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On some mixed variational principles in magneto-elastostatics

Franziska Vogel [a,*], Roger Bustamante [b], Paul Steinmann [a]

[a] Chair of Applied Mechanics, Friedrich-Alexander University Erlangen-Nuremberg, Egerlandstr. 5, 91058 Erlangen, Germany
[b] Departamento de Ingeniería Mecánica, Universidad de Chile, Beaucheff 850, Santiago, Chile

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ABSTRACT

Some mixed variational formulations are presented for the problem of a deforming and magneto-active body completely surrounded by free space. The possibility of large elastic deformations is considered, in particular due to the application of magnetic loads. One-, two- and three-field functionals are studied, for which either the magnetic vector potential or the magnetic field strength serve as leading variable. The relations between the different formulations are explained in terms of Legendre transformations and the Lagrange multiplier method.

1. Introduction

In the last years there has been a growing interest in the development of theories to study the behavior of magneto-sensitive bodies. Applications of such theories lie among others in the modeling of magneto-active materials investigated experimentally, for example, in [1–6]. Several aspects of the theory of magneto-elastic interactions considering large deformations had been developed several decades ago. Among the most important early contributions we mention the papers by Brown [7,8] and Tiersten [9], the monographs by Brown [10], by Hutter and van de Ven [11] and the review article by Pao [12] (for the general case of electromagnetic interactions with continua). Furthermore, we recall the books by Maugin [13] and by Maugin and Eringen [14].

The recent development of some new rubber-like materials, which can react to magnetic fields [1–6], has prompted the revision of the theories presented in the classical works mentioned previously, and triggered the development of new alternative formulations. Examples for a discussion considering large deformations can be found in [15–25].

For the case of modeling the behavior of a magneto-active elastic body surrounded by free space in the quasi-static case, it is necessary to solve a system of three non-linear partial differential equations (see, for example, [16,17]). Two of these equations have to be solved inside the body (in order to determine the displacement field and the magnetic field or magnetic induction), whereas the simplified form of the Maxwell equations must be solved in the space surrounding the body. All three equations are coupled and their solutions must satisfy some continuity conditions across the boundary of the body [14,17,26–28]. The possibility of obtaining large deformations, the coupling of the displacement with the solutions of the Maxwell equations inside and outside the body, the need to satisfy a number of continuity conditions for the different variables and finally, the necessity to solve two non-linear partial differential equations inside the body imply that solving this boundary value problem is in general a daunting task, even for simple geometries and boundary conditions (see, for example, [29]). Therefore, the application of numerical methods is particularly important while modeling the behavior of such magneto-active bodies. Among the different numerical methods, the finite element method is one of the most widely used. The finite element method can be developed using variational formulations as a starting point.

In this work we study in detail mixed variational formulations for three classes of problems. In the first case we explore the well known problem of determining the magnetic field distribution for a portion of free space. In the second case we extend the previous results by adding the presence of a rigid magneto-active body immersed in vacuum, considering the presence of the magnetic field inside and outside the body. In the third and last case, we modify the derived mixed variational formulations to be valid for a deformable magneto-elastic body surrounded by free space. There are different options to choose the set of independent variables for each functional associated with the respective variational formulation. In this work the main aim is to develop master functionals, which depend on all three magnetic

* Corresponding author. Tel.: +49 9131 85 28508; fax: +49 9131 85 28503.
E-mail address: franziska.vogel@ltm.uni-erlangen.de (F. Vogel).

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quantities, such as magnetic field strength, magnetic induction, and magnetic vector potential. From these master functionals, sets of three-, two- and one-field functionals are then derived subsequently. For the equivalent electro-elastic problem see, for example, [30].

An important topic addressed in this work is the use of the magnetic vector potential as one of the independent variables of the functionals. In some recent works on the numerical modeling of magneto-sensitive elastomers, a magnetic scalar potential has been defined, from which the magnetic field can be calculated as its negative gradient (see, for example, [29]). Although the use of such a magnetic scalar potential simplifies the implementation of the numerical work, its lack of clear physical meaning implies that it is necessary to consider other possibilities. Alternatively, the magnetic vector potential can be used, from which the magnetic induction is derived as the curl of such a potential, and which has a reasonable physical significance [31].

The use of the magnetic vector potential leads for the three-dimensional case to a lack of uniqueness: if we add the gradient of a scalar field to such a vector potential, we obtain the same magnetic induction field. Different techniques have been developed to address this issue theoretically as well as from a numerical point of view, see, for example, [32–44]. A proper study of the subject requires, in particular, a careful analysis of the functional spaces to which the different variables in the functionals belong. Such a detailed study is absent in most of the works reviewed so far, and so another of the objectives of this work is to do such a detailed study for the different mixed formulations proposed.

Although a lot of work has already been done in the field of variational principles for magneto-elasticity, most of the formulations do not include these three major aspects:

1. Physical setting of a body carrying a volume current immersed into vacuum.
2. Continuum mechanical setting for a geometrically non-linearly deforming body.
3. Functional analytical setting for the independent variables.

Regarding variational principles, one of the early works that can be mentioned is the article by Brown [8], which presents a two-field functional in the magnetization and the magnetic field strength together with the resulting Euler–Lagrange equations for a deforming body in finite strain theory. The surrounding free space and the corresponding jump conditions were not considered in this work. In [9], Tiersten presents an internal energy which was a function of the magnetization, the gradient of the magnetization and the deformation gradient. The variational principle considered a non-linearly deforming body surrounded by free space. In [45], Maugin defines a Lagrangian density depending on the deformation gradient, the magnetic moment and the magnetic moment's gradient. A variation following the Hamiltonian principles leads to the Euler–Lagrange equations that coincide with the magnetostatic Maxwell's equations in terms of magnetic induction and the magnetic couple density. In [46], Penman proposed a functional for a rigid body, which coincides with our primal master functional, however, he considered only linearized constitutive equations and no surrounding space. Rikabi [47] extended the results presented in [46], using the concept of complementary variational principles. As functional, Rikabi used the error in the constitutive relation between the magnetic field and the magnetic induction, which may also be coupled non-linearly.

For the case of a deforming ferromagnets, we would like to mention the works by James [48,49], DeSimone [50,51], Rybka [52] and Bielski [53]. Besides the formulation of functionals including the surface and exchange energy for ferromagnetic material, they also made some statements about the uniqueness and existence of minimizers. However, as stated in [52], energy minimizing solutions without surface energy and exchange energy do not generally exist. To overcome this problem, microstructures defined by minimizing sequences are studied in [49]. For a two-dimensional problem without surface and exchange energy and surface energy considering certain boundary conditions, the existence of minimizers was proved in [50]. Bielski [53] follows the model studied in [48,50–52] and extends it to weakly incompressible and incompressible materials.

Consequently with some of the objectives described previously, this work is divided in the following sections. In Section 2 we review in detail the boundary value problem for the magnetostatic case in free (vacuum) space and a number of variational principles are explored. In Section 3 we extent the analysis given in Section 2, by considering a rigid magneto-active body surrounded by free space. In Section 4 we finally consider the full-blown problem of an elastic magneto-sensitive body, surrounded by free space, which can undergo large deformations and displacements due to the application of magnetic and mechanical loads. For the mixed variational formulations, one of the objectives is to obtain master functionals, from which a number of special cases could be derived systematically, and to study the connections between those functionals by obtaining a chart depicted in Fig. 1. Finally, in Section 5 we discuss the results presented and we outline some future lines of research.

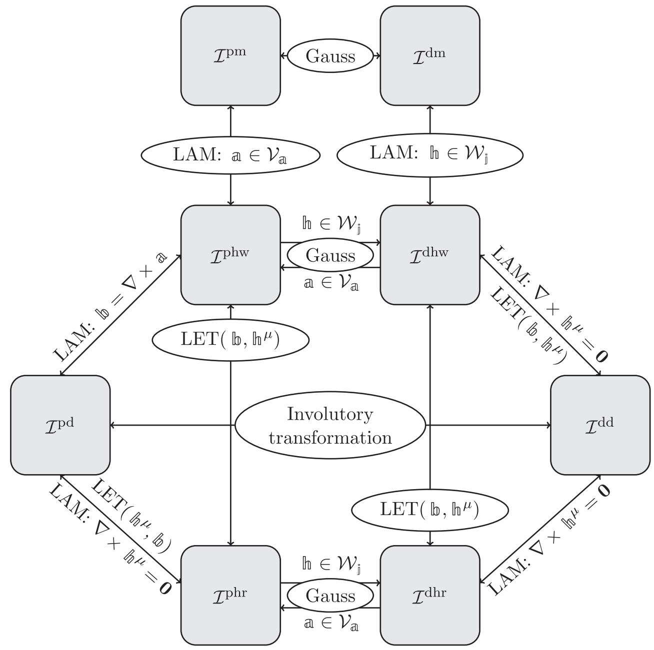

Fig. 1. Extended illustration of the formulations under investigation together with the corresponding transformation procedures. Following the arrows explains how to transform one functional into the other. 'LAM' denotes a Lagrange multiplier procedure together with the respective constraint that has to be added or omitted is displayed. 'LET(•, •)' denotes a Legendre transformation from the first argument to the second argument. The application of the theorem of Gauss is marked by the word 'Gauss'.

2. Free space

Before we consider the case of free space (or vacuum) occupying the domain $\mathcal{S}$ as depicted in Fig. 2, we briefly recall some basic aspects of magnetostatics. In general, the quasi-static

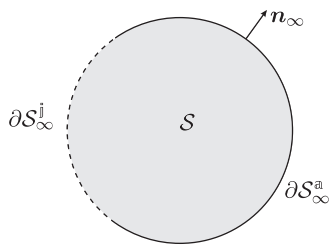

Fig. 2. Illustration of the bounded open and simply connected geometry of free space $\mathcal{S}$ with a Lipschitz-continuous exterior boundary $\partial\mathcal{S}_\infty$.

field equations for the magnetic field $\mathbb{h}$ and the magnetic induction $\mathbb{b}$ read as [14,26–28,54]

$$\nabla\times\mathbb{h}=\mathbb{j}^{\mathrm{f}},\quad \nabla\cdot\mathbb{b}=0, \tag{1}$$

where $\mathbb{j}^{\mathrm{f}}$ denotes a possible volume free current density.

As discussed, for example, in [41], Maxwell's equations are naturally posed in unbounded domains together with the continuity or jump conditions for the magnetic field and the magnetic induction along and across surfaces of material discontinuities [14,26–28,54]

$$\boldsymbol{n}\times[\![\mathbb{h}]\!]=\hat{\mathbb{j}}^{\mathrm{f}},\quad [\![\mathbb{b}]\!]\cdot\boldsymbol{n}=0 \text{ at a surface of material discontinuity,} \tag{2}$$

where $\hat{\mathbb{j}}^{\mathrm{f}}$ (A/m) denotes a surface free current density, and the jump $[\![(\bullet)]\!]$ is defined as the difference of a certain value with regard to the normal vector pointing outwards: $[\![(\bullet)]\!]=(\bullet)_{outside}-(\bullet)_{inside}$.

In order to establish a framework which allows for a numerical approximation of the partial differential equations, a bounded domain with explicit boundary conditions for the unknown magnetic fields has to be defined. To do this, we consider a bounded open and simply connected geometry $\mathcal{S}$ with a Lipschitz-continuous exterior boundary $\partial\mathcal{S}_\infty$ [41,55] as illustrated in Fig. 2.

The surface $\partial\mathcal{S}_\infty$ has to be regarded as an interface that truncates the computational domain $\mathcal{S}$ from the complementary infinite outer domain. Imposing certain behavior for the magnetic field and the magnetic induction in the complementary region, denoted by $\mathbb{h}_\infty$ and $\mathbb{b}_\infty$, results in prescribing boundary conditions for the tangential component of the magnetic field and for the normal component of the magnetic induction [14,26–28,54]

$$\begin{aligned}&\boldsymbol{n}_\infty\times\mathbb{h}=-\hat{\mathbb{j}}^{\mathrm{f}}+\boldsymbol{n}_\infty\times\mathbb{h}_\infty\quad\text{on }\partial\mathcal{S}_\infty,\\&\mathbb{b}\cdot\boldsymbol{n}_\infty=\mathbb{b}_\infty\cdot\boldsymbol{n}_\infty\quad\text{on }\partial\mathcal{S}_\infty.\end{aligned} \tag{3}$$

For the special case of vacuum (free space), the basic magnetic quantities are coupled linearly by the permeability of vacuum $\mu_0=4\pi\times10^{-7}$ V s/A m [26–28]

$$\mathbb{h}^\mu=\frac{1}{\mu_0}\mathbb{b},\quad \mathbb{b}=\mu_0\mathbb{h}^\mu. \tag{4}$$

To emphasize this linear relation in vacuum notation-wise, we will denote the magnetic field in free space by $\mathbb{h}^\mu$ instead of $\mathbb{h}$. Furthermore, we can set the surface current densities equal to zero [28,56] and assume no external sources in vacuum, i.e. $\hat{\mathbb{j}}^{\mathrm{f}}_\infty=\mathbb{j}^{\mathrm{f}}_\infty=\mathbf{0}$. The boundary $\partial\mathcal{S}_\infty$ is split into two connected non-overlapping surfaces,

$$\partial\mathcal{S}_\infty=\partial\mathcal{S}_\infty^{\mathbb{j}}\cup\partial\mathcal{S}_\infty^{\mathbb{a}},\quad \partial\mathcal{S}_\infty^{\mathbb{j}}\cap\partial\mathcal{S}_\infty^{\mathbb{a}}=\emptyset, \tag{5}$$

where the tangential component of $\mathbb{h}^\mu$ or the normal component of $\mathbb{b}$ are prescribed, respectively (see [38,40,41,54]). As a summary we have

$$\nabla\times\mathbb{h}^\mu=\mathbf{0}\quad\text{in }\mathcal{S}, \tag{6a}$$

$$\nabla\cdot\mathbb{b}=0\quad\text{in }\mathcal{S}, \tag{6b}$$

$$\boldsymbol{n}_\infty\times\mathbb{h}^\mu=\boldsymbol{n}_\infty\times\mathbb{h}^\mu_\infty\quad\text{on }\partial\mathcal{S}_\infty^{\mathbb{j}}, \tag{6c}$$

$$\mathbb{b}\cdot\boldsymbol{n}_\infty=\mathbb{b}_\infty\cdot\boldsymbol{n}_\infty\quad\text{on }\partial\mathcal{S}_\infty^{\mathbb{a}}. \tag{6d}$$

As pointed out in [41], for the case $\partial\mathcal{S}_\infty^{\mathbb{j}}=\emptyset$, the hypothesis $\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\mathbb{b}_\infty\cdot\boldsymbol{n}_\infty\,\mathrm{d}a=0$ is needed due to compatibility reasons.

Since the divergence of the magnetic induction is equal to zero, we can introduce a vector potential $\mathbb{a}$ such that [26–28,54–56]

$$\mathbb{b}=\nabla\times\mathbb{a}\quad\text{in }\mathcal{S}. \tag{7}$$

Of course, the vector potential is defined only up to a gradient field, unless a gauge transformation is imposed, which leads to a unique definition [26–28,54–56]. This is taken as divergence free condition on the magnetic vector potential known as the Coulomb gauge [26,27,54–56]

$$\nabla\cdot\mathbb{a}=0\quad\text{in }\mathcal{S}. \tag{8}$$

The introduction of a magnetic vector potential leads together with the normal continuity condition $(2)_2$ on $\mathbb{b}$ to a tangential continuity condition for $\mathbb{a}$ (see, for example, [28,34,35,54,55])

$$\boldsymbol{n}\times[\![\mathbb{a}]\!]=\mathbf{0}\quad\text{across a surface of material discontinuities.} \tag{9}$$

In view of the boundary condition (6d) on $\mathbb{b}$, an essential (Dirichlet) boundary condition on the tangential component of the magnetic potential $\mathbb{a}$ has to be imposed [34,35,54,55][1]

$$\boldsymbol{n}_\infty\times\mathbb{a}=\boldsymbol{n}_\infty\times\mathbb{a}_\infty=:\mathbb{a}_\infty^{\tan}\quad\text{on }\partial\mathcal{S}_\infty^{\mathbb{a}}. \tag{10}$$

The natural (Neumann) boundary conditions on the tangential component of the magnetic vector potential are given by the boundary condition (6c) on $\mathbb{h}^\mu$.

