]

On the other hand, for all \( \epsilon \) sufficiently small, the functions \( Z_{ij} \) are linearly independent, hence the constants \( c_{ij} \) have unique, computable expressions in terms of \( \phi \). We will prove that problem (3.6)–(3.7) has a unique small solution \( \phi = \Phi(q) \). In that way we will get a solution to the full problem (3.4) if we can find a value of \( q \) such that \( c_{ij}(\Phi(q)) = 0 \) for all \( i, j \). In order to build \( \Phi(q) \) we need a theory of solvability for associated linear operator in suitable spaces. This is what we develop in the next section.

4. Linear theory

We consider the linear problem of finding \( \phi \in H^2_2(\mathbb{R}^N) \) such that for certain constants \( c_{ij} \) we have
\[
(\Delta^s \phi + V(\epsilon x)\phi - pW_q^{p-1}(x)\phi + g(x) = \sum_{i=1}^N \sum_{i=1}^k c_{ij}Z_{ij},
\]
(4.1)
\[
\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j.
\]
(4.2)
The constants $c_{ij}$ are uniquely determined in terms of $\phi$ and $g$ when $\epsilon$ is sufficiently small, from the linear system

$$
\sum_{i,j} c_{ij} \int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \int_{\mathbb{R}^N} Z_{lk} \left[ (-\Delta)^s \phi + V(\epsilon x) \phi - p W_{q}^{p-1}(x) \phi + g \right].
$$

(4.3)

Taking into account that

$$
\int_{\mathbb{R}^N} Z_{lk} (-\Delta)^s \phi = \int_{\mathbb{R}^N} \phi (-\Delta)^s Z_{lk} = \int_{\mathbb{R}^N} (p w_i^{p-1} - \lambda_l) Z_{lk} \phi,
$$

we find

$$
c_{ij} \int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \int_{\mathbb{R}^N} g Z_{lk} + (p w_i^{p-1} - p W_{q}^{p-1} + V(\epsilon x) - \lambda_l) Z_{lk} \phi.
$$

(4.4)

On the other hand, we check that

$$
\int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \alpha_l \delta_{ijkl} + O \left( d^{-N} \right)
$$

where the numbers $\alpha_l$ are positive, and independent of $\epsilon$, and

$$
d = \min \{|q_i - q_j| / i \neq j\} \gg 1.
$$

Then, we see that relations (4.4) define a uniquely solvable (nearly diagonal) linear system, provided that $\epsilon$ is sufficiently small. We assume this last fact in what follows, and hence that the numbers $c_{ij} = c_{ij}(\phi, g)$ are defined by relations (4.4).

Moreover, we have that

$$
\left| (p w_i^{p-1} - p W_{q}^{p-1} + V(\epsilon x) - \lambda_l) Z_{lk}(x) \right| \leq C \left( R^{-N} + \epsilon |x - q_j| \right) \left( 1 + |x - q_j| \right)^{-N-s}
$$

and then from expression (4.4) we obtain the following estimate.

**Lemma 4.1.** The numbers $c_{ij}$ in (4.1) satisfy:

$$
c_{ij} = \frac{1}{\alpha_l} \int_{\mathbb{R}^N} g Z_{ij} + \theta_{ij},
$$

where

$$
|\theta_{ij}| \leq C (\epsilon + d^{-N}) \left[ \|\phi\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2(\mathbb{R}^N)} \right].
$$

In the rest of this section we shall build a solution to problem (4.1)–(4.2).
**Proposition 4.1.** Given $k \geq 1$, $\frac{N}{2} < \mu < N + 2s$, $C > 0$, there exist positive numbers $d_0$, $\varepsilon_0$, $C$ such that for any points $q_1, \ldots, q_k$ and any $\varepsilon$ with

$$\sum_{i=1}^{k} |q_i| \leq \frac{C}{\varepsilon}, \quad R := \min\{|q_i - q_j| / i \neq j\} > R_0, \quad 0 < \varepsilon < \varepsilon_0$$

there exists a solution $\phi = T[g]$ of (4.1)–(4.2) that defines a linear operator of $g$, provided that

$$\|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)} < +\infty, \quad \rho(x) = \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu}.$$ 

**Besides**

$$\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \leq C \|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)}.$$ 

To prove this result we require several steps. We begin with corresponding a priori estimates.

**Lemma 4.2.** Under the conditions of Proposition 4.1, there exists a $C > 0$ such that for any solution of (4.1)–(4.2) with $\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} < +\infty$ we have the validity of the a priori estimate

$$\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \leq C \|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)}.$$ 

**Proof.** Let us assume the a priori estimate does not hold, namely there are sequences $\varepsilon_n \to 0$, $q_{jn}$, $j = 1, \ldots, k$, with

$$\min\{|q_i - q_{jn}| / i \neq j\} \to \infty$$

and $\phi_n$, $g_n$ with

$$\|\rho_n(x)^{-1}\phi_n\|_{L^\infty(\mathbb{R}^N)} = 1, \quad \|\rho_n(x)^{-1}g_n\|_{L^\infty(\mathbb{R}^N)} \to 0,$$

where

$$\rho_n(x) = \sum_{j=1}^{k} \frac{1}{(1 + |x - q_{jn}|)^\mu},$$

with $\phi_n$, $g_n$ satisfying (4.1)–(4.2). We claim that for any fixed $R > 0$ we have that

$$\sum_{j=1}^{k} \|\phi_n\|_{L^\infty(B_R(q_{jn}))} \to 0.$$ (4.5)
Indeed, assume that for a fixed \( j \) we have that \( \| \phi_n \|_{L^\infty(B_R(q_{jn}))} \geq \gamma > 0 \). Let us set \( \tilde{\phi}_n(x) = \phi_n(q_{jn} + x) \). We also assume that \( \lambda_j^n = V(q_{jn}) \to \bar{\lambda} > 0 \) and

\[
(-\Delta)^s \tilde{\phi}_n + V(q_{jn} + \varepsilon_n x) \tilde{\phi}_n + p \left( w_{\lambda_j}(x) + \theta_n(x) \right)^{p-1} \tilde{\phi}_n = \tilde{g}_n
\]

where

\[
\tilde{g}_n(x) = g_n(q_{jn} + x) - \sum_{l=1}^k \sum_{i=1}^n c_{ln} \partial_i w_{\lambda_j}(q_{jn} - q'_{ln} + x).
\]

