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Letting `Alice` and `Bob` choose which problem to solve: Implications to the study of cellular automata☆

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ABSTRACT

In previous works we found necessary conditions for a cellular automaton (CA) in order to be intrinsically universal (a CA is said to be intrinsically universal if it can simulate any other). The idea was to introduce different canonical communication problems, all of them parameterized by a CA. The necessary condition was the following: if Ψ is an intrinsically universal CA then the communication complexity of all the canonical problems, when parameterized by Ψ, must be maximal. In this paper, instead of introducing a new canonical problem, we study the setting where they can all be used *simultaneously*. Roughly speaking, when Alice and Bob – the two parties of the communication complexity model – receive their inputs they may choose *online* which canonical problem to solve. We give results showing that such freedom makes this new problem, that we call Ovrl, a very strong filter for ruling out CAs from being intrinsically universal. More precisely, there are some CAs having high complexity in all the canonical problems but have much lower complexity in Ovrl.

1. Introduction

Universality and completeness are central issues in the theories of computation and computational complexity. In fact, understanding universality and self-reproduction in cellular automata became a key problem since the pioneering work of John von Neumann [28]. A one-dimensional cellular automaton (CA) is said to be intrinsically universal if it is able to simulate any other (see [25] for a survey). On the other hand, a CA is said to be Turing universal if it can simulate a universal Turing machine. Of course, if a CA is intrinsically universal then it is also Turing universal. In contrast with the Turing universality notion – for which there is no consensus on its formal definition [10] – the intrinsic universality notion can be completely formalized [10,21,25]. Therefore, proving negative results appears to be a much more approachable problem.

In the quest for small intrinsically universal CAs [23], Ollinger and Richard built an intrinsically universal CA having four states and radius one [26]. On the other hand, Cook proved that the elementary CA Rule 110 (two states, radius one) is Turing universal [8] (the proof is based in the simulation of cyclic tag systems). Despite the fact that being intrinsically universal can be a very common property among the CAs [3,27], the existence of an elementary intrinsically universal CA remains open.

By using results and tools of communication complexity theory, we have previously introduced an approach to prove negative results (i.e., to rule out particular CAs from being intrinsically universal) [11–14]. The idea of applying communication complexity has also been used for proving lower bounds in other models of computation: Turing machines [1,2], VLSI circuits [20], boolean circuits [15,18], decision trees [16], and more.

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In previous works [13,4] we developed the following technique. We defined a computational problem P_Φ parameterized by a CA Φ. We split the input into two parts: one given to Alice and the other given to Bob. Then, we viewed such problem as a communication problem. We proved that the existence of a CA Ψ for which the communication complexity of P_Ψ is greater than the one of P_Φ corresponds to a *certificate* of the fact that Φ is not intrinsically universal.

Five of such canonical problems (which must satisfy some technical properties) have been useful for ruling out different CAs from being intrinsically universal: Pred, Cycl, SInv, TInv and CInv.

1.1. Our contribution

Roughly, it is clear that the main goal of our approach is to find a problem P having a small set of CAs Ψ's for which the communication complexity of P_Ψ is maximal. In such a way, P will be a good filter for ruling out CAs from being intrinsically universal. Instead of finding new problems like P, the idea developed in this paper is to use the canonical ones simultaneously. More precisely, we give the following freedom to Alice and Bob: depending on the input they receive, they choose the problem to solve. By definition, this new problem – which we denote Ovrl – will be much simpler (in terms of communication complexity) than all the canonical ones. Therefore, for a non intrinsically universal CA Φ it will be much more likely to obtain a result saying that Ovrl_Φ has a small communication complexity (and this result will serve as a certificate). In fact, given an input, in order to solve Ovrl_Φ it suffices to find *any* canonical problem P for which P_Φ is simple.

It is known that a necessary condition for a CA Φ to be intrinsically universal is the P-completeness of the prediction problem Pred_Φ when viewed as a classical computational problem [24]. It was a very important result the one obtained by Neary and Woods [22] in which they proved that Pred is P-complete for the elementary CA Rule 110. But it is not known yet whether CA Rule 110 is intrinsically universal. Since it is not difficult to find non intrinsically universal CAs for which Pred is P-complete [9], we think that our approach is a very promising alternative for proving negative results. In fact, there exist CAs whose prediction problem is P-complete but for which the communication complexity is not maximal [13].

In addition, we also would like to point out that the idea of letting Alice and Bob choose the problem they solve is, to our knowledge, new in the communication complexity area.

1.2. Basic definitions

1.2.1. Communication complexity (see [19])

For a function $f : X \times Y \to Z$, the main question in the communication complexity setting is how much information do Alice and Bob need to exchange, in the worst case, in order to compute $f(x, y)$, with Alice knowing only $x \in X$ and Bob only $y \in Y$. This communication problem f is solved by a protocol, which specifies, at each step of the communication between Alice and Bob, who speaks (Alice or Bob), and what she/he says (a bit, 0 or 1), as a function of her/his respective input.

Formally, a protocol $\mathcal{P}$ over a domain $X \times Y$ with range Z is a binary tree where each internal node v is labeled either by a map $a_v : X \to \{0, 1\}$ or by a map $b_v : Y \to \{0, 1\}$, and each leaf ℓ is labeled either by a map $A_\ell : X \to Z$ or by a map $B_\ell : Y \to Z$.

The *value* of protocol $\mathcal{P}$ on input $(x, y) \in X \times Y$ is given by $A_\ell(x)$ (or $B_\ell(y)$) where A_ℓ (or B_ℓ) is the label of the leaf reached by walking on the tree from the root, turning left if $a_v(x) = 0$ (or $b_v(y) = 0$), and right otherwise. We say that a protocol computes a function $f : X \times Y \to Z$ if, for every $(x, y) \in X \times Y$, its value on input (x, y) is $f(x, y)$.

Intuitively, each internal node specifies a bit to be communicated either by Alice or by Bob, whereas at the leaves either Alice or Bob determines the value of f when she/he has received enough information from the other party.

We denote by $\mathbf{cc}(f)$ the (deterministic) communication complexity of a function $f : X \times Y \to Z$. It is the minimal depth of a protocol tree computing f.

Definition 1. Given a function $f : X \times Y \to Z$, a subset $R = A \times B \subseteq X \times Y$ is called f-monochromatic rectangle (in short, monochromatic rectangle) if f is constant on R.

One approach for proving lower bounds on the communication complexity of an arbitrary function f is based on the so-called *fooling sets*.

Definition 2. Given a function $f : X \times Y \to Z$, a set $\mathcal{F} \subseteq X \times Y$ is a *fooling set* for f if there exists $z \in Z$ such that:

1. For every $(x, y) \in \mathcal{F}$, $f(x, y) = z$,
2. For every distinct pairs (x_1, y_1) and (x_2, y_2) in $\mathcal{F}$, either $f(x_1, y_2) \neq z$ or $f(x_2, y_1) \neq z$.

