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EXISTENCE OF MINIMIZERS ON DROPS*

R. CORREA†, P. GAJARDO‡, L. THIBAULT§, AND D. ZAGRODNY¶

Abstract. For a boundedly generated drop $[a, E]$ (a property which holds, for instance, whenever E is bounded), where a belongs to a real Banach space X and $E \subset X$ is a nonempty convex set, we show that for every lower semicontinuous function $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ that satisfies $\sup_{\delta>0} \inf_{x \in E+\delta B_X} h(x) > h(a)$ (B_X being the unitary open ball in X), there exists $\bar{x} \in [a, E]$ such that $h(a) \geq h(\bar{x})$ and $\bar{x}$ is a strict minimizer of h on the drop $[\bar{x}, E]$.

Key words. variational principle, variational inequalities, drop, minimizers

AMS subject classifications. 65K10, 49J40

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1. Introduction. Classical results related to the existence of minimizers require compactness of the set of restrictions (Weierstrass theorem) or boundedness from below of the objective function (Ekeland variational principle [6]). The purpose of this work is to show that there exist minimizers in some particular contexts where the mentioned assumptions do not necessarily hold. Namely, for a given lower semicontinuous function (lsc) $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined on a Banach space X, if we consider a nonempty convex set $E \subset X$ (closed or open) and a point $a \in X$ such that $h(a) < h(x)$ for all x in a neighborhood V of E, and the drop $[a, E] := \{ta + (1 - t)y \mid t \in [0, 1], \ y \in E\}$ is *boundedly generated*, which holds, for instance, whenever E is bounded, then we prove that there exists $\bar{x} \in [a, E] \setminus V$ such that $h(\bar{x}) \leq h(a)$ and $h(\bar{x}) < h(x)$ for all $x \in [\bar{x}, E] \setminus \{\bar{x}\}$. In particular, this result says that for some point $\bar{x}$ in the drop $[a, E]$, the function h has its strict minimum at $\bar{x}$ on every segment $[\bar{x}, e]$ with $e \in E$. This is an existence result for a minimizer to be at a vertex of a truncated cone generated by the set E. The result is obtained even for functions which are not bounded from below on the drop. The arguments of the proof are based on the maximality principle established by Gajek and Zagrodny in [7].

In the next section we define the boundedly generated drop property, giving some examples and preliminary properties of such sets, and afterward we establish the main result of this work: existence of a minimizer on a boundedly generated drop. In section 3 we apply this result to obtain the Daneš drop theorem (see [4], [1]) which is equivalent to the Ekeland variational principle, the Krasnoselskii–Zabreiko renorming theorem, the Caristi fixed point theorem, and the Brézis–Browder generalization of the Bishop–Phelps theorem as it was observed in [5] (see also [8], [9] for results on

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†Centro de Modelamiento Matemático (CNRS UMI 2807) and Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170/3, Correo 3, Santiago, Chile (rcorrea@dim.uchile.cl).
‡Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile (pedro.gajardo@usm.cl).
§Département des Sciences Mathématiques, Université de Montpellier II, 34095 Montpellier Cedex 5, France (thibault@math.univ-montp2.fr).
¶Faculty of Mathematics and Natural Science, College of Science, Cardinal Stefan Wyszyński University, Warsaw, Poland (d.zagrodny@uksw.edu.pl).

the drop properties), establishing a clear link between our result and these classical variational principles.

Throughout the paper X stands for a real normed space and B_X for the unitary open ball in X centered at zero. As usual, we denote by $B(x,r)$ (resp., $\overline{B}(x,r)$) the open (resp., closed) ball with center $x \in X$ and radius $r > 0$.

By cl we denote the topological closure of set, i.e., the closure with respect to the norm. For $a \in X$ and a nonempty set E of X, we define the drops

$$[a,E] = \{ta + (1-t)y \ \mid \ t \in [0,1], \quad y \in E\};$$

$$]a,E] = \{ta + (1-t)y \ \mid \ t \in [0,1[, \quad y \in E\}.$$

Given a nonempty set $S \subset X$, we write $d_S(\cdot)$ for the distance function to S.

For an extended real-valued function ϕ defined on X and a nonempty subset $C \subset X$, we define the *limit of infimums* of ϕ over enlargements of C by (see [2])

$$\operatorname*{linf}_{v\in C} \phi(v) := \lim_{\delta \to 0^+} \inf_{v \in C + \delta B_X} \phi(v) = \sup_{\delta > 0} \inf_{v \in C + \delta B_X} \phi(v). \tag{1.1}$$

It is not difficult to prove (see [2]) that the equality

$$\operatorname*{linf}_{v\in C} \phi(v) = \inf_{v \in C} \phi(v)$$

holds whenever the set C is compact and ϕ is lsc. This equality also holds true for every nonempty set $C \subset X$ when the function ϕ is uniformly continuous on $C + sB_X$ for some $s > 0$.

In some developments we will deal with extended real-valued functions $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined on a general set Y (eventually not necessarily normed). The domain of such functions will be denoted by $\operatorname{dom} g$, that is,

$$\operatorname{dom} g = \{y \in Y \ \mid \ g(y) < +\infty\}.$$

2. Existence of minimizer in a drop. The following section is devoted to showing, for a given lsc $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, the existence of an element $\bar{x}$ in a boundedly generated (see Definition 2.1 below) drop $[a,E]$ such that $h(\bar{x}) \le h(a)$ and

$$h(\bar{x}) < h(y) \qquad \forall y \in [\bar{x}, E] \setminus \{\bar{x}\},$$

where $E \subset X$ is a nonempty convex set and $a \in X$ satisfies the inequality $h(a) < \operatorname{linf}_{x\in E} h(x)$.

This existence result requires neither the compactness assumption on the drop nor the boundedness from below assumption on h. It seems that only the shape of the drop, besides the completeness of the space, guarantees the existence of the minimizer. Thus, concerning the feasible set of minimization, neither the assumption of the Weierstrass theorem nor that of the Ekeland variational principle is fulfilled.