The free magnetic field energy density M expressed in terms of the magnetic induction is defined as [26–28]

$$M(\mathbb{b})=\frac{1}{2\mu_0}\mathbb{b}\cdot\mathbb{b}, \tag{11}$$

while its Legendre transformation, as described for example in [57] and applied to magneto-elasticity for example in [58], is defined as

$$\mathrm{LET}(\mathbb{h}^\mu,\mathbb{b}):\quad M^*(\mathbb{h}^\mu)=\inf_{\mathbb{b}}\{M(\mathbb{b})+\mathbb{h}^\mu\cdot\mathbb{b}\}=-\frac{1}{2}\mu_0\mathbb{h}^\mu\cdot\mathbb{h}^\mu. \tag{12}$$

With this at hand, we define the following constitutive laws for the magnetic field quantities, which coincide with the coupling relation (4):

$$\mathbb{h}^\mu=\frac{\partial M}{\partial\mathbb{b}},\quad \mathbb{b}=-\frac{\partial M^*}{\partial\mathbb{h}^\mu}\text{ in }\mathcal{S}. \tag{13}$$

2.1. Variational principles in free space

The goal of this section is to define several functionals whose stationary points are solutions of the system of partial differential equations described in (6a)–(10). We start with the definition of a Dirichlet type functional, for which we can also state existence and uniqueness of a solution with the help of the Brezzi theorem (see Appendix B). The Dirichlet type functional is then extended to a more general master functional, which is achieved by dropping certain constraints on the unknown variables and incorporating these conditions directly into the functional. In order to do this within a functional analytic framework, a proper

[1] We favor the definition of the boundary conditions (6c) and (10) in terms of the vector product, such that we are able to prescribe functional analytical requirements on $\mathbb{a}$ and $\mathbb{h}$ with the least regularity. (see also Remark 1 in Section 2.1.1).

definition of the involved functional spaces is necessary, which can be found in Definition 1 of Appendix A.

2.1.1. Master functionals

We define the primal Dirichlet-type functional

$$\mathcal{I}^{\mathrm{pd}}(\mathbb{a}) = \int_{\mathcal{S}} M(\mathbb{b})\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathrm{j}}_\infty} \mathbb{h}^\mu_\infty \cdot [\boldsymbol{n}_\infty \times \mathbb{a}]\,\mathrm{d}a \tag{14}$$

for $\mathbb{a} \in \mathcal{V}_{\mathbb{a}}(\mathcal{S})$ and with the constraint $\mathbb{b} = \nabla \times \mathbb{a}$. Under the assumption that $\mathbb{a}$ fulfills the Dirichlet boundary condition (10), and assuming enough regularity for $\mathbb{a}$ and $\delta\mathbb{a}$ and in consequence for $\mathbb{h}^\mu$, we can easily derive that the Euler–Lagrange equations of the functional (14) coincide with Eqs. (6a) and (6c). A theory for existence and uniqueness of the primal Dirichlet-problem is given by the Brezzi theorem. For further details see also Appendix B.

Remark 1 (*Functional space for* $\mathbb{h}^\mu_\infty$). The trace theorem for $\mathrm{H(curl;}\mathcal{S})$ (see [55, Theorem 2.11]) guarantees that the tangential component along the boundary for every function in $\mathrm{H(curl;}\mathcal{S})$ is in $\mathrm{H}^{-1/2}(\partial\mathcal{S}_\infty)$. This means that for each magnetic vector potential $\mathbb{a}$ in $\mathrm{H(curl;}\mathcal{S})$, its tangential component $\boldsymbol{n}_\infty \times \mathbb{a}$ is well-defined. To ensure this also for the boundary integral in (14), we propose for the prescribed boundary term $\mathbb{h}^\mu_\infty \in \mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathrm{j}}_\infty)$. With regard to Footnote 1, it should be observed here that a boundary integral of the form $\int_{\partial\mathcal{S}^{\mathrm{j}}_\infty} [\boldsymbol{n}_\infty \times \mathbb{h}^\mu_\infty] \cdot \mathbb{a}\,\mathrm{d}a$ would call for $\mathbb{a} \in \mathrm{H}^{1,2}(\mathcal{S})$, which is not the desired choice.

Using the primal Dirichlet functional as a starting point, it is our goal to derive further one-, two- and three-field functionals whose stationary points represent a solution to the magnetostatic problem. To pursue this intention, we develop in this section two types of master functionals. From these master functionals, the desired one-, two- and three-field functionals can be derived later on in a systematic top-down procedure, depending on the number of strong assumptions proposed directly onto the master functionals.

Adding the constraint $\mathbb{b} = \nabla \times \mathbb{a}$ (7) and the Dirichlet boundary condition on $\mathbb{a}$, $\boldsymbol{n}_\infty \times \mathbb{a} = \boldsymbol{n}_\infty \times \mathbb{a}_\infty$ on $\partial\mathcal{S}^{\mathbb{a}}_\infty$ (see Eq. (10)), with the Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ to the primal Dirichlet functional (14), renders the preliminary functional:

$$\begin{aligned}\mathcal{I}(\mathbb{a},\mathbb{b},\boldsymbol{\lambda},\boldsymbol{\mu}) = &\int_{\mathcal{S}} M(\mathbb{b})\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathrm{j}}_\infty} \mathbb{h}^\mu_\infty \cdot [\boldsymbol{n}_\infty \times \mathbb{a}]\,\mathrm{d}a \\ &- \int_{\mathcal{S}} \boldsymbol{\lambda} \cdot [\mathbb{b} - \nabla \times \mathbb{a}]\,\mathrm{d}v + \int_{\partial\mathcal{S}^{\mathbb{a}}_\infty} \boldsymbol{\mu} \cdot [\boldsymbol{n}_\infty \times \mathbb{a} - \boldsymbol{n}_\infty \times \mathbb{a}_\infty]\,\mathrm{d}a.\end{aligned} \tag{15}$$

The variation of the preliminary functional $\mathcal{I}^{\mathrm{pm}}$ leads to a set of Euler–Lagrange equations which give a meaningful interpretation to the Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. The imposition of these Lagrange multipliers directly into the preliminary functional $\mathcal{I}$ leads to the following primal master functional $\mathcal{I}^{\mathrm{pm}}$:

$$\begin{aligned}\mathcal{I}^{\mathrm{pm}}(\mathbb{a},\mathbb{b},\mathbb{h}^\mu) = &\int_{\mathcal{S}} M(\mathbb{b})\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathrm{j}}_\infty} \mathbb{h}^\mu_\infty \cdot [\boldsymbol{n}_\infty \times \mathbb{a}]\,\mathrm{d}a \\ &- \int_{\mathcal{S}} \mathbb{h}^\mu \cdot [\mathbb{b} - \nabla \times \mathbb{a}]\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathbb{a}}_\infty} \mathbb{h}^\mu \cdot [\boldsymbol{n}_\infty \times \mathbb{a} - \boldsymbol{n}_\infty \times \mathbb{a}_\infty]\,\mathrm{d}a\end{aligned} \tag{16}$$

for $\mathbb{a} \in \mathrm{H(curl;}\mathcal{S})$, $\mathbb{b} \in \mathrm{L}^2(\mathcal{S})$, $\mathbb{h}^\mu \in \mathrm{L}^2(\mathcal{S})$ and $\mathbb{h}^\mu \in \mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathbb{a}}_\infty)$.

Remark 2 (*Functional space for* $\mathbb{a}_\infty$ *and* $\mathbb{h}^\mu$). We propose that $\mathbb{h}^\mu \in \mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathbb{a}}_\infty)$ with the same argument as in Remark 1. In consequence, this leads to the assumption that $\mathbb{a}_\infty \in \mathrm{H(curl;}\mathcal{S})$. The restriction on $\mathbb{h}^\mu$ is obviously fulfilled if we assume more regularity on $\mathbb{h}^\mu$, as for example, $\mathbb{h}^\mu \in \mathrm{H}^{1,2}(\mathcal{S})$. Then, the trace theorem for $\mathrm{H}^{1,2}(\mathcal{S})$ guarantees that the boundary values of $\mathbb{h}^\mu$ are well-defined without further ado.

In order to exchange the weak Dirichlet boundary condition appearing in the boundary integral, we apply the Gauss theorem to the primal master functional $\mathcal{I}^{\mathrm{pm}}$, which leads to the following dual master functional:

$$\begin{aligned}\mathcal{I}^{\mathrm{dm}}(\mathbb{a},\mathbb{b},\mathbb{h}^\mu) = &\int_{\mathcal{S}} M(\mathbb{b})\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathrm{j}}_\infty} \mathbb{a} \cdot [\boldsymbol{n}_\infty \times \mathbb{h}^\mu - \boldsymbol{n}_\infty \times \mathbb{h}^\mu_\infty]\,\mathrm{d}a \\ &- \int_{\mathcal{S}} [\mathbb{h}^\mu \cdot \mathbb{b} - [\nabla \times \mathbb{h}^\mu] \cdot \mathbb{a}]\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathbb{a}}_\infty} \mathbb{a}_\infty \cdot [\boldsymbol{n}_\infty \times \mathbb{h}^\mu]\,\mathrm{d}a\end{aligned} \tag{17}$$

for $\mathbb{a} \in \mathrm{L}^2(\mathcal{S})$ and $\mathbb{a} \in \mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathrm{j}}_\infty)$, $\mathbb{b} \in \mathrm{L}^2(\mathcal{S})$, $\mathbb{h}^\mu \in \mathrm{H(curl;}\mathcal{S})$. The functional spaces for $\mathbb{a}$ and $\mathbb{h}^\mu$ now switched roles in contrast to the primal case. The given boundary values now have to belong to the spaces $\mathbb{a}_\infty \in \mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathbb{a}}_\infty)$ and $\mathbb{h}^\mu_\infty \in \mathrm{H(curl;}\mathcal{S})$.

Table 1
Classification of results according to weak and strong satisfaction of field equations and boundary conditions in free space.

In $\mathcal{S}$	phw	phr	pd	dhw	dhr	dd	Eq.
$\nabla \times \mathbb{h}^\mu = \mathbf{0}$	Weak	Weak	Weak	Weak	Weak	Strong	(6a)
$\mathbb{b} = \nabla \times \mathbb{a}$	Weak	Weak	Strong	Weak	Weak	Weak	(7)
$\boldsymbol{n}_\infty \times \mathbb{h}^\mu = \boldsymbol{n}_\infty \times \mathbb{h}^\mu_\infty$	Weak	Weak	Weak	Strong	Strong	Strong	(6c)
$\boldsymbol{n}_\infty \times \mathbb{a} = \mathbb{a}^{\tan}_\infty$	Strong	Strong	Strong	Weak	Weak	Weak	(10)
$\mathbb{h}^\mu = \dfrac{\partial M}{\partial \mathbb{b}}$	Weak	–	Strong	Weak	–	–	$(13)_1$
$\mathbb{b} = -\dfrac{\partial M^*}{\partial \mathbb{h}^\mu}$	–	Strong	–	–	Strong	Strong	$(13)_2$

2.1.2. Three-, two- and one field functionals

From the master functionals, Hu–Washizu-, Hellinger–Reissner- and Dirichlet-type functionals can now be derived in a systematic top-down procedure. The primal functionals result from the primal master functional and the dual functionals from the dual master functional, respectively. Table 1 summarizes the respective choices of the effective assumptions to obtain the different variational formulations. We present the different functionals in the following paragraphs:

(a) *Primal Hu–Washizu Principle* (*phw*): Starting with the primal master functional (16) and enforcing that $\mathbb{a} \in \mathcal{V}_{\mathbb{a}}(\mathcal{S})$, we obtain the primal Hu–Washizu functional:

$$\begin{aligned}\mathcal{I}^{\mathrm{phw}}(\mathbb{a},\mathbb{b},\mathbb{h}^\mu) = &\int_{\mathcal{S}} [M(\mathbb{b}) - \mathbb{h}^\mu \cdot [\mathbb{b} - \nabla \times \mathbb{a}]]\,\mathrm{d}v \\ &- \int_{\partial\mathcal{S}^{\mathrm{j}}_\infty} \mathbb{h}^\mu_\infty \cdot [\boldsymbol{n}_\infty \times \mathbb{a}]\,\mathrm{d}a,\end{aligned} \tag{18}$$

for $\mathbb{a} \in \mathcal{V}_{\mathbb{a}}(\mathcal{S})$, $\mathbb{b} \in \mathrm{L}^2(\mathcal{S})$, $\mathbb{h}^\mu \in \mathrm{L}^2(\mathcal{S})$ and $\delta\mathbb{a} \in \mathcal{V}_0(\mathcal{S})$, $\delta\mathbb{b} \in \mathrm{L}^2(\mathcal{S})$, $\delta\mathbb{h}^\mu \in \mathrm{L}^2(\mathcal{S})$.

Let $(\mathbb{a},\mathbb{b},\mathbb{h}^\mu)$ be a saddle point[2] of $\mathcal{I}^{\mathrm{phw}}$. Exploiting the boundary condition of $\mathcal{V}_0(\mathcal{S})$ for $\delta\mathbb{a}$ and supposing enough regularity of $\mathbb{h}^\mu$ and $\delta\mathbb{a}$, this triple is the result of the variation

$$\begin{aligned}\delta\mathcal{I}^{\mathrm{phw}}(\mathbb{a},\mathbb{b},\mathbb{h}^\mu) &= \int_{\mathcal{S}} \left[\frac{\partial M(\mathbb{b})}{\partial \mathbb{b}} \cdot \delta\mathbb{b} - \delta\mathbb{h}^\mu \cdot [\mathbb{b} - \nabla \times \mathbb{a}] - \mathbb{h}^\mu \cdot \delta\mathbb{b}\right]\mathrm{d}v \\ &\quad + \int_{\mathcal{S}} \mathbb{h}^\mu \cdot [\nabla \times \delta\mathbb{a}]\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathrm{j}}_\infty} \mathbb{h}^\mu_\infty \cdot [\boldsymbol{n}_\infty \times \delta\mathbb{a}]\,\mathrm{d}a \\ &= \int_{\mathcal{S}} \left[\left[\frac{\partial M(\mathbb{b})}{\partial \mathbb{b}} - \mathbb{h}^\mu\right] \cdot \delta\mathbb{b} - \delta\mathbb{h}^\mu \cdot [\mathbb{b} - \nabla \times \mathbb{a}]\right]\mathrm{d}v \\ &\quad + \int_{\mathcal{S}} [\nabla \times \mathbb{h}^\mu] \cdot \delta\mathbb{a}\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathrm{j}}_\infty} \delta\mathbb{a} \cdot [\boldsymbol{n}_\infty \times \mathbb{h}^\mu - \boldsymbol{n}_\infty \times \mathbb{h}^\mu_\infty]\,\mathrm{d}a.\end{aligned}$$

It is possible to see that Eqs. (6a), (6c), (7) and $(13)_1$ are weakly fulfilled by the solution of the saddle point problem.

[2] We assume that the stationary points of the two- and three field principles are saddle points, because solving for the magnetic vector potential $\mathbb{a}$ necessitates the formulation of a saddle point problem already within the primal Dirichlet framework.

(b) *Primal Hellinger–Reissner principle* (*phr*): Performing a Legendre transformation on the Hu–Washizu functional reduces the number of independent variables by one, and yields the Hellinger–Reissner functional:

$$\mathcal{I}^{\text{phr}}(\mathbb{a},\mathbb{h}^\mu)=\int_{\mathcal{S}}[M^*(\mathbb{h}^\mu)+\mathbb{h}^\mu\cdot[\nabla\times\mathbb{a}]]\,\mathrm{d}v-\int_{\partial\mathcal{S}_\infty^{\mathbb{j}}}\mathbb{h}_\infty^\mu\cdot[\boldsymbol{n}_\infty\times\mathbb{a}]\,\mathrm{d}a, \tag{19}$$

for $\mathbb{a}\in\mathcal{V}_{\mathbb{a}}(\mathcal{S})$, $\mathbb{h}^\mu\in\mathrm{L}^2(\mathcal{S})$ and $\delta\mathbb{a}\in\mathcal{V}_0(\mathcal{S})$, $\delta\mathbb{h}^\mu\in\mathrm{L}^2(\mathcal{S})$.