The usefulness of fooling sets is given by the following lemma.

Lemma 3. *If $\mathcal{F}$ is a fooling set of size t for f then* $\mathbf{cc}(f) \geq \log_2(t)$.

Previous notions can be generalized to relations.

Definition 4. A relation $\mathcal{R}$ is a subset $\mathcal{R} \subseteq X \times Y \times Z$. The associated communication problem is the following: Alice receives $x \in X$, Bob receives $y \in Y$ and they have to find a $z \in Z$ such that $(x, y, z) \in \mathcal{R}$.

A protocol $\mathcal{P}$ computes a relation $\mathcal{R}$ if for every legal input $(x, y) \in X \times Y$ the protocol reaches a leaf marked by a value z such that $(x, y, z) \in \mathcal{R}$. Note that an input (x, y) is called legal if there exists at least one $z \in Z$ such that $(x, y, z) \in R$ (otherwise, (x, y) is called illegal).

We denote by $\mathbf{cc}(R)$ the (deterministic) communication complexity of a relation $R \subseteq X \times Y \times Z$. It is the minimal depth of a protocol tree computing R.

Definition 5. Given a relation $\mathcal{R} \subseteq X \times Y \times Z$, a subset $R = A \times B \subseteq X \times Y$ is called monochromatic rectangle if there exists a value z such that for every $(x, y) \in A \times B$ either $(x, y, z) \in R$ or (x, y) is illegal.

Now we introduce two classical communication problems used in next section: EQ, DISJ : $\{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$. $\text{EQ}(x, y) = 1$ iff $x = y$ and $\text{DISJ}(x, y) = 1$ iff $x_i \cdot y_i = 0$ for all $1 \le i \le n$. It is well-known that the best possible protocol for both problems is the one consisting in sending the whole input from one party to the other. In other words, their communication complexity is $\Theta(n)$.

1.2.2. Intrinsic universality in CAs (see [23])

A (one-dimensional) CA is defined by its local rule $\phi : A^{2r+1} \to A$ (where A corresponds to the set of states and r denotes the radius of the local rule). We denote by $\Phi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ the global rule induced by ϕ following the classical definition $\Phi(x)_i = \phi(x_{i-r}, \dots, x_{i+r})$. The t-step iteration of the global function is denoted by $\Phi^t : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$. Note that a global function Φ can be represented by different local functions. All properties considered in this paper depend only on Φ and are not sensitive to the choice of a particular local function. However, to avoid useless formalism, we use the following notion of *canonical* local representation: (ϕ, r) is the canonical local representation of Φ if ϕ has radius r and it is the local function of smallest radius having Φ as its associated global function. We say that a CA Φ_1 is a *sub-automaton* of a CA Φ_2, and we denote $\Phi_1 \sqsubseteq \Phi_2$ if, after renaming the states, we can identify the transitions of Φ_1 in Φ_2. Formally, $\Phi_1 \sqsubseteq \Phi_2$ if there is an injective map ι from A_1 to A_2 such that $\bar{\iota} \circ \Phi_1 = \Phi_2 \circ \bar{\iota}$, where $\bar{\iota} : A_1^{\mathbb{Z}} \to A_2^{\mathbb{Z}}$ denotes the *uniform extension* of ι and A_i is the set of states of the CA Φ_i. Note that $\bar{\iota}$ is the uniform extension of ι if $\bar{\iota}(\cdots x_{-1}x_0x_1\cdots) = \cdots \iota(x_{-1})\iota(x_0)\iota(x_1)\cdots$, for every $(x_i)_{i\in\mathbb{Z}} \in A_1^{\mathbb{Z}}$.

We say that a CA Φ_2 simulates a CA Φ_1 if some *rescaling* of Φ_2 is a sub-automaton of some *rescaling* of Φ_1. The ingredients of the rescalings are simple: packing cells into blocks, iterating the rule and composing with a translation.

Formally, given any state set A and any $m \ge 1$, we define the bijective packing map $\gamma_m : A^{\mathbb{Z}} \to \left(A^m\right)^{\mathbb{Z}}$ by:

$$\forall i \in \mathbb{Z} : \big(\gamma_m(x)\big)(i) = \big(x(mi), \dots, x(mi + m - 1)\big),$$

for all $x \in A^{\mathbb{Z}}$. We define the *shift* map as $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, where $\sigma(x)_i = x_{i+1}$, for each configuration $x \in A^{\mathbb{Z}}$.

The rescaling $\langle m, t, z\rangle$ of Φ by parameters m (packing), $t \ge 1$ (iterating) and $z \in \mathbb{Z}$ (shifting) is the CA with set of states A^m and global rule:

$$\gamma_m \circ \sigma^z \circ \Phi^t \circ \gamma_m^{-1}.$$

The fact that the above function is the global rule of a CA follows from Curtis–Lyndon–Hedlund theorem [17] because it is continuous and commutes with the shift. With these definitions, we have the following.

Definition 6. We say that Φ_2 *simulates* Φ_1, denoted $\Phi_1 \preccurlyeq \Phi_2$, if there exist rescaling parameters $m_1, m_2, t_1, t_2 \in \mathbb{N}$ and $z_1, z_2 \in \mathbb{Z}$ such that

$$\Phi_1^{\langle m_1,t_1,z_1\rangle} \sqsubseteq \Phi_2^{\langle m_2,t_2,z_2\rangle}.$$

We can now naturally define the notion of universality associated to this simulation relation.

Definition 7. Ψ is *intrinsically universal* if for all Φ it holds that $\Phi \preccurlyeq \Psi$.

2. Overlapping in the communication complexity model

We start this section by formalizing the idea of letting several parties (in particular, Alice and Bob) choose which problem to solve.

Definition 8. Let $\{f_i : X \times Y \to Z_i\}_{i=1}^k$ be a family of functions. We define the *overlapping* $f_1 \uplus \cdots \uplus f_k$ of such family as the relation that follows:

$$(x, y, (z, i)) \in f_1 \uplus \cdots \uplus f_k \iff f_i(x, y) = z.$$

In other words, $f_1 \uplus \cdots \uplus f_k$ asks about some index i pointing towards a problem f_i together with the answer $z \in Z_i$ to such problem. The communication complexity of $f_1 \uplus \cdots \uplus f_k$ corresponds to the amount of information Alice and Bob need to exchange in order to find a correct answer. Obviously, $\mathbf{cc}(f_1 \uplus \cdots \uplus f_k) \le \min_{i=1,\dots,k} \mathbf{cc}(f_i)$.

We introduce now a generalization of the classical fooling set notion.

