USE OF SOME THEOREMS RELATED WITH THE TENSOR EQUATION $AX + XA = H$ FOR SOME CLASSES OF IMPLICIT CONSTITUTIVE RELATIONS

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Summary

The implicit function theorem is used to obtain some necessary and sufficient conditions, for which an implicit constitutive relation can be solved in terms of the stresses and strains, in the sense of expressing, for example, the strains in terms of the stresses or vice versa. For an isotropic body some exact solutions for the strains in term of the stresses are presented, using the theory of tensor equations of the form $AX + XA = H$. For the exact solutions considered, conditions for the existence and the uniqueness are given.

1. Introduction

In some recent works, Rajagopal and coworkers (**1**, **2**, **3**) have proposed constitutive relations for elastic bodies which cannot be classified as either Cauchy or Green elastic bodies (**4**). If $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor and $\mathbf{C}$ is the right Cauchy–Green strain tensor, one such relation is of the form

$$\boldsymbol{f}(\mathbf{S}, \mathbf{C}) = \mathbf{0}, \tag{1.1}$$

where $\boldsymbol{f}$ is an implicit tensor relation. Special classes of (1.1) correspond to Cauchy elastic bodies, where

$$\mathbf{S} = \boldsymbol{h}(\mathbf{C}), \tag{1.2}$$

and the relatively new class of elastic bodies defined by the constitutive equation (**5**)

$$\mathbf{C} = \boldsymbol{g}(\mathbf{S}). \tag{1.3}$$

An interesting question is the following: under which conditions on $\boldsymbol{f}$ could we solve (1.1) in order to obtain $\mathbf{S}$ as an explicit function of $\mathbf{C}$ or vice versa? A similar question has attracted the attention of researchers in the classical theory of elasticity. Let us consider, for example, the case of the linear

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theory of elasticity, where for isotropic homogeneous bodies the constitutive equation is of the form

$$\mathbf{T} = \lambda\,\mathrm{tr}(\boldsymbol{\epsilon})\mathbf{I} + 2\mu\boldsymbol{\epsilon}, \tag{1.4}$$

where λ and μ are the Lamé constants, and $\mathbf{T}$ and $\boldsymbol{\epsilon}$ are the Cauchy stress tensor and the linearised strain tensor, respectively. It is easy to obtain the inversion of (1.4), considering that it is a linear relation, in which case we obtain

$$\boldsymbol{\epsilon} = \frac{1}{2\mu}\mathbf{T} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}(\mathrm{tr}\mathbf{T})\mathbf{I}, \tag{1.5}$$

or equivalently

$$\boldsymbol{\epsilon} = \frac{(1+\nu)}{E}\mathbf{T} - \frac{\nu}{E}(\mathrm{tr}\mathbf{T})\mathbf{I}, \tag{1.6}$$

where $E > 0$ and ν are the Young modulus and the Poisson ratio, respectively, and we have the connections

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}.$$

It is interesting to point out that in order to invert (1.4), some restrictions are necessary for the Lamé constants, namely that $\mu > 0$ and $3\lambda + 2\mu > 0$, that is, the inversion cannot be performed for arbitrary μ and λ. See (**6**, sections 22 and 24) for a detailed discussion on these topics, and also (**7**, footnote 3).

Another interesting fact to mention is that the relation (1.6) is used in experiments in order to characterise a material, where the constants E and ν have a clearer physical meaning than μ and λ. In order to gain some insight into the physical significance of the elastic constants, the usual procedure is to assume some uniform state of stress, such as simple tension or pure shear, which would satisfy the equilibrium equations, and then using (1.6) to obtain the corresponding values of $\boldsymbol{\epsilon}$ (which should not be complicated to measure experimentally, especially on the surface of a body).