We observe that \( \tilde{g}_n(x) \to 0 \) uniformly on compact sets. From the uniform Hölder estimates (2.6), we also obtain equicontinuity of the sequence \( \tilde{\phi}_n \). Thus, passing to a subsequence, we may assume that \( \tilde{\phi}_n \) converges, uniformly on compact sets, to a bounded function \( \tilde{\phi} \) which satisfies \( \| \tilde{\phi} \|_{L^\infty(B_R(0))} \geq \gamma \). In addition, we have that

\[
\left\| (1 + |x|)^\mu \tilde{\phi} \right\|_{L^\infty(\mathbb{R}^N)} \leq 1
\]

and that \( \tilde{\phi} \) solves the equation

\[
(-\Delta)^s \tilde{\phi} + \bar{\lambda} \tilde{\phi} + p w_{\bar{\lambda}}^{p-1} \tilde{\phi} = 0.
\]

Let us notice that \( \tilde{\phi} \in L^2(\mathbb{R}^N) \), and hence the nondegeneracy result in [15] applies to yield that \( \tilde{\phi} \) must be a linear combination of the partial derivatives \( \partial_i w_{\bar{\lambda}} \). But the orthogonality conditions pass to the limit, and yield

\[
\int_{\mathbb{R}^N} \partial_i w_{\bar{\lambda}} \tilde{\phi} = 0 \quad \text{for all } i = 1, \ldots, N.
\]

Thus, necessarily \( \tilde{\phi} = 0 \). We have obtained a contradiction that proves the validity of (4.5). This and the a priori estimate in Lemma 2.5 shows that also, \( \| \rho_n(x)^{-1} \phi_n \|_{L^\infty(\mathbb{R}^N)} \to 0 \), again a contradiction that proves the desired result. \( \square \)

Next we construct a solution to problem (4.1)–(4.2). To do so, we consider first the auxiliary problem

\[
(-\Delta)^s \phi + V \phi = g + \sum_{i=1}^k \sum_{j=1}^N c_{ij} Z_{ij}, \quad \text{(4.6)}
\]

\[
\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j, \quad \text{(4.7)}
\]

where \( V \) is our bounded, continuous potential with

\[
\inf_{\mathbb{R}^N} V = m > 0.
\]
Lemma 4.3. For each \( g \) with \( \| \rho^{-1} g \|_\infty < +\infty \), there exists a unique solution of problem (4.1)–(4.2), \( \phi =: A[g] \in H^2(\mathbb{R}^N) \). This solution satisfies
\[
\| \rho^{-1} A[g] \|_{L^\infty(\mathbb{R}^N)} \leq C \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)}.
\] (4.8)

Proof. First we write a variational formulation for this problem. Let \( X \) be the closed subspace of \( H \) defined as
\[
X = \left\{ \tilde{\phi} \in H / \int_{\mathbb{R}^N} \phi Z_{ij} = 0 \text{ for all } i, j \right\}.
\]
Then, given \( g \in L^2 \), we consider the problem of finding a \( \tilde{\phi} \in X \) such that
\[
\langle \tilde{\phi}, \tilde{\psi} \rangle := \int_{\mathbb{R}^{N+1}_+} \nabla \tilde{\phi} \nabla \tilde{\psi} y^{1-2s} + \int_{\mathbb{R}^N} V \phi \psi = \int_{\mathbb{R}^N} g \psi \text{ for all } \psi \in X.
\] (4.9)

We observe that \( \langle \cdot, \cdot \rangle \) defines an inner product in \( X \) equivalent to that of \( H \). Thus existence and uniqueness of a solution follows from Riesz’s theorem. Moreover, we see that
\[
\| \phi \|_{L^2(\mathbb{R}^N)} \leq C \| g \|_{L^2(\mathbb{R}^N)}.
\]

Next we check that this produces a solution in strong sense. Let \( Z \) be the space spanned by the functions \( Z_{ij} \). We denote by \( \Pi[g] \) the \( L^2(\mathbb{R}^N) \) orthogonal projection of \( g \) onto \( Z \) and by \( \tilde{\Pi}[g] \) its natural \( s \)-harmonic extension. For a function \( \tilde{\varphi} \in H \) let us write \( \tilde{\psi} = \tilde{\varphi} - \tilde{\Pi}[\varphi] \) so that \( \tilde{\psi} \in X \). Substituting this \( \tilde{\psi} \) into (4.9) we obtain
\[
\int_{\mathbb{R}^{N+1}_+} \nabla \tilde{\phi} \nabla \tilde{\psi} y^{1-2s} + \int_{\mathbb{R}^N} V \phi \tilde{\psi} = \int_{\mathbb{R}^N} g \tilde{\psi} + \int_{\mathbb{R}^N} [V \phi - g] \Pi[\varphi] + \int_{\mathbb{R}^N} \phi(-\Delta)^s \Pi[\varphi].
\]

Here we have used that \( \tilde{\Pi}[\varphi] \) is regular and
\[
\int_{\mathbb{R}^{N+1}_+} \nabla \phi \nabla \tilde{\Pi}[\varphi] y^{1-2s} = \int_{\mathbb{R}^N} \phi(-\Delta)^s \Pi[\varphi].
\]