For a given saddle point solution $(\mathbb{a},\mathbb{h}^\mu)$ with enough regularity, the variation of $\mathcal{I}^{\text{phr}}$ can be rearranged to

$$\begin{aligned}\delta\mathcal{I}^{\text{phr}}(\mathbb{a},\mathbb{h}^\mu)&=\int_{\mathcal{S}}\left[\frac{\partial M^*(\mathbb{h}^\mu)}{\partial\mathbb{h}^\mu}\cdot\delta\mathbb{h}^\mu+\delta\mathbb{h}^\mu\cdot[\nabla\times\mathbb{a}]+\mathbb{h}^\mu\cdot[\nabla\times\delta\mathbb{a}]\right]\mathrm{d}v\\&\quad-\int_{\partial\mathcal{S}_\infty^{\mathbb{j}}}\mathbb{h}_\infty^\mu\cdot[\boldsymbol{n}_\infty\times\delta\mathbb{a}]\,\mathrm{d}a\\&=\int_{\mathcal{S}}[[-\mathbb{b}+\nabla\times\mathbb{a}]\cdot\delta\mathbb{h}^\mu+[\nabla\times\mathbb{h}^\mu]\cdot\delta\mathbb{a}]\,\mathrm{d}v\\&\quad-\int_{\partial\mathcal{S}_\infty^{\mathbb{j}}}\delta\mathbb{a}\cdot[\boldsymbol{n}_\infty\times\mathbb{h}^\mu-\boldsymbol{n}_\infty\times\mathbb{h}_\infty^\mu]\,\mathrm{d}a\end{aligned}$$

assuming the strong identity $(13)_2$ and the boundary condition of $\mathcal{V}_0(\mathcal{S})$ for $\delta\mathbb{a}$. The remaining equations (6a), (6c) and (7) hold for this principle in a weak sense.

(c) *Primal Dirichlet principle* (*pd*): The primal Dirichlet functional defined in Eq. (14) can be derived from the primal Hu–Washizu functional omitting the Lagrange multiplier $\mathbb{h}^\mu$ and instead assuming Eq. (7) in a strong sense. Alternatively, we can start with the primal Hellinger–Reissner functional proposing the strong validity of Eq. (7) and performing subsequently a second Legendre transformation. This leads back to the original free space energy $M(\mathbb{b})$. For a discussion of its variation we refer to Appendix B. Within the primal Dirichlet principle, Eqs. (6a) and (6c) are weakly fulfilled, whereas Eqs. (7), (10) and $(13)_1$ are valid in a strong sense.

(d) *Dual Hu–Washizu Principle* (*dhw*): In analogy to the primal case, we start now with the dual master functional and incorporate the boundary condition $\mathbb{h}^\mu\in\mathcal{W}_{\mathbb{j}}(\mathcal{S})$ directly, which results in the dual Hu–Washizu functional

$$\mathcal{I}^{\text{dhw}}(\mathbb{a},\mathbb{b},\mathbb{h}^\mu)=\int_{\mathcal{S}}[M(\mathbb{b})-\mathbb{h}^\mu\cdot\mathbb{b}+[\nabla\times\mathbb{h}^\mu]\cdot\mathbb{a}]\,\mathrm{d}v-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{h}^\mu]\,\mathrm{d}s \tag{20}$$

for $\mathbb{a}\in\mathrm{L}^2(\mathcal{S})$, $\mathbb{b}\in\mathrm{L}^2(\mathcal{S})$, $\mathbb{h}^\mu\in\mathcal{W}_{\mathbb{j}}(\mathcal{S})$ and $\delta\mathbb{a}\in\mathrm{L}^2(\mathcal{S})$, $\delta\mathbb{b}\in\mathrm{L}^2(\mathcal{S})$, $\delta\mathbb{h}^\mu\in\mathcal{W}_0(\mathcal{S})$.

Let $(\mathbb{a},\mathbb{b},\mathbb{h}^\mu)$ be a saddle point of $\mathcal{I}^{\text{dhw}}$. Assuming that the boundary condition of $\mathcal{V}_0(\mathcal{S})$ holds strongly for $\delta\mathbb{a}$ and supposing enough regularity of $\mathbb{h}^\mu$ and $\delta\mathbb{a}$, this triple is the result of the variation

$$\begin{aligned}\delta\mathcal{I}^{\text{dhw}}(\mathbb{a},\mathbb{b},\mathbb{h}^\mu)&=\int_{\mathcal{S}}\left[\frac{\partial M(\mathbb{b})}{\partial\mathbb{b}}\cdot\delta\mathbb{b}-\delta\mathbb{h}^\mu\cdot\mathbb{b}-\mathbb{h}^\mu\cdot\delta\mathbb{b}+[\nabla\times\mathbb{h}^\mu]\cdot\delta\mathbb{a}\right]\mathrm{d}v\\&\quad+\int_{\mathcal{S}}[\nabla\times\delta\mathbb{h}^\mu]\cdot\mathbb{a}\,\mathrm{d}v-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\delta\mathbb{h}^\mu]\,\mathrm{d}s\\&=\int_{\mathcal{S}}\left[\left[\frac{\partial M(\mathbb{b})}{\partial\mathbb{b}}-\mathbb{h}^\mu\right]\cdot\delta\mathbb{b}+[\nabla\times\mathbb{a}-\mathbb{b}]\cdot\delta\mathbb{h}^\mu\right]\\&\quad+\int_{\mathcal{S}}[\nabla\times\mathbb{h}^\mu]\cdot\delta\mathbb{a}\,\mathrm{d}v-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}[\boldsymbol{n}_\infty\times\mathbb{a}-\boldsymbol{n}_\infty\times\mathbb{a}_\infty]\cdot\delta\mathbb{h}^\mu\,\mathrm{d}s.\end{aligned}$$

Therefore, (6a), (7), (10) and $(13)_1$ are the Euler–Lagrange equations of this principle. Eq. (6c), in contrast, is the only equation that holds in a strong sense.

(e) *Dual Hellinger–Reissner principle* (*dhr*): In order to decrease the number of independent variables from three to two fields, the dual Hu–Washizu principle is Legendre transformed. As a result, the Hellinger–Reissner functional reads as

$$\mathcal{I}^{\text{dhr}}(\mathbb{a},\mathbb{h}^\mu)=\int_{\mathcal{S}}[M^*(\mathbb{h}^\mu)+[\nabla\times\mathbb{h}^\mu]\cdot\mathbb{a}]\,\mathrm{d}v-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{h}^\mu]\,\mathrm{d}s$$

for $\mathbb{a}\in\mathrm{L}^2(\mathcal{S})$, $\mathbb{h}^\mu\in\mathcal{W}_{\mathbb{j}}(\mathcal{S})$ and $\delta\mathbb{a}\in\mathrm{L}^2(\mathcal{S})$, $\delta\mathbb{h}^\mu\in\mathcal{W}_0(\mathcal{S})$.

Assuming the strong identity $(13)_2$ and the boundary condition of $\mathcal{W}_0(\mathcal{S})$ for $\delta\mathbb{h}^\mu$, the variation of $\mathcal{I}^{\text{dhr}}$ can be manipulated to the following expression:

$$\begin{aligned}\delta\mathcal{I}^{\text{dhr}}(\mathbb{a},\mathbb{h}^\mu)&=\int_{\mathcal{S}}\left[\frac{\partial M^*(\mathbb{h}^\mu)}{\partial\mathbb{h}^\mu}\cdot\delta\mathbb{h}^\mu+[\nabla\times\delta\mathbb{h}^\mu]\cdot\mathbb{a}+[\nabla\times\mathbb{h}^\mu]\cdot\delta\mathbb{a}\right]\mathrm{d}v\\&\quad-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\delta\mathbb{h}^\mu]\,\mathrm{d}s\\&=\int_{\mathcal{S}}\left[\left[\frac{\partial M^*(\mathbb{h}^\mu)}{\partial\mathbb{h}^\mu}+\mathbb{b}\right]\cdot\delta\mathbb{h}^\mu+[\nabla\times\mathbb{h}^\mu]\cdot\delta\mathbb{a}\right]\mathrm{d}v\\&\quad-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\delta\mathbb{h}^\mu\cdot[\boldsymbol{n}_\infty\times\mathbb{a}-\boldsymbol{n}_\infty\times\mathbb{a}_\infty]\,\mathrm{d}s.\end{aligned}$$

The Euler–Lagrange equations of the dual Hellinger–Reissner principle coincide with Eqs. (6a), (7) and (10).

(f) *Dual Dirichlet principle* (*dd*): The dual Dirichlet functional can be derived from the dual Hellinger–Reissner functional omitting the Lagrange multiplier $\mathbb{a}$ and assuming instead Eq. (6a) in a strong sense. Alternatively, we can start with the dual Hu–Washizu functional proposing the strong validity of Eq. (6a) and performing subsequently a Legendre transformation:

$$\mathcal{I}^{\text{dd}}(\mathbb{h}^\mu)=\int_{\mathcal{S}}M^*(\mathbb{h}^\mu)\,\mathrm{d}v-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{h}^\mu]\,\mathrm{d}s$$

for $\mathbb{h}^\mu\in\widetilde{\mathcal{W}}_{\mathbb{j}}(\mathcal{S})$ and $\delta\mathbb{h}^\mu\in\widetilde{\mathcal{W}}_0(\mathcal{S})$.

For a given solution $\mathbb{h}^\mu$ of the dual Dirichlet principle with enough regularity, we can derive the first variation as

$$\begin{aligned}\delta\mathcal{I}^{\text{dd}}(\mathbb{h}^\mu)&=\int_{\mathcal{S}}\frac{\partial M^*(\mathbb{h}^\mu)}{\partial\mathbb{h}^\mu}\cdot\delta\mathbb{h}^\mu\,\mathrm{d}v-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\delta\mathbb{h}^\mu]\,\mathrm{d}s\\&=\int_{\mathcal{S}}[\nabla\times\mathbb{a}-\mathbb{b}]\cdot\delta\mathbb{h}^\mu\,\mathrm{d}v-\int_{\partial\mathcal{S}_\infty^{\mathbb{a}}}\delta\mathbb{h}^\mu\cdot[\boldsymbol{n}_\infty\times\mathbb{a}-\boldsymbol{n}_\infty\times\mathbb{a}_\infty]\,\mathrm{d}s.\end{aligned}$$

For the dual Dirichlet principle, Eqs. (6a) and (6c) and the constitutive relation $(13)_2$ are strongly fulfilled, whereas (7) and (10) are satisfied in a weak sense.

3. Rigid body embedded into free space

We now introduce a rigid body $\mathcal{B}$ to our geometry, as depicted in Fig. 3. The most important difference with the formulation in

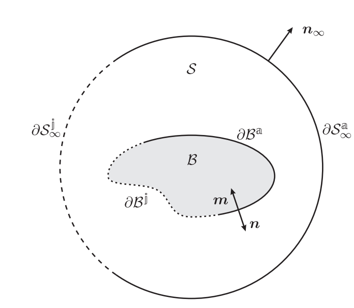

Fig. 3. Geometry of rigid body immersed in free space. The body $\mathcal{B}$ is bounded by the C^1-boundary $\partial\mathcal{B}$. The Lipschitz-continuous exterior boundary $\partial\mathcal{S}_\infty$ denotes the truncation of the computational domain. Along the interface and the exterior boundary external magnetic loads are applied.

mere free space is the careful investigation of the physical phenomena arising on the surface $\partial\mathcal{B}$ separating the body $\mathcal{B}$ from the outer space $\mathcal{S}$. The surface $\partial\mathcal{B}$ of the body $\mathcal{B}$ is divided into two parts,

$$\partial\mathcal{B} = \partial\mathcal{B}^{\mathbb{j}} \cup \partial\mathcal{B}^{\mathbb{a}}, \tag{21}$$

such that $\partial\mathcal{B}^{\mathbb{j}} \cap \partial\mathcal{B}^{\mathbb{a}} = \emptyset$. The body $\mathcal{B}$ is bounded and we assume $\partial\mathcal{B}$ is C^1. As in the case of free space, $\partial\mathcal{S}_\infty$ denotes the truncation of the computational domain where external magnetic loads are applied. The boundary conditions (6c) and (10) still hold in vacuum all relations shown in the previous section.

Inside the body a possible non-linear coupling between the magnetic field and the magnetic induction is assumed. This can be described by an additional variable, the magnetization $\mathbb{m}$ of the material, where [26–28]

$$\mathbb{h} = \frac{1}{\mu_0}\mathbb{b} - \mathbb{m} \quad \text{in } \mathcal{B}. \tag{22}$$

For the special case of vacuum, $\mathbb{h}$ coincides with $(1/\mu_0)\mathbb{b}$ since there is no magnetization in free space, i.e. $\mathbb{m} = \mathbf{0}$.

Additionally to the free space equations (6a) and (6b), the static field equation for the magnetic field and the magnetic induction have to hold within the body, which read as [14,26–28,54]

$$\nabla \times \mathbb{h} = \mathbb{j}^{\mathrm{f}}, \quad \nabla \cdot \mathbb{b} = 0 \text{ in } \mathcal{B}. \tag{23}$$

Although when working with a magneto-active polymer no free currents exist, we keep the term $\mathbb{j}^{\mathrm{f}}$ to make our formulations more general. Since we assume the quasistatic case of the Maxwell's equations, the free current density has to be invariant in time. Eq. $(23)_2$ is satisfied by setting [26–28,54–56]

$$\mathbb{b} = \nabla \times \mathbb{a} \quad \text{in } \mathcal{B}. \tag{24}$$

Across the boundary of the body, the continuity conditions already introduced in Section 2 as Eqs. $(2)_1$ and (9) are relevant. Assuming that only $\partial\mathcal{B}^{\mathbb{j}}$ carries a free surface current and only along $\partial\mathcal{B}^{\mathbb{a}}$ a prescribed value for the magnetic vector potential is given, the continuity conditions can be stated more precisely as [14,26–28,34,35,54][3]

$$\boldsymbol{n} \times [\![\mathbb{h}]\!] = \boldsymbol{n} \times \mathbb{j}^{\mathrm{f,p}} =: \hat{\mathbb{j}}^{\mathrm{f}} \quad \text{on } \partial\mathcal{B}^{\mathbb{j}}, \tag{25a}$$

$$\boldsymbol{n} \times \mathbb{a}_{\mathrm{o}} = \boldsymbol{n} \times \mathbb{a}_{\mathrm{i}} = \boldsymbol{n} \times \mathbb{a}^{\mathrm{p}} =: \mathbb{a}^{\mathrm{tan,p}} \quad \text{on } \partial\mathcal{B}^{\mathbb{a}}, \tag{25b}$$

where the indices i and o refer to inside and outside the bulk, respectively. The index p indicates a prescribed value. Since we assume the quasistatic case of Maxwell's equations, the free surface current density has to be invariant in time.

The magnetic quantities are coupled by the constitutive law obtained by the derivatives of the total internal potential energy density W, where W consists of the free field magnetic energy density M and the free energy density ψ associated with the magnetization of the material, i.e.

$$W(\mathbb{b}) = M(\mathbb{b}) + \psi(\mathbb{b}). \tag{26}$$

Following a procedure similar to the method developed by Coleman and Noll, we can write

$$\mathbb{h} = \frac{\partial W}{\partial \mathbb{b}} \quad \text{in } \mathcal{B}, \tag{27}$$

where we implicitly defined the magnetization as

$$\mathbb{m} = -\frac{\partial \psi}{\partial \mathbb{b}}. \tag{28}$$

After a Legendre transformation of $W(\mathbb{b})$ into the complementary energy density $W^*(\mathbb{h})$, we obtain the alternative constitutive relation

$$\mathbb{b} = -\frac{\partial W^*}{\partial \mathbb{h}} \quad \text{in } \mathcal{B}. \tag{29}$$

3.1. Variational principles for a rigid body in free space

Several concepts and expressions obtained in Section 2 are also of relevance here. The structure of weak and strong satisfaction for each variational principle remains unchanged for a rigid body, in comparison to the case of mere free space as in Table 1. Therefore, we present most results briefly with only some details, however, the adaption of the function spaces to the new geometry is necessary. A proper definition of the involved functional spaces can be found in Definition 2 of Appendix A.The most important difference is the incorporation of the jump conditions along the interface of the body with the surrounding free space (Table 2).

The tangential continuity across material discontinuities, proposed for the magnetic vector potential $\mathbb{a}$ in (9), is guaranteed through the trace theorem for functions in $\mathcal{V}(\mathcal{B} \cup \mathcal{S})$ [55].