Definition 9. Let $\{f_i : X \times Y \to Z_i\}_{i=1}^k$ be a family of functions. $\mathcal{F} \subseteq X \times Y$ is called a *fooling set* if, for all $1 \le i \le k$, there exists a value $z_i \in Z_i$ such that:

- For every $(x, y) \in \mathcal{F}$, $f_i(x, y) = z_i$.
- For every two distinct pairs (x_1, y_1) and (x_2, y_2) in $\mathcal{F}$, either $f_i(x_1, y_2) \neq z_i$ or $f_i(x_2, y_1) \neq z_i$.

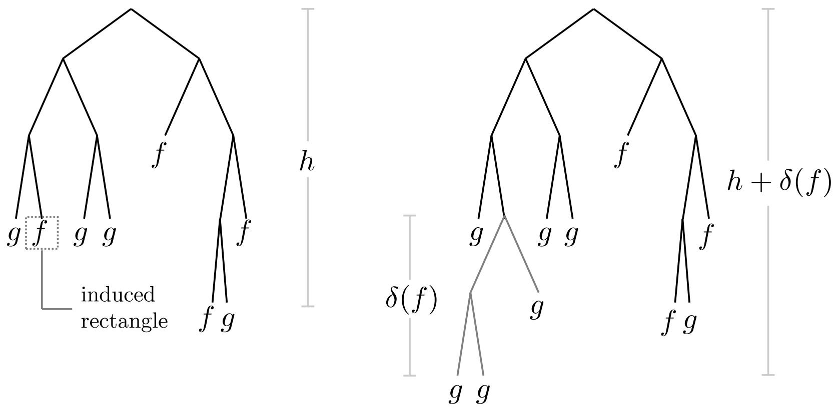

Fig. 1. Construction of protocol $\tilde{\mathcal{P}}$.

Proposition 10. *If $\{f_i : X \times Y \to Z_i\}_{i=1}^{k}$ has a fooling set $\mathcal{F}$ of size t, then:*

$$\mathbf{cc}(f_1 \uplus \cdots \uplus f_k) \geq \log_2 t.$$

Proof. Analogous to the case $k = 1$ [19]. □

Example 11. Consider functions EQ and DISJ which were defined in the Introduction. Recall that $\mathbf{cc}(\mathrm{EQ}), \mathbf{cc}(\mathrm{DISJ}) \in \Theta(n)$. In order to clarify our definition we are going to show that $\mathbf{cc}(\mathrm{EQ} \uplus \mathrm{DISJ}) \in \Theta(\log n)$.

For the upper bound consider the following protocol. If $x = 0 \dots 0$ then Alice sends a 0 to Bob; otherwise she sends a 1. If Bob received a 0 or if $y = 0 \dots 0$ then he answers $\mathrm{DISJ}(x, y) = 1$; otherwise he sends the position i corresponding to the leftmost 1 in y. Finally, Alice compares x_i with y_i. If $x_i = y_i$ then she answers $\mathrm{DISJ}(x, y) = 0$; otherwise she answers $\mathrm{EQ}(x, y) = 0$. Obviously, the complexity of the protocol is $O(\log n)$ because of the number of bits needed to encode index i.

For the lower bound, we are going to prove that the set $\mathcal{F} = \{(x, x) \in \{0, 1\}^n \times \{0, 1\}^n : \sum_{i=1}^{n} x_i = 1\}$ is a fooling set. Let $(x, x), (x', x') \in \mathcal{F}$ such that $x \neq x'$. Then, $\mathrm{DISJ}(x, x) = \mathrm{DISJ}(x', x') = 0$ and $\mathrm{EQ}(x, x) = \mathrm{EQ}(x', x') = 1$, but $\mathrm{DISJ}(x, x') = 1$ and $\mathrm{EQ}(x, x') = 0$. Note that $|\mathcal{F}| = n$ and therefore $\mathbf{cc}\,(\mathrm{EQ} \uplus \mathrm{DISJ}) \in \Theta(\log n)$.

A natural question arises from previous example: Is it true that for every function f with $\mathbf{cc}(f) \in \Theta(n)$ there exists another function g with $\mathbf{cc}(g) \in \Theta(n)$ such that $\mathbf{cc}(f \uplus g) \ll \mathbf{cc}(f)$? For answering this question we introduce the parameter $\delta(f)$.

Definition 12. Given a function $f : X \times Y \to Z$, we define the parameter $\delta(f) = \log_2 \max\{|A| : A \times B \subseteq X \times Y$ is a monochromatic square$\}$, where $A \times B$ is a monochromatic square if it is a monochromatic rectangle and $|A| = |B|$.

Proposition 13. *Let $f : X \times Y \to Z_f$ and $g : X \times Y \to Z_g$. It follows that $\mathbf{cc}(g) \leq \mathbf{cc}\,(f \uplus g) + \delta(f)$.*

Proof. Let $\mathcal{P}$ be a protocol for $f \uplus g$. We can see such protocol as a tree of height h where the set of leaves $L = L_f \cup L_g$ is such that L_f are the answers to f and L_g the answers to g. Obviously, the set of inputs $R_\ell \subseteq X \times Y$ that ends in the leaf $\ell \in L$ corresponds to a monochromatic rectangle of the function it answers.

Now, from $\mathcal{P}$, we can construct another protocol $\tilde{\mathcal{P}}$ that solves g (see Fig. 1). Suppose that with $\mathcal{P}$ we arrive to a leaf ℓ that answers g (i.e., $\ell \in L_g$). If this is the case, no modification is done. In the other case, we know that $\ell \in L_f$ and R_ℓ is an f-monochromatic rectangle (this is a well-known property of protocol trees [19]). Note that with respect to g the rectangle R_l is not necessarily monochromatic. Such rectangle has length or width less or equal to $2^{\delta(f)}$. Then, the complexity of the subproblem $g|_{R_\ell}$ is less or equal to the logarithm of the smaller side of R_ℓ with the trivial protocol that communicates the whole input. Replacing every leaf that answers g with that subprotocol, we construct a new protocol $\tilde{\mathcal{P}}$ where all the leaves belong to L_g. Such protocol solves g and the height of its tree is less or equal to $h + \delta(f)$. Taking the minimum over all the protocols that solve $f \uplus g$ we get that $\mathbf{cc}(g) \leq \mathbf{cc}(f \uplus g) + \delta(f)$. □

Now we are in position to give an answer to the question whether for every function f such that $\mathbf{cc}(f) \in \Theta(n)$ there exists another function g such that $\mathbf{cc}(g) \in \Theta(n)$ and $\mathbf{cc}(f \uplus g) \ll \mathbf{cc}(f)$. The answer, as it is stated in Proposition 15, is negative. More precisely, we are going to prove the existence of a function f^* such that $\delta(f^*) \leq \log n + 1$. For proving this we are going to use (by relaxing and manipulating upper and lower bounds) a well-known, non trivial result from Ramsey theory. More precisely, we are going to identify a monochromatic square with a bipartite monochromatic complete subgraph.