In the classical theory of non-linear elasticity similar issues have attracted the attention of researchers. If $\mathbf{P}$ is the nominal stress tensor and $\mathbf{F}$ is the deformation gradient, for a Green elastic body we have a constitutive equation of the form

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \tag{1.7}$$

where $W = W(\mathbf{F})$ is the energy function. The problem of inverting (1.7) has been studied extensively in the past, see, for example (**8**, **9**). It is well known that in general (1.7) cannot be inverted for every $\mathbf{F}$ and W to obtain, for example, $\mathbf{F}$ as a function of $\mathbf{P}$ (**8**). The same conclusion can be reached for (1.2). Despite this, the problem of proposing a given stress state and then studying the corresponding strain field is also important as in the linearised theory. For example, Batra (**10**) considered the problem of assuming a uniform stress field, and found that in such a case a uniform strain field is obtained if W satisfied certain inequalities (compare this with the previous discussion about the restrictions on μ and λ). A similar problem has been treated by Destrade *et al.* (**11**), where it was found that a distribution of uniform shear stress produces a triaxial shear stretch superposed on a simple shear deformation for a hyperelastic body.

In the present work, we are interested in determining conditions on $\boldsymbol{f}$, such that we can express either $\mathbf{S}$ as an explicit function of $\mathbf{C}$ or vice versa; to do so, the implicit function theorem (see, for example (**12**, **13**)) is used in order to give some conditions for the local solvability of the implicit constitutive relation (1.1). Solvability means that the strains can be written as functions of the stresses as in (1.2), or vice versa as in (1.3). Locality is meant in the sense of a neighbourhood of a point in the 12-dimensional space of stresses and strains. That is, the aforementioned equations are valid only on a small neighbourhood of the 12-dimensional stress–strain space.

Special attention is given to the case where $\boldsymbol{f}$ describes the behaviour of an isotropic body. In such a case, there are some situations, where the implicit constitutive relation can be written in the form of the tensor equation $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$. For those cases, some solutions using standard approaches on the topic are presented (see, for example, (**14**, **15**, **16**)). The solutions are accompanied by necessary and sufficient conditions for their existence and uniqueness. When these conditions fail, one may not express the strains as functions of the stresses and vice versa, and we are in the realm of a real implicit constitutive framework. The main novelty of the present article lies in the fact that such conditions have never been reported in the literature, owing to the very recent study of implicit constitutive relations by Rajagopal and coworkers.

2. Local solvability for the generic case

The key point is the application of the implicit function theorem. The idea is to find conditions under which an implicit constitutive relation of the form

$$\boldsymbol{f}(\mathbf{S}, \mathbf{C}) = \mathbf{0} \tag{2.1}$$

can be solved, that is, there exist functions $\boldsymbol{g}$ or $\boldsymbol{h}$ such that

$$\mathbf{C} = \boldsymbol{g}(\mathbf{S}), \quad \mathbf{S} = \boldsymbol{h}(\mathbf{C}). \tag{2.2}$$

Let us give a short summary of the implicit function theorem. For more details see, for example, (**12**, p. 230) or (**13**, p. 13).

THEOREM. Let $\mathcal{U} \subset \mathcal{X}$, $\mathcal{V} \subset \mathcal{Y}$ be open subsets and $f : \mathcal{U} \times \mathcal{V} \to \mathcal{Z} \in C^r$, $r \geq 1$. Given $x_0 \in \mathcal{U}$, $y_0 \in \mathcal{Y}$ assume that $D_2 f(x_0, y_0)$ is an isomorphism. Then, there are neighbourhoods $\mathcal{U}_0$ of x_0, $\mathcal{Y}_0$ of y_0 and $\mathcal{W}_0$ of $f(x_0, y_0)$ and a unique C^r mapping $g : \mathcal{U}_0 \times \mathcal{W}_0 \to \mathcal{Y}_0$ such that for all $(x, w) \in \mathcal{U}_0 \times \mathcal{W}_0$

$$f(x, g(x, w)) = w. \tag{2.3}$$

Setting $w = 0$, the above theorem means that $f(x, y) = 0$ can be solved locally for y as a function of x, if $D_2 f(x_0, y_0)$ is an isomorphism.