Let us start by defining the concept of boundedly generated drop.

DEFINITION 2.1. *Given a nonempty convex set $E \subset X$ and an element $a \notin E$, we say that the drop $[a,E]$ is boundedly generated if $[a,E] \setminus E$ is bounded.*

An example of a boundedly generated drop is shown in Figure 2.1.

A useful result in order to deal with boundedly generated drops is given in the following lemma.

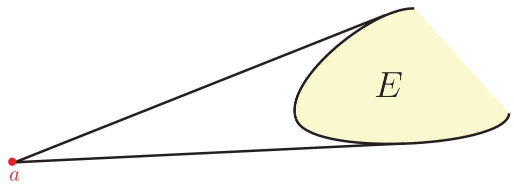

FIG. 2.1. *Example of a boundedly generated drop.*

LEMMA 2.2. *Let $E \subset X$ be a nonempty convex set, and let $a \notin E$. Then the drop $[a, E]$ is boundedly generated if and only if there exists a convex bounded subset $Q \subset E$ such that*

$$[a, E] \setminus E = [a, Q] \setminus E. \tag{2.1}$$

Proof. Let us prove only the nontrivial implication, i.e., the existence of a convex bounded subset $Q \subset E$ such that (2.1) holds whenever the drop $[a, E]$ is boundedly generated. In that case, for $\delta > 0$ let us define the following bounded convex set:

$$Q := E \ \cap \ \mathrm{co}\left(([a, E] \setminus E) + \delta B_X\right),$$

where co stands for the convex hull.

Take $x \in [a, E]\setminus E$ different from a (the case $[a, E]\setminus E = \{a\}$ ensures the equalities $[a, E] \setminus E = [a, Q] \setminus E = \{a\}$, and the proof is finished). Then there exist $t \in \,]0, 1[$ and $e \in E$ such that

$$x = (1 - t)a + te.$$

Define $t_e := \inf\{s \in [0, 1] \mid (1 - s)a + se \in E\}$. By the convexity of E we get $t_e \geq t$; otherwise $x \in E$, which contradicts the choice of $x \notin E$. From the definition of t_e there are $\alpha \in \ [t_e, 1]$ and $\beta \in \ [0, t_e]$ such that

$$(\alpha - \beta)\|a - e\| < \delta,$$

along with $z_\alpha := (1 - \alpha)a + \alpha e \in E$ and $z_\beta := (1 - \beta)a + \beta e \notin E$.

Hence,

$$d_{[a,E]\setminus E}(z_\alpha) \leq \|z_\alpha - z_\beta\| = (\alpha - \beta)\|a - e\| < \delta$$

and $z_\alpha \in E$, which implies $z_\alpha \in Q$. Since $\alpha > t$ (if $\alpha = t$ then $x \in E$, which contradicts $x \notin E$), we conclude the proof from the equality

$$x = \left(1 - \frac{t}{\alpha}\right) a + \frac{t}{\alpha} z_\alpha \in [a, Q] \setminus E. \qquad \square$$

Example 2.3. Let Q be a bounded convex set, $a \in X$, and $E := \{x = t(q-a)+a \mid t \geq 1, q \in Q\}$. Then $[a, E]$ is a boundedly generated drop. This example is slightly pathological. For instance, if $Q = B(0, 1)$ and $a = 0$, one has $E = X$. This observation puts in evidence that we have to assume $a \notin E$ (in Definition 2.1) in order to have an interesting class of sets.

Example 2.4. If E is bounded, then $[a, E]$ is boundedly generated for all $a \in X$.

Example 2.5 (Figure 2.2). If $X = \mathbb{R}^2$ and $E = \{(x, y) \in \mathbb{R}^2 \ \mid \ xy \geq 1, \ x > 0, \ y > 0\}$, for $a = (0, 0)$ the drop $[a, E]$ is not boundedly generated but, if $a = (x_0, y_0) \in \mathbb{R}^2_+$ (where $\mathbb{R}_+ = [0, +\infty[$) with $0 < x_0y_0 < 1$, the drop $[a, E]$ is boundedly generated.

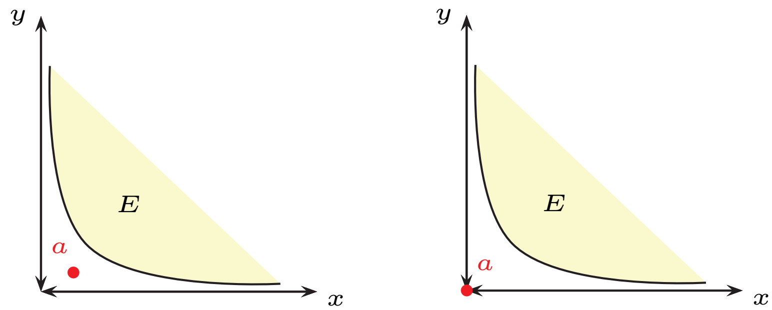

FIG. 2.2. *Example 2.5: Configuration of a and E that implies (left) and does not imply (right) a boundedly generated drop.*

This example shows that the boundedly generated property can be very unstable with respect to the position of the point $a \in X$.

Remark 2.6. If $X = \mathbb{R}^2$, $a = (0,0)$, and $E = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, then $[a,E] = E \cup \{a\}$ and for any bounded subset $Q \subset E$ we get $[a,Q] \setminus E = [a,E] \setminus E = \{a\}$. For some results, we will need $a \notin \operatorname{cl} E$, and therefore, this example does not fit in this context.