Proposition 14 ([7]). *For all k sufficiently large:*

$$\log k - 1 \leq \log \log b(k) \leq \log k + 1,$$

where $b(k)$ denotes the minimum number such that for every edge bicoloring of the graph $K_{b(k),b(k)}$ there exists a monochromatic subgraph $K_{k,k}$.

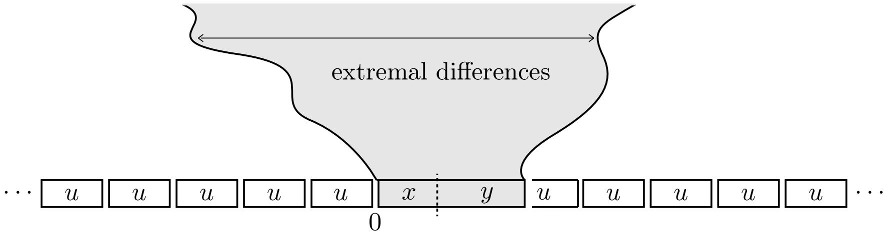

Fig. 2. $\mathrm{SInv}^u_\Phi(x, y)$.

Proposition 15. *There exists a function* $f^* : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ *such that, for every other function* $g : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ *with* $\mathbf{cc}(g) \in \Theta(n)$, *we have* $\mathbf{cc}(f^* \uplus g) \in \Theta(n)$.

Proof. We use the lower bound for $b(k)$ of Proposition 14. Then, identifying functions $f : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ with bicolorings of the graph $K_{2^n,2^n}$, $b(k)$ with 2^n and $\log k$ with $\delta(f)$, we have that there exists f^* such that $\delta(f^*) \leq \log n + 1$ (for all n sufficiently large). Then, by Proposition 13, $\mathbf{cc}(f^* \uplus g) \in \Omega(n - \log n)$ and the result follows. □

Remark 1. By using the upper bound for $b(k)$ from Proposition 14, and the same identification of the proof above, we have that every $f : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ satisfies that $\delta(f) \geq \log n - 1$ (for all n sufficiently large).

3. Communication problems in CAs

We first consider classical computational input–output problems of the form $\mathsf{P} : A^+ \to Z$ whose inputs are words over some alphabet A and outputs are elements of a finite set Z. Given such type of problems P, we define, for any n, its restriction to words of length n, i.e., we consider the restricted problem $\mathsf{P}|_n : A^n \to Z$. The key idea of the communication approach is to *split* the input into two parts. More precisely, for any $1 \leq i < n$, we define $\mathsf{P}|_n^i : A^i \times A^{n-i} \to Z$. Therefore, for every $x \in A^i$ and $y \in A^{n-i}$, we have $\mathsf{P}|_n^i(x, y) = \mathsf{P}|_n(xy)$. Then, we can consider the communication complexity $\mathbf{cc}\left(\mathsf{P}|_n^i\right)$ of the ith split function $\mathsf{P}|_n^i$. Note that, when the lengths of x and y are known, we simply write $\mathsf{P}(x, y)$ instead of $\mathsf{P}|_n^i(x, y)$. Now we can define the communication complexity of P as follows.

Definition 16. Let $\mathsf{P} : A^+ \to Z$ be a computational problem. The *communication complexity* of P, denoted $\mathrm{CC}(\mathsf{P})$, is the function $n \mapsto \max_{1 \leq i < n} \mathbf{cc}\left(\mathsf{P}|_n^i\right)$.

Having this, we proceed to define five problems induced by CAs. These problems are related to prediction, existence of cycles, spatial-invasion, temporal-invasion and controlled-invasion.

Definition 17 (Pred^l_Φ *[13]*)**.** (The CA Φ and $l \in \mathbb{N}$ are fixed parameters). The input of Pred^l_Φ is a word $x \in A^+$. The output is the word $z \in A^+$ that results after iterating $\left\lfloor \frac{|x|-l}{2r} \right\rfloor$ steps (where r is the radius) the CA Φ starting from x. Intuitively, we apply Φ to the finite word x until ending up with a word shorter than $2rl + 1$.

Definition 18 (Cycl^k_Φ *[13]*)**.** (The CA Φ and $k \in \mathbb{N}$ are fixed parameters). The input of Cycl^k_Φ is a word $x \in A^+$. Let $p_x = \ldots xxx \ldots \in A^{\mathbb{Z}}$ be the x-periodic configuration. Clearly, the evolution of Φ starting from p_x becomes periodic (in time) after a finite number of steps. The output of Cycl^k_Φ consists in determining whether the length of this ultimate (temporal) period is less or equal to k (the answer 1 means yes and the answer 0 means no).

Definition 19 (SInv^u_Φ *[13]*)**.** (The CA Φ and $u \in A^+$ are fixed parameters). The input of SInv^u_Φ is a word $x \in A^+$. Let $p_u = \ldots uuu \ldots \in A^{\mathbb{Z}}$ and let $p_u(x) \in A^{\mathbb{Z}}$ be the configuration obtained by putting the word x at the origin over p_u. The output of SInv^u_Φ consists in determining whether the differences between p_u and $p_u(x)$ will expand to an infinite width as times tends to infinity when applying Φ (the answer 1 means yes and the answer 0 means no). See Fig. 2.

Definition 20 (TInv^u_Φ *[4]*)**.** (The CA Φ and $u \in A^+$ are fixed parameters). The input of TInv^u_Φ is a word $x \in A^+$. Let p_u and $p_u(x)$ be defined as in Definition 19. The output of TInv^u_Φ consists in determining whether the differences between p_u and $p_u(x)$ persist forever when applying Φ (the answer 1 means yes and the answer 0 means no).

Definition 21 (CInv^u_Φ *[4]*)**.** (The CA Φ and $u \in A^+$ are fixed parameters). The input of CInv^u_Φ is a word $x \in A^+$. Let p_u and $p_u(x)$ be defined as in Definition 19. The output of CInv^u_Φ consists in determining whether the differences between p_u and $p_u(x)$ persist forever but remain bounded to a finite width $1 \leq w < \infty$ when applying Φ (the answer 1 means yes and the answer 0 means no).

Remark 2. For all CA Φ and for all word u:

$$\mathrm{CInv}^u_\Phi(x) = \left[\mathrm{TInv}^u_\Phi(x) \wedge \neg \mathrm{SInv}^u_\Phi(x)\right].$$

Table 1
Elementary CA Rule 184.

x_i^{t+1}	1	0	1	1	1	0	0	0
$x_{i-1}^t x_i^t x_{i+1}^t$	111	110	101	100	011	010	001	000

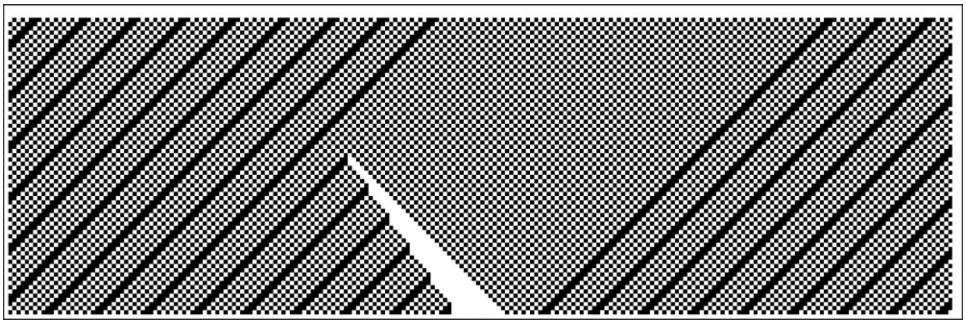

Fig. 3. Case $\#_{00}(u) > \#_{11}(u)$ (time goes up).