Let us observe that for \( f \in L^2(\mathbb{R}^N) \) the functional
\[
\ell(f) = \int_{\mathbb{R}^N} \phi(-\Delta)^s \Pi[f]
\]
satisfies
\[ |\ell(f)| \leq C \|\phi\|_{L^2(\mathbb{R}^N)} \|\psi\|_{L^2(\mathbb{R}^N)}, \]
hence there is an \( h(\phi) \in L^2(\mathbb{R}^N) \) such that
\[ \ell(\psi) = \int_{\mathbb{R}^N} h(\psi). \]
If \( \phi \) was a priori known to be in \( H^{2s}(\mathbb{R}^N) \) we would have precisely that
\[ h(\phi) = \Pi [(-\Delta)^s \phi]. \]
Since \( \Pi \) is a self-adjoint operator in \( L^2(\mathbb{R}^N) \) we then find that
\[
\int_{\mathbb{R}^{N+1}_+} \nabla \tilde{\varphi} \nabla \tilde{\psi} + \int_{\mathbb{R}^N} V \varphi \psi = \int_{\mathbb{R}^N} \tilde{g} \varphi
\]
where
\[ \tilde{g} = g + \Pi [V \phi - g] + h(\phi). \]
Since \( \tilde{g} \in L^2(\mathbb{R}^N) \), it follows then that \( \phi \in H^{2s}(\mathbb{R}^N) \) and it satisfies
\[ (-\Delta)^s \phi + V \phi - g = \Pi [(-\Delta)^s \phi + V \phi - g] \in \mathcal{Z}, \]
hence Eqs. (4.6)–(4.7) are satisfied. To establish estimate (4.8), we use just Corollary 2.1, observing that
\[
\|\rho^{-1} \Pi [(-\Delta)^s \phi + V \phi - g]\|_{L^\infty(\mathbb{R}^N)} \leq C \left( \|\phi\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2(\mathbb{R}^N)} \right)
\leq C \|g\|_{L^2(\mathbb{R}^N)}
\leq \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)}.
\]
The proof is concluded. \( \square \)

**Proof of Proposition 4.1.** Let us solve now problem (4.1)–(4.2). Let \( Y \) be the Banach space
\[
Y := \{ \phi \in C(\mathbb{R}^N) / \|\phi\|_Y := \|\rho^{-1} \phi\|_{L^\infty(\mathbb{R}^N)} < +\infty \}. \quad (4.10)
\]
Let \( A \) be the operator defined in Lemma 4.3. Then we have a solution to problem (4.1)–(4.2) if we solve
\[
\phi - A[p W_q^{p-1} \phi] = A[g], \quad \phi \in Y. \quad (4.11)
\]
We claim that


defines a compact operator in \( Y \). Indeed. Let us assume that \( \phi_n \) is a bounded sequence in \( Y \). We observe that for some \( \sigma > 0 \) we have

\[ |W_q^{p-1}\phi_n| \leq C\|\phi_n\|_Y \rho^{1+\sigma}. \]

If \( \sigma \) is sufficiently small, it follows that \( f_n := B[\phi_n] \) satisfies

\[ |\rho^{-1}f_n| \leq C\rho^\sigma. \]

Besides, since \( f_n = T_m((V - m)f_n + g_n) \) we use estimate (2.6) to get that for some \( \alpha > 0 \)

\[ \sup_{x \neq y} \frac{|f_n(x) - f_n(y)|}{|x - y|^\alpha} \leq C. \]

Arzela’s theorem then yields the existence of a subsequence of \( f_n \) which we label the same way, that converges uniformly on compact sets to a continuous function \( f \) with

\[ |\rho^{-1}f| \leq C\rho^\sigma. \]

Let \( R > 0 \) be a large number. Then we estimate

\[ \|\rho^{-1}(f_n - f)\|_{L^\infty(\mathbb{R}^N)} \leq \|\rho^{-1}(f_n - f)\|_{L^\infty(B_R(0))} + C \max_{|x| > R} \rho^\sigma(x). \]

Since

\[ \max_{|x| > R} \rho^\sigma(x) \to 0 \quad \text{as} \quad R \to \infty \]

we conclude then that \( \|f_n - f\|_\infty \to 0 \) and the claim is proven.

Finally, the a priori estimate tells us that for \( g = 0 \), Eq. (4.11) has only the trivial solution. The desired result follows at once from Fredholm’s alternative. \( \square \)

We conclude this section by analyzing the differentiability with respect to the parameter \( q \) of the solution \( \phi = T_q[g] \) of (4.1)–(4.2). As in the proof above we let \( Y \) be the space in (4.10), so that \( T_q \in \mathcal{L}(Y) \)

**Lemma 4.4.** The map \( q \mapsto T_q \) is continuously differentiable, and for some \( C > 0 \),

\[ \|\partial_q T_q\|_{\mathcal{L}(Y)} \leq C \quad (4.12) \]

for all \( q \) satisfying constraints (3.3).
Proof. Let us write $q = (q_1, \ldots, q_k)$, $q_i = (q_{i1}, \ldots, q_{iN})$, $\phi = T_q[g]$, and (formally)

$$
\psi = \partial_{q_{ij}} T_q[g], \quad d_{lk} = \partial_{q_{ij}} c_{lk}.
$$

Then, by differentiation of Eqs. (4.1)–(4.2), we get

$$
(-\Delta)^s \psi + V(\varepsilon x) \psi - p W_q^{p-1} \psi = p \partial_{q_{ij}} W_q^{p-1} \phi + \sum_{l,k} c_{lk} \partial_{q_{ij}} Z_{lk} + \sum_{l,k} d_{lk} Z_{lk}, \quad (4.13)
$$

$$
\int_{\mathbb{R}^N} \psi Z_{lk} = - \int_{\mathbb{R}^N} \phi \partial_{q_{ij}} Z_{lk} \quad \text{for all } l, k. \quad (4.14)
$$

We let

$$
\tilde{\psi} = \psi - \Pi[\psi]
$$

where, as before, $\Pi[\psi]$ denotes the orthogonal projection of $\psi$ onto the space spanned by the $Z_{lk}$. Writing

$$
\Pi[\psi] = \sum_{l,k} \alpha_{lk} Z_{lk} \quad (4.15)
$$

and relations (4.14) as

$$
\int_{\mathbb{R}^N} \Pi[\psi] Z_{lk} = - \int_{\mathbb{R}^N} \phi \partial_{q_{ij}} Z_{lk} \quad \text{for all } l, k. \quad (4.16)
$$

we get

$$
|\alpha_{lk}| \leq C \|\phi\|_Y \leq C \|g\|_Y. \quad (4.17)
$$

From (4.13) we have then that

$$
(-\Delta)^s \tilde{\psi} + V(\varepsilon x) \tilde{\psi} - p W_q^{p-1} \tilde{\psi} = \tilde{g} + \sum_{l,k} d_{lk} Z_{lk}, \quad (4.18)
$$

or $\tilde{\psi} = T_q[\tilde{g}]$ where

$$
\tilde{g} = p \partial_{q_{ij}} W_q^{p-1} \phi + \sum_{l,k} c_{lk} \partial_{q_{ij}} Z_{lk} - \left[(-\Delta)^s + V(\varepsilon x) - p W_q^{p-1}\right] \Pi[\psi], \quad (4.19)
$$