Example 2.5 can be also used to observe that if a drop $[a,E]$ is of the form $a + C$, i.e., $[a,E] = a + C$, where C is a cone, then we cannot expect that the drop be boundedly generated. However, whenever the cone C has a compact base, a small perturbation of the set E is enough to get a boundedly generated drop; i.e., the compactness of the base of the cone maintains the drop $[a, E + \varepsilon B_X]$ boundedly generated for all $\varepsilon > 0$, as we will see in the next proposition.

PROPOSITION 2.7. *Let X be a Banach space, let $a \in X$, and let $D \subset X$ be a compact subset such that $0 \notin D$ and $C = [0,\infty[\,D$. Assume that $E \subset X$ is a convex subset such that $[a,E] \subset a + C$ and $a + D \subset [a, E + \delta B_X]$ for every $\delta > 0$. Then for every $\varepsilon > 0$ the drop $[a, E + \varepsilon B_X]$ is boundedly generated.*

Proof. Fix any $\varepsilon > 0$. Observe that

$$[0, E - a + \varepsilon B_X] \setminus (E - a + \varepsilon B_X) = \Big([a, E + \varepsilon B_X] \setminus (E + \varepsilon B_X)\Big) - a,$$

so in order to prove the boundedness it is enough to do it for $a = 0$, since the boundedness is invariant relative to the shifting. So let $a = 0$. Let us suppose the contrary, that is there is a sequence $\{x_k\}_{k\in\mathbb{N}} \subset [0, E + \varepsilon B_X] \setminus (E + \varepsilon B_X)$ such that $\lim_{k\longrightarrow\infty} \|x_k\| = \infty$. For every $k \in \mathbb{N}$ there are $t_k \in [0,1]$, $\alpha_k \in [0,\infty[$, $d_k \in D$, $e_k \in E$, $b_k \in B_X$ such that

$$x_k = t_k(e_k + \varepsilon b_k), \;\; t_k e_k = \alpha_k d_k.$$

By the compactness assumption we may assume that $d_k \longrightarrow d \in D$; otherwise we choose a proper subsequence. Of course $\alpha_k \longrightarrow \infty$, since $\|x_k\| \longrightarrow \infty$. It follows from the inclusion $D \subset [0, E + \delta B_X]$ for every $\delta > 0$ that there is $M > 0$ such that $Md \in E + \frac{\varepsilon}{2} B_X$. Now define a sequence $\{s_k\}_{k\in\mathbb{N}} \subset \,]0,1[$ as follows:

$$s_k := \min\left\{1, \frac{M}{\alpha_k}\right\}.$$

We have $s_k \longrightarrow 0$, $s_k \alpha_k = M$ for k large enough. Put $y_k := s_k x_k$. Observe that $y_k \in [0, E + \varepsilon B_X]$ and $[0, x_k[\cap (E + \varepsilon B_X) = \emptyset$, by the convexity of $(E + \varepsilon B_X)$ and the choice of the sequence $\{x_k\}_{k\in\mathbb{N}} \subset [0, E+\varepsilon B_X] \setminus (E+\varepsilon B_X)$. In fact, if there exists $z \in [0, x_k[\cap (E+\varepsilon B_X)$, then $x_k \in [z, E+\varepsilon B_X] = E+\varepsilon B_X$, which contradicts the choice of x_k. Thus

$$y_k \in [0, E+\varepsilon B_X] \setminus (E+\varepsilon B_X) \tag{2.2}$$

for every $k \in \mathbb{N}$. On the other hand, for k large enough we have $s_k\alpha_k = M$, and therefore $s_k\alpha_k(d_k - d) + s_k t_k \varepsilon b_k \longrightarrow 0$, implying

$$y_k = Md + s_k\alpha_k(d_k - d) + s_k t_k \varepsilon b_k \in (E+\varepsilon B_X),$$

which is a contradiction with (2.2). □

In the next lemma we show that the boundedly generated property can be inherited by some included drops. Namely, if $[a, E]$ is a boundedly generated drop, then $[x, E]$ is boundedly generated too whenever $x \in [a, E] \setminus E$; that is (see Lemma 2.2), there exists a convex bounded set Q_1 such that $[x, E] \setminus E = [x, Q_1] \setminus E$. Moreover, we show below that the set Q_1 is independent from the choice of $x \in [a, E] \setminus E$.

LEMMA 2.8. *Let $E \subset X$ be a nonempty convex set and let $a \notin \operatorname{cl} E$. Assume that a convex subset $Q \subset E$ is such that $[a, E] \setminus E = [a, Q] \setminus E$, and define $Q_1 := [a, Q] \cap E$. Then we have the following properties:*

1. $[a, Q_1] \setminus E = [a, E] \setminus E$.
2. *For all $x \in [a, E] \setminus E$, one has*

$$[x, E] \setminus E = [x, Q_1] \setminus E.$$

Proof. The proof of the first statement is straightforward, so let us prove the second one. Fix $x \in [a, E] \setminus E = [a, Q_1] \setminus E$ and take $y \in [x, E] \setminus E$. Then there exist $\alpha_x, \alpha_y \in [0, 1[$ such that

$$\begin{aligned} x &= (1-\alpha_x)a + \alpha_x q_x && \text{for some } q_x \text{ in } Q_1, \\ y &= (1-\alpha_y)x + \alpha_y b_y && \text{for some } b_y \text{ in } E. \end{aligned}$$

We claim that

$$[a, b_y] \cap Q_1 \neq \emptyset. \tag{2.3}$$

Indeed, if $b_y \in Q_1$, then obviously $b_y \in [a, b_y] \cap Q_1$. If we suppose that $b_y \notin Q_1$, this yields

$$b_y \notin [a, Q] \tag{2.4}$$

because $b_y \in E$ and $Q_1 = [a, Q] \cap E$. Furthermore, the relation $a \notin \operatorname{cl} E$ ensures that $[a, z] \cap E = \emptyset$ for some $z \in]a, b_y]$, and hence

$$[a, z] \subset [a, E] \setminus E = [a, Q_1] \setminus E \subset [a, Q_1],$$

where the equality is due to the first assertion of the lemma. The inclusions $[a, z] \subset [a, Q_1]$ and $z \in]a, b_y]$ provide some $\lambda,\ \mu \in [0, 1[$ such that

$$z = \lambda a + (1-\lambda)b_y = \mu a + (1-\mu)q_1$$

for some $q_1 \in Q_1$. If $\lambda \geq \mu$, we obtain that $q_1 \in [a, b_y] \cap Q_1$, thus proving (2.3). If $\lambda < \mu$, we conclude that $b_y \in [a, Q_1] \subset [a, Q]$, which is a contradiction with (2.4). Therefore, (2.3) holds.