4. Elementary CA Rule 184

The goal of this section is to illustrate with a concrete example the power of the overlapping operation. For such purpose we are going to consider the elementary CA Rule 184, denoted by Φ_{184}, which has been used as a model for traffic flow and ballistic annihilation [6], among others.

We are going to prove that, despite the fact that:

$$\exists \overline{u} \in A^+, \quad \text{CC}\left(\text{SInv}^{\overline{u}}_{\Phi_{184}}\right), \text{CC}\left(\text{TInv}^{\overline{u}}_{\Phi_{184}}\right) \in \Theta(\log n),$$

when we overlap the two problems the communication complexity decreases dramatically. More precisely,

$$\forall u \in A^+, \quad \text{CC}\left(\text{SInv}^{u}_{\Phi_{184}} \uplus \text{TInv}^{u}_{\Phi_{184}}\right) \in O(1).$$

The CA Φ_{184} has a set of states $A = \{0, 1\}$ and its local rule is defined in Table 1. We can understand space–time diagrams of Φ_{184} as particles and antiparticles moving with speed 1 in a background $p_{01} = {}^{\infty}01^{\infty} = \ldots 010101\ldots$ [5]. Formally, a particle is the interstice (the gap) between two consecutive 1s (white cells) and an antiparticle is the interstice (the gap) between two consecutive 0s (black cells). Therefore, as shown in Figs. 3–5, a particle can be seen as a pattern 11 which moves to the left while an antiparticle can be seen as pattern 00 moving to the right. Note that a block of ℓ consecutive 1s (0s) corresponds to a block of $\ell - 1$ consecutive particles (antiparticles) moving to the left (right). The key property of this CA is that when a particle collides with an antiparticle both signals annihilate. Therefore, a key property of the initial configuration is the number and position of particles and antiparticles.

Let u be a word in A^+. We denote by $\#_{11}(u)$ the number of particles in u and by $\#_{00}(u)$ the number of antiparticles in u, where we consider u with cyclic boundary (for instance, if $u = 0110$, then $\#_{00}(0110) = \#_{11}(0110) = 1$). Next, consider the set of balanced patterns $\mathcal{B} = \{u \in A^+ : \#_{00}(u) = \#_{11}(u)\}$. It can be verified by induction that, for t large enough (more precisely, for $t > |u|$), any periodic configuration p_u satisfies:

- $\#_{00}(u) = \#_{11}(u) \Rightarrow \Phi^t_{184}(p_u) = p_{01}$.
- $\#_{00}(u) > \#_{11}(u) \Rightarrow \Phi^t_{184}(p_u) = p_v$, for some $v \in A^+$ s.t. $\#_{00}(v) > \#_{11}(v) = 0$.
- $\#_{00}(u) < \#_{11}(u) \Rightarrow \Phi^t_{184}(p_u) = p_v$, for some $v \in A^+$ s.t. $\#_{11}(v) > \#_{00}(v) = 0$.

Considering this, we have the following proposition.

Proposition 22. $\forall u \in A^+, \text{CC}\left(\text{SInv}^{u}_{\Phi_{184}}\right), \text{CC}\left(\text{TInv}^{u}_{\Phi_{184}}\right) \in O(\log n)$.

Proof. We prove the case $\text{SInv}^{u}_{\Phi_{184}}$ (the proof for $\text{TInv}^{u}_{\Phi_{184}}$ is almost the same). The protocol is the following (recall that u is known to both parties):

Case $u \notin \mathcal{B}$**.** $\text{SInv}^{u}_{\Phi_{184}}(x, y) = 0$. W.l.g. suppose that $\#_{00}(u) > \#_{11}(u)$. Since there are infinite antiparticles in p_u any perturbation will be stopped by the antiparticles (and restricted to a width proportional to the size of the input, see Fig. 3).

Case $u \in \mathcal{B}$**.** Note that if we iterate p_u alone then we end up, after $|u|$ steps, with the alternating background $\cdots 01010 \cdots$. Therefore, after $|u|$ steps, the initial configuration $p_u(xy)$ will be transformed into a new one which can be seen as $\cdots 010101x'by'010101 \cdots$ with b being the central bit and where Alice knows $\cdots 010101x'b$ and Bob knows $by'010101 \cdots$ (by sharing the central bit they are sure to count a possible central particle or antiparticle which could be formed by the rightmost cell of Alice with the leftmost cell of Bob in the initial configuration). For having this, Alice and Bob only needs to exchange a number of bits proportional to $|u|$ that do not depend on the size of x and y (see Fig. 4).

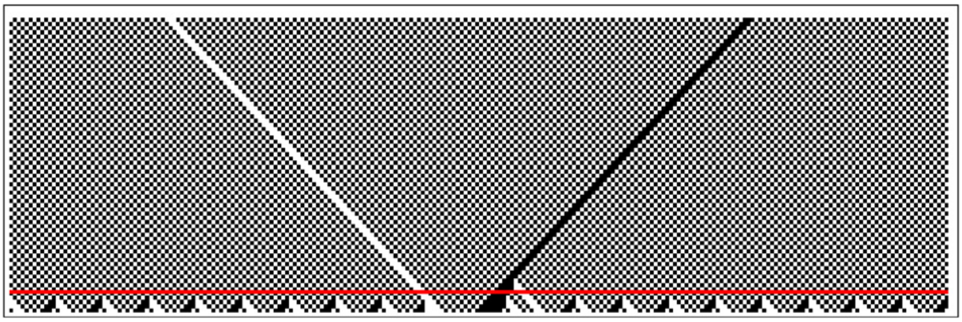

Fig. 4. Case $u \in \mathcal{B}$ and both (1) and (2) occur.