3.1.1. Master functionals

The primal master functional for the free space (16) is amended by the contributions within the bulk $\mathcal{B}$, as well as by the Dirichlet boundary condition and the energy stored by the surface free current density along the interface $\partial\mathcal{B}$. The new two master functionals (with the superscripts pm and dm having the same meaning as in Section 2.1.1) read as

$$\begin{aligned}\mathcal{I}^{\mathrm{pm}}(\mathbb{a},\mathbb{b},\mathbb{h}) = &\int_{\mathcal{S}} M(\mathbb{b})\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathbb{j}}_\infty} \mathbb{h}_\infty \cdot [\boldsymbol{n}_\infty \times \mathbb{a}]\,\mathrm{d}a \\ &- \int_{\mathcal{S}} \mathbb{h} \cdot [\mathbb{b} - \nabla \times \mathbb{a}]\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathbb{a}}_\infty} \mathbb{h} \cdot [\boldsymbol{n}_\infty \times \mathbb{a} - \boldsymbol{n}_\infty \times \mathbb{a}_\infty]\,\mathrm{d}a \\ &+ \int_{\mathcal{B}} [W(\mathbb{b}) - \mathbb{j}^{\mathrm{f}} \cdot \mathbb{a}]\,\mathrm{d}v - \int_{\partial\mathcal{B}^{\mathbb{j}}} \mathbb{j}^{\mathrm{f,p}} \cdot [\boldsymbol{n} \times \mathbb{a}]\,\mathrm{d}a \\ &- \int_{\mathcal{B}} \mathbb{h} \cdot [\mathbb{b} - \nabla \times \mathbb{a}]\,\mathrm{d}v + \int_{\partial\mathcal{B}^{\mathbb{a}}} [\![\mathbb{h}]\!] \cdot [\boldsymbol{n} \times \mathbb{a} - \boldsymbol{n} \times \mathbb{a}^{\mathrm{p}}]\,\mathrm{d}a\end{aligned} \tag{30}$$

for $\mathbb{a} \in \mathcal{V}(\mathcal{B} \cup \mathcal{S})$, $\mathbb{b} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S})$, $\mathbb{h} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S})$ and $\mathbb{h} \in \mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathbb{a}}_\infty)$ as well as $\mathbb{h} \in \mathrm{H}^{1/2}(\partial\mathcal{B}^{\mathbb{a}})$.

$$\mathcal{I}^{\mathrm{dm}}(\mathbb{a},\mathbb{b},\mathbb{h}) = \int_{\mathcal{S}} M(\mathbb{b})\,\mathrm{d}v - \int_{\partial\mathcal{S}^{\mathbb{j}}_\infty} \mathbb{a} \cdot [\boldsymbol{n}_\infty \times \mathbb{h} - \boldsymbol{n}_\infty \times \mathbb{h}_\infty]\,\mathrm{d}a$$

Table 2
Classification of results according to weak and strong satisfaction of field equations and boundary conditions for a rigid body immersed in free space. The classification of the equations valid in free space can be found in Table 1.

In $\mathcal{B}$	phw	phr	pd	dhw	dhr	dd	Eq.
$\nabla \times \mathbb{h} = \mathbb{j}^{\mathrm{f}}$	Weak	Weak	Weak	Weak	Weak	Strong	$(23)_1$
$\mathbb{b} = \nabla \times \mathbb{a}$	Weak	Weak	Strong	Weak	Weak	Weak	(24)
$\boldsymbol{n} \times [\![\mathbb{h}]\!] = \boldsymbol{n} \times \mathbb{j}^{\mathrm{f,p}}$	Weak	Weak	Weak	Strong	Strong	Strong	(25a)
$\boldsymbol{n} \times \mathbb{a}_{\mathrm{o}} = \boldsymbol{n} \times \mathbb{a}_{\mathrm{i}} = \boldsymbol{n} \times \mathbb{a}^{\mathrm{p}}$	Strong	Strong	Strong	Weak	Weak	Weak	(25b)
$\mathbb{h} = \frac{\partial W}{\partial \mathbb{b}}$	Weak	–	Strong	Weak	–	–	(27)
$\mathbb{b} = -\frac{\partial W^*}{\partial \mathbb{h}}$	–	Strong	–	–	Strong	Strong	(29)

[3] We favour the definition of the jump conditions (25a) and (25b) in terms of the vector product, such that we are able to prescribe functional analytical requirements on $\mathbb{a}$ and $\mathbb{h}$ with the least regularity. See also Remark 3 in Section 3.1.1.

$$
\begin{aligned}
&-\int_{\mathcal{S}}[\mathbb{h}\cdot\mathbb{b}-[\nabla\times\mathbb{h}]\cdot\mathbb{a}]\,\mathrm{d}v-\int_{\partial\mathcal{S}^{\mathbb{a}}_\infty}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{h}]\,\mathrm{d}a\\
&+\int_{\mathcal{B}}[W(\mathbb{b})-\mathbb{j}^{\mathrm{f}}\cdot\mathbb{a}]\,\mathrm{d}v+\int_{\partial\mathcal{B}^{\mathbb{j}}}\mathbb{a}\cdot[\boldsymbol{n}\times[\![\mathbb{h}]\!]-\mathbb{j}^{\mathrm{f,p}}\times\boldsymbol{n}]\,\mathrm{d}a\\
&-\int_{\mathcal{B}}[\mathbb{h}\cdot\mathbb{b}-[\nabla\times\mathbb{h}]\cdot\mathbb{a}]\,\mathrm{d}v+\int_{\partial\mathcal{B}^{\mathbb{a}}}\mathbb{a}^{\mathrm{p}}\cdot[\boldsymbol{n}\times[\![\mathbb{h}]\!]]\,\mathrm{d}a \qquad (31)
\end{aligned}
$$

for $\mathbb{a}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$ and $\mathbb{a}\in\mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathbb{a}}_\infty)$ as well as $\mathbb{a}\in\mathrm{H}^{1/2}(\partial\mathcal{B}^{\mathbb{a}})$, $\mathbb{b}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\mathbb{h}\in\mathcal{V}(\mathcal{B}\cup\mathcal{S})$.

Remark 3 (*Functional spaces of* $\mathbb{j}^{\mathrm{f,p}}$ *and* $\mathbb{a}^{\mathrm{p}}$). We propose for the primal case that $\mathbb{j}^{\mathrm{f,p}}\in\mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathbb{j}}_\infty)$ and $\mathbb{a}^{\mathrm{p}}\in\mathrm{H}(\mathrm{curl};\partial\mathcal{B}^{\mathbb{a}})$. For the dual case we propose that $\mathbb{j}^{\mathrm{f,p}}\in\mathrm{H}(\mathrm{curl};\mathcal{B})$ and $\mathbb{a}^{\mathrm{p}}\in\mathrm{H}^{1/2}(\partial\mathcal{S}^{\mathbb{a}}_\infty)$. Observe here that a boundary integral of the form $\int_{\partial\mathcal{B}^{\mathbb{j}}}\hat{\mathbb{j}}^{\mathrm{f}}\cdot\mathbb{a}\,\mathrm{d}a$ in (30) would call for $\mathbb{a}\in\mathrm{H}^{1,2}(\mathcal{B}\cup\mathcal{S})$ and analogously, the boundary integral $\int_{\partial\mathcal{B}^{\mathbb{a}}}[\![h]\!]\cdot\mathbb{a}^{\mathrm{tan,p}}\,\mathrm{d}a$ in (31) for $\mathbb{h}\in\mathrm{H}^{1,2}(\mathcal{B}\cup\mathcal{S})$.

3.1.2. Three-, two- and one field functionals

Once we derive the functionals $\mathcal{I}^{\mathrm{pm}}$ and $\mathcal{I}^{\mathrm{dm}}$ in their most general form, we can follow all steps from Section 2 without any changes in order to derive the one-, two- and three-field functionals. Therefore, for the sake of brevity, only the final expressions are presented:

(a) *Primal Hu–Washizu-type functional*

$$
\begin{aligned}
\mathcal{I}^{\mathrm{phw}}(\mathbb{a},\mathbb{b},\mathbb{h})=&\int_{\mathcal{S}}[M(\mathbb{b})-\mathbb{h}\cdot[\mathbb{b}-\nabla\times\mathbb{a}]]\,\mathrm{d}v-\int_{\partial\mathcal{S}^{\mathbb{j}}_\infty}\mathbb{h}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{a}]\,\mathrm{d}a\\
&+\int_{\mathcal{B}}[W(\mathbb{b})-\mathbb{j}^{\mathrm{f}}\cdot\mathbb{a}-\mathbb{h}\cdot[\mathbb{b}-\nabla\times\mathbb{a}]]\,\mathrm{d}v-\int_{\partial\mathcal{B}^{\mathbb{j}}}\mathbb{j}^{\mathrm{f,p}}\cdot[\boldsymbol{n}\times\mathbb{a}]\,\mathrm{d}a
\end{aligned}
\qquad (32)
$$

for $\mathbb{a}\in\mathcal{V}_{\mathbb{a}}(\mathcal{B}\cup\mathcal{S})$, $\mathbb{b}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\mathbb{h}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, and $\delta\mathbb{a}\in\mathcal{V}_0(\mathcal{B}\cup\mathcal{S})$, $\delta\mathbb{b}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\delta\mathbb{h}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$. By the solution of the saddle point problem, Eqs. $(23)_1$, (24), (25a) and (27) are weakly fulfilled, whereas Eq. (25b) is strongly fulfilled.

(b) *Primal Hellinger–Reissner-type functional*

$$
\begin{aligned}
\mathcal{I}^{\mathrm{phr}}(\mathbb{a},\mathbb{h})=&\int_{\mathcal{S}}[M^*(\mathbb{h})\,\mathrm{d}v+\mathbb{h}\cdot[\nabla\times\mathbb{a}]]-\int_{\partial\mathcal{S}^{\mathbb{j}}_\infty}\mathbb{h}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{a}]\,\mathrm{d}a\\
&+\int_{\mathcal{B}}[W^*(\mathbb{b})\,\mathrm{d}v-\mathbb{j}^{\mathrm{f}}\cdot\mathbb{a}+\mathbb{h}\cdot[\nabla\times\mathbb{a}]]\,\mathrm{d}v-\int_{\partial\mathcal{B}^{\mathbb{j}}}\mathbb{j}^{\mathrm{f,p}}\cdot[\boldsymbol{n}\times\mathbb{a}]\,\mathrm{d}a
\end{aligned}
\qquad (33)
$$

for $\mathbb{a}\in\mathcal{V}(\mathcal{B}\cup\mathcal{S})$, $\mathbb{h}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, and $\delta\mathbb{a}\in\mathcal{V}_0(\mathcal{B}\cup\mathcal{S})$, $\delta\mathbb{h}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$. Assuming the strong identities (29) and the boundary condition (25b), the remaining equations $(23)_1$, (24) and (25a) hold for this principle in a weak sense.

(c) *Primal Dirichlet-Type functional*:

$$
\begin{aligned}
\mathcal{I}^{\mathrm{pd}}(\mathbb{a})=&\int_{\mathcal{S}}M(\mathbb{b})\,\mathrm{d}v-\int_{\partial\mathcal{S}^{\mathbb{j}}_\infty}\mathbb{h}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{a}]\,\mathrm{d}a\\
&+\int_{\mathcal{B}}[W(\mathbb{b})-\mathbb{j}^{\mathrm{f}}\cdot\mathbb{a}]\,\mathrm{d}v-\int_{\partial\mathcal{B}^{\mathbb{j}}}\mathbb{j}^{\mathrm{f,p}}\cdot[\boldsymbol{n}\times\mathbb{a}]\,\mathrm{d}a
\end{aligned}
\qquad (34)
$$

for $\mathbb{a}\in\mathcal{V}_{\mathbb{a}}(\mathcal{B}\cup\mathcal{S})$ and $\delta\mathbb{a}\in\mathcal{V}_0(\mathcal{B}\cup\mathcal{S})$. Within the primal Dirichlet principle, Eqs. $(23)_1$ and (25a) are weakly fulfilled, whereas Eqs. (24), (25b) and (27) are valid in a strong sense.

(d) *Dual Hu–Washizu-type functional*:

$$
\begin{aligned}
\mathcal{I}^{\mathrm{dhw}}(\mathbb{a},\mathbb{b},\mathbb{h})=&\int_{\mathcal{S}}[M(\mathbb{b})-\mathbb{h}\cdot\mathbb{b}+[\nabla\times\mathbb{h}]\cdot\mathbb{a}]\,\mathrm{d}v-\int_{\partial\mathcal{S}^{\mathbb{a}}_\infty}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{h}]\,\mathrm{d}a\\
&+\int_{\mathcal{B}}[W(\mathbb{b})-\mathbb{j}^{\mathrm{f}}\cdot\mathbb{a}-\mathbb{h}\cdot\mathbb{b}+[\nabla\times\mathbb{h}]\cdot\mathbb{a}]\,\mathrm{d}v+\int_{\partial\mathcal{B}^{\mathbb{a}}}\mathbb{a}^{\mathrm{p}}\cdot[\boldsymbol{n}\times[\![\mathbb{h}]\!]]\,\mathrm{d}a
\end{aligned}
\qquad (35)
$$

for $\mathbb{a}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\mathbb{b}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\mathbb{h}\in\mathcal{W}_{\mathbb{j}}(\mathcal{B}\cup\mathcal{S})$, and $\delta\mathbb{a}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\delta\mathbb{b}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\delta\mathbb{h}\in\mathcal{W}_0(\mathcal{B}\cup\mathcal{S})$. Eqs. $(23)_1$, (24), (25b), (27) are the Euler–Lagrange equations of this principle. Eq. (25a), in contrast, is the only equation that holds in a strong sense.

(e) *Dual Hellinger–Reissner-type functional*

$$
\begin{aligned}
\mathcal{I}^{\mathrm{dhr}}(\mathbb{a},\mathbb{h})=&\int_{\mathcal{S}}[M^*(\mathbb{h})+[\nabla\times\mathbb{h}]\cdot\mathbb{a}]\,\mathrm{d}v-\int_{\partial\mathcal{S}^{\mathbb{a}}_\infty}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{h}]\,\mathrm{d}a\\
&+\int_{\mathcal{B}}[W^*(\mathbb{h})-\mathbb{j}^{\mathrm{f}}\cdot\mathbb{a}+[\nabla\times\mathbb{h}]\cdot\mathbb{a}]\,\mathrm{d}v+\int_{\partial\mathcal{B}^{\mathbb{a}}}\mathbb{a}^{\mathrm{p}}\cdot[\boldsymbol{n}\times[\![\mathbb{h}]\!]]\,\mathrm{d}a \qquad (36)
\end{aligned}
$$

for $\mathbb{a}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\mathbb{h}\in\mathcal{W}_{\mathbb{j}}(\mathcal{B}\cup\mathcal{S})$, and $\delta\mathbb{a}\in\mathrm{L}^2(\mathcal{B}\cup\mathcal{S})$, $\delta\mathbb{h}\in\mathcal{W}_0(\mathcal{B}\cup\mathcal{S})$. Assuming the strong identity (25a) and the constitutive law (29), the Euler–Lagrange equations of the dual Hellinger–Reissner principle coincide with Eqs. $(23)_1$, (24) and (25b).

(f) *Dual Dirichlet-type functional*:

$$
\begin{aligned}
\mathcal{I}^{\mathrm{dd}}(\mathbb{a},)=&\int_{\mathcal{S}}M^*(\mathbb{h})\,\mathrm{d}v-\int_{\partial\mathcal{S}^{\mathbb{a}}_\infty}\mathbb{a}_\infty\cdot[\boldsymbol{n}_\infty\times\mathbb{h}]\,\mathrm{d}a\\
&+\int_{\mathcal{B}}W^*(\mathbb{h})\,\mathrm{d}v-\int_{\mathcal{B}}\mathbb{j}^{\mathrm{f}}\cdot\mathbb{a}\,\mathrm{d}v+\int_{\partial\mathcal{B}^{\mathbb{a}}}\mathbb{a}^{\mathrm{p}}\cdot[\boldsymbol{n}\times[\![\mathbb{h}]\!]]\,\mathrm{d}a \qquad (37)
\end{aligned}
$$

for $\mathbb{h}\in\widetilde{\mathcal{W}}_{\mathbb{j}}(\mathcal{B}\cup\mathcal{S})$ and $\delta\mathbb{h}\in\widetilde{\mathcal{W}}_0(\mathcal{B}\cup\mathcal{S})$. For the dual Dirichlet principle, Eqs. $(23)_1$, (25a) and the constitutive relation (29) are strongly fulfilled, whereas (24) and (25b) are satisfied in a weak sense.

4. Geometrically non-linear deforming body embedded into free space

The final step is to investigate a non-linearly deforming body as depicted in Fig. 4, which is subjected to a magnetic field. At a first glance, the reader will notice that the functionals that we are going to present in the next Section 4.1, do not differ much from those presented in Section 3.1. Basically, they are only expanded by the external contributions of the body forces inside the bulk and mechanical tractions on the surface. This simple adaptation of the functionals, of course, should not mask the fact that the major challenge, when observing magnetic and mechanical effects simultaneously, lies in a suitable formulation of the balance of linear momentum and the stress tensor used therein.