Now, for any $q \in [a, b_y] \cap Q_1$ consider $\alpha_q \in [0,1]$ such that $q = (1-\alpha_q)a + \alpha_q b_y$. Observe that $[q, q_x] \cap [x, b_y] \neq \emptyset$. Indeed, since $x = (1-\alpha_x)a + \alpha_x q_x$, with $\alpha_x < 1$, if we put

$$\alpha := \frac{(1-\alpha_x)\alpha_q}{1-\alpha_x\alpha_q}, \qquad r := \frac{(1-\alpha_q)\alpha_x}{1-\alpha_x\alpha_q},$$

we see

$$(1-r)q + rq_x \in [q, q_x] \subset Q_1, \qquad (1-\alpha)x + \alpha b_y \in [x, b_y]$$

and

$$(1-r)q + rq_x = (1-\alpha)x + \alpha b_y,$$

proving $[q, q_x] \cap [x, b_y] \neq \emptyset$. Choose q_y in the latter intersection. Then $q_y \in [q, q_x] \subset Q_1 \subset E$, so $[q_y, b_y] \subset E$ (since $b_y \in E$). Combining this with the relations $y \in [x, b_y] \setminus E$ and $q_y \in [x, b_y]$ yields $y \in [x, q_y]$; thus $y \in [x, Q_1]$, which proves the inclusion of the first member of assertion 2 into the second one. Since the converse inclusion is evident, the proof is finished. ∎

If we have a boundedly generated drop $[a, E]$, then any drop of the form $[a, E+C]$ is also boundedly generated, whenever C is bounded convex with $0 \in C$. This is explained in the lemma below.

LEMMA 2.9. *Let $C \subset X$ be a bounded convex set with $0 \in C$. Let $E \subset X$ be another convex set and let $a \notin \operatorname{cl} E$. If the drop $[a, E]$ is boundedly generated, then the drop $[a, E+C]$ is boundedly generated too.*

Proof. Choose by Lemma 2.2 a bounded convex set $Q \subset E$ such that $[a, E] \setminus E = [a, Q] \setminus E$. Take Q_1 as in Lemma 2.8, i.e., $Q_1 := [a, Q] \cap E$. Of course, $Q_1 + C$ is convex and bounded. We claim that

$$[a, Q_1 + C] \setminus (E + C) = [a, E + C] \setminus (E + C).$$

In order to prove this equality, let us fix $z \in [a, E+C] \setminus (E+C)$, not equal to a, where $z = (1-t)a + t(e+c)$, with $t \in]0,1[,\ e \in E,\ c \in C$. If $(1-t)a + te \in E$, then $z \in E + C$, which contradicts the choice of z. Then, from the first assertion of Lemma 2.8, there exist $t_1 \in]0,1]$ and $q_1 \in Q_1$ such that $(1-t)a + te = (1-t_1)a + t_1 q_1$. If $t_1 \geq t$, then

$$z = (1-t_1)a + t_1 q_1 + tc = (1-t_1)a + t_1\left(q_1 + \frac{t}{t_1}c\right) \in [a, Q_1 + C]\,.$$

If $t_1 < t$, one obtains

$$e = \left(1 - \frac{t_1}{t}\right)a + \frac{t_1}{t}q_1 \in [a, Q_1]\,,$$

and hence, $e \in [a, Q_1] \cap E \subset [a, Q] \cap E = Q_1$, which proves $z = (1-t)a + t(e+c) \in [a, Q_1 + C]$. ∎

Drops have the nesting property; namely, if we take $y_2 \in [y_1, E]$, then $[y_2, E] \subset [y_1, E]$. Moreover, assuming that $[y_1, E]$ is boundedly generated and E is open or closed, in the next result we show that $\operatorname{cl}\,[y_2, E] \setminus \operatorname{cl} E \subset [y_1, E] \setminus E$ whenever $y_2 \in]y_1, E]$.

PROPOSITION 2.10. *Let $a \in X$ and $E \subset X$ be a nonempty convex set such that $[a, E]$ is a boundedly generated drop and $a \notin \operatorname{cl} E$. If $E \subset X$ satisfies the hypothesis*

$$\forall\ \bar{c} \in \operatorname{cl} E, \quad c \in E \Longrightarrow\ [c, \bar{c}[\ \subset E, \tag{2.5}$$

then, for any $y_1 \in [a, E] \setminus E$ and $y_2 \in\ ]y_1, E] \setminus E$, one has

$$\operatorname{cl}[y_2, E] \setminus \operatorname{cl} E \subset [y_1, E] \setminus E.$$

Proof. Let us take any bounded subset $Q \subset E$ such that $[a, E] \setminus E = [a, Q] \setminus E$ (see Lemma 2.2). It follows from Lemma 2.8 that the bounded convex set $Q_1 := [a, Q] \cap E \subset E$ is such that $[a, E] \setminus E = [a, Q_1] \setminus E$ and, for all $y \in [a, E] \setminus E$, one has $[y, E] \setminus E \subset [y, Q_1] \setminus E$.