For our problem the starting point is the implicit constitutive relation (1.1). We assume that $\boldsymbol{f} \in \mathcal{C}^r$, $r \geq 1$ and we denote the set of all non-singular tensors of second order by $\mathcal{S}$, and the set of all symmetric non-singular tensors by $\mathcal{S}_{\text{sym}}$. The second Fréchet derivative of $\boldsymbol{f}$ with respect to the strains as a mapping has the form

$$D_{\mathbf{CC}}\boldsymbol{f} : \mathcal{S}_{\text{sym}} \to \mathcal{S} \times \mathcal{S} \times \mathcal{S}. \tag{2.4}$$

When evaluated at a specific point $\mathbf{S}_0$, $\mathbf{C}_0$ it has the form

$$D_{\mathbf{CC}}\boldsymbol{f}[\mathbf{S}_0, \mathbf{C}_0] : \mathcal{S}_{\text{sym}} \to \mathcal{S}. \tag{2.5}$$

So, following the implicit function theorem, when $D_{\mathbf{CC}}\boldsymbol{f}[\mathbf{S}_0, \mathbf{C}_0]$ is an isomorphism then the implicit relation $\boldsymbol{f}(\mathbf{S}, \mathbf{C}) = \mathbf{0}$ can be solved in a neighbourhood of the point $(\mathbf{S}_0, \mathbf{C}_0)$ uniquely. The solution renders, for example, the strains as functions of the stresses in the form $\mathbf{C} = \boldsymbol{g}(\mathbf{S})$. Therefore, the strains can be written as functions of the stresses locally, when $\boldsymbol{f} \in C^r$, $r \geq 1$ and $D_{\mathbf{CC}}[\mathbf{S}_0, \mathbf{C}_0]$ is an isomorphism. The solution can be extended to the whole space by partitions of unity on open coverings of the 12-dimensional space of strain and stress.

In order to proceed further we assume that the function $\boldsymbol{f}$ is isotropic. In this case the second Fréchet derivative of $\mathbf{f}$ as a mapping has the form

$$D_{\mathbf{CC}}\boldsymbol{f}[\mathbf{S}_0, \mathbf{C}_0] : S_{\text{sym}} \to S_{\text{sym}}. \tag{2.6}$$

In this case, the range and the domain space are identical finite vector spaces, so isomorphic as well. Thus, the mapping being linear as a derivative it will be an isomorphism if and only if its kernel is trivial (see, for example (**17**, p. 6))

$$D_{\mathbf{CC}}\boldsymbol{f}[\mathbf{S}_0, \mathbf{0}] = \mathbf{0}. \tag{2.7}$$

For writing stresses as functions of strains we interchange their role in the above analysis. So, provided $D_{\mathbf{SS}}\mathbf{f}[\mathbf{S}_0, \mathbf{C}_0]$ is an isomorphism we have the local existence of a function $\boldsymbol{h} \in \mathcal{C}^r$, $r \geq 1$ such that $\mathbf{S} = \boldsymbol{h}(\mathbf{C})$ in a neighbourhood of the point $(\mathbf{S}_0, \mathbf{C}_0)$. The analogous condition to (2.7) will now be $D_{\mathbf{SS}}\boldsymbol{f}[\mathbf{0}, \mathbf{C}_0] = \mathbf{0}$, but the assumptions of isotropy for $\boldsymbol{f}$ and symmetry for $\mathbf{S}$ are not needed.

3. Some classes of solvable constitutive laws and their solutions

For the case where the relation $\boldsymbol{f}(\mathbf{S}, \mathbf{C})$ is an isotropic function of its arguments the implicit constitutive law is written as (**2**)

$$\begin{aligned}\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_2\mathbf{C} + \alpha_3\mathbf{S}^2 + \alpha_4\mathbf{C}^2 + \alpha_5(\mathbf{SC} + \mathbf{CS}) + \alpha_6(\mathbf{S}^2\mathbf{C} + \mathbf{CS}^2)\\ + \alpha_7(\mathbf{C}^2\mathbf{S} + \mathbf{SC}^2) + \alpha_8(\mathbf{S}^2\mathbf{C}^2 + \mathbf{C}^2\mathbf{S}^2) = 0,\end{aligned} \tag{3.1}$$

where the scalar functions α_i, $i = 0, 1, \ldots, 8$ are of the form

$$\alpha_i = \alpha_i\left(\varrho, \text{tr}\mathbf{S}, \text{tr}\mathbf{C}, \text{tr}(\mathbf{S}^2), \text{tr}(\mathbf{C}^3), \text{tr}(\mathbf{S}^3), \text{tr}(\mathbf{SC}), \text{tr}(\mathbf{S}^2\mathbf{C}), \text{tr}(\mathbf{SC}^2), \text{tr}(\mathbf{S}^2\mathbf{C}^2)\right). \tag{3.2}$$

We study some specific cases of the above relation where the strain $\mathbf{C}$ can be written as a function of the stress $\mathbf{S}$. In all the development, one may interchange the role played by strain and stress by making an analogous assumption.