Then we see that

$$
\|\tilde{\psi}\|_Y \leq C \|\tilde{g}\|_Y.
$$
Using (4.17) and Lemma 4.1, we see also that
\[ \| \tilde{g} \|_Y \leq C \| g \|_Y, \quad \| \Pi[\psi'] \| \leq C \| g \|_Y \]
and thus
\[ \| \psi' \| \leq C \| g \|_Y. \] (4.20)

Let us consider now, rigorously, the unique \( \psi = \tilde{\psi} + \Pi[\psi] \) that satisfies Eqs. (4.14) and (4.19). We want to show that indeed
\[ \psi = \partial_{q_{ij}} T_q[g]. \]

To do so, \( q_i' = q_i + te_j \) where \( e_j \) is the \( j \)-th element of the canonical basis of \( \mathbb{R}^N \), and set
\[ q' = (q_1, \ldots, q_{i-1}, q_i', \ldots, q_k). \]

For a function \( f(q) \) we denote
\[ D_{ij}^t f = t^{-1}(f(q') - f(q)) \]
we also set
\[ \phi' := T_{q'[g]}, \quad D_{ij}^t T_q[g] =: \psi' = \tilde{\psi}' + \Pi[\tilde{\psi}'] \]
so that
\[ (-\Delta)^s \tilde{\psi}' + V(\varepsilon x)\tilde{\psi}' - pW_q^{p-1} \tilde{\psi}' = \tilde{g}' + \sum_{i,k} d_{ik}' Z_{lk}, \]
where
\[ \tilde{g}' = pD_{ij}[W_q^{p-1}] \phi + \sum_{l,k} c_{lk} D_{ij}^t Z_{lk} - [(-\Delta)^s + V(\varepsilon x) - pW_q^{p-1}] \Pi[\psi'], \]
\[ d_{ik}' = D_{ij}^t c_{lk} \]
and
\[ \Pi[\psi'] = \sum_{l,k} \alpha_{lk}' Z_{lk}, \]
where the constants \( \alpha_{lk}' \) are determined by the relations
\[ \int_{\mathbb{R}^N} \Pi[\psi'] Z_{lk} = - \int_{\mathbb{R}^N} \phi D_{ij}^t Z_{lk}. \]
Comparing these relations with (4.15), (4.16), (4.18) defining $\psi$, we obtain that

$$\lim_{t \to 0} \| \psi_t - \psi \|_Y = 0$$

which by definition tells us $\psi = \partial_{q_{ij}} T_q[g]$. The continuous dependence in $q$ is clear from that of the data in the definition of $\psi$. Estimate (4.12) follows from (4.20). The proof is concluded. □

5. Solving the nonlinear projected problem

In this section we solve the nonlinear projected problem

$$(-\Delta)^s \phi + V(\varepsilon x) \phi - p W_q^{p-1} \phi = E + N(\phi) + \sum_{i=1}^k \sum_{j=1}^N c_{ij} Z_{ij},$$

(5.1)

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j.$$  (5.2)

We have the following result.

**Proposition 5.1.** Assuming that $\|E\|_Y$ is sufficiently small problem (5.1)–(5.2) has a unique small solution $\phi = \Phi(q)$ with

$$\|\Phi(q)\|_Y \leq C \|E\|_Y.$$  

The map $q \mapsto \Phi(q)$ is of class $C^1$, and for some $C > 0$

$$\|\partial_q \Phi(q)\|_Y \leq C \left[ \|E\|_Y + \|\partial_q E\|_Y \right].$$  (5.3)

for all $q$ satisfying constraints (3.3).

**Proof.** Problem (5.1)–(5.2) can be written as the fixed point problem

$$\phi = T_q(E + N(\phi)) =: K_q(\phi), \quad \phi \in Y.$$  (5.4)