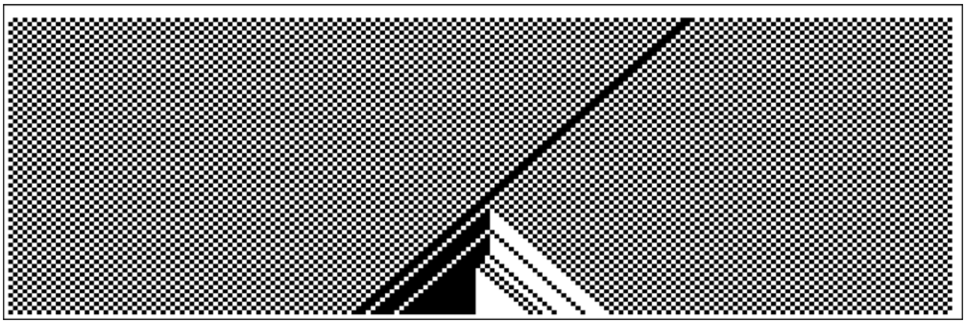

Fig. 5. Case $u \in \mathcal{B}$ and neither (1) nor (2) occurs.

Consider the following two possible situations: (1) from the "side" of Alice (of the form $\cdots 010101x'b$) there exists a particle that propagates infinitely far to the left; (2) from the "side" of Bob (of the form $by'010101\cdots$) there exists an antiparticle that propagates infinitely far to the right.

If (1) and (2) occur, then $\textsc{SInv}^{u}_{\Phi_{184}}(x, y) = 1$, since the gap between the rightmost difference and the leftmost difference grows to infinity (see Fig. 4, time $t = |u|$ is marked with an horizontal line).

If neither (1) nor (2) occurs, then $\textsc{SInv}^{u}_{\Phi_{184}}(x, y) = 0$ (see Fig. 5).

Let suppose, w.l.g, that only (1) occurs. In that case, it is sufficient for Alice to send the number of antiparticles that will cross the border between $x'b$ and by' (more precisely, the number of antiparticles that cross the origin assuming that y' does not differ with the background). With that information, Bob is able to decide whether the distance between the differences will grow to infinity. This last step has a logarithmic cost of information. Therefore, $\mathrm{CC}(\textsc{SInv}^{u}_{\Phi_{184}}) \in O(\log n)$. $\square$

Proposition 23. $\exists \overline{u} \in A^+$, *s.t.* $\mathrm{CC}\left(\textsc{SInv}^{\overline{u}}_{\Phi_{184}}\right), \mathrm{CC}\left(\textsc{TInv}^{\overline{u}}_{\Phi_{184}}\right) \in \Theta(\log n)$.

Proof. We prove the case $\textsc{TInv}^{u}_{\Phi_{184}}$ (the proof for $\textsc{SInv}^{u}_{\Phi_{184}}$ is similar). Consider $\overline{u} = 10$ and the set $\mathcal{F}_n = \left\{\left((00)^{n-i}(10)^i, (10)^i(11)^{n-i}\right) : 0 \le i < n\right\}$. Every $(x, y) \in \mathcal{F}_n$ satisfy that $\textsc{TInv}^{\overline{u}}_{\Phi_{184}}(x, y) = 0$. Let $(x_1, y_1), (x_2, y_2) \in \mathcal{F}_n$ be such that $(x_1, y_1) \neq (x_2, y_2)$. It is clear that in this case $\textsc{TInv}^{\overline{u}}_{\Phi_{184}}(x_1, y_2) = 1$ and, hence, $\mathcal{F}_n$ is a fooling set. Note that $|\mathcal{F}_n| = n$. Therefore, $\mathrm{CC}(\textsc{TInv}^{\overline{u}}_{\Phi_{184}}) \in \Omega(\log n)$. $\square$

Proposition 24. $\forall u \in A^+$, $\mathrm{CC}\left(\textsc{SInv}^{u}_{\Phi_{184}} \uplus \textsc{TInv}^{u}_{\Phi_{184}}\right) \in O(1)$.

Proof. Case $u \notin \mathcal{B}$**.** $\textsc{SInv}^{u}_{\Phi_{184}}(x, y) = 0$. This case corresponds exactly to the first one of Proposition 22. Obviously, no information must be exchanged.

Case $u \in \mathcal{B}$**.** Consider now situations (1) and (2) of the second case of Proposition 22. If (1) and (2) occur, then $\textsc{SInv}^{u}_{\Phi_{184}}(x, y) = 1$. If neither (1) nor (2) occurs, then $\textsc{SInv}^{u}_{\Phi_{184}}(x, y) = 0$. If only one of them occurs, then, instead of answering $\textsc{SInv}^{u}_{\Phi_{184}}$, Alice and Bob *answer the other problem*, $\textsc{TInv}^{u}_{\Phi_{184}}$. In fact, they already know that $\textsc{TInv}^{u}_{\Phi_{184}}(x, y) = 1$ since the particle (or antiparticle) which starts propagating from one side persists in time. Therefore, the amount of communication needed in order to know the case in which they are is constant. $\square$

Remark 3. All these protocols also work for CA Rule 56. The only difference between Rule 56 and Rule 184 is the evaluation of the pattern 111, but this pattern does not have any antecedent, so it disappears after one step. Then, after iterating just one step the initial configuration (this is possible with only two bits of communication), the previous protocols work for CA Rule 56 (however, the fooling set should be modified to obtain a lower bound).

5. Intrinsic universality in CAs: a new tool for proving negative results

We denote $\Phi_1 \preccurlyeq \Phi_2$ when the CA Φ_2 simulates the CA Φ_1. We say that a CA Ψ is *intrinsically universal* if $\Phi \preccurlyeq \Psi$ for every CA Φ. Formal definitions appear in [23]. Finding strong necessary conditions for universality is one of the most challenging problems in theoretical computer science. For tackling that issue we proved in previous works the following result:

Proposition 25 ([13,4])**.** *Let Ψ be an intrinsically universal CA. Then, there exist l, k, u_1, u_2 and u_3 such that:*

$$n \prec \mathrm{CC}(\textsc{Pred}^l_\Psi),\ \mathrm{CC}(\textsc{Cycl}^k_\Psi),\ \mathrm{CC}(\mathrm{S}\textsc{Inv}^{u_1}_\Psi),\ \mathrm{CC}(\mathrm{C}\textsc{Inv}^{u_2}_\Psi),\ \mathrm{CC}(\mathrm{T}\textsc{Inv}^{u_3}_\Psi),$$

where $f_1 \prec f_2$ if there exist non-constant affine functions $\alpha, \beta, \gamma, \delta : \mathbb{N} \to \mathbb{N}$ such that $\alpha \circ f_1 \circ \beta \leq \gamma \circ f_2 \circ \delta$.[1]

Definition 26. Let Φ be a CA. Given parameters $k \in \mathbb{N}$ and $u \in A^+$, we define the problem:

$$\textsc{Ovrl}^{l,k,u}_\Phi := \textsc{Pred}^l_\Phi \uplus \textsc{Cycl}^k_\Phi \uplus \mathrm{S}\textsc{Inv}^u_\Phi \uplus \mathrm{T}\textsc{Inv}^u_\Phi \uplus \mathrm{C}\textsc{Inv}^u_\Phi.$$

The main goal of this section is to obtain the same result of Proposition 25 but for Ovrl. This is a much stronger result because the complexity of Ovrl is always smaller that the canonical problems that we are overlapping. Moreover, as it can be seen in next proposition, the decrease in the complexity could be dramatic.