In what follows we assume that all the non-vanishing α_i are functions of the following form

$$\alpha_i = \alpha_i\left(\varrho, \text{tr}\mathbf{S}, \text{tr}(\mathbf{S}^2), \text{tr}(\mathbf{S}^3)\right). \tag{3.3}$$

If we see (3.3) in (**2**), the class of problems considered in the present work is rather limited, but nevertheless it is still a rather wide class of elastic bodies. For example, plane problems related with materials that fall under this class are considered by Bustamante and Rajagopal (**18**) while the same authors treat the inhomogeneous shearing as well as some boundary value problems (**19**, **20**). It is interesting to notice that in the equations shown in Section 2 of (**11**), we have expressions for the left Cauchy Green deformation tensor $\mathbf{B}$ in terms of the Cauchy stress tensor, but the scalar functions β_0, β_1 still depend on the invariants in $\mathbf{B}$. In our case, under the assumption (3.3), we can find completely explicit expressions for either $\mathbf{S}$ in terms of $\mathbf{C}$ or vice versa.

Let us consider the following cases:

(a) $\alpha_0\alpha_1\alpha_2\alpha_3 \neq 0$. In this case the implicit constitutive law (3.1) can be solved at once as

$$\mathbf{C} = -\frac{1}{\alpha_2}\left(\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_3\mathbf{S}^2\right). \tag{3.4}$$

(b) $\alpha_0\alpha_1\alpha_3\alpha_4 \neq 0$. In this case the implicit constitutive law (3.1) gives

$$\mathbf{C}^2 = -\frac{1}{\alpha_4}\left(\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_3\mathbf{S}^2\right). \tag{3.5}$$

So, when the right-hand side of this equation is positive definite, one can solve for $\mathbf{C}$ by taking the square root. One sufficient condition for the positive definiteness of the right-hand side term is the stress tensor to be positive definite and the fractions of the form $-\alpha_0/\alpha_4$, $-\alpha_1/\alpha_4$ and $-\alpha_3/\alpha_4$ to be positive when evaluated at admissible stress states.

(c) $\alpha_0\alpha_1\alpha_3\alpha_5 \neq 0$. In this case the implicit constitutive law (3.1) renders

$$\mathbf{SC} + \mathbf{CS} = -\frac{1}{\alpha_5}\left(\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_3\mathbf{S}^2\right). \tag{3.6}$$

By setting the right-hand side equal to $\mathbf{H}$ this equation can be written as a tensor equation of the form (see (**14**, **15**, **16**))

$$\mathbf{AX} + \mathbf{XA} = \mathbf{H}, \tag{3.7}$$

where the unknown is the tensor $\mathbf{X}$. Systems of the form (3.7) have received some attention in the literature (**14**, **15**, **16**).

The stress tensor $\mathbf{S}$ is symmetric, therefore necessary and sufficient conditions for existence and uniqueness of solution of (3.6) in terms of $\mathbf{C}$ are (see, for example, (**15**, p. 3462))

$$\mathit{III}_{\mathbf{S}} \neq 0, \quad I_{\mathbf{S}}\mathit{II}_{\mathbf{S}} - \mathit{III}_{\mathbf{S}} \neq 0, \tag{3.8}$$

where $I_{\mathbf{S}}$, $\mathit{II}_{\mathbf{S}}$ and $\mathit{III}_{\mathbf{S}}$ are the principal invariants of the tensor $\mathbf{S}$.