Let

$$B = \{ \phi \in Y / \|\phi\|_Y \leq \rho \}.$$  

If $\phi \in B$ we have that

$$|N(\phi)| \leq C|\phi|^\beta, \quad \beta = \min\{p, 2\},$$

and hence

$$\|N(\phi)\|_Y \leq C\|\phi\|^2.$$
It follows that
\[ \| K_q(\phi) \|_Y \leq C_0 \left[ \| E \| \rho^2 \right] \]
for a number \( C_0 \), uniform in \( q \) satisfying (3.3). Let us assume
\[ \rho := 2C_0 \| E \|, \quad \| E \| \leq \frac{1}{2C_0}. \]
Then
\[ \| K_q(\phi) \|_Y \leq C_0 \left[ \frac{1}{2C_0} \rho + \rho^2 \right] \leq \rho \]
so that \( K_q(B) \subset B \). Now, we observe that
\[ |N(\phi_1) - N(\phi_2)| \leq C[|\phi|^{\beta-1} + |\phi|^{\beta-1}]|\phi_1 - \phi_2| \]
and hence
\[ \| N(\phi_1) - N(\phi_2) \|_Y \leq C \rho^{\beta-1} \| \phi_1 - \phi_2 \|_Y \]
and
\[ \| K_q(\phi_1) - K_q(\phi_2) \| \leq C \rho^{\beta-1} \| \phi_1 - \phi_2 \|_Y. \]
Reducing \( \rho \) if necessary, we obtain that \( K_q \) is a contraction mapping and hence has a unique solution of Eq. (5.4) exists in \( B \). We denote it as \( \phi = \Phi(q) \). We prove next that \( \Phi \) defines a \( C^1 \) function of \( q \). Let
\[ M(\phi, q) := \phi - T_q \left( E + N(\phi) \right). \]
Let \( \phi_0 = \Phi(q_0) \). Then \( M(\phi_0, q_0) = 0 \). On the other hand,
\[ \partial_\phi M(\phi, q)[\psi] = \psi - T_q \left( N'(\phi) \psi \right) \]
where \( N'(\phi) = \rho[(W + \phi)^{p-1} - W^{p-1}] \), so that
\[ \| N'(\phi) \psi \|_Y \leq C \rho^{\beta-1} \| \psi \|_Y. \]
If \( \rho \) is sufficiently small we have then that \( D_\phi M(\phi_0, q_0) \) is an invertible operator, with uniformly bounded inverse. Besides
\[ \partial_q M(\phi, q) = (\partial_q T_q)(E + N(\phi)) + T_q \left( \partial_q E + \partial_q N(\phi) \right). \]
Both partial derivatives are continuous in their arguments. The implicit function applies in a small neighborhood of \( (\phi_0, q_0) \) to yield existence and uniqueness of a function \( \phi = \phi(q) \) with \( \phi(q_0) = \)
\( \phi_0 \) defined near \( q_0 \) with \( M(\phi(q), q) = 0 \). Besides, \( \phi(q) \) is of class \( C^1 \). But, by uniqueness, we must have \( \phi(q) = \Phi(q) \). Finally, we see that
\[
\partial_q \Phi(q) = -D_\phi M(\Phi(q), q)^{-1}[(\partial_q T_q)(E + N(\Phi(q))) + T_q(\partial_q E + \partial_q N(\Phi(q)))]
\]
and hence
\[
\| (\partial_q N)(\Phi(q)) \|_Y \leq C \| \Phi(q) \|_Y^{\beta} \leq C \| E \|_Y^{\beta}.
\]
From here, the above expressions and the bound of Lemma 4.4 we finally get the validity of estimate (5.3).

5.1. An estimate of the error

Here we provide an estimate of the error \( E \) defined in (3.5),
\[
E := \sum_{j=1}^{k} (\lambda_j - V(\varepsilon x)) w_j + \left( \sum_{j=1}^{k} w_j \right)^p - \sum_{j=1}^{k} w_j^p
\]
in the norm \( \| \cdot \|_Y \). Here we need to take \( \mu \in \left( \frac{N}{2}, \frac{N+2s}{2} \right) \). We denote
\[
R = \min_{i \neq j} |q_i - q_j| \gg 1.
\]
The first term in \( E \) can be easily estimated as
\[
\left| \rho^{-1}(x) \sum_{j=1}^{k} (\lambda_j - V(\varepsilon x)) w_j \right| \leq C \varepsilon^{\min(2s,1)}.
\]
To estimate the interaction term in \( E \), we divide the \( \mathbb{R}^N \) into the \( k \) sub-domains
\[
\Omega_j = \{ w_j \geq w_i, \forall i \neq j \}, \quad j = 1, \ldots, k.
\]
In \( \Omega_j \), we have
\[
\left| \left( \sum_{j=1}^{k} w_j \right)^p - \sum_{j=1}^{k} w_j^p \right| \leq C w_j^{p-1} \sum_{i \neq j} \frac{1}{|x - q_i|^{N+2s}}
\]
\[
\leq C \frac{1}{(1 + |x - q_j|)(N+2s)(p-1)+\mu} \sum_{i \neq j} \frac{1}{|q_j - q_i|^{N+2s-\mu}}
\]
\[
\leq C \rho(x) R^{\mu-N-2s}.
\]
In summary, we conclude that

$$\|E\|_Y \leq C\varepsilon^{2s} + CR^{\mu - N - 2s}. \quad (5.5)$$

As a consequence of Proposition 5.1 and the estimate (5.5), we obtain that

$$\|\Phi(q)\|_Y \leq C\varepsilon^{\min(2s, 1)} + CR^{\mu - N - 2s}.$$ 

Let us now take

$$\tau = C\varepsilon^{\min(2s, 1)} + CR^{\mu - N - 2s}.$$ 

6. The variational reduction

We will use the above introduced ingredients to find existence results for the equation

$$(-\Delta)^s v + V(\varepsilon x)v - v^p_+ = 0. \quad (6.1)$$

An energy whose Euler–Lagrange equation corresponds formally to (6.1) is given by

$$J_\varepsilon(\tilde{v}) := \frac{1}{2} \int_{\mathbb{R}^N} v(-\Delta)^s v + V(\varepsilon x)v^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} V(\varepsilon x)v^2.$$ 

We want to find a solution of (6.1) with the form

$$v = v_q := W_q + \Phi(q)$$

where $\Phi(q)$ is the function in Proposition 5.1. We observe that

$$(-\Delta)^s v_q + V(\varepsilon x)v_q - (v_q)^p_+ = \sum_{i,j} c_{ij} Z_{ij} \quad (6.2)$$

hence what we need is to find points $q$ such that $c_{ij} = 0$ for all $i, j$. This problem can be formulated variationally as follows.

**Lemma 6.1.** Let us consider the function of points $q = (q_1, \ldots, q_k)$ given by

$$I(q) := J_\varepsilon(W_q + \Phi(q)),$$

where $W_q + \Phi(q)$ is the unique $s$-harmonic extension of $W_q + \Phi(q)$. Then in (6.2), we have $c_{ij} = 0$ for all $i, j$ if and only if

$$\partial_q I(q) = 0.$$
**Proof.** Let us write \( v_q = W_q + \phi(q) \). We observe that

\[
\partial_{q_{ij}} I(q) = \int_{\mathbb{R}^{N+1}} \nabla \bar{v}_q \nabla (\partial_{q_{ij}} \bar{v}_q) y^{1-2s} + \int_{\mathbb{R}^N} V(\varepsilon x) v_q \partial_{q_{ij}} v_q - \int_{\mathbb{R}^N} (v_q)^{p-1} \partial_{q_{ij}} v_q
\]

\[
= \int_{\mathbb{R}^N} \left[ (-\Delta)^s v_q + V(\varepsilon x) v_q - (v_q)^p \right] \partial_{q_{ij}} v_q
\]

\[
= \sum_{k,l} c_{kl} \int_{\mathbb{R}^N} Z_{kl} \partial_{q_{ij}} v_q. \tag{6.3}
\]

