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Resonance phenomenon for a Gelfand-type problem ☆

Wenjing Chen *, Juan Dávila

Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile

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ABSTRACT

We consider positive radially symmetric solutions of

$$-\Delta u = \lambda(e^u - 1), \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B,$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$ is a parameter. We establish infinite multiplicity of regular solutions for $3 \leq N \leq 9$ and some λ, and we obtain a bound for the Morse index and the number of solutions when $N \geq 10$.

1. Introduction

In this article, we are interested in the structure of the solution set of the boundary value problem

$$\begin{cases} -\Delta u = \lambda(e^u - 1), \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$ is a parameter. Smooth solutions to (1.1) are radially symmetric and decreasing by the classical result of Gidas, Ni and Nirenberg [1].

Problem (1.1) is related to the following *Gelfand* problem:

$$\begin{cases} -\Delta u = \lambda e^u, & \text{in } B; \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.2}$$

Barenblatt [2] and Joseph and Lundgren [3], using phase-plane analysis, gave a complete description of the classical solutions to (1.2), which are again radially symmetric [1].

Proposition 1.1. *Assume $N \geq 1$, then there exists $\lambda^* = \lambda^*(N) > 0$, such that*

- *for $0 < \lambda < \lambda^*$, (1.2) has the minimal solution u_λ;*
- *for $\lambda = \lambda^*$, (1.2) has a unique solution;*
- *for $\lambda > \lambda^*$, (1.2) has no solution (even in the weak sense).*

Moreover, we have the following.

(a) *if $N = 1, 2$, then for $0 < \lambda < \lambda^*$, there are exactly two solutions to (1.2), one of them is the minimal solution u_λ. The other one, denoted by U_λ, has Morse index 1.*

* Corresponding author. Tel.: +56 2 9784554.
E-mail addresses: wjchen1102@yahoo.com.cn, wjchen1102@gmail.com (W. Chen), jdavila@dim.uchile.cl (J. Dávila).

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(b) *If* $3 \le N \le 9$, *then* $\lambda^* > 2(N-2)$. *For* $0 < \lambda < \lambda^*$, $\lambda \ne 2(N-2)$, (1.2) *has finitely many solutions; for* $\lambda = 2(N-2)$, (1.2) *has infinitely many solutions; for* λ *close to* $2(N-2)$, (1.2) *has a large number of solutions that converge to* $-2\log|x|$.

(c) *If* $N \ge 10$, *then* $\lambda^* = 2(N-2)$ *and* $u_* = -2\log|x|$. *Moreover* (1.2) *has a unique minimal solution* u_λ *for each* $\lambda \in (0, \lambda^*)$.

Nagasaki and Suzuki [4] classified the solutions of (1.2) according to their Morse index. In a few words, the family of regular solutions of (1.2) can be described as a curve $(u(s), \lambda(s))$ with $s \in [0, \infty)$, such that $(u(s), \lambda(s)) \to (0, 0)$ as $s \to 0$ and $(u(s), \lambda(s)) \to (u_\sigma, \lambda_\sigma)$ as $s \to \infty$, where $u_\sigma(r) = -2\log(r)$, $\lambda_\sigma = 2(N-2)$ is a singular solution of (1.2). In dimensions $3 \le N \le 9$, $\lambda(s)$ oscillates around $2(N-2)$ as $s \to \infty$ and the Morse index of $u(s)$ increases by one in each oscillation. In dimensions $N \ge 10$, $\lambda(s)$ is monotone, $u(s)$ is monotone and is stable for each s. We refer the reader to the book of L. Dupaigne [5] for further references on problem (1.2). Moreover, Berchio, Gazzola and Pierotti in [6] studied Gelfand type elliptic problems under Steklov boundary conditions.

A problem analogous to (1.1) is

$$\begin{cases} -\Delta u = u^p + \lambda u, \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.3}$$

where $p > 1$ and $\lambda > 0$ is a parameter. According to classical bifurcation theory [7], the point $(\mu_1, 0)$ is a bifurcation point from which emanates an unbounded branch $\mathcal{C}$ of solutions of (1.3), where μ_1 is the first eigenvalue of the negative Laplacian operator under Dirichlet boundary condition in B.