The unique solution in this case is (see (**15**, p. 3473))

$$\begin{aligned}
2[I_{\mathbf{S}}\mathit{II}_{\mathbf{S}} - \mathit{III}_{\mathbf{S}}]\mathit{III}_{\mathbf{S}}\mathbf{C} &= 2(I_{\mathbf{S}}^2 - \mathit{II}_{\mathbf{S}})\mathit{III}_{\mathbf{S}}\mathbf{H} - 2\mathit{III}_{\mathbf{S}}(\mathbf{S}^2\mathbf{H} + \mathbf{HS}^2) \\
&+ (I_{\mathbf{S}}\mathit{II}_{\mathbf{S}}^2 + \mathit{II}_{\mathbf{S}}\mathit{III}_{\mathbf{S}} - I_{\mathbf{S}}^2\mathit{III}_{\mathbf{S}})\mathrm{tr}(\mathbf{H})\mathbf{I} - I_{\mathbf{S}}^2\mathit{II}_{\mathbf{S}}[\mathrm{tr}(\mathbf{H})\mathbf{S} + \mathrm{tr}(\mathbf{HS})\mathbf{I}] \\
&+ [I_{\mathbf{S}}\mathit{II}_{\mathbf{S}} + \mathit{III}_{\mathbf{S}}][\mathrm{tr}(\mathbf{H})\mathbf{S}^2 + \mathrm{tr}(\mathbf{HS}^2)\mathbf{I}] + I_{\mathbf{S}}^3\mathit{III}_{\mathbf{S}}\mathrm{tr}(\mathbf{HS})\mathbf{S} \\
&+ -I_{\mathbf{S}}^2[\mathrm{tr}(\mathbf{HS})\mathbf{S}^2 + \mathrm{tr}(\mathbf{HS}^2)\mathbf{S}] + I_{\mathbf{S}}\mathrm{tr}(\mathbf{HS}^2)\mathbf{S}^2.
\end{aligned} \tag{3.9}$$

(d) $\alpha_0\alpha_1\alpha_3\alpha_5\alpha_6 \neq 0$. In this case the implicit constitutive law (3.1) can be written as

$$[\alpha_5\mathbf{S} + \alpha_6\mathbf{S}^2]\mathbf{C} + \mathbf{C}[\alpha_5\mathbf{S} + \alpha_6\mathbf{S}^2] = -[\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_3\mathbf{S}^2]. \tag{3.10}$$

Setting $[\alpha_5\mathbf{S} + \alpha_6\mathbf{S}^2] = \mathbf{A}$ and $-[\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_3\mathbf{S}^2] = \mathbf{H}$, (3.10) is of the form $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$. Therefore, we can write similar necessary and sufficient conditions for existence and uniqueness of solutions of $\mathbf{C}$ in terms of $\mathbf{S}$ as in the previous case and also evaluate the unique solution.

(e) $\alpha_0\alpha_1\alpha_3\alpha_8 \neq 0$. In this case the implicit constitutive law (3.1) can be written as

$$\mathbf{S}^2\mathbf{C}^2 + \mathbf{C}^2\mathbf{S}^2 = -\frac{1}{\alpha_8}[\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_3\mathbf{S}^2]. \qquad (3.11)$$

If we set $\mathbf{C}^2 = \mathbf{X}$ and for the right-hand side $\mathbf{H} = -\alpha_8^{-1}[\alpha_0\mathbf{I} + \alpha_1\mathbf{S} + \alpha_3\mathbf{S}^2]$, then we obtain an equation of the form $\mathbf{S}^2\mathbf{X} + \mathbf{X}\mathbf{S}^2 = \mathbf{H}$, which is solvable for $\mathbf{X}$ as a function of the stress. Then one has the square of the strain as a function of the stress. Thus, by taking the square root of the result one can find the strain as a function of the stress by making suitable assumptions in order for $\mathbf{X}$ to be positive definite.

By analogous reasoning one may search for solutions of the stress as a function of the strain using similar techniques.

4. Conclusions

Some results have been reported on the solvability of some special classes of implicit constitutive relations. Using the implicit function theorem, we presented some generic conditions for the solvability of stresses in terms of strains (or vice versa). For the assumption of an isotropic function and for some specific cases, we gave necessary and sufficient conditions for the solvability, and we presented the solutions as well. Violation of the conditions reported here means that solvability fails, so one has to use the full implicit constitutive relation (1.1) (or (3.1) in the case of isotropic bodies).

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