Proposition 27. *There exist a CA Φ and $l, k \in \mathbb{N}$, $u_1, u_2, u_3 \in A^+$ such that:*

$$\mathrm{CC}(\textsc{Pred}^l_\Phi),\ \mathrm{CC}(\textsc{Cycl}^k_\Phi),\ \mathrm{CC}(\mathrm{S}\textsc{Inv}^{u_1}_\Phi),\ \mathrm{CC}(\mathrm{C}\textsc{Inv}^{u_2}_\Phi),\ \mathrm{CC}(\mathrm{T}\textsc{Inv}^{u_3}_\Phi) \in \Theta(n),$$

and, for all $l, k \in \mathbb{N}$, $u \in A^+$:

$$\mathrm{CC}(\textsc{Ovrl}^{l,k,u}_\Phi) \in O(1).$$

Proof. Given two CAs Φ^1 (with set of states A_1 and local rule ϕ^1) and Φ^2 (with set of states A_2 and local rule ϕ^2), we define the *sum* between them as a new CA $\Phi^1 \oplus \Phi^2$ such that its set of states is the disjoint union of A_1 and A_2 plus an extra symbol #, and its local rule $\phi^1 \oplus \phi^2$ is defined by:

$$\phi^1 \oplus \phi^2(u_{-r}\cdots u_r) = \begin{cases} \phi^1(u_{-r}\cdots u_r) & \text{if } u_{-r}\cdots u_r \in A_1^{2r+1}, \\ \phi^2(u_{-r}\cdots u_r) & \text{if } u_{-r}\cdots u_r \in A_2^{2r+1}, \\ \# & \text{otherwise,} \end{cases}$$

where the radius r is the maximum between the radii of Φ^1 and Φ^2. Roughly speaking, this CA behaves like Φ^1 or Φ^2 if all the states belong to one of the two sets or it erases everything if they are mixed. A basic but important observation is that Φ^i is a sub-automaton of $\Phi^1 \oplus \Phi^2$, for $i = 1, 2$.

On the other hand, we know from [13,4] that there exist CAs $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ and Φ_5 such that:

- $\exists l \in \mathbb{N}$, $\mathrm{CC}(\textsc{Pred}^l_{\Phi_1}) \in \Theta(n)$ and $\forall u \in A_1^+$, $\mathrm{CC}(\mathrm{T}\textsc{Inv}^u_{\Phi_1}) \in O(1)$.
- $\exists u_5 \in A_5^+$, $\mathrm{CC}(\mathrm{T}\textsc{Inv}^{u_5}_{\Phi_5}) \in \Theta(n)$ and $\forall u \in A_5^+$, $\mathrm{CC}(\mathrm{C}\textsc{Inv}^u_{\Phi_5}) \in O(1)$.
- $\exists u_4 \in A_4^+$, $\mathrm{CC}(\mathrm{C}\textsc{Inv}^{u_4}_{\Phi_4}) \in \Theta(n)$ and $\forall u \in A_4^+$, $\mathrm{CC}(\mathrm{S}\textsc{Inv}^u_{\Phi_4}) \in O(1)$.
- $\exists u_3 \in A_3^+$, $\mathrm{CC}(\mathrm{S}\textsc{Inv}^{u_3}_{\Phi_3}) \in \Theta(n)$ and $\forall k \in \mathbb{N}$, $\mathrm{CC}(\textsc{Cycl}^k_{\Phi_3}) \in O(1)$.
- $\exists k \in \mathbb{N}$, $\mathrm{CC}(\textsc{Cycl}^k_{\Phi_2}) \in \Theta(n)$ and $\forall l \in \mathbb{N}$, $\mathrm{CC}(\textsc{Pred}^l_{\Phi_2}) \in O(1)$.

We assert that $\Phi := \Phi_1 \oplus \Phi_2 \oplus \Phi_3 \oplus \Phi_4 \oplus \Phi_5$ satisfies the conditions of the proposition. In fact, we have that $\Phi_i \sqsubseteq \Phi_1 \oplus \Phi_2 \oplus \Phi_3 \oplus \Phi_4 \oplus \Phi_5$ for all i. Since Φ_i is hard for the ith problem, it follows by transitivity of communication complexity under $\sqsubseteq$, that Φ is hard for all the problems. Now, we only have to verify that $\forall l, k \in \mathbb{N}$, $\forall u \in A^+$, $\mathrm{CC}(\textsc{Ovrl}^{l,k,u}_\Phi) \in O(1)$. If the input and the background only have states from a single CA, then we only have to consider the CA with such states and use the protocol of the problem for which it is easy. If not (if there are states from more than one CA), the dynamic becomes trivial because everything is invaded and, in particular, the complexity of $\mathrm{S}\textsc{Inv}^u_\Phi$ is constant. □

The usefulness of Ovrl as a filter for ruling out CAs from being intrinsically universal (Corollary 30) is the result of:

1. The compatibility of Ovrl with our simulation notion (by compatibility we mean the following: if Φ_2 simulates Φ_1 then the communication complexity of $\textsc{Ovrl}_{\Phi_2}$ is greater than or equal to the one of $\textsc{Ovrl}_{\Phi_1}$, see Proposition 28).
2. The existence of a specific CA Φ such that $\textsc{Ovrl}_\Phi$ has high communication complexity (see Proposition 29).

These results are a little bit technical due to the incompatibility of Cycl with the shift (a CA could have different communication complexity for Cycl with respect to a shifted version of itself). In other words, it cannot be proved that the communication complexity is preserved by simulations that use the shift if we want to include the Cycl problem in the overlapping (all the other problems satisfy that). However, as for the Cycl problem itself, we can prove a strongest statement that leads to the same conclusion (Proposition 29).

[1] Note that $\mathrm{CC}(\mathrm{P}) \in \Omega(n)$ implies $n \prec \mathrm{CC}(\mathrm{P})$.

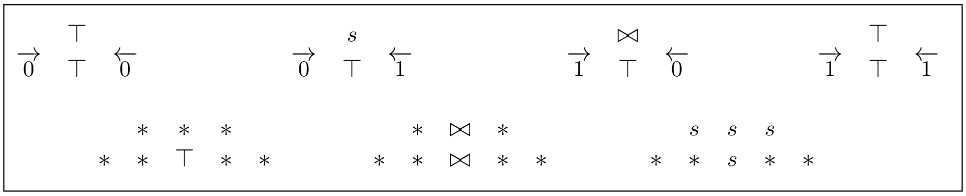

Fig. 6. Some essential values from the local rule of Φ.