Fix $y_1 \in [a, E] \setminus E$ and $y_2 \in\ ]y_1, E] \setminus E$, and take any $\bar{x} \in \operatorname{cl}[y_2, E] \setminus \operatorname{cl} E$. To prove $\bar{x} \in [y_1, E] \setminus E$, in the nontrivial case where $\bar{x} \neq y_2$, consider a sequence

$$\{x_k\}_{k \in \mathbb{N}} \subset [y_2, E] \setminus \operatorname{cl} E \subset [y_2, E] \setminus E\ \subset [y_2, Q_1] \setminus E$$

converging to $\bar{x}$. Write $x_k = \lambda_k y_2 + (1 - \lambda_k) c_k$ for some $c_k \in Q_1$ and $\lambda_k \in [0, 1]$. Without loss of generality we may assume $\lambda_k \longrightarrow \lambda$ for some $\lambda \in [0, 1]$. The limit λ must be lower than one because, if $\lambda_k \longrightarrow 1$, then, from the boundedness of Q_1, we have $\bar{x} = y_2$. Hence, $\bar{x} = \lambda y_2 + (1 - \lambda)\bar{c}$ for some $\bar{c} \in \operatorname{cl} E$ and $\lambda \in [0, 1[$. Observe also that $\lambda > 0$ because $\bar{x} \notin \operatorname{cl} E$.

On the other hand, from Lemma 2.8 we have $y_2 = t_2 y_1 + (1 - t_2)c'$ for some $t_2 \in [0, 1[$ ($t_2 < 1$ because $y_2 \neq y_1$), $c' \in Q_1 \subset E$. From (2.5) together with $t_2 < 1$ and $\lambda > 0$ we have $\left(\frac{\lambda(1-t_2)}{1-\lambda t_2} c' + \frac{(1-\lambda)}{1-\lambda t_2} \bar{c}\right) \in E$ and therefore

$$\bar{x} = \lambda t_2 y_1 + (1 - \lambda t_2) \left(\frac{\lambda(1 - t_2)}{1 - \lambda t_2} c' + \frac{(1 - \lambda)}{1 - \lambda t_2} \bar{c} \right) \in [y_1, E]. \qquad \square$$

Evidently if the convex set E is open or closed, the hypothesis (2.5) in the previous proposition is satisfied. Of course there are sets which are neither open nor closed for which the hypothesis (2.5) is satisfied too. For example, if D is an open convex set of X such that $\operatorname{bd} D = \operatorname{Ext}(\operatorname{cl} D)$, where $\operatorname{bd} D$ stands for the boundary of D and $\operatorname{Ext}(\operatorname{cl} D)$ is the set of extreme points of $\operatorname{cl} D$, then any convex set E of the form $E = D \cup R$ with $R \subset \operatorname{Ext}(\operatorname{cl} D)$ satisfies (2.5).

The next theorem, established in [7], will be crucial for proving the existence of a minimizer in a drop.

THEOREM 2.11 (see [7]). *Let Y be a nonempty set and let S be a nonempty subset of $Y \times Y$ such that there exists a function $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ where $(x, y) \in S \Rightarrow g(y) < g(x)$. Assume that the following condition is satisfied:*

$$\begin{cases} \forall\ \{y_k\}_{k \in \mathbb{N}} \quad \textit{with} \quad (y_k, y_{k+1}) \in S \\ \exists\ y \in Y \quad \textit{and} \quad k_0 \in \mathbb{N} \quad \textit{such that}\ (y_k, y) \in S \qquad \forall\ k \geq k_0. \end{cases} \tag{2.6}$$

Then there exists $y \in Y$ such that for all $y' \in Y$ one has $(y, y') \notin S$.

In the theorem below we show that for a given drop $[a, E]$ satisfying (2.5) and a given function h, we can find a point $\bar{x} \in [a, E]$, which is a minimizer of h on the drop $[\bar{x}, E]$.

THEOREM 2.12. *Let X be a real Banach space, $E \subset X$ be a nonempty convex set, $a \in X$, and $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $h(a) < \operatorname{linf}_{x \in E} h(x)$. Assume that the drop $[a, E]$ is boundedly generated, the set E satisfies property* (2.5) *in Proposition* 2.10, *and h is lsc on $\operatorname{cl}[a, E]$. Then there exists $\bar{x} \in [a, E] \setminus E$ such that*

1. $h(\bar{x}) \leq h(a)$;
2. $h(\bar{x}) < h(y)$ *for all* $y \in]\bar{x}, E]$.

Proof. Since the drop $[a, E]$ is boundedly generated, we have the set $[a, E] \setminus E$ bounded. Indeed (see Lemma 2.2), there exists a bounded convex set $Q \subset E$ such that $[a, E] \setminus E = [a, Q] \setminus E$. Moreover, for $Q_1 = [a, Q] \cap E$ and $x \in [a, E] \setminus E$ (see Lemma 2.8), one has $x \in [a, Q_1] \setminus E$ and

$$[x, E] \setminus E \subset [x, Q_1] \setminus E. \tag{2.7}$$

In order to use Theorem 2.11 for proving the statements, let us define

$$Y := \{u \in [a, E] \mid h(u) \leq h(a)\},$$
$$S := \{(u, v) \in Y \times Y \mid h(u) \geq h(v) \quad \text{and} \quad v \in]u, E]\}.$$