Due to the magnetic couple acting on the bulk, the Cauchy stress tensor is not symmetric anymore. This was already stated in the pioneering works of Tiersten [59] and Brown [8]. From those early times on, it was known that for the case of static equilibrium, the antisymmetric part of the Cauchy stress tensor (named as elastic or local stress in the equivalent work on non-linear dielectrics by Toupin [60]) is balanced by the antisymmetric part of a ponderomotive stress tensor. This additional contribution to the elastic stress measure is usually referred to as the Maxwell stress (see, for example, [10,15,18,17,21,59,60]) or (electro)magnetic stress [45] and results in a Cauchy-like symmetric total stress tensor [16,18,19,28,61].

We will adopt the term ponderomotive stress here and use the terms magnetization stress and Maxwell stress for the non-symmetric and symmetric parts of the ponderomotive stress, respectively. This notation may at first seem to be a further complication of the predominant use in the literature. However, we take it as a handy tool to distinguish immediately between the non-symmetric part of the ponderomotive stress tensors, which exists only inside the body, and the symmetric part of the ponderomotive stress tensor, which exists also outside the body. The same split can be found in [19,23] and is in analogy to our previous works [25,30].

The surrounding free space may not be neglected when it comes to the investigation of a body immersed by vacuum, because the magnetic fields inside and outside the body influence each other and the need to satisfy the continuity conditions for the magnetic variables, generates non-homogeneous fields inside a finite body. Due to the existence of a magnetic field outside the body, parts of the total stress tensor do not vanish in vacuum.

The ponderomotive contribution of stress has to be taken into account in cases, where the magnetic field surrounding the body does not decrease abruptly, when reaching the interface but continues gradually. This occurs in the case of magneto-active material with a small relative permeability. This class of magneto-rheological elastomers was investigated in experiments documented, for example, in [1–3,62], where relative permeabilities of order $\mathcal{O}(1)$ for particle concentrations up to 50% were reported. Fluid counterparts with the same order of relative permeability can be found in experimental works like [5,6,63].

Before presenting the variational formulations for the non-linearly deforming body, we will repeat briefly the basic concepts of non-linear continuum mechanics, and the necessary alterations of the magnetostatics equations from the previous sections.

Our body may undergo large deformations. Therefore, we have to distinguish between an undeformed configuration $\mathcal{B}_0$ and a deformed configuration $\mathcal{B}_t$. For the material as well as for the spatial setting, the surface of the body is divided into different subsets:

$$\partial\mathcal{B} = \partial\mathcal{B}^{\mathbb{j}} \cup \partial\mathcal{B}^{\mathbb{a}}, \quad \partial\mathcal{B} = \partial\mathcal{B}^{\mathrm{d}} \cup \partial\mathcal{B}^{\mathrm{t}}, \tag{38}$$

such that

$$\partial\mathcal{B}^{\mathbb{j}} \cap \partial\mathcal{B}^{\mathbb{a}} = \emptyset, \quad \partial\mathcal{B}^{\mathrm{d}} \cap \partial\mathcal{B}^{\mathrm{t}} = \emptyset. \tag{39}$$

Obviously, $\mathcal{B}$ is bounded and we assume $\partial\mathcal{B}$ is C^1. Let $\boldsymbol{\varphi}$ denote the deformation map, $\boldsymbol{F}$ the deformation gradient, i.e. $\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{\varphi}$ and J the determinant of $\boldsymbol{F}$. Here, $\nabla_{\boldsymbol{X}}$ denotes the nabla operator with respect to $\boldsymbol{X}$. The position vector and the magnetic field quantities are designated by capital letters in the material setting and by small letters in the spatial setting, respectively. The push-forward operations for the magnetic field quantities read as [13,16]

$$\mathbb{b} = \mathbb{B} \cdot \mathrm{cof}\, \boldsymbol{F}^{-1}, \quad \mathbb{h} = \mathbb{H} \cdot \boldsymbol{F}^{-1}, \quad \mathbb{m} = \mathbb{M} \cdot \boldsymbol{F}^{-1}. \tag{40}$$

From the magnetostatic point of view, the equations to be solved are Maxwell's equations as we have seen them before in Section 3. Due to the deformation of the body, we now have to specify them for a formulation in the material configuration:

$$\nabla \times \mathbb{H} = \mathbb{J}^{\mathrm{f}}, \quad \nabla \cdot \mathbb{B} = 0 \text{ in } \mathcal{B}_0, \tag{41}$$

where $\mathbb{J}^{\mathrm{f}}$ denotes the free current density within the bulk per undeformed unit volume. Eq. $(41)_2$ is satisfied by

$$\mathbb{B} = \nabla_{\boldsymbol{X}} \times \mathbb{A} \quad \text{in } \mathcal{B}_0, \tag{42}$$

where for the magnetic vector potential the following push-forward operation holds

$$\mathbb{a} = \mathbb{A} \cdot \boldsymbol{F}^{-1}. \tag{43}$$

In free space we have[4]

$$\nabla \times \mathbb{H} = \mathbf{0}, \quad \nabla \cdot \mathbb{B} = 0 \text{ in } \mathcal{S}_0. \tag{44}$$

Eq. $(44)_2$ is satisfied by

$$\mathbb{B} = \nabla_{\boldsymbol{X}} \times \mathbb{A} \quad \text{in } \mathcal{S}_0. \tag{45}$$

Observe here that in (22) we have a relation for the three magnetic field variables, which is, however, valid only for the deformed configuration. In the material setting this translates to [17]

$$\mathbb{H} = \frac{1}{J\mu_0}\boldsymbol{C} \cdot \mathbb{B} - \mathbb{M} \quad \text{in } \mathcal{B}_0, \tag{46}$$

which reduces to $\mathbb{H} = [\mu_0 J]^{-1}\boldsymbol{C} \cdot \mathbb{B} =: \mathbb{H}^{\mu}$ in free space $\mathcal{S}_0$, where $\mathbb{M} = \mathbf{0}$.

[4] We have extended the displacement field $\boldsymbol{\varphi}$ outside the body in vacuum, and as a result we can define $\mathbb{H}$ and $\mathbb{B}$, the magnetic field and magnetic induction for free space.

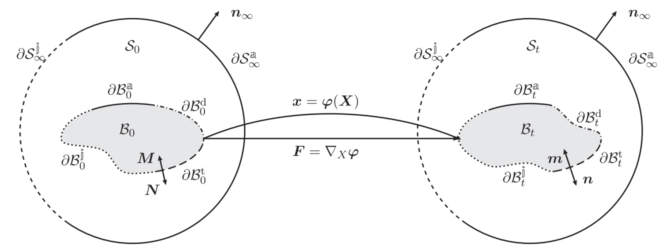

Fig. 4. Geometry of geometrically non-linearly deforming body immersed in free space with reference configuration on the left and deformed configuration on the right. Besides the magnetic loading along the interface and the exterior boundary, mechanical loads are applied along the interface.

In addition to the magnetic field equations that have been discussed previously, one has to solve the balance of linear momentum [17]

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{P} + \boldsymbol{b}_0^{\mathrm{pon}} + \boldsymbol{b}_0 = \nabla_{\boldsymbol{X}} \cdot \boldsymbol{P}^{\mathrm{tot}} + \boldsymbol{b}_0 = \mathbf{0} \quad \text{in } \mathcal{B}_0, \tag{47}$$

where $\boldsymbol{b}_0^{\mathrm{pon}}$ denotes the ponderomotive body force, and $\boldsymbol{b}_0$ denotes the mechanical body force. $\boldsymbol{P}^{\mathrm{tot}}$ symbolizes a total stress tensor which contains the contributions of the ponderomotive body force. A Piola transformation of $\boldsymbol{P}^{\mathrm{tot}}$ leads to the Cauchy-like symmetric total stress tensor $\boldsymbol{\sigma}^{\mathrm{tot}}$, which was mentioned in the introductory part of this Section 4 in the deformed configuration

$$\boldsymbol{\sigma}^{\mathrm{tot}} = \boldsymbol{P}^{\mathrm{tot}} \cdot \mathrm{cof}\, \boldsymbol{F}^{-1}. \tag{48}$$

This transformations together with (40) allows a formulation of the set of partial differential equations in the deformed configuration. For the balance of linear momentum we obtain

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} + \boldsymbol{b}_t^{\mathrm{pon}} + \boldsymbol{b}_t = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma}^{\mathrm{tot}} + \boldsymbol{b}_t = \mathbf{0} \quad \text{in } \mathcal{B}_t. \tag{49}$$

As mentioned before, $\boldsymbol{P}^{\mathrm{tot}}$ and $\boldsymbol{\sigma}^{\mathrm{tot}}$ not only consist of the Piola stress $\boldsymbol{P}$ or the Cauchy stress $\boldsymbol{\sigma}$, respectively, but also of a part arising from the magnetic field effects. To be more precise, $\boldsymbol{P}^{\mathrm{tot}}$ and $\boldsymbol{\sigma}^{\mathrm{tot}}$ are split additively into

$$\boldsymbol{P}^{\mathrm{tot}} = \boldsymbol{P} + \boldsymbol{P}^{\mathrm{pon}} = \boldsymbol{P} + \boldsymbol{P}^{\mathrm{mag}} + \boldsymbol{P}^{\mathrm{max}} \quad \text{in } \mathcal{B}_0, \tag{50a}$$

$$\boldsymbol{\sigma}^{\mathrm{tot}} = \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{pon}} = \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{mag}} + \boldsymbol{\sigma}^{\mathrm{max}} \quad \text{in } \mathcal{B}_t. \tag{50b}$$

This construction of the different stress contributions is exemplified for the spatial configuration as follows, and can easily be understood with the choice for the ponderomotive body force[5] as

$$\boldsymbol{b}_t^{\mathrm{pon}} = \mathbb{m} \cdot \nabla\mathbb{b} + \mathbb{j}^{\mathrm{f}} \times \mathbb{b}, \tag{51}$$

which coincides with the force term introduced by Pao [12] when neglecting all electric fields and the polarization. The motivation for the ponderomotive stress is given by the intention to express the ponderomotive body force in terms of a divergence:

$$\boldsymbol{b}_t^{\mathrm{pon}} = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma}^{\mathrm{pon}} \quad \text{with } \boldsymbol{\sigma}^{\mathrm{pon}} = \frac{1}{2\mu_0}[\mathbb{b} \cdot \mathbb{b}]\boldsymbol{i} - [\mathbb{h} \cdot \mathbb{b}]\boldsymbol{i} + \mathbb{h} \otimes \mathbb{b}. \tag{52}$$

With relation to Eq. (46), the ponderomotive stress $\boldsymbol{\sigma}^{\mathrm{pon}}$ is separated into its non-symmetric part, the magnetization stress

$$\boldsymbol{\sigma}^{\mathrm{mag}} = [\mathbb{m} \cdot \mathbb{b}]\boldsymbol{i} - \mathbb{m} \otimes \mathbb{b}, \tag{53}$$

and its symmetric part, the Maxwell stress

$$\boldsymbol{\sigma}^{\mathrm{max}} = -M_t\boldsymbol{i} + \frac{1}{\mu_0}\mathbb{b} \otimes \mathbb{b}. \tag{54}$$

[5] The definition of the ponderomotive body force is not unique, see, for example, [7,45,59]. Each choice for the force leads to alternative decompositions of the ponderomotive stress tensor, see, for example Table 1 of [18] for an overview of the possible forms. We have chosen a force in terms of the magnetic induction $\mathbb{b}$, because it plays well along with the magnetic free field energy defined in (11), and with the variational functionals in terms of the magnetic vector potential $\mathbb{a}$.

The magnetization stress $\boldsymbol{\sigma}^{\mathrm{mag}}$ compensates the antisymmetric part of the elastic Cauchy stress $\boldsymbol{\sigma}$. Adding all stress contributions together leads to the symmetric Cauchy-like total stress tensor $\boldsymbol{\sigma}^{\mathrm{tot}}$. The corresponding expressions for $\boldsymbol{P}^{\mathrm{pon}}$, $\boldsymbol{P}^{\mathrm{mag}}$ and $\boldsymbol{P}^{\mathrm{max}}$ are obtained via the Piola transformation (48), however, their summation does not lead to a symmetric stress tensor. The total Piola stress tensor $\boldsymbol{P}^{\mathrm{tot}}$ is non-symmetric in the material configuration, as it holds for the Piola stress tensor in pure elasticity as well.

It should be observed that the Maxwell stress also exists in vacuum, whereas all other stress contributions vanish in outer space. Furthermore, it can also be derived directly as a derivative of the free field magnetic energy with respect to the deformation gradient

$$\boldsymbol{P}^{\mathrm{max}} = \frac{\partial M_0}{\partial \boldsymbol{F}}. \tag{55}$$

The spatial magnetic free field energy M_t is given as in (11), its material counterpart is calculated from the relation $M_0 = JM_t$ and results in

$$M_0(\boldsymbol{F},\mathbb{B}) = \frac{1}{2\mu_0} J^{-1}\boldsymbol{C} : [\mathbb{B}\otimes\mathbb{B}]. \tag{56}$$

Without loss of generality, we assume that there is no deformation at the far-away boundary $\partial\mathcal{S}_\infty$, therefore

$$\boldsymbol{\varphi}(\boldsymbol{X}) - \boldsymbol{X} = \boldsymbol{0} \quad \text{on } \partial\mathcal{S}_\infty, \tag{57}$$

and we do not have to consider the Maxwell stress or any mechanical tractions at that boundary:

$$\boldsymbol{P}^{\mathrm{max}} \cdot \boldsymbol{n}_\infty = \boldsymbol{0} \quad \text{on } \partial\mathcal{S}_\infty. \tag{58}$$

The existence of such an extension of $\boldsymbol{\varphi} \in \mathrm{H}^{1,2}(\mathcal{B}_0)$ to $\boldsymbol{\varphi} \in \mathrm{H}^{1,2}(\mathcal{B}_0 \cup \mathcal{S}_0)$, which preserves the weak derivatives across $\partial\mathcal{B}_0$ and has support only in a subdomain of $\mathcal{B}_0 \cup \mathcal{S}_0$, is guaranteed by the extension theorem, see, for example, [57]. Since $\boldsymbol{\varphi} \in \mathrm{H}^{1,2}(\mathcal{B}_0 \cup \mathcal{S}_0)$, the continuity of $\boldsymbol{\varphi}$ across the boundary $\partial\mathcal{B}_0$ is obtained due to the uniqueness of the trace operator [55]. Of course, one can prescribe Dirichlet values for the deformation map $\boldsymbol{\varphi}$ along the boundary $\partial\mathcal{B}_0$

$$\boldsymbol{\varphi}_{\mathrm{o}} = \boldsymbol{\varphi}_{\mathrm{i}} = \boldsymbol{\varphi}^{\mathrm{p}} \quad \text{on } \partial\mathcal{B}_0^{\mathrm{d}}. \tag{59}$$

The jump condition regarding the stress along the boundary of the body is written as

$$[\![\boldsymbol{P}^{\mathrm{tot}}]\!] \cdot \boldsymbol{N} = -\boldsymbol{t}_0 \quad \text{on } \partial\mathcal{B}_0^{\mathrm{t}}, \tag{60}$$

where $\boldsymbol{t}_0$ is a prescribed mechanical traction and $\mathbf{N}$ is the outwards pointing normal vector to the surface of $\partial\mathcal{B}_0$. For the magnetic continuity and boundary conditions we refer to Eqs. (6c), (10), (25a), and (25b).

As a final step to complete the boundary value problem, we introduce constitutive relations for the total stress tensor and the magnetic field strength following closely [17]. In analogy with what was done for the free energy used in Sections 2 and 3, we keep the magnetic induction $\mathbb{B}$ as the independent variable which is complemented by the deformation gradient $\boldsymbol{F}$. We define an energy density U_0 related to matter per referential unit volume, which consists of an internal contribution W_0 and an external contribution V_0. The internal contribution W_0 is equivalent to the amended energy density presented in [17] and is split into two parts: the free field magnetic energy density M_0 and an energy density ψ_0, which is associated with the magnetization as well as the strain energy density of the material.

$$\begin{aligned} U_0(\mathbb{A},\mathbb{B},\mathbb{H};\boldsymbol{F},\boldsymbol{\varphi}) &= W_0(\mathbb{B};\boldsymbol{F}) + V_0(\mathbb{A},\mathbb{H};\boldsymbol{\varphi}) \\ &= M_0(\mathbb{B};\boldsymbol{F}) + \psi_0(\mathbb{B};\boldsymbol{F}) + V_0(\mathbb{A},\mathbb{H};\boldsymbol{\varphi}). \end{aligned} \tag{61}$$

Based on the above energy densities the total Piola stress $\boldsymbol{P}^{\mathrm{tot}}$ and the referential magnetic field strength $\mathbb{H}$ are defined as [17]

$$\boldsymbol{P}^{\mathrm{tot}} = \frac{\partial U_0}{\partial \boldsymbol{F}} = \frac{\partial W_0}{\partial \boldsymbol{F}}, \quad \mathbb{H} = \frac{\partial U_0}{\partial \mathbb{B}} = \frac{\partial W_0}{\partial \mathbb{B}}, \tag{62}$$

and the magnetization as

$$\mathbb{M} = -\frac{\partial \psi_0}{\partial \mathbb{B}}. \tag{63}$$

After a Legendre transformation of $W_0(\mathbb{B})$ into the complementary energy density $W_0^*(\mathbb{H})$, we obtain the alternative constitutive relation

$$\mathbb{B} = -\frac{\partial W_0^*}{\partial \mathbb{H}} \quad \text{in } \mathcal{B}_0. \tag{64}$$

4.1. Variational principles for a geometrically non-linear deforming body in free space

In order to investigate variational formulations of a deforming body in free space, the definition of two additional functional spaces for the deformation map $\boldsymbol{\varphi}$ is necessary, see Definition 3 in Appendix A.