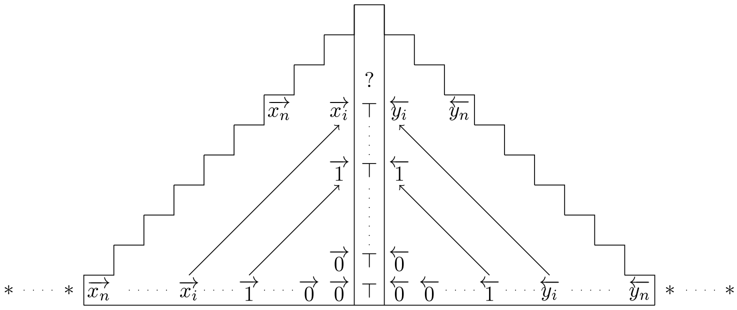

Fig. 7. Hard instances for Φ.

Proposition 28. *If Φ_1 and Φ_2 have set of states of A_1 and A_2, respectively, and*

$$\Phi_1^{\langle m_1,t_1,0\rangle} \sqsubseteq \Phi_2^{\langle m_2,t_2,0\rangle},$$

for some $m_1, m_2, t_1, t_2 \in \mathbb{N}$, then, for all $l \in \mathbb{N}$, $k_0 \in \mathbb{N}$ and $u \in A_1^+$, there exist $l' \in \mathbb{N}$, $k, k' \geq k_0$ and $v \in A_2^+$ such that:

$$\mathrm{CC}(\textsc{Ovrl}_{\Phi_1}^{l,k,u}) \prec \mathrm{CC}(\textsc{Ovrl}_{\Phi_2}^{l',k',v}).$$

Proof. The proof of this proposition comes from the fact that each problem preserves communication complexity under the sub-automaton, packing and iteration transformations modulo change of parameters l, k, and u. However, since the $\prec$ relation means $\leq$ under subsequences, we have to take a common subsubsequence to the subsequence given by each problem, which in this case is possible (see [13,4]). □

Proposition 29. *There exists a specific CA Φ and parameters l, k_0 and u such that:*

$$\mathrm{CC}\left(\textsc{Ovrl}_{\sigma^z\circ\Phi}^{l,k_0,u}\right) \in \Theta(n),$$

for every $z \in \mathbb{Z}$.

Proof. We focus our proof on the case $z = 0$. The difficulty comes from the fact that the three invasion problems (SInv, TInv, and CInv) are related in a logical way such that, generally, when a CA is hard for two of them, the third one becomes easy. To solve this, we consider a CA Φ with set of states $A = \left\{\overrightarrow{0}, \overrightarrow{1}, \overleftarrow{0}, \overleftarrow{1}, \top, *, \bowtie, s\right\}$. The idea is that, given $x, y \in \{0, 1\}^n$, the CA Φ represent in its dynamic a test for $\mathrm{EQ}(x, y)$ but also tests for $\mathrm{GT}(x, y)$ and $\mathrm{GT}(y, x)$. The *greater than* function [19] $\mathrm{GT}(x, y)$, is defined to be 1 if $x > y$ and 0 in another case, when x and y are considered as n-bit integers $0 \leq x, y < 2^n$. To do this, we consider signals carrying 0s and 1s in both directions and a special state $\top$ that do the tests.

In Fig. 6, we define local rules in order to represent the results of the test in the dynamics of Φ, where $\bowtie$ is a wall and s is a spreading state. This rule guarantees that the CA behaves differently according to the value that x and y represent when interpreted in binary notation.

Then, in an instance with a test like in Fig. 7, there are three cases and their respective consequences: $x = y$, $x < y$ and $x > y$. If $l = 1$, $k = 1$, and $u = *$.

Considering an input like $\overrightarrow{x_n} \;\cdots\; \overrightarrow{x_1} \;\top\; \overleftarrow{y_1} \;\cdots\; \overleftarrow{y_n}$, we have the following results:

	$x = y$	$x > y$	$x < y$
$\textsc{Pred}_\Phi^1$	$*$	$\bowtie$	s
$\textsc{SInv}_\Phi^*$	0	0	1
$\textsc{TInv}_\Phi^*$	0	1	1
$\textsc{CInv}_\Phi^*$	0	1	0

Finally, to include a hard instance for the cycle length problem, we only have to add new symbols $\leftarrow$ and $\rightarrow$ which play the role of a signal that is sent in the $x > y$ case and rebounds when it encounters the $\bowtie$ symbol. Then, the cases are the following:

	$x = y$	$x > y$	$x < y$
$\textsc{Cycl}^1_\Phi$	1	0	1

This represents that in the case $x > y$ the signal rebounds having a cycle of length proportional to $\Theta(n)$. On the other hand, in the cases $x = y$ and $x < y$, the cycle have length equal to 1 (in the first case, all is annihilated and only the $\top$ symbol prevails; in the second case, all is erased by the spreading state).

Considering the set $\mathcal{F} = \left\{\left(\overrightarrow{x_n} \cdots \overrightarrow{x_1} \top, \overleftarrow{x_1} \cdots \overleftarrow{x_n}\right) : x_1 \cdots x_n \in \{0, 1\}^n\right\}$, we have that any monochromatic rectangle cannot have two elements of it. In other words, given two elements $x \neq y$ in $\{0, 1\}^n$, the rectangle given by $(\overrightarrow{x} \top, \overleftarrow{x})$, $(\overrightarrow{x} \top, \overleftarrow{y})$, $(\overrightarrow{y} \top, \overleftarrow{x})$ and $(\overrightarrow{y} \top, \overleftarrow{y})$ is not monochromatic for every problem. Then, $\mathcal{F}$ is a fooling set for $\textsc{Ovrl}^{1,1,*}_\Phi$, and $|\mathcal{F}| = 2^n$. Therefore, by Proposition 10, the complexity is in $\Theta(n)$.

Finally, note that for every $z \in \mathbb{Z}$, $\sigma^z \circ \Phi$ has high communication complexity for $\textsc{Ovrl}$. This comes from the fact that: (1) the result of each invasion problem is shift-invariant; (2) the complexity of $\textsc{Pred}_\Phi$ and $\textsc{Pred}_{\sigma^z \circ \Phi}$ is modified by a constant that depend on $|z|$, due to the definition of CC that consider the maximum along every possible partition; (3) the length of cycles is 1 in the case where the spreading state is triggered or $\Omega(n)$ in other case (due to cycle of the signal, or the wall in the shifted case). $\square$

Corollary 30. *Let Ψ be an intrinsically universal CA. Then, there exist l, k and u such that:*

$$n \prec \mathrm{CC}(\textsc{Ovrl}^{l,k,u}_\Psi).$$

Proof. We conclude by using the last two propositions and the fact that every intrinsically universal CA can simulate any other CA without using the shift, but shifting the simulated one (see [9]). $\square$

Open question 1. *Is there any list of problems, each of which is hard for some CA, but such that the overlapping of them becomes easy for any CA?*

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