- If $p < \frac{N+2}{N-2} (N \ge 3)$, for $\lambda < \mu_1$, there is a positive solution of (1.3) by a standard constrained minimization procedure involving compactness of the Sobolev embedding. Moreover, by Pohozaev's identity [8], problem (1.3) has no solutions for $\lambda \le 0$ whenever $p \ge \frac{N+2}{N-2}$.
- If $p = \frac{N+2}{N-2}$, which is the classical Brezis–Nirenberg problem [9], problem (1.3) has a solution for $0 < \lambda < \mu_1$ if $N \ge 4$, and for $\frac{1}{4}\mu_1 < \lambda < \mu_1$ if $N = 3$.
- If $p > \frac{N+2}{N-2}$, Dolbeault and Flores found that if $p > \frac{N+2}{N-2}$, and $p < \frac{N-2\sqrt{N-1}}{N-2\sqrt{N-1}-4}$ or $N \le 10$, then there is a unique number $\lambda_* > 0$, such that for λ close to λ_*, a large number of classical solutions of (1.3) exist. In particular, there are infinitely many classical solutions for $\lambda = \lambda_*$. Recently, Guo and Wei in [10] showed that the structure of the branch $\mathcal{C}$ changes for $p \ge p_c$ and $\frac{N+2}{N-2} < p < p_c$, where $p_c = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}$ if $N \ge 11$; and $p_c = \infty$ if $2 \le N \le 10$. Moreover, they established that for $\frac{N+2}{N-2} < p < p_c$, $\mathcal{C}$ turns infinitely many times around $\lambda_* \in (0, \mu_1)$. For $p \ge p_c$, all solutions have a finite Morse index, and for $N \ge 12$ and $p > p_c$ sufficiently large all solutions have exactly Morse index one.

This paper is devoted to the study of the structure of solutions to problem (1.1). We start with some general remarks. First, classical solutions of (1.1) can exist only for λ in some interval.

Proposition 1.2. *Let* μ_1 *be the first eigenvalue of the* $-\Delta$ *under Dirichlet boundary condition in B. Then there exists* $\lambda_0 > 0$, *such that a necessary condition for existence of classical solutions to problem* (1.1) *is* $\lambda \in (\lambda_0, \mu_1)$.

See a proof in the Appendix. By classical bifurcation theory [11,7] we have that $(\mu_1, 0)$ is a bifurcation point of solutions to (1.1). Both observations are also valid if we replace the ball by a bounded smooth domain (star shaped in the case of Proposition 1.2).

We are interested also in weak solutions, allowing for possible singularities.

Definition 1.3. We say that $u \in H_0^1(B)$ is a weak solution of (1.1) if $e^u \in L^1(B)$ and

$$\int_B \nabla u \nabla \varphi = \lambda \int_B (e^u - 1)\varphi \quad \text{for all } \varphi \in C_0^\infty(B). \tag{1.4}$$

We say that a weak solution u of (1.1) is regular (resp., singular) if $u \in L^\infty(B)$ (resp., $u \notin L^\infty(B)$).

We say that a radial weak solution u of (1.1) is a weakly singular solution if it is singular and $\lim_{r\to 0} ru'(r)$ exists.

We first study singular solutions to (1.1).

Theorem 1.4. *Assume* $N \ge 3$. *Let* $\lambda > 0$ *and suppose that* $u \in C^2(B\backslash\{0\})$, $u \ge 0$ *is a radial solution of*

$$-\Delta u = \lambda(e^u - 1) \quad \text{in } B\backslash\{0\}. \tag{1.5}$$

Then either

(a) *u can be extended as a function in* $C^\infty(B)$ *and* (1.5) *holds in B,* *or*

(b) *u is singular at $r = 0$ and satisfies*

$$\lim_{r\to 0}(u(r) + 2\log r) = \log \frac{2(N-2)}{\lambda},$$
$$\lim_{r\to 0} ru'(r) = -2.$$

As a consequence, u is a radial singular weak solution to (1.1) if and only if u is a weakly singular solution.

Theorem 1.5. *For $N \geq 3$, there exists a unique $\lambda_* > 0$, such that (1.1) admits a radial singular solution for $\lambda = \lambda_*$, and the radial singular solution is unique.*

By Theorem 1.4 the singular solution is weakly singular.

Next, we consider the question of multiplicity of solutions to (1.1).

Theorem 1.6. *If $3 \leq N \leq 9$, then problem (1.1) has infinitely many regular radial solutions for $\lambda = \lambda_*$. For $\lambda \neq \lambda_*$ but close to λ_*, there is a large number of regular radial solutions for (1.1).*

For a weak solution (λ, u) of (1.1) we define the Morse index of u as the largest dimension k of a subspace $Y \subset C_c^\infty(B)$ such that

$$Q_u(\varphi) = \int_B |\nabla\varphi|^2 - \lambda e^u \varphi^2 < 0 \quad \forall\, \varphi \in Y\backslash\{0\}.$$

If u is a regular solution this is the number of negative eigenvalues, counting multiplicity, of the operator $-\Delta - \lambda e^u$. By Theorem 3 of Dancer and Farina [12], if $3 \leq N \leq 9$, for a sequence of solutions $(\lambda_n, u_{\lambda_n})$ to (1.1) with $\|u_n\|_{L^\infty(B)} \to \infty$ as $n \to \infty$, then the Morse index of u_{λ_n} goes to infinity as $n \to \infty$.