At this point we refer to [64] for a mathematically detailed discussion on non-linear elasticity theory including the definition of the necessary Sobolev space, i.e. $\mathrm{W}^{1,p}$ for $p \geq 2$, and the sufficient growth conditions on the strain energy function to ensure the existence of a minimizer for the variational functional. In the following sections, we choose for simplicity $p = 2$.

4.1.1. Three-, two- and one field functionals

For the sake of simplicity, we study here the transformations in the independent variables only for the magnetic quantities. For different variational settings in the mechanical quantities, we refer to standard procedures that can be found, for example, in [65]. Thus, $\boldsymbol{\varphi}$ remains as independent variable with $\boldsymbol{\varphi} \in \mathcal{U}_\varphi(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\boldsymbol{\varphi} \in \mathcal{U}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$ and the identity $\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{\varphi}$ still holds. All principles satisfy the balance of linear momentum (47) and the boundary conditions on the Piola total stress tensor, (58) and (60), in a weak sense, whereas the Dirichlet boundary conditions on the deformation, (57) and (59), as well as the constitutive relations for the Piola total stress inside and the Maxwell stress outside the body, (55) and $(62)_1$, are fulfilled strongly. Table 3 gives an overview of the strong an weak validity of all equations that hold within the bulk and the surrounding free space.

Since we have studied master functionals dealing with a rigid body in Section 3.1, we have basically only to amend them with the external contributions from the body forces and the mechanical tractions. The additional terms are easily identified as the two ultimate integrals in the following list of functionals.

(a) *Primal Hu–Washizu-type functional*

$$\begin{aligned} &\mathcal{I}^{\mathrm{phw}}(\mathbb{A},\mathbb{B},\mathbb{H};\boldsymbol{\varphi}) \\ &= \int_{\mathcal{S}_0} [M_0(\mathbb{B};\boldsymbol{F}) - \mathbb{H}\cdot[\mathbb{B} - \nabla_{\boldsymbol{X}}\times\mathbb{A}]]\,\mathrm{d}V - \int_{\partial\mathcal{S}_\infty^{\mathrm{j}}} \mathbb{H}_\infty\cdot[\boldsymbol{N}_\infty\times\mathbb{A}]\,\mathrm{d}A \\ &\quad + \int_{\mathcal{B}_0} [W_0(\mathbb{B};\boldsymbol{F}) - \mathbb{J}^{\mathrm{f}}\cdot\mathbb{A} - \mathbb{H}\cdot[\mathbb{B} - \nabla_{\boldsymbol{X}}\times\mathbb{A}]]\,\mathrm{d}V \\ &\quad - \int_{\partial\mathcal{B}_0^{\mathrm{j}}} \mathbb{J}^{\mathrm{f,p}}\cdot[\mathbf{N}\times\mathbb{A}]\,\mathrm{d}A - \int_{\mathcal{B}_0} \boldsymbol{\varphi}\cdot\boldsymbol{b}_0\,\mathrm{d}V - \int_{\partial\mathcal{B}_0^{\mathrm{t}}} \boldsymbol{\varphi}\cdot\boldsymbol{t}_0^{\mathrm{p}}\,\mathrm{d}A \end{aligned} \tag{65}$$

for $\mathbb{A} \in \mathcal{V}_{\mathbb{a}}(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\mathbb{B} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\mathbb{H} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\boldsymbol{\varphi} \in \mathcal{U}_\varphi(\mathcal{B}_0 \cup \mathcal{S}_0)$, and $\delta\mathbb{A} \in \mathcal{V}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\mathbb{B} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\mathbb{H} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\boldsymbol{\varphi} \in \mathcal{U}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$. By the solution of the saddle point problem, Eqs. $(44)_1$, $(41)_1$, (45) and (42), the boundary conditions on $\mathbb{H}$ (6c) and (25a), and the constitutive laws for $\mathbb{H}$ $(13)_1$ and $(62)_2$ are weakly

Table 3
Classification of results according to weak and strong satisfaction of field equations and boundary conditions for a deforming body immersed in free space.

In $\mathcal{S}$	In $\mathcal{B}$	phw	phr	pd	dhw	dhr	dd
$\nabla \times \mathbb{H}^{\mu} = \mathbf{0}$	$\nabla \times \mathbb{H} = \mathbb{J}^{\mathrm{f}}$	Weak	Weak	Weak	Weak	Weak	Strong
$\mathbb{B} = \nabla_{\boldsymbol{X}} \times \mathbb{A}$	$\mathbb{B} = \nabla_{\boldsymbol{X}} \times \mathbb{A}$	Weak	Weak	Strong	Weak	Weak	Weak
$\boldsymbol{n}_{\infty} \times \mathbb{H}^{\mu} = \boldsymbol{n}_{\infty} \times \mathbb{H}^{\mu}_{\infty}$	$\boldsymbol{N} \times [\![\mathbb{H}]\!] = \boldsymbol{n} \times \mathbb{J}^{\mathrm{f,p}}$	Weak	Weak	Weak	Strong	Strong	Strong
$\boldsymbol{n}_{\infty} \times \mathbb{A} = \mathbb{A}^{\tan}_{\infty}$	$\boldsymbol{N} \times \mathbb{A}_{\mathrm{o}} = \boldsymbol{N} \times \mathbb{A}_{\mathrm{i}} = \boldsymbol{N} \times \mathbb{A}^{\mathrm{p}}$	Strong	Strong	Strong	Weak	Weak	Weak
$\mathbb{H}^{\mu} = \dfrac{\partial M_0}{\partial \mathbb{B}}$	$\mathbb{H} = \dfrac{\partial W_0}{\partial \mathbb{B}}$	Weak	–	Strong	Weak	–	–
$\mathbb{B} = -\dfrac{\partial M_0^*}{\partial \mathbb{H}^{\mu}}$	$\mathbb{B} = -\dfrac{\partial W_0^*}{\partial \mathbb{H}}$	–	Strong	–	–	Strong	Strong
$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{P}^{\max} = \mathbf{0}$	$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{P}^{\mathrm{tot}} + \boldsymbol{b}_0 = \mathbf{0}$	Weak	Weak	Weak	Weak	Weak	Weak
$\boldsymbol{\varphi}(\boldsymbol{X}) - \boldsymbol{X} = \mathbf{0}$	$\boldsymbol{\varphi} = \boldsymbol{\varphi}^{\mathrm{p}}$	Strong	Strong	Strong	Strong	Strong	Strong
$\boldsymbol{P}^{\max} \cdot \boldsymbol{n}_{\infty} = \mathbf{0}$	$[\![\boldsymbol{P}^{\mathrm{tot}}]\!] \cdot \boldsymbol{N} = -\boldsymbol{t}_0$	Weak	Weak	Weak	Weak	Weak	Weak
$\boldsymbol{P}^{\max} = \dfrac{\partial M_0}{\partial \boldsymbol{F}}$	$\boldsymbol{P}^{\mathrm{tot}} = \dfrac{\partial W_0}{\partial \boldsymbol{F}}$	Strong	Strong	Strong	Strong	Strong	Strong

fulfilled, whereas the boundary conditions on $\mathbb{A}$ (10) and (25b) are strongly fulfilled.

(b) *Primal Hellinger–Reissner-type functional*

$$
\begin{aligned}
&\mathcal{I}^{\mathrm{phr}}(\mathbb{A},\mathbb{H};\boldsymbol{\varphi}) \\
&= \int_{\mathcal{S}_0} [M_0^*(\mathbb{H};\boldsymbol{F})\,\mathrm{d}V + \mathbb{H}\cdot(\nabla_{\boldsymbol{X}} \times \mathbb{A})] - \int_{\partial\mathcal{S}^{\mathrm{j}}_{\infty}} \mathbb{H}_{\infty} \cdot [\boldsymbol{N}_{\infty} \times \mathbb{A}]\,\mathrm{d}A \\
&\quad + \int_{\mathcal{B}_0} [W_0^*(\mathbb{B};\boldsymbol{F})\,\mathrm{d}V - \mathbb{J}^{\mathrm{f}} \cdot \mathbb{A} + \mathbb{H} \cdot [\nabla_{\boldsymbol{X}} \times \mathbb{A}]]\,\mathrm{d}V \\
&- \int_{\partial\mathcal{B}^{\mathrm{j}}_0} \mathbb{J}^{\mathrm{f,p}} \cdot [\mathbf{N} \times \mathbb{A}]\,\mathrm{d}A - \int_{\mathcal{B}_0} \boldsymbol{\varphi} \cdot \boldsymbol{b}_0\,\mathrm{d}V - \int_{\partial\mathcal{B}^{\mathrm{t}}_0} \boldsymbol{\varphi} \cdot \boldsymbol{t}^{\mathrm{p}}_0\,\mathrm{d}A
\end{aligned} \tag{66}
$$

for $\mathbb{A} \in \mathcal{V}(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\mathbb{H} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\boldsymbol{\varphi} \in \mathcal{U}_{\varphi}(\mathcal{B}_0 \cup \mathcal{S}_0)$, and $\delta\mathbb{A} \in \mathcal{V}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\mathbb{H} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\boldsymbol{\varphi} \in \mathcal{U}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$. Assuming the strong validity of the constitutive law for $\mathbb{B}$, $(13)_2$ and (64), as well as the boundary conditions on $\mathbb{A}$, (10) and (25b), then hold the remaining equations $(44)_1$, $(41)_1$, (45) and (42), as well as boundary conditions on $\mathbb{H}$, (6c) and (25a), for this principle in a weak sense.

(c) *Primal Dirichlet-type functional*

$$
\begin{aligned}
\mathcal{I}^{\mathrm{pd}}(\mathbb{A};\boldsymbol{\varphi}) &= \int_{\mathcal{S}_0} M_0(\mathbb{B};\boldsymbol{F})\,\mathrm{d}V - \int_{\partial\mathcal{S}^{\mathrm{j}}_{\infty}} \mathbb{H}_{\infty} \cdot [\boldsymbol{N}_{\infty} \times \mathbb{A}]\,\mathrm{d}A \\
&\quad + \int_{\mathcal{B}_0} [W_0(\mathbb{B};\boldsymbol{F}) - \mathbb{J}^{\mathrm{f}} \cdot \mathbb{A}]\,\mathrm{d}V - \int_{\partial\mathcal{B}^{\mathrm{j}}_0} \mathbb{J}^{\mathrm{f,p}} \cdot [\mathbf{N} \times \mathbb{A}]\,\mathrm{d}A \\
&\quad - \int_{\mathcal{B}_0} \boldsymbol{\varphi} \cdot \boldsymbol{b}_0\,\mathrm{d}V - \int_{\partial\mathcal{B}^{\mathrm{t}}_0} \boldsymbol{\varphi} \cdot \boldsymbol{t}^{\mathrm{p}}_0\,\mathrm{d}A
\end{aligned} \tag{67}
$$

for $\mathbb{A} \in \mathcal{V}_{\mathbb{a}}(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\boldsymbol{\varphi} \in \mathcal{U}_{\varphi}(\mathcal{B}_0 \cup \mathcal{S}_0)$, and $\delta\mathbb{A} \in \mathcal{V}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\boldsymbol{\varphi} \in \mathcal{U}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$. Within the primal Dirichlet principle, Eqs. $(44)_1$, $(41)_1$, and the boundary conditions on $\mathbb{H}$, (6c) and (25a), are weakly fulfilled, whereas Eqs. (45) and (42), the constitutive law for $\mathbb{H}$ $(13)_1$ and $(62)_2$, and the boundary conditions on $\mathbb{A}$ (10) and (25b) are valid in a strong sense.

(d) *Dual Hu–Washizu-type functional*:

$$
\begin{aligned}
&\mathcal{I}^{\mathrm{dhw}}(\mathbb{A},\mathbb{B},\mathbb{H};\boldsymbol{\varphi}) \\
&= \int_{\mathcal{S}_0} [M_0(\mathbb{B};\boldsymbol{F}) - \mathbb{H} \cdot \mathbb{B} + [\nabla_{\boldsymbol{X}} \times \mathbb{H}] \cdot \mathbb{A}]\,\mathrm{d}V - \int_{\partial\mathcal{S}^{\mathrm{a}}_{\infty}} \mathbb{A}_{\infty} \cdot [\boldsymbol{N}_{\infty} \times \mathbb{H}]\,\mathrm{d}A \\
&\quad + \int_{\mathcal{B}_0} [W_0(\mathbb{B};\boldsymbol{F}) - \mathbb{J}^{\mathrm{f}} \cdot \mathbb{A} - \mathbb{H} \cdot \mathbb{B} + [\nabla_{\boldsymbol{X}} \times \mathbb{H}] \cdot \mathbb{A}]\,\mathrm{d}V \\
&\quad + \int_{\partial\mathcal{B}^{\mathrm{a}}_0} \mathbb{A}^{\mathrm{p}} \cdot [\mathbf{N} \times [\![\mathbb{H}]\!]]\,\mathrm{d}A - \int_{\mathcal{B}_0} \boldsymbol{\varphi} \cdot \boldsymbol{b}_0\,\mathrm{d}V - \int_{\partial\mathcal{B}^{\mathrm{t}}_0} \boldsymbol{\varphi} \cdot \boldsymbol{t}^{\mathrm{p}}_0\,\mathrm{d}A
\end{aligned} \tag{68}
$$

for $\mathbb{A} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\mathbb{B} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\mathbb{H} \in \mathcal{W}_{\mathbb{j}}(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\boldsymbol{\varphi} \in \mathcal{U}_{\varphi}(\mathcal{B}_0 \cup \mathcal{S}_0)$, and $\delta\mathbb{A} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\mathbb{B} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\mathbb{H} \in \mathcal{W}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\boldsymbol{\varphi} \in \mathcal{U}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$. Eqs. $(44)_1$, $(41)_1$, (45) and (42), the boundary conditions on $\mathbb{A}$, (10) and (25b), and the constitutive law for $\mathbb{H}$, $(13)_1$ and $(62)_2$, are the Euler–Lagrange equations of this principle. The boundary conditions on $\mathbb{H}$, (6c) and (25a), in contrast, are the only equations that hold in a strong sense.

(e) *Dual Hellinger–Reissner-type functional*:

$$
\begin{aligned}
&\mathcal{I}^{\mathrm{dhr}}(\mathbb{a},\mathbb{h};\boldsymbol{\varphi}) \\
&= \int_{\mathcal{S}_0} [M_0^*(\mathbb{H};\boldsymbol{F}) + [\nabla_{\boldsymbol{X}} \times \mathbb{H}] \cdot \mathbb{A}]\,\mathrm{d}A - \int_{\partial\mathcal{S}^{\mathrm{a}}_{\infty}} \mathbb{A}_{\infty} \cdot [\boldsymbol{N}_{\infty} \times \mathbb{H}]\,\mathrm{d}A \\
&\quad + \int_{\mathcal{B}_0} [W^*(\mathbb{H};\boldsymbol{F}) - \mathbb{J}^{\mathrm{f}} \cdot \mathbb{A} + [\nabla_{\boldsymbol{X}} \times \mathbb{H}] \cdot \mathbb{A}]\,\mathrm{d}V + \int_{\partial\mathcal{B}^{\mathrm{a}}_0} \mathbb{A}^{\mathrm{p}} \cdot [\mathbf{N} \times [\![\mathbb{H}]\!]]\,\mathrm{d}A \\
&\quad - \int_{\mathcal{B}_0} \boldsymbol{\varphi} \cdot \boldsymbol{b}_0\,\mathrm{d}V - \int_{\partial\mathcal{B}^{\mathrm{t}}_0} \boldsymbol{\varphi} \cdot \boldsymbol{t}^{\mathrm{p}}_0\,\mathrm{d}A
\end{aligned} \tag{69}
$$

for $\mathbb{A} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\mathbb{H} \in \mathcal{W}_{\mathbb{j}}(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\boldsymbol{\varphi} \in \mathcal{U}_{\varphi}(\mathcal{B}_0 \cup \mathcal{S}_0)$, and $\delta\mathbb{A} \in \mathrm{L}^2(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\mathbb{H} \in \mathcal{W}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\boldsymbol{\varphi} \in \mathcal{U}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$. Assuming the strong identity of the boundary conditions on $\mathbb{H}$, (6c) and (25a), and the constitutive law for $\mathbb{B}$, $(13)_2$ and (64), the Euler–Lagrange equations of the dual Hellinger–Reissner principle coincide with Eqs. $(44)_1$, $(41)_1$, (45) and (42), and the boundary conditions for $\mathbb{A}$, (10) and (25b).