We observe that

\[
\partial_{q_{ij}} v_q = -Z_{ij} + O(\varepsilon^\rho) + \partial_{q_{ij}} \Phi(q).
\]

Since, according to Proposition 5.1

\[
\| \partial_{q} \Phi(q) \|_Y = O(\| E \|_Y + \| \partial_{q} E \|_Y)
\]

and this quantity gets smaller as the number \( \delta \) in (3.3) is reduced, and the functions \( Z_{kl} \) are linearly independent (in fact nearly orthogonal in \( L^2 \)), it follows that the quantity in (6.3) equals zero for all \( i, j \) if and only if \( c_{ij} = 0 \) for all \( i, j \). The proof is concluded. \( \square \)

Our task is therefore to find critical points of the functional \( I(q) \). Useful to this end is to achieve expansions of the energy in special situations.

**Lemma 6.2.** Assume that the numbers \( \delta \) and \( R \) in the definition of \( \Gamma \) in (3.3) is taken so small that

\[
\| E \|_Y + \| \partial_{q} E \| \leq \tau \ll 1.
\]

Then

\[
I_\varepsilon(q) = J_\varepsilon(W_q) + O(\tau^2)
\]

and

\[
\partial_{q} I_\varepsilon(q) = \partial_{q} J_\varepsilon(W_q) + O(\tau^2)
\]

uniformly on points \( q \) in \( \Gamma \).

**Proof.** Let us estimate

\[
I(q) = J_\varepsilon(v_q), \quad v_q = W_q + \Phi(q).
\]
We have that
\[ I(\xi) = \frac{1}{2} \int_{\mathbb{R}^N} v_q (-\Delta)^s v_q + V v_q^2 - \frac{1}{p+1} \int v_q^{p+1}. \]

Thus we can expand
\[ I(q) = J_\varepsilon(W_q) + \int_{\mathbb{R}^N} \Phi \left[ (-\Delta)^s v_q + V v_q - v_q^p \right] + \frac{1}{2} \int \Phi (-\Delta)^s \Phi + V \Phi^2 \]
\[ - \frac{1}{p+1} \int (W_q + \Phi)^{p+1} - W_q^{p+1} - (p+1) W_q^p \Phi. \]

Since, \( \|E\|_Y \leq \tau \) then \( \|\Phi\|_Y = O(\tau) \), and from the equation satisfied by \( \Phi \), also \( \|(-\Delta)^s \Phi\|_Y = O(\tau) \). This implies
\[ \left| \frac{1}{2} \int \Phi (-\Delta)^s \Phi + V \Phi^2 \right| \leq C \int \rho^{2\mu} \tau^2 \leq C \tau^2 \]
and
\[ \left| \int (W_q + \Phi)^{p+1} - W_q^{p+1} - (p+1) W_q^p \Phi \right| \leq C \int \rho^{2\mu} \tau^2 \leq C \tau^2. \]

Here we have used the fact that \( \mu \in (\frac{N}{2}, \frac{N+2s}{2}) \).

On the other hand the second term in the above expansion equals 0, since by definition
\[ (-\Delta)^s v_q + V v_q - v_q^p \in Z \]
and \( \Phi \) is \( L^2 \)-orthogonal to that space. We arrive to the conclusion that
\[ I(q) = J_\varepsilon(W_q) + O(\tau^2) \]
uniformly for \( q \) in a bounded set. By differentiation we also have that
\[ \partial_q I(q) = \partial_q J_\varepsilon(W_q) + \int_{\mathbb{R}^N} \partial_q \Phi (-\Delta)^s \Phi + V \Phi \partial_q \Phi \]
\[ + \int_{\mathbb{R}^N} \left[ (W_q + \Phi)^p - W_q^p - p W_q^{p-1} \Phi \right] \partial_q W_q + \left[ (W_q + \Phi)^p - W_q^p \right] \partial_q \Phi. \]

Since we also have \( \|\partial_q \Phi\|_Y = O(\tau) \), then the second and third term above are of size \( O(\varepsilon^2) \). Thus,
\[ \partial_q I(q) = \partial_q J_\varepsilon(W_q) + O(\rho^2) \]

uniformly on \( q \in \Gamma \) and the proof is complete. \( \square \)

Next we estimate \( J_\varepsilon(W_q) \) and \( \partial_q J_\varepsilon(W_q) \). We begin with the simpler case \( k = 1 \). Here it is always the case that

\[ \| E \|_Y + \| \partial_q E \|_Y \leq \tau. \]

Let us also set \( \xi = \varepsilon q \). We have now that

\[ W_q(x) = w_\lambda(x - q), \quad \lambda = V(\xi). \]

We compute

\[ J_\varepsilon(W_q) = J^\lambda(w_\lambda) + \frac{1}{2} \int_{\mathbb{R}^N} (V(\xi + \varepsilon x) - V(\xi)) w_\lambda^2(x) \, dx \]

where

\[ J^\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{\lambda}{2} \int_{\mathbb{R}^N} v^2 - \frac{1}{p + 1} \int_{\mathbb{R}^N} v^{p+1}. \]

We recall that

\[ w_\lambda(x) := \lambda^{\frac{1}{p+1}} w(\lambda^{\frac{1}{p+1}} x) \]

satisfies the equation

\[ (-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0 \quad \text{in} \ \mathbb{R}^N, \]

where \( w = w_1 \) is the unique radial least energy solution of

\[ (-\Delta)^s w + w - w^p = 0 \quad \text{in} \ \mathbb{R}^N. \]