Theorem 1.7. *Assume $N \geq 10$. Then there exists $K < \infty$ such that the Morse index of any radial solution (λ, u_λ) of (1.1) (regular or singular) is bounded by K. The number of intersections of any regular solution and the radial singular solution is uniformly bounded by $2K + 1$. Moreover, for each $\lambda \in (\lambda_0, \mu_1)$, the number of regular solutions to (1.1) is bounded by $(K + 1)^2$.*

A natural conjecture for $N \geq 10$, which is observed in numerical calculations, is that the Morse index of any radial solution of (1.1) (regular or singular) is 1, the number of intersections of any regular solution and the radial singular solution is 1, and that for each $\lambda \in (\lambda_*, \mu_1)$ there is a unique solution.

To obtain multiplicity of solutions to problem (1.1) we use geometric theory of dynamical systems in three-dimensional phase space, which was applied in [13], and subsequently in [14–16]. There are some analogies between the results and techniques of this work and [17–21] on fourth order problems involving the exponential nonlinearity. In Section 2 we give some preliminaries. In Section 3 we prove Theorem 1.4, namely that radial solutions either are regular or weakly singular. Theorem 1.5, which is about the existence and uniqueness of a singular solution is proved in Section 4. In Section 5 we prove Theorem 1.6 on the multiplicity of solutions in dimensions $3 \leq N \leq 9$. In Section 6 we analyze the Morse index of solutions to problem (1.1), give the structure of the branch of solutions to (1.1), and prove Theorem 1.7. Finally, we give the proof of Proposition 1.2 in the Appendix.

2. Preliminary results

Let u satisfy (1.1) and make the change of variables

$$v(t) = u(r) \quad \text{with } r = e^t, \text{ for } t \in (-\infty, 0). \tag{2.1}$$

Then problem (1.1) becomes

$$\begin{cases} -v''(t) + (2-N)v'(t) = \lambda e^{2t}(e^{v(t)} - 1), & t \in (-\infty, 0) \\ v(0) = 0, & \lim_{t\to-\infty} e^{-t}v'(t) = 0. \end{cases} \tag{2.2}$$

Define

$$\begin{cases} v_1(t) = \dfrac{\lambda}{2(N-2)} e^{v(t)+2t}, \\ v_2(t) = v'(t), \\ v_3(t) = \lambda e^{2t}. \end{cases} \tag{2.3}$$

We find that (v_1, v_2, v_3) satisfies the following differential system

$$\begin{cases} v_1' = v_1(v_2 + 2), \\ v_2' = -2(N-2)v_1 - (N-2)v_2 + v_3, \\ v_3' = 2v_3, \end{cases} \tag{2.4}$$

with the condition

$$v_3(0) = 2(N-2)v_1(0). \tag{2.5}$$

System (2.4) has two stationary points

$$P_1 = (0, 0, 0) \quad \text{and} \quad P_2 = (1, -2, 0).$$

The linearization of (2.4) around P_1 is given by $X' = M_1 X$, with

$$M_1 = \begin{bmatrix} 2 & 0 & 0 \\ -2(N-2) & 2-N & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

The eigenvalues of M_1 are $\tilde{\nu}_1 = \tilde{\nu}_2 = 2$, $\tilde{\nu}_3 = 2 - N$. Thus for $N \geq 3$, $P_1 = (0, 0, 0)$ is a hyperbolic point, which has a 2-dimensional unstable manifold $W^u(P_1)$ and a 1-dimensional stable manifold $W^s(P_1)$.

The linearization of (2.4) around P_2 is given by $X' = M_2 X$, with

$$M_2 = \begin{bmatrix} 0 & 1 & 0 \\ -2(N-2) & 2-N & 1 \\ 0 & 0 & 2 \end{bmatrix}. \tag{2.6}$$

The eigenvalues of M_2 are given by

$$\nu_1 = 2, \qquad \nu_{2,3} = \frac{(2-N) \pm \sqrt{(N-2)(N-10)}}{2}. \tag{2.7}$$

For $3 \leq N \leq 9$, ν_2 and ν_3 are complex conjugates and $\mathrm{Re}(\nu_2) = \mathrm{Re}(\nu_3) = \frac{2-N}{2} < 0$. For $N \geq 10$, all the eigenvalues are real and $\nu_1 > 0$, $\nu_2 < 0$, $\nu_3 < 0$. Thus for all $N \geq 3$, $P_2 = (1, -2, 0)$ is a hyperbolic point, which has a 1-dimensional unstable manifold $W^u(P_2)$ and a 2-dimensional stable manifold $W^s(P_2)$. Actually $W^s(P_2)$ is contained in the plane $\{v_3 = 0\}$, which is invariant for (2.4).