(f) *Dual Dirichlet-type functional*:

$$
\begin{aligned}
\mathcal{I}^{\mathrm{dd}}(\mathbb{A};\boldsymbol{\varphi}) &= \int_{\mathcal{S}} M_0^*(\mathbb{H};\boldsymbol{F})\,\mathrm{d}V - \int_{\partial\mathcal{S}^{\mathrm{a}}_{\infty}} \mathbb{A}_{\infty} \cdot [\boldsymbol{N}_{\infty} \times \mathbb{H}]\,\mathrm{d}A \\
&\quad + \int_{\mathcal{B}} W_0^*(\mathbb{h};\boldsymbol{F})\,\mathrm{d}V - \int_{\mathcal{B}_0} \mathbb{J}^{\mathrm{f}} \cdot \mathbb{A}\,\mathrm{d}V + \int_{\partial\mathcal{B}^{\mathrm{a}}_0} \mathbb{A}^{\mathrm{p}} \cdot [\mathbf{N} \times [\![\mathbb{H}]\!]]\,\mathrm{d}A \\
&\quad - \int_{\mathcal{B}_0} \boldsymbol{\varphi} \cdot \boldsymbol{b}_0\,\mathrm{d}V - \int_{\partial\mathcal{B}^{\mathrm{t}}_0} \boldsymbol{\varphi} \cdot \boldsymbol{t}^{\mathrm{p}}_0\,\mathrm{d}A
\end{aligned} \tag{70}
$$

for $\mathbb{H} \in \widetilde{\mathcal{W}}_{\mathbb{j}}(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\boldsymbol{\varphi} \in \mathcal{U}_{\varphi}(\mathcal{B}_0 \cup \mathcal{S}_0)$, and $\delta\mathbb{H} \in \widetilde{\mathcal{W}}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$, $\delta\boldsymbol{\varphi} \in \mathcal{U}_0(\mathcal{B}_0 \cup \mathcal{S}_0)$. For the dual Dirichlet principle, Eqs. $(44)_1$ and $(41)_1$, the constitutive relations for $\mathbb{B}$, $(13)_2$ and (64), as well as the boundary conditions on $\mathbb{H}$, (6c) and (25a), are strongly fulfilled, whereas (45) and (42), and the boundary conditions on $\mathbb{A}$, (10) and (25b), are satisfied in a weak sense.

5. Conclusions

In this communication we have presented different variational formulations for the problem of a magneto-active body exhibiting large elastic deformations under the stimulation of external mechanical loads and magnetic fields. The body is assumed to be surrounded completely by free space, which calls for a careful consideration of the continuity conditions of the magnetic field quantities. The magnetic vector potential is used as one of the independent magnetic variables. The main interest has been to propose systematically mixed formulations (for the magnetic part of the problem), and to study in detail the functional spaces to which the different variables in these formulations belong. A summary of the important results corresponds to Fig. 1, where the relations between the presented mixed formulations are illustrated.

As remarked in the Introduction, the modeling of the behavior of bodies undergoing large elastic deformation under the coupled effect of magnetic fields and mechanical loads is a particularly complex challenge considering the non-linearity of the equations involved and the different couplings to be considered. There is a need for numerical methods of solutions, and the finite element method can be one of the best options for the simulation. It is important to have different mixed variational formulations with a number of different variables for the functionals, since a standard formulation, where we would only consider the displacement field and the magnetic vector potential as the main variables, may not be the best option from the numerical point of view. In particular, in order to deal with the different problems and the continuity conditions between the body and the surrounding free space, a larger variety in the choice of approximation functions for the unknown vector fields might be advantageous.

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Appendix A. Definitions from functional analysis

Definition 1 (*Function spaces in free space*). Let $\mathbb{v}$ be an arbitrary vector field defined on the domain $\Omega \in \mathbb{R}^3$, $d \in \{1,2,3\}$, with boundary Γ. We define the following function spaces:

$$\mathrm{L}^2(\Omega) = \left\{ \mathbb{v} : \Omega \to \mathbb{R}^d \,\middle|\, \|\mathbb{v}\|_{\mathrm{L}^2(\Omega)} = \left(\int_\Omega |\mathbb{v}|^2 \,\mathrm{d}v \right)^{1/2} < \infty \right\},$$

$$\mathrm{H}^{1,2}(\Omega) = \{ \mathbb{v} \in \mathrm{L}^2(\Omega) \,|\, \nabla \mathbb{v} \in \mathrm{L}^2(\Omega) \},$$

$$\mathrm{H}_0^{1,2}(\Omega) = \{ \mathbb{v} \in \mathrm{H}^{1,2}(\Omega) \,|\, \mathbb{v} = \mathbf{0} \text{ on } \Gamma \},$$

$$\mathrm{H}(\mathrm{curl};\Omega) = \{ \mathbb{v} \in \mathrm{L}^2(\Omega) \,|\, \nabla \times \mathbb{v} \in \mathrm{L}^2(\Omega) \},$$

$$\mathrm{H}_0(\mathrm{curl};\Omega) = \{ \mathbb{v} \in \mathrm{H}(\mathrm{curl};\Omega) \,|\, \boldsymbol{n}_\infty \times \mathbb{v} = \mathbf{0} \text{ on } \Gamma \},$$

$$\widetilde{\mathrm{H}}(\mathrm{curl};\Omega) = \{ \mathbb{v} \in \mathrm{H}(\mathrm{curl};\Omega) \,|\, \nabla \times \mathbb{v} = \mathbf{0} \},$$

$$\mathcal{V}_{\mathbb{a}}(\mathcal{S}) = \{ \mathbb{v} \in \mathrm{H}(\mathrm{curl};\mathcal{S}) \,|\, \boldsymbol{n}_\infty \times \mathbb{v} = \mathbb{a}_\infty^{\tan} \text{ on } \partial\mathcal{S}_\infty^{\mathbb{a}} \},$$

$$\mathcal{V}_0(\mathcal{S}) = \{ \mathbb{v} \in \mathrm{H}(\mathrm{curl};\mathcal{S}) \,|\, \boldsymbol{n}_\infty \times \mathbb{v} = \mathbf{0} \text{ on } \partial\mathcal{S}_\infty^{\mathbb{a}} \},$$

$$\mathcal{W}_{\mathbb{j}}(\mathcal{S}) = \{ \mathbb{v} \in \mathrm{H}(\mathrm{curl};\mathcal{S}) \,|\, \boldsymbol{n}_\infty \times \mathbb{v} = \boldsymbol{n}_\infty \times \mathbb{h}_\infty^{\mu} \text{ on } \partial\mathcal{S}_\infty^{\mathbb{j}} \},$$

$$\mathcal{W}_0(\mathcal{S}) = \{ \mathbb{v} \in \mathrm{H}(\mathrm{curl};\mathcal{S}) \,|\, \boldsymbol{n}_\infty \times \mathbb{v} = \mathbf{0} \text{ on } \partial\mathcal{S}_\infty^{\mathbb{j}} \},$$

$$\widetilde{\mathcal{W}}_{\mathbb{j}}(\mathcal{S}) = \{ \mathbb{v} \in \widetilde{\mathrm{H}}(\mathrm{curl},\mathcal{S}) \,|\, \boldsymbol{n}_\infty \times \mathbb{v} = \boldsymbol{n}_\infty \times \mathbb{h}_\infty^{\mu} \text{ on } \partial\mathcal{S}_\infty^{\mathbb{j}} \},$$

$$\widetilde{\mathcal{W}}_0(\mathcal{S}) = \{ \mathbb{v} \in \widetilde{\mathrm{H}}(\mathrm{curl},\mathcal{S}) \,|\, \boldsymbol{n}_\infty \times \mathbb{v} = \mathbf{0} \quad \text{on } \partial\mathcal{S}_\infty^{\mathbb{j}} \},$$

and the proper norms for $\mathrm{H}^{1,2}(\Omega)$ and $\mathrm{H}(\mathrm{curl};\Omega)$:

$$\|\mathbb{v}\|_{\mathrm{H}^{1,2}(\Omega)} = (\|\mathbb{v}\|_{\mathrm{L}^2(\Omega)}^2 + \|\nabla \mathbb{v}\|_{\mathrm{L}^2(\Omega)}^2)^{1/2},$$

$$\|\mathbb{v}\|_{\mathrm{H}(\mathrm{curl};\Omega)} = (\|\mathbb{v}\|_{\mathrm{L}^2(\Omega)}^2 + \|\nabla \times \mathbb{v}\|_{\mathrm{L}^2(\Omega)}^2)^{1/2}.$$

For a detailed definition of the space of traces $\mathrm{H}^{1/2}(\Gamma)$ of $\mathrm{H}^{1,2}$, we refer to [55,66,67]. Its dual space is denoted by $\mathrm{H}^{-1/2}(\Gamma) = [\mathrm{H}^{1/2}(\Gamma)]'$.

$\mathrm{L}^2(\Omega)$, $\mathrm{H}^{1,2}(\Omega)$ and $\mathrm{H}(\mathrm{curl};\Omega)$ are Hilbert spaces equipped with the inner products

$$(\mathbb{u},\mathbb{v})_{\mathrm{L}^2(\Omega)} = \int_\Omega \mathbb{u} \cdot \mathbb{v} \,\mathrm{d}v,$$

$$(\mathbb{u},\mathbb{v})_{\mathrm{H}^{1,2}(\Omega)} = (\mathbb{u},\mathbb{v})_{\mathrm{L}^2(\Omega)} + (\nabla \mathbb{u}, \nabla \mathbb{v})_{\mathrm{L}^2(\Omega)},$$

$$(\mathbb{u},\mathbb{v})_{\mathrm{H}(\mathrm{curl};\Omega)} = (\mathbb{u},\mathbb{v})_{\mathrm{L}^2(\Omega)} + (\nabla \times \mathbb{u}, \nabla \times \mathbb{v})_{\mathrm{L}^2(\Omega)}.$$

The same holds for $\mathrm{H}_0^{1,2}(\Omega)$ (as well as $\mathcal{V}_0(\mathcal{S})$) and $\mathrm{H}_0(\mathrm{curl};\Omega)$ together with the inner product of $\mathrm{H}^{1,2}(\Omega)$ and $\mathrm{H}(\mathrm{curl};\Omega)$, respectively.

Definition 2 (*Function spaces for the body and free space*). Let $\mathbb{v}$ be an arbitrary vector field. We define the following function spaces:

$$\mathcal{V}(\mathcal{B} \cup \mathcal{S}) = \{ \mathbb{v} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S}) \,|\, \mathbb{v}|_{\mathcal{S}} \in \mathrm{H}(\mathrm{curl};\mathcal{S}),\ \mathbb{v}|_{\mathcal{B}} \in \mathrm{H}(\mathrm{curl};\mathcal{B}) \},$$

$$\mathcal{V}_{\mathbb{a}}(\mathcal{B} \cup \mathcal{S}) = \{ \mathbb{v} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S}) \,|\, \mathbb{v}|_{\mathcal{S}} \in \mathcal{V}_{\mathbb{a}}(\mathcal{S}), \mathbb{v}|_{\mathcal{B}} \in \mathrm{H}(\mathrm{curl};\mathcal{B}),\ \boldsymbol{n} \times \mathbb{v} = \mathbb{a}^{\tan,\mathrm{p}} \text{ on } \partial\mathcal{B}^{\mathbb{a}} \},$$

$$\mathcal{V}_0(\mathcal{B} \cup \mathcal{S}) = \{ \mathbb{v} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S}) \,|\, \mathbb{v}|_{\mathcal{S}} \in \mathcal{V}_0(\mathcal{S}), \mathbb{v}|_{\mathcal{B}} \in \mathrm{H}(\mathrm{curl};\mathcal{B}),\ \boldsymbol{n} \times \mathbb{v} = \mathbf{0} \text{ on } \partial\mathcal{B}^{\mathbb{a}} \},$$

$$\mathcal{W}_{\mathbb{j}}(\mathcal{B} \cup \mathcal{S}) = \{ \mathbb{v} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S}) \,|\, \mathbb{v}|_{\mathcal{S}} \in \mathcal{W}_{\mathbb{j}}(\mathcal{S}), \mathbb{v}|_{\mathcal{B}} \in \mathrm{H}(\mathrm{curl};\mathcal{B}),\ \boldsymbol{n} \times [\![\mathbb{v}]\!] = \hat{\mathbb{j}}^{\mathrm{f}} \text{ on } \partial\mathcal{B}^{\mathbb{j}} \},$$

$$\mathcal{W}_0(\mathcal{B} \cup \mathcal{S}) = \{ \mathbb{v} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S}) \,|\, \mathbb{v}|_{\mathcal{S}} \in \mathcal{W}_{\mathbb{j}}(\mathcal{S}), \mathbb{v}|_{\mathcal{B}} \in \mathrm{H}(\mathrm{curl};\mathcal{B}),\ \boldsymbol{n} \times [\![\mathbb{v}]\!] = \mathbf{0} \text{ on } \partial\mathcal{B}^{\mathbb{j}} \},$$

$$\widetilde{\mathcal{W}}_{\mathbb{j}}(\mathcal{B} \cup \mathcal{S}) = \{ \mathbb{v} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S}) \,|\, \mathbb{v}|_{\mathcal{S}} \in \widetilde{\mathcal{W}}_{\mathbb{j}}(\mathcal{S}), \mathbb{v}|_{\mathcal{B}} \in \widetilde{\mathrm{H}}(\mathrm{curl};\mathcal{B}),\ \boldsymbol{n} \times [\![\mathbb{v}]\!] = \hat{\mathbb{j}}^{\mathrm{f}} \text{ on } \partial\mathcal{B}^{\mathbb{j}} \},$$

$$\widetilde{\mathcal{W}}_0(\mathcal{B} \cup \mathcal{S}) = \{ \mathbb{v} \in \mathrm{L}^2(\mathcal{B} \cup \mathcal{S}) \,|\, \mathbb{v}|_{\mathcal{S}} \in \widetilde{\mathcal{W}}_{\mathbb{j}}(\mathcal{S}), \mathbb{v}|_{\mathcal{B}} \in \widetilde{\mathrm{H}}(\mathrm{curl};\mathcal{B}),\ \boldsymbol{n} \times [\![\mathbb{v}]\!] = \mathbf{0} \text{ on } \partial\mathcal{B}^{\mathbb{j}} \}.$$

Definition 3 (*Functional spaces for the deformation*). Let $\boldsymbol{v}$ be an arbitrary vector field. We define

$$\mathcal{U}_\varphi(\mathcal{B}_0 \cup \mathcal{S}_0) = \{ \boldsymbol{v} \in \mathrm{W}^{1,\mathrm{p}}(\mathcal{B}_0 \cup \mathcal{S}_0) \,|\, \boldsymbol{v}|_{\partial\mathcal{S}_\infty} = \mathbf{0}, \boldsymbol{v}|_{\partial\mathcal{B}_0^{\mathrm{d}}} = \boldsymbol{\varphi}^{\mathrm{p}} \},$$

$$\mathcal{U}_0(\mathcal{B}_0 \cup \mathcal{S}_0) = \{ \boldsymbol{v} \in \mathrm{W}^{1,p}(\mathcal{B}_0 \cup \mathcal{S}_0) \,|\, \boldsymbol{v}|_{\partial\mathcal{S}_\infty} = \mathbf{0}, \boldsymbol{v}|_{\partial\mathcal{B}_0^{\mathrm{d}}} = \mathbf{0} \},$$

for $p \geq 2$ and where $\boldsymbol{v}|_{\partial\mathcal{S}_\infty}$ denotes the trace of $\boldsymbol{v}$ along the boundary $\partial\mathcal{S}_\infty$.