Then, after a change of variables we find

\[ J^\lambda(w_\lambda) = \frac{1}{2} \int_{\mathbb{R}^N} w_\lambda(-\Delta)^s w_\lambda + \frac{\lambda}{2} \int_{\mathbb{R}^N} w_\lambda^2 - \frac{1}{p + 1} \int_{\mathbb{R}^N} w_\lambda^{p+1} = \lambda^{\frac{p+1}{p+1}} \frac{N}{2} \int_{\mathbb{R}^N} J^1(w). \]

Now since \( w \) is radial, we find

\[ \int_{\mathbb{R}^N} x_i w_\lambda(x) \, dx = 0. \]
Thus,
\[
\int_{\mathbb{R}^N} \left( V(\xi + \varepsilon x) - V(\xi) \right) w^2_\lambda(x) \, dx = \nabla V(\xi) \cdot \int_{\mathbb{R}^N} x \, w_\lambda + O(\varepsilon^2) = O(\varepsilon^2).
\]

On the other hand
\[
\partial_q \int_{\mathbb{R}^N} \left( V(\xi + \varepsilon x) - V(\xi) \right) w^2_\lambda(x) \, dx
= \varepsilon \int_{\mathbb{R}^N} \left( \nabla V(\xi + \varepsilon x) - \nabla V(\xi) \right) w^2_\lambda(x) \, dx + 2 \int_{\mathbb{R}^N} \left( V(\xi + \varepsilon x) - V(\xi) \right) w_\lambda \partial_q w_\lambda \, dx
= O(\varepsilon^2).
\]

Lemma 6.3. Let \( \theta = \frac{p+1}{p-1} - \frac{N}{2s} \), \( c_* = J_1(w) \) and \( k = 1 \). Then the following expansions hold:
\[
I(q) = c_* V^\theta(\xi) + O\left(\varepsilon^{\min(4s,2)}\right),
\]
\[
\nabla_q I(q) = c_* \varepsilon \nabla (V^\theta)(\xi) + O\left(\varepsilon^{\min(4s,2)}\right).
\]

For the case \( k > 1 \) and \( \min_i \neq j |q_i - q_j| \geq R \gg 1 \), we observe that, also, \( \|E\|_Y = O(\tau) \) and hence we also have
\[
I(q) = J_\varepsilon(W_q) + O(\tau^2), \quad \partial_q I(q) = \partial_q J_\varepsilon(W_q) + O(\tau^2).
\]

By expanding \( I(q) \) we get the validity of the following estimate.

Lemma 6.4. Letting \( \xi = \varepsilon q \) we have that
\[
I(q) = c_* \sum_{i=1}^k V^\theta(\xi_i) - \sum_{i \neq j} \frac{c_{ij}}{|q_i - q_j|^{N+2s}} + O\left(\varepsilon^{\min(4s,2)} + \frac{1}{R^{2(N+2s-\mu)}}\right),
\]
\[
\nabla_q I(q) = c_* \varepsilon \nabla (V^\theta)(\xi) \left[ \sum_{i=1}^k V^\theta(\xi_i) - \sum_{i \neq j} \frac{c_{ij}}{|q_i - q_j|^{N+2s}} \right] + O\left(\varepsilon^{\min(4s,2)} + \frac{1}{R^{2(N+2s-\mu)}}\right)
\]
where \( c_* \) and \( c_{ij} = c_0(V(\xi_i))^\alpha(V(\xi_j))^\beta \) are positive constants.

Proof. It suffices to expand \( J_\varepsilon(W_q) \). We see that, denoting \( w_i(x) := w_{\lambda_i}(x - q_i) \),
\[
J_\varepsilon(W_q) = J_\varepsilon \left( \sum_{i=1}^k w_i \right)
\]
\[
\begin{align*}
&= \sum_{i=1}^{k} J_{\varepsilon}(w_i) + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} w_i(-\Delta)^{s} w_j + \int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j \\
&\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} w_i \right)^{p+1} - \sum_{i=1}^{k} w_i^{p+1}.
\end{align*}
\] (6.4)

We estimate for \( i \neq j \),
\[
\int_{\mathbb{R}^N} w_i(-\Delta)^{s} w_j + \int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j = \int_{\mathbb{R}^N} w_i w_j^p + \int_{\mathbb{R}^N} (V(\varepsilon x) - \lambda_j) w_i w_j
\]
\[
= (c_{ij} + o(1)) \frac{1}{|q_i - q_j|^{N+2s}} + O\left( \frac{\varepsilon^{2s}}{\mathbb{R}^{N+2s-\mu}} \right) \quad (6.5)
\]

where \( c_{ij} = c_0 (V(\xi_i))^\alpha (V(\xi_j))^\beta \) and \( c_0, \alpha, \beta \) are constants depending on \( p, s \) and \( N \) only. Indeed,
\[
w_i(x) = \lambda_i^{\frac{1}{p-1}} w\left( \lambda_i^{\frac{1}{2s}} (x - q_i) \right)
\]

and it is known that
\[
w(x) = \frac{c_0}{|x|^{N+2s}} (1 + o(1)) \quad \text{as} \quad |x| \to \infty.
\]

Then, we have
\[
\int_{\mathbb{R}^N} w_i^p w_j = \lambda_i^{\frac{1}{p-1} - \frac{n+2s}{2s}} \lambda_j^{\frac{p}{p-1} - \frac{n}{2s}} \left( \int_{\mathbb{R}^N} w_i^p \right) \frac{c_0}{|q_i - q_j|^{N+2s}},
\]

and hence
\[
c_{ij} = c_0 \lambda_i^\alpha \lambda_j^\beta
\]

where
\[
\lambda_i = V(\xi_i), \quad \lambda_j = V(\xi_j), \quad \alpha = \frac{1}{p-1} - \frac{n+2s}{2s}, \quad \beta = \frac{p}{p-1} - \frac{n}{2s}.
\]