Also we note that solutions of system (2.4) restricted to $\{v_3 = 0\}$ are related to radial solutions of the equation

$$-\Delta u = \lambda e^u \tag{2.8}$$

by exactly the same change of variables (2.1) and the first two equations in (2.3). This yields immediately a heteroclinic connection from P_1 to P_2, which is associated to the unique radial solution of (2.8) with $\lambda = 2(N-2)$ and initial condition $u(0) = u'(0) = 0$.

Proposition 2.1. *For $N \geq 3$, system (2.4) has a heteroclinic orbit from P_1 to P_2, which is contained in the plane $\{v_3 = 0\}$.*

Thanks to a result of Belickiĭ [22], we have the following lemma.

Lemma 2.2. *The system (2.4) is C^1-conjugate to its linearization around $P_2 = (1, -2, 0)$.*

Proof. We just need to check that none of the following relations

$$\mathrm{Re}(\nu_i) = \mathrm{Re}(\nu_j) + \mathrm{Re}(\nu_k), \tag{2.9}$$

holds for different indices $i, j, k \in \{1, 2, 3\}$ such that $\mathrm{Re}(\nu_j) < 0$ and $\mathrm{Re}(\nu_k) > 0$, where ν_1, ν_2, ν_3 are corresponding eigenvalues of M_2. It is easy to check this by calculation for $N \geq 3$. $\square$

Lemma 2.3. *Let $v^{(1)}, v^{(2)}, v^{(3)}$ be the eigenvectors of M_2 associated to ν_1, ν_2, ν_3. Then $v^{(k)} = (1, \nu_k, \nu_k(\nu_k-(2-N))+2(N-2))$ and $v^{(1)}$ is always real; for $3 \leq N \leq 9$, $v^{(2)}, v^{(3)}$ are complex conjugates. In particular the components of $v^{(1)} = (1, 2, 4(N-1))$ are positive.*

Proof. By direct calculation, $v^{(k)} = (1, \nu_k, \nu_k(\nu_k - (2-N)) + 2(N-2))$ is an eigenvector associated to ν_k. $\square$

3. Characterization of weakly singular solutions

In this section our aim is to prove Theorem 1.4. We assume that $u \in C^2(0, 1)$, $u \geq 0$ satisfies

$$-\Delta u = 2(N-2)(e^u - 1) \quad \text{in } (0, 1), \tag{3.1}$$

where we assume, by using a scaling, that $\lambda = 2(N-2)$. The scaling changes the length of the interval where the solution is defined, but this is not relevant for the next arguments, so we assume that the interval is $(0, 1)$.

Define $v(t) = u(e^t)$, $w(t) = v(t) + 2t$ for $t \leq 0$. Then w satisfies

$$-w''(t) + (2-N)w'(t) = 2(N-2)\left(e^{w(t)} - e^{2t} - 1\right) \quad \text{for all } t \leq 0. \tag{3.2}$$

We also let v_1, v_2, v_3 be defined in (2.3).

By similar arguments as in [20], we have the following results.

Lemma 3.1. *One has*

$$\liminf_{t\to-\infty} w(t) \le 0. \tag{3.3}$$

Proof. We follow [23]. Let $L := \liminf_{t\to-\infty} w(t)$ and suppose by contradiction that $L > 0$. Then there exists $T_0 > 0$, such that $w(t) \ge L/2$ for all $t \le -T_0$. Let ϕ be a smooth cut-off function in $\mathbb{R}$ such that $0 \le \phi(t) \le 1$, $\phi(t) = 0$ for $t \le -(T_0+3)$ and $t \ge -T_0$; $\phi(t) = 1$ for $t \in [-(T_0+2), -(T_0+1)]$, and for $i = 1, 2$

$$\int_{-(T_0+3)}^{-T_0} \frac{(\phi^{(i)})^2}{\phi}\, dt := c_i < +\infty.$$

Let $\tau > 1$ and $\phi_\tau(t) = \phi(\frac{t}{\tau})$. Multiplying (3.2) by ϕ_τ and integrating, we get