Note that if $(u, v) \in Y \times Y$ and $(u, v) \notin S$, then $h(u) < h(v)$ whenever $v \in]u, E]$; moreover, $u \in Y$ and $v \notin Y$ imply that $h(u) < h(v)$ whenever $v \in]u, E]$. Hence in order to prove the statement, it is enough to find $\bar{x} \in Y$ such that $(\bar{x}, y) \notin S$ for every $y \in]\bar{x}, E]$. If the set S is empty, then the element $\bar{x} := a$ satisfies the conclusions of the theorem. Consider then the case $S \neq \emptyset$. Since $h(a) < \operatorname{linf}_{x \in E} h(x) \leq \inf_{x \in E} h(x)$ we get $Y \subset [a, E] \setminus E$. Therefore, Y is bounded, and furthermore

$$\delta := \inf_{u \in Y} d_E(u) > 0 \tag{2.8}$$

according to the inequality $h(a) < \operatorname{linf}_{x \in E} h(x)$ again. We have also

$$(x, y) \in S \Longrightarrow d_E(x) > d_E(y).$$

Indeed, take any $(x, y) \in S$. On one hand, $h(x) \leq h(a)$ since $x \in Y$, and since

$$h(a) < \operatorname*{linf}_{u \in E} h(u) \leq \inf_{u \in \operatorname{cl} E} h(u),$$

we have $h(x) < \inf_{u \in \operatorname{cl} E} h(u)$; thus $x \notin \operatorname{cl} E$. On the other hand, $y \in [x, E] \setminus \{x\}$ by definition of S; then there are $t \in [0, 1[$ and $e \in E$ such that $y = tx + (1 - t)e$. From the convexity of d_E we get

$$d_E(y) \leq t d_E(x) + (1 - t) d_E(e) = t d_E(x) < d_E(x).$$

Now the idea is to apply Theorem 2.11 with d_E in place of g. In order to do that, let us see that condition (2.6) is satisfied: take any sequence $\{y_k\}_{k \in \mathbb{N}} \subset Y$ such that

$$(y_k, y_{k+1}) \in S.$$

For this sequence, we have

$$d_E(y_k) \geq \delta\ , \quad h(y_k) \geq h(y_{k+1})\ , \quad y_{k+1} \in]y_k, E], \tag{2.9}$$

and hence

$$[a, E] \supset [y_1, E] \supset [y_2, E] \supset \cdots \supset [y_k, E] \supset [y_{k+1}, E]. \tag{2.10}$$

This ensures by (2.7) that there exist sequences $\{t_k\}_{k \in \mathbb{N}} \subset]0, 1[$ and $\{q_k\}_{k \in \mathbb{N}} \subset Q_1 \subset E$ such that

$$y_{k+1} = y_k + t_k(q_k - y_k). \tag{2.11}$$

Hence, from the convexity of the function $d_E(\cdot)$, we get

$$d_E(y_{k+1}) \leq (1-t_k)d_E(y_k) < d_E(y_k), \tag{2.12}$$

and then, for every $m \geq 1$, we obtain

$$\begin{aligned} d_E(y_{m+1}) &\leq d_E(y_m) - t_m d_E(y_m) \\ &\leq d_E(y_{m-1}) - t_{m-1}d_E(y_{m-1}) - t_m d_E(y_m) \\ &\leq d_E(y_1) - \sum_{k=1}^{m} t_k d_E(y_k). \end{aligned}$$

The latter inequality combined with the inequality $\inf_{u\in Y} d_E(u) > 0$ (see (2.8)) implies that the series $\sum_{k=1}^{m} t_k$ converges when $m \to +\infty$. On the other hand, from (2.11) we can write

$$y_{n+m} - y_n = \sum_{k=n}^{n+m-1} t_k(q_k - y_k),$$

and hence

$$\|y_{n+m} - y_n\| \leq \sum_{k=n}^{n+m-1} t_k(\|q_k\| + \|y_k\|).$$

From the boundedness of the sequences $\{y_k\}_{k\in\mathbb{N}} \subset Y$ and $\{q_k\}_{k\in\mathbb{N}} \subset Q_1$, we conclude that $\{y_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, and then there exists $y \in \operatorname{cl} Y \subset \operatorname{cl}[a, E]$ such that $y_k \to y$.

By (2.9), for each $k \in \mathbb{N}$ we have $h(y_k) \geq h(y_n)$ for all $n \geq k$; thus $h(y_k) \geq h(y)$, since h is lsc on $\operatorname{cl}[a,E]$. Let us prove that $y \in Y$ and $y \in \,]y_k, E]$, so $(y_k, y) \in S$ for all k. Since $\delta = \inf_{u\in Y} d_E(u) > 0$, for $k \geq 1$ we have by (2.10) that y belongs to $\operatorname{cl}[y_{k+1}, E] \setminus (E + \delta B_X)$. Moreover, $y_{k+1} \in \,]y_k, E]$ (see (2.9)), and therefore from Proposition 2.10 we get

$$y \in \operatorname{cl}[y_{k+1}, E] \setminus (E + \delta B_X) \subset \operatorname{cl}[y_{k+1}, E] \setminus \operatorname{cl} E \subset [y_k, E] \setminus E \qquad \forall\, k \geq 1.$$

It follows from (2.12) that $d_E(y) < d_E(y_k)$, so we have $y \neq y_k$ for all $k \geq 1$. Hence, $h(y) \leq h(y_k)$ and $y \in \,]y_k, E]$ for all $k \geq 1$, and therefore

$$(y_k, y) \in S \qquad \forall\, k \geq 1.$$

This means that condition (2.6) in Theorem 2.11 is fulfilled, which ensures the existence of some $\bar{x} \in Y$ (in Theorem 2.11 take $\bar{x} = y$) such that

$$(\bar{x}, x) \notin S \qquad \forall\, x \in Y,$$

thus proving the statements of the theorem. ∎

Remark 2.13. One of the referees[1] pointed out to us that Theorem 3.1 in [3] or Theorem 1 in [10] could be used in place of Theorem 2.11. To do that we must follow strictly the same development as in the proof of Theorem 2.12 given above. In order

[1] We thank the referee for this suggestion and the references.

to make everything clear, we first have to reformulate Theorem 3.1 in [3] or Theorem 1 in [10].