To estimate the last term we note that
\[
\int_{\mathbb{R}^N} \left( \left( \sum_{i=1}^{k} w_i \right)^{p+1} - \sum_{i=1}^{k} w_i^{p+1} \right)^{p+1}
\]
\[
= \sum_{j=1}^{k} \int_{\Omega_j} \left( \left( \sum_{i=1}^{k} w_i \right)^{p+1} - \sum_{i=1}^{k} w_i^{p+1} \right)
\]
\[
= \sum_{j=1}^{k} \sum_{i \neq j} (p+1) w_j^p \left( \sum_{i \neq j} w_i \right) + O \left( w_j^{\min(p-1,1)} \left( \sum_{i \neq j} w_i \right)^2 \right)
\]
\[
= \sum_{j=1}^{K} \sum_{i \neq j} (p+1) \int_{\mathbb{R}^N} w_j^p w_i + O \left( \frac{1}{R^{2(N+2s-\mu)}} \right)
\]
\[
= \sum_{j=1}^{K} \sum_{i \neq j} (p+1) \frac{c_{ij} + o(1)}{|q_i - q_j|^{N+2s}} + O \left( \frac{1}{R^{2(N+2s-\mu)}} \right). \tag{6.6}
\]

Substituting (6.5) and (6.6) into (6.4) and using the estimate of \( J_\varepsilon(w_i) \) in the proof of Lemma 6.3, we have estimated \( J_\varepsilon(w_i) \), and we have proven the lemma.

7. The proofs of Theorems 1–3

Based on the asymptotic expansions in Lemma 6.4, we present the proofs of Theorems 1–3.

**Proof of Theorems 1 and 2.** Let us consider the situation in Remark 1.1, which is more general than that of Theorem 1. Then, in the definition of the configuration space \( \Gamma \) (3.3), we can take a fixed \( \delta \) and \( R \sim \varepsilon^{-1} \) and achieve that \( \Lambda \subset \varepsilon \Gamma \). Then we get

\[
\|E\|_Y + \|\partial_q E\|_Y = O(\varepsilon^{\min\{2s,1\}}).
\]

Letting

\[
\tilde{I}(\xi) := I(\varepsilon q)
\]

we need to find a critical point of \( \tilde{I} \) inside \( \Lambda \). By Lemma 6.4, we see then that

\[
\tilde{I}(\xi) - c_* \varphi(\xi) = o(1), \quad \nabla_{\xi} \tilde{I}(\xi) - c_* \nabla_{\xi} \varphi(\xi) = o(1),
\]

uniformly in \( \xi \in \Lambda \) as \( \varepsilon \to 0 \), where \( \varphi \) is the functional in (1.10). It follows, by the assumption on \( \varphi \) that for all \( \varepsilon \) sufficiently small there exists a \( \xi^\varepsilon \in \Lambda \) such that \( \nabla \tilde{I}(\xi^\varepsilon) = 0 \), hence Lemma 6.1 applies and the desired result follows.

Theorem 2 follows in the same way. We just observe that because of the \( C^1 \)-proximity, the same variational characterization of the numbers \( c \), for the functional \( \tilde{I}(\xi) \) holds. This means that the critical value predicted in that form is indeed close to \( c \). The proof is complete.

**Proof of Theorem 3.** Finally we prove Theorem 3. Following the argument in [18], we choose the following configuration space

\[
\Lambda = \left\{ (\xi_1, \ldots, \xi_k) / \xi_j \in \Gamma, \min_{i \neq j} |\xi_i - \xi_j| > \varepsilon^{1-s} \right\} \tag{7.1}
\]
with $\Gamma$ given by (3.3), and we prove the following Claim and then Theorem 3 follows from Lemma 6.1:

**Claim.** Letting $\xi = \varepsilon q$, the problem

$$\max_{(\xi_1, \ldots, \xi_k) \in \Lambda} I(q)$$

admits a maximizer $(\xi^e_1, \ldots, \xi^e_k) \in \Lambda$.

We shall prove this by contradiction. First, by continuity of $I(q)$, there is a maximizer $\xi^e = (\xi^e_1, \ldots, \xi^e_k) \in \bar{\Lambda}$. We need to prove that $\xi^e \in \Lambda$. Let us suppose, by contradiction, that $\xi^e \notin \Lambda$, hence it lies on its boundary. Thus there are two possibilities: either there is an index $i$ such that $\xi^e_i \in \partial \Gamma$, or there exist indices $i \neq j$ such that

$$|\xi^e_i - \xi^e_j| = \min_{i \neq j} |\xi_i - \xi_j| = \varepsilon^{1-\delta}.$$

Denoting $q^e = \frac{\xi^e}{\varepsilon}$, and using Lemma 6.4, we have in the first case that

$$I(q^e) \leq c_* V^\theta(\xi^e_i) + c_* \sum_{j \neq i} V^\theta(\xi^e_j) + C \varepsilon^{2\delta}$$

$$\leq c_* k \max_{\Gamma} V^\theta(x) + c_* \left( \max_{\partial \Gamma} V^\theta(x) - \max_{\Gamma} V^\theta(x) \right) + C \varepsilon^{2\delta}. \quad (7.3)$$

In the second case, we invoke again Lemma 6.4 and obtain

$$I(q^e) \leq c_* k \max_{\Gamma} V^\theta(x) - c_2 \varepsilon^{\frac{\delta}{3}} + C \varepsilon^{2\delta}$$

for some $c_2 > 0$. On the other hand, we can get an upper bound for $I(q^e)$ as follows. Let us choose a point $\xi_0$ such that $V(\xi_0) = \max_{\Gamma} V(x)$ and let

$$\xi_j = \xi_0 + \varepsilon^{1-\frac{\delta}{3}}(1, 0, \ldots, 0), \quad j = 1, \ldots, k.$$

It is easy to see that $(\xi_1, \ldots, \xi_k) \in \Lambda$. Now, we compute by Lemma 6.4:

$$I(q^e) = \max_{\Lambda} I(q) \geq c_* k \max_{\Gamma} V^\theta(x) - c_3 \varepsilon^{\frac{\delta}{3}}. \quad (7.4)$$

For $\varepsilon$ sufficiently small, a contradiction follows immediately from $(7.3)-(7.4)$. $\square$
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- After completion of this work we have learned about the paper [4] in which the result of Corollary 1.1 is found for $k = 1$ under further constraints in the space dimension $N$ and the values of $s$ and $p$.
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References