Appendix B. Existence and uniqueness of a stationary point for the primal Dirichlet-type functional

Existence and uniqueness of a stationary point for the primal Dirichlet-type functional (14) can be proved with the help of Brezzi's theorem [55,68,69]. The Brezzi theorem guarantees existence and uniqueness of a solution of saddle point problems with divergence-free conditions as, for the example, in the classical Stokes problem. We adopt this structure for the magnetostatic problem and define the following bilinear forms:

$$a(\mathbb{u},\mathbb{v}) := \int_{\mathcal{S}} \frac{1}{\mu_0}[\nabla \times \mathbb{u}] \cdot [\nabla \times \mathbb{v}]\, \mathrm{d}v$$

and

$$b(\mathbb{v},q) := \int_{\mathcal{S}} \mathbb{v} \cdot \nabla q\, \mathrm{d}v,$$

and the linear form

$$f(\mathbb{v}) := \int_{\partial \mathcal{S}_\infty^{\mathrm{i}}} \mathbb{h}_\infty^{\mu} \cdot [\boldsymbol{n}_\infty \times \mathbb{v}]\, \mathrm{d}a.$$

Note here that the bilinear form a is not coercive in general and thus, an application of the Lax–Milgram lemma is not possible. Furthermore, the divergence free condition on $\mathbb{a}$ is needed in order to guarantee uniqueness of the magnetic vector potential, which is incorporated by the bilinear form b. For the test functions $q \in \mathrm{H}_0^{1,2}(\mathcal{S})$, the elements $\mathbb{v}$ of the kernel of $b(\mathbb{v},q)$ are divergence free, since

$$b(\mathbb{v},q) = \int_{\mathcal{S}} \mathbb{v} \cdot \nabla q\, \mathrm{d}v = -\int_{\mathcal{S}} [\nabla \cdot \mathbb{v}]\varphi\, \mathrm{d}v = 0 \quad \forall q \in \mathrm{H}_0^{1,2}(\mathcal{S})$$
$$\Leftrightarrow \nabla \cdot \mathbb{v} = 0 \quad \text{in } \mathcal{S}.$$

As prerequisites for the Brezzi theorem, one has to show the continuity of a and b which is achieved straightforwardly with the Cauchy–Schwartz inequality. Furthermore, it is assumed that a is coercive on the kernel of b which can be shown with the help of the Friedrichs'-type inequality. As last assumption, b has to satisfy the Ladyzhenskaya–Babuška–Brezzi-condition, which holds trivially for $\mathbb{v} = \nabla q$. A final application of the Poincaré inequality leads to the desired condition.

The mixed variational problem of the saddle point problem reads as follows: given $\widetilde{\mathbb{a}} \in \mathcal{V}_{\mathbb{a}}(\mathcal{S})$, find $(\mathbb{a},\varphi) \in \mathcal{V}_{\mathbb{a}}(\mathcal{S}) \times \mathrm{H}_0^{1,2}(\mathcal{S})$ such that $(\mathbb{a}-\widetilde{\mathbb{a}},\varphi) \in \mathcal{V}_0(\mathcal{S}) \times \mathrm{H}_0^{1,2}(\mathcal{S})$ and

$$a(\mathbb{a}-\widetilde{\mathbb{a}},\delta\mathbb{a}) + b(\delta\mathbb{a},\varphi) = f(\delta\mathbb{a}), \quad \forall \delta\mathbb{a} \in \mathcal{V}_0(\mathcal{S}),$$

$$b(\mathbb{a}-\widetilde{\mathbb{a}},\delta\varphi) = 0, \quad \forall \delta\varphi \in \mathrm{H}_0^{1,2}(\mathcal{S}).$$

With $\widetilde{f}(\mathbb{v}) := f(\mathbb{v}) + a(\widetilde{\mathbb{a}},\mathbb{v})$ and $\widetilde{g}(q) := b(\widetilde{\mathbb{a}},q)$, this can be rewritten as

$$\begin{aligned} &a(\mathbb{a},\delta\mathbb{a}) + b(\delta\mathbb{a},\varphi) = \widetilde{f}(\delta\mathbb{a}), \quad \forall \delta\mathbb{a} \in \mathrm{H}_0(\mathrm{curl};\mathcal{S}),\\ &b(\mathbb{a},\delta\varphi) = \widetilde{g}(\delta\varphi), \quad \forall \delta\varphi \in \mathrm{H}_0^{1,2}(\mathcal{S}). \end{aligned} \tag{71}$$

For problem (71), existence and uniqueness is given by the Brezzi theorem.

References

[1] Yousef Razouk, Eric Duhayon, Bertrand Nogarede, Magnetoactivated elastomer: an alternative to electroactive polymer, Solid State Phenomena 144 (2009) 244–249.
[2] M. Kallio, T. Aalto, T. Lindroos, E. Jarvinen, T. Karna, T. Meiander, Preliminary test on a MRE device, Proceedings of AMAS Workshop on Smart Materials, Jadwisin, 2003, p. 353–360.
[3] K. Danas, S.V. Kankanala, N. Triantafyllidis, Experiments and modeling of iron-particle-filled magnetorheological elastomers, Journal of the Mechanics and Physics of Solids 60 (1) (2012) 120–138.
[4] M. R Jolly, J.D. Carlson, B.C. Munoz, A model of the behaviour of magnetorheological materials, Smart Materials and Structures 5 (5) (1996) 607–614.
[5] J. David Carlson, Mark R. Jolly, MR fluid, foam and elastomer devices, Mechatronics 10 (2000) 555–569.
[6] T.M. Simon, F. Reitich, M.R. Jolly, K. Ito, H.T. Banks, The effective magnetic properties of magnetorheological fluids, Mathematical and Computer Modelling 33 (1–3) (2001) 273–284.
[7] William Fuller Brown, Electric and magnetic forces: a direct calculation. I, American Journal of Physics 19 (5) (1951) 290–304.
[8] William Fuller Brown, Theory of magnetoelastic effects in ferromagnetism, Journal of Applied Physics 36 (3) (1965) 994–1000.
[9] H.F. Tiersten, Variational principle for saturated magnetoelastic insulators, Journal of Mathematical Physics 6 (5) (1965) 779–787.
[10] William Fuller Brown, Magnetoelastic Interactions, Springer, Berlin, Heidelberg, New York, 1966.
[11] K. Hutter, A. van de Ven, Field Matter Interactions in Thermoelastic Solids: A Unification of Existing Theories of Electro-magneto-mechanical Interactions, Lecture Notes in Physics, vol. 88, Springer, Berlin, Heidelberg, 1978.
[12] Y. H Pao, Electromagnetic forces in deformable continua, in: S. Nemat-Nasser (Ed.), Mechanics Today, Pergamon Press, New York, 1978, pp. 209–306.
[13] G.A. Maugin, Continuum Mechanics of Electromagnetic Solids, North-Holland, Amsterdam, 1988.
[14] Gerard Maugin, A. Cemal Eringen, Electrodynamics of Continua, Springer, Berlin, 1990.
[15] Antonio DeSimone, Paolo Podio-Guidugli, On the continuum theory of deformable ferromagnetic solids, Archive for Rational Mechanics and Analysis 136 (1996) 201–233.
[16] A. Dorfmann, R.W. Ogden, Nonlinear magnetoelastic deformations of elastomers, Acta Mechanica 167 (1–2) (2003) 13–28.
[17] A. Dorfmann, R.W. Ogden, Nonlinear magnetoelastic deformations, Quarterly Journal of Mechanics and Applied Mathematics 57 (4) (2004) 599–622.
[18] S.V. Kankanala, N. Triantafyllidis, On finitely strained magnetorheological elastomers, Journal of the Mechanics and Physics of Solids 52 (12) (2004) 2869–2908.
[19] D.J. Steigmann, Equilibrium theory for magnetic elastomers and magnetoelastic membranes, International Journal of Non-Linear Mechanics 39 (7) (2004) 1193–1216.
[20] M. Barham, D.J. Steigmann, M. McElfresh, R.E. Rudd, Finite deformation of a pressurized magnetoelastic membrane in a stationary dipole field, Acta Mechanica 191 (1–2) (2007) 1–19.
[21] Roger Bustamante, Mathematical Modelling of Non-linear Magneto- and Electro-active Rubber-like Materials, Ph.D. Thesis, University of Glasgow, Glasgow, 2007.
[22] D.K. Vu, P. Steinmann, Nonlinear electro- and magneto-elastostatics: material and spatial settings, International Journal of Solids and Structures 44 (24) (2007) 7891–7905.
[23] Gerard Maugin, On modelling electromagnetomechanical interactions in deformable solids, International Journal of Advances in Engineering Sciences and Applied Mathematics 1 (1) (2009) 25–32.
[24] D.K. Vu, P. Steinmann, Material and spatial motion problems in nonlinear electro- and magneto-elastostatics, Mathematics and Mechanics of Solids 15 (2) (2010) 239–257.
[25] Paul Steinmann, Computational nonlinear electro-elasticity–getting started: mechanics and electrodynamics of magneto- and electro-elastic materials, in: Ray W. Ogden, David J. Steigmann (Eds.), Mechanics and Electrodynamics of Magneto- and Electro-elastic Materials, CISM Courses and Lectures, vol. 527, Springer, Vienna, 2011.
[26] David J. Griffiths, Introduction to Electrodynamics, 2nd edition, Prentice-Hill Inc., Englewood Cliffs, NJ, 1989.
[27] J.D. Jackson, Classical Electrodynamics, 3rd edition, Wiley, 1999.
[28] Attay Kovetz, Electromagnetic Theory, Oxford University Press, Oxford, New York, 2000.
[29] R. Bustamante, A. Dorfmann, R.W. Ogden, Numerical solution of finite geometry boundary-value problems in nonlinear magnetoelasticity, International Journal of Solids and Structures 48 (2011) 874–883.
[30] F. Vogel, R. Bustamante, P. Steinmann, On some mixed variational principles in electro-elastostatics, International Journal of Non-Linear Mechanics 47 (2) (2012) 341–354.
[31] Mark D. Semon, Thoughts on the magnetic vector potential, American Journal of Physics 64 (11) (1996) 1361.
[32] C.J. Carpenter, Comparison of alternative formulations of 3-dimensional magnetic field and eddy current problems at power frequencies, Proceedings of the Institution of Electrical Engineers 124 (11) (1977) 1026–1034.
[33] Z.J. Cendes, J. Weiss, S. Hoole, Alternative vector potential formulations of 3-D magnetostatic field problems, IEEE Transactions on Magnetics 18 (2) (1982) 367–372.
[34] T. Morisue, A new formulation of the magnetic vector potential method for three dimensional magnetostatic field problems, IEEE Transactions on Magnetics 21 (6) (1985) 2192–2195.
[35] O. Biro, K. Preis, On the use of the magnetic vector potential in the finite-element analysis of three-dimensional eddy currents, IEEE Transactions on Magnetics 25 (4) (1989) 3145–3159.
[36] R. Albanese, G. Rubinacci, Magnetostatic field computations in terms of two-component vector potentials, International Journal for Numerical Methods in Engineering 29 (1990) 515–532.

[37] R.C. Mesquita, J.P.A. Bastos, An incomplete gauge formulation for 3D nodal finite-element magnetostatics, IEEE Transactions on Magnetics 28 (2) (1992) 1044–1047.

[38] K. Preis, I. Bardi, O. Biro, C. Magele, G. Vrisk, K.R. Richter, Different finite element formulations of 3D magnetostatic fields, IEEE Transactions on Magnetics 28 (2) (1992) 1056–1059.

[39] N.A. Golias, T.D. Tsiboukis, Magnetostatics with edge elements: a numerical investigation in the choice of the tree, IEEE Transactions on Magnetics 30 (5) (1994) 2877–2880.

[40] O. Biro, K. Preis, K.R. Richter, On the use of the magnetic vector potential in the nodal and edge finite element analysis of 3D magnetostatic problems, IEEE Transactions on Magnetics 32 (3) (1996) 651–654.

[41] I. Perugia, A mixed formulation for 3D magnetostatic problems: theoretical analysis and face-edge finite element approximation, Numerische Mathematik 84 (2) (1999) 305–326.

[42] H. Igarashi, On the property of the curl–curl matrix in finite element analysis with edge elements, IEEE Transactions on Magnetics 37 (5) (2001) 3129–3132.

[43] C. Daveau, J. Laminie, Mixed and hybrid formulations for the three-dimensional magnetostatic problem, Numerical Methods for Partial Differential Equations 18 (1) (2002) 85–104.

[44] John B. Manges, Zoltan J. Cendes, Generalized tree-cotree gauge for magnetic field computation, IEEE Transactions on Magnetics 31 (3 pt 1) (1995) 1342–1347.

[45] G.A. Maugin, Deformable magnetically saturated media. I. Field equations, Journal of Mathematical Physics 13 (2) (1972) 143.

[46] J. Penman, J. Fraser, Complementary and dual energy finite element principles in magnetostatics, IEEE Transactions on Magnetics 18 (2) (1982) 319–324.

[47] J. Rikabi, C.F. Bryant, E.M. Freeman, An error-based approach to complementary formulations of static field solutions, International Journal for Numerical Methods in Engineering 26 (9) (1988) 1963–1987.

[48] R.D. James, D. Kinderlehrer, Frustration in ferromagnetic materials, Continuum Mechanics and Thermodynamics 2 (3) (1990) 215–239.

[49] R.D. James, D. Kinderlehrer, Theory of magnetostriction with applications to $Tb_xDy_{1-x}Fe_2$, Philosophical Magazine Part B 68 (2) (1993) 237–274.

[50] Antonio DeSimone, Georg Dolzmann, Existence of minimizers for a variational problem in two-dimensional nonlinear magnetoelasticity, Archive for Rational Mechanics and Analysis 144 (2) (1998) 107–120.

[51] A. DeSimone, R.D. James, A constrained theory of magnetoelasticity, Journal of the Mechanics and Physics of Solids 50 (2) (2002) 283–320.

[52] Piotr Rybka, Mitchell Luskin, Existence of energy minimizers for magnetostrictive materials, SIAM Journal on Mathematical Analysis 36 (6) (2005) 2004.

[53] W. Bielski, B. Gambin, Relationship between existence of energy minimizers of incompressible and nearly incompressible magnetostrictive materials, Reports on Mathematical Physics 66 (2) (2010) 147–157.

[54] Arnulf Kost, Numerische Methoden in der Berechnung elektromagnetischer Felder, Springer, Berlin, Heidelberg, 1994.

[55] Vivette Girault, Pierre-Arnaud Raviart, Finite Element Methods for Navier–Stokes Equations, Springer Series in Computational Mathematics, Springer, Berlin, 1986.

[56] Peter Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press, Clarendon, 2003.

[57] Lawrence C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[58] Roger Bustamante, Luis Dorfmann, Ray Ogden, On variational formulations in nonlinear magnetoelastostatics, Mathematics and Mechanics of Solids 13 (8) (2008) 725–745.

[59] H.F. Tiersten, Coupled magnetomechanical equations for magnetically saturated insulators, Journal of Mathematical Physics 5 (9) (1964) 1298–1318.

[60] Richard A. Toupin, The elastic dielectric, Journal of Rational Mechanics and Analysis 5 (1956) 849–915.

[61] G.A. Maugin, A continuum theory of deformable ferrimagnetic bodies. I. Field equations, Journal of Mathematical Physics 17 (9) (1975) 1727–1738.

[62] Halit S. Gokturk, Thomas J. Fiske, Dilhan M. Kalyon, Electric and magnetic properties of a thermoplastic elastomer incorporated with ferromagnetic powders, IEEE Transactions on Magnetics 29 (6 pt 3) (1993) 4170–4176.

[63] M.R. Jolly, J.D. Carlson, B.C. Munoz, T.A. Bullions, The magnetoviscoelastic response of elastomer composites consisting of ferrous particles embedded in a polymer matrix, Journal of Intelligent Material Systems and Structures 7 (6) (1996) 613–622.

[64] Philippe G. Ciarlet, Mathematical Elasticity, vol. 1, Elsevier, Amsterdam, 1998.

[65] R.W. Ogden, Non-linear Elastic Deformations, Dover Publications, Inc, Mineola, 1997.

[66] Hans Triebel, Theory of Function Spaces II, Birkhäuser Verlag, Basel, Boston, 1992.

[67] Thomas Runst, Winfried Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, Walter de Gruyter, Berlin, New York, 1996.

[68] Franco Brezzi, Michel Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

[69] Dietrich Braess, Finite Elemente: Finite Elemente Finite Elemente Theorie, Schnelle Löser und Anwendungen in der Elastizitätstheorie, Springer, Berlin, Heidelberg, 2004.