THEOREM 2.14 (see [3], [10]). *Let (Y, d) be a metric space and let $\Phi : Y \rightrightarrows Y$ be a set-valued mapping such that*

$$y \in \Phi(y) \quad \text{and} \quad \Phi(\Phi(y)) \subset \Phi(y) \quad \forall\, y \in Y. \tag{2.13}$$

Assume that any Picard sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y (that is, $y_{n+1} \in \Phi(y_n)$) satisfies $d(y_{n+1}, y_n) \to 0$ as $n \to \infty$, and that any Picard sequence $\{y_n\}_{n\in\mathbb{N}}$ with the Cauchy property converges to some point in $\bigcap_{n\in\mathbb{N}} \Phi(y_n)$. Then there is some $\bar{x} \in Y$ such that $\Phi(\bar{x}) = \{\bar{x}\}$.

Although the statement of Theorem 3.1 in [3] (resp., Theorem 1 in [10]) assumes the completeness (resp., order-completeness) of the space Y and the closedness (resp., order-closedness) of the images $\Phi(x)$, the proofs given in [3] and [10] work for the above statement too.

For applying the previous result, let us consider $Y := [a, E] \cap \{u \in X \mid h(u) \leq h(a)\}$ (with the above notation) and define $\Phi(y) := \{u \in [y, E] \mid h(u) \leq h(y)\}$. Obviously we have $y \in \Phi(y)$ and $\Phi(\Phi(y)) \subset \Phi(y)$ for all $y \in Y$. Further, part of the proof of Theorem 2.12, where the convergence of the sequence $\{y_k\}_{k\in\mathbb{N}}$ to $y \in \bigcap_{k\in\mathbb{N}}[y_k, E]$ such that $h(y) \leq h(y_k)$ is shown, can be used to get that any Picard sequence $\{y_k\}_{k\in\mathbb{N}}$ of Φ converges to some $y \in \bigcap_{k\in\mathbb{N}} \Phi(y_k)$; so, it follows from Theorem 2.14 that there is some $\bar{x} \in Y$ such that $\Phi(\bar{x}) = \{\bar{x}\}$. This point $\bar{x}$ enjoys the desired property in Theorem 2.12.

A consequence of Theorem 2.12 is that, for a given drop and a given function, we can find minimizers of the function on some enlargements of the drop.

THEOREM 2.15. *Let X be a real Banach space and let $C \subset X$ be a nonempty convex set. Consider $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $a \in \operatorname{dom} h$ such that $h(a) < \operatorname{linf}_{x\in C} h(x)$ and the drop $[a, C + \varepsilon B_X]$ is boundedly generated for every $\varepsilon > 0$. If h is lsc on a neighborhood of the drop $[a, C]$, then for every $\varepsilon > 0$ there exist $\bar{\varepsilon} \in (0, \varepsilon]$ and $\bar{x}_\varepsilon \in [a, C + \bar{\varepsilon} B_X]$ such that*

1. *$h(\bar{x}_\varepsilon) \leq h(a)$ and $h(\bar{x}_\varepsilon) < h(y)$ for all $y \in\,]\bar{x}_\varepsilon, C + \bar{\varepsilon} B_X]$;*
2. *$h(\bar{x}_\varepsilon) = \operatorname{linf}_{x\in[\bar{x}_\varepsilon, C+\delta B_X]} h(x)$ for all $\delta \in [0, \bar{\varepsilon})$.*

Proof. For $\varepsilon > 0$ take $\bar{\varepsilon} \in (0, \varepsilon]$ such that h is lsc at each point of the drop $[a, C + \bar{\varepsilon} B_X]$ and

$$h(a) < \inf_{x\in C+\bar{\varepsilon} B_X} h(x), \tag{2.14}$$

in particular $a \notin C + \bar{\varepsilon} B_X$.

Applying Theorem 2.12 to the open convex set $E = C + \bar{\varepsilon} B_X$ (which verifies the hypothesis (2.5) in Proposition 2.10) and the point a, we obtain that there exists $\bar{x}_\varepsilon$ in the boundedly generated drop $[a, C + \bar{\varepsilon} B_X]$ such that

$$h(\bar{x}_\varepsilon) \leq h(a); \tag{2.15}$$

$$h(\bar{x}_\varepsilon) < h(y) \quad \forall y \in\,]\bar{x}_\varepsilon, C + \bar{\varepsilon} B_X]. \tag{2.16}$$

Of course (2.15) and (2.16) provide the first assertion of the theorem. In order to get the second assertion, take any $\delta \in [0, \bar{\varepsilon})$ and let us prove that $h(\bar{x}_\varepsilon) = \operatorname{linf}_{x\in[\bar{x}_\varepsilon, C+\delta B_X]} h(x)$. Suppose that the previous equality does not hold. There is

$\varepsilon' \in]\delta, \bar{\varepsilon}[$ such that $\bar{x}_\varepsilon \in [a, C + \varepsilon' B_X]$, and there exist a sequence $\{z_k\}_{k\in\mathbb{N}} \subset X$ and a positive number $\nu > 0$ such that

$$d_{D_\delta}(z_k) \to 0 \qquad \text{where} \quad D_\delta := [\bar{x}_\varepsilon, C + \delta B_X],$$

$$h(z_k) < h(\bar{x}_\varepsilon) - \nu. \tag{2.17}$$

In that case, for each $k \in \mathbb{N}$ there are $c_k \in C$, $u_k \in B_X$, $t_k \in [0, 1]$, and $\{w_k\}_{k\in\mathbb{N}} \subset X$, with $w_k \to 0$, such that

$$z_k = (1 - t_k)\bar{x}_\varepsilon + t_k(c_k + \delta u_k) + w_k.$$

Observe that if

$$(1 - t_k)\bar{x}_\varepsilon + t_k(c_k + \delta u_k) \in C + \varepsilon' B_X$$

for infinitely many $k \in \mathbb{N}$, then $z_k \in C + \bar{\varepsilon} B_X$ for infinitely many $k \in \mathbb{N}$ too. It follows from (2.17) that this is impossible because it contradicts the inequality (2.16). Thus, for k large enough we have

$$(1 - t_k)\bar{x}_\varepsilon + t_k(c_k + \delta u_k) \notin C + \varepsilon' B_X.$$