$$\int_{-(T_0+3)\tau}^{-T_0\tau} (e^{w(t)} - 1)\phi_\tau dt = \sum_{i=1}^{2} a_i \int_{-(T_0+3)\tau}^{-T_0\tau} w\phi_\tau^{(i)}\, dt + \int_{-(T_0+3)\tau}^{-T_0\tau} e^{2t}\phi_\tau dt, \tag{3.4}$$

where $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{2(N-2)}$. Using Young's inequality with $\varepsilon_1 > 0$ to be fixed later on, we have

$$\begin{aligned}\left|\int_{-(T_0+3)\tau}^{-T_0\tau} w\phi_\tau^{(i)}\, dt\right| &\le \varepsilon_1 \int_{-(T_0+3)\tau}^{-T_0\tau} w^2\phi_\tau\, dt + C_{\varepsilon_1} \int_{-(T_0+3)\tau}^{-T_0\tau} \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}\, dt \\ &\le \varepsilon_1 \int_{-(T_0+3)\tau}^{-T_0\tau} w^2\phi_\tau\, dt + C_{\varepsilon_1} c_i \tau^{1-2i}. \end{aligned} \tag{3.5}$$

We also have

$$\int_{-(T_0+3)\tau}^{-T_0\tau} e^{2t}\phi_\tau\, dt \le \frac{1}{2}e^{-2T_0\tau}. \tag{3.6}$$

From (3.4)–(3.6) we get

$$\int_{-(T_0+3)\tau}^{-T_0\tau} \left[e^{w(t)} - 1 - \varepsilon_1 K w(t)^2\right]\phi_\tau\, dt \le C_{\varepsilon_1} K \max_{i=1,2} c_i \tau^{1-2i} + \frac{1}{2}e^{-2T_0\tau}$$

with $K = |a_1| + |a_2|$. Since $w(t) \ge L/2 > 0$ for all $t \le -T_0$, we can choose $\varepsilon_1 > 0$ small, such that $e^{w(t)} - 1 - \varepsilon_1 K w(t)^2 \ge \varrho$ for $t \le -T_0$, where $\varrho > 0$ is fixed. Then

$$\varrho\tau \le \int_{-(T_0+3)\tau}^{-T_0\tau} \left[e^{w(t)} - 1 - \varepsilon_1 K w(t)^2\right]\phi_\tau\, dt \le C_{\varepsilon_1} K \max_{i=1,2} c_i \tau^{1-2i} + \frac{1}{2}e^{-2T_0\tau},$$

which is impossible for $\tau > 1$ large. □

Lemma 3.2. *We have*

$$\limsup_{t\to-\infty} w(t) < +\infty.$$

Proof. Assume by contradiction that $\limsup_{t\to-\infty} w(t) = +\infty$. Then there is a sequence $t_k \to -\infty$ such that $w(t_k) \to +\infty$. Furthermore we can assume that for all $k \ge 1$ we have $t_{k+1} + \log 2 < t_k$, $w(t_{k+1}) \ge w(t_k)$.

Set $M_k = w(t_k)$, $r_k = e^{t_k}$ and $\rho_k = \frac{r_{k+1}}{r_k}$. Note that $0 < \rho_k < \frac{1}{2}$. Let $\eta_k(r) = \frac{N-2}{N} r_k^2(1-r^2)$ so that it satisfies

$$-\Delta\eta_k = 2(N-2)r_k^2 \quad \text{in } B, \qquad \eta_k = 0 \quad \text{on } \partial B.$$

Define

$$u_k(r) = u(rr_k) - M_k + 2\log(r_k) + \eta_k(r).$$

Then we have

$$-\Delta u_k(r) = 2(N-2)r_k^2 e^{u(r_k r)} = 2(N-2)e^{M_k - \eta_k(r)}e^{u_k(r)}, \quad \text{for } 0 < r < r_k^{-1}.$$

Since η_k is bounded from above,

$$-\Delta u_k \ge C_0 e^{M_k} e^{u_k} \quad \forall 0 < r < r_k^{-1}, \tag{3.7}$$

for some $C_0 > 0$ independent of k. Also note that

$$u_k(1) = u(r_k) - M_k + 2t_k = 0,$$
$$u_k(\rho_k) = M_{k+1} - M_k + 2(t_k - t_{k+1}) + \eta_k(\rho_k) \geq 0.$$

Let $\lambda_{1,k}$ be the first eigenvalue for $-\Delta$ with Dirichlet boundary condition in the annulus $B\backslash B_{\rho_k}$ and $\phi_k > 0$ be the corresponding eigenfunction, that is,

$$\begin{cases} -\Delta\phi_k = \lambda_{1,k}\phi_k, \quad \phi_k > 0 & \text{in } B\backslash B_{\rho_k}; \\ \phi_k