Lemma 2.2 ensures the existence of a bounded convex set $Q \subset C + \varepsilon' B_X$ such that

$$[a, C + \varepsilon' B_X] \setminus (C + \varepsilon' B_X) = [a, Q] \setminus (C + \varepsilon' B_X),$$

and it follows from (2.14) that $a \notin \operatorname{cl}(C + \varepsilon' B_X)$. Since $\bar{x}_\varepsilon \notin C + \varepsilon' B_X$ for

$$h(\bar{x}_\varepsilon) < \inf_{x \in C + \bar{\varepsilon} B_X} h(x) \ ,$$

applying Lemma 2.8 to the drop $[a, C + \varepsilon' B_X]$ and to the point $\bar{x}_\varepsilon$ in place of x, we obtain for $Q_1 = [a, Q] \cap (C + \varepsilon' B_X)$ that

$$[\bar{x}_\varepsilon, C + \varepsilon' B_X] \setminus (C + \varepsilon' B_X) \subset [\bar{x}_\varepsilon, Q_1] \setminus (C + \varepsilon' B_X).$$

Thus, recalling that $\delta < \varepsilon'$, for k large enough we have

$$y_k := (1 - t_k)\bar{x}_\varepsilon + t_k(c_k + \delta u_k) \in [\bar{x}_\varepsilon, Q_1] \setminus (C + \varepsilon' B_X), \tag{2.18}$$

and this yields $y_k = (1 - s_k)\bar{x}_\varepsilon + s_k q_k$ for some $s_k \in [0, 1]$ and $q_k \in Q_1$. Hence,

$$z_k = (1 - s_k)\bar{x}_\varepsilon + s_k q_k + w_k = y_k + w_k.$$

We may suppose that $\{s_k\}_{k\in\mathbb{N}}$ converges to some $s \in [0, 1]$. If $s = 0$, then the boundedness of the sequence $\{q_k\}_{k\in\mathbb{N}}$ implies $z_k \to \bar{x}_\varepsilon$ and by the lsc property of h at $\bar{x}_\varepsilon \in [a, C + \bar{\varepsilon} B_X]$ and (2.17) we obtain

$$h(\bar{x}_\varepsilon) \le \liminf_{k\to\infty} h(z_k) \le h(\bar{x}_\varepsilon) - \nu,$$

which is a contradiction since $h(\bar{x}_\varepsilon)$ is finite by (2.15). Consequently, $s > 0$. Therefore, for k large enough we may write

$$z_k = (1 - s_k)\bar{x}_\varepsilon + s_k \left(q_k + \frac{w_k}{s_k}\right)$$

and

$$q_k + \frac{w_k}{s_k} \in Q_1 + \frac{w_k}{s_k} \subset C + \varepsilon' B_X + \frac{w_k}{s_k} \subset C + \bar{\varepsilon} B_X,$$

implying, by (2.16) and (2.17), that

$$h(\bar{x}_\varepsilon) \le h(z_k) < h(\bar{x}_\varepsilon) - \nu,$$

which is a contradiction. This completes the proof. □

3. Link with other variational principles. The result established in Theorem 2.12 implies the drop theorem obtained in [4], which is equivalent to the Ekeland variational principle [6], the Krasnosselskii–Zabreiko renorming theorem, the Caristi fixed point principle, and the generalization of the Bishop–Phelps theorem as it was proven in [5]; see also [3], [8], [10] for some other related results.

Below, we will give a simple and short proof of the drop theorem using only statements of Theorem 2.12.

THEOREM 3.1 (drop theorem [4]). *Let X be a Banach space, $S \subset X$ a nonempty closed set, $x_0 \in X \setminus S$, $\varepsilon > 0$, and $r \in]0, R[$, where $R = d_S(x_0)$. Then there exists $\bar{x} \in S$ such that*

$$\|\bar{x} - x_0\| < R + \varepsilon \quad \text{and} \quad S \cap \left[\bar{x}, \overline{B}(x_0, r)\right] = \{\bar{x}\}.$$

Proof. Define the function $h(x) = d_S(x)$ and the closed convex bounded set $E = \overline{B}(x_0, r)$. For $\varepsilon > 0$ take $a \in S$ such that

$$\|x_0 - a\| < R + \varepsilon.$$

It is clear that $[a, E]$ is boundedly generated and the set E satisfies property (2.5) in Proposition 2.10 (because E is a bounded closed convex set). Also, one has

$$h(a) = 0 < R - r \leq \liminf_{x \in E} h(x).$$

Then, from Theorem 2.12, there exists $\bar{x} \in [a, E] \setminus E$ such that

- $0 \leq h(\bar{x}) \leq h(a) = 0 \Rightarrow \ \bar{x} \in S$;
- $0 = h(\bar{x}) < h(y)$ for all $y \in [\bar{x}, E] \setminus \{\bar{x}\} \Rightarrow \ S \cap \left[\bar{x}, \overline{B}(x_0, r)\right] = \{\bar{x}\}$.

Finally, since $\bar{x} \in [a, E] \setminus E$, there exists $t \in]0, 1]$ and $u \in \overline{B}(0, 1)$ such that

$$\bar{x} = ta + (1 - t)(x_0 + ru), \qquad \text{i.e., } \bar{x} - x_0 = t(a - x_0) + (1 - t)ru,$$

which implies

$$\|\bar{x} - x_0\| \leq t\|a - x_0\| + (1 - t)r < t(R + \varepsilon) + (1 - t)r \leq R + \varepsilon,$$

thus concluding the desired result. ∎

Remark 3.2. In the proof of the above result, one can also use an lsc function $h : X \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ such that $h(x) = 0$ for all $x \in S$ and $h(x) > 0$ for all $x \in X \setminus S$. An example different from d_S is the indicator function of the closed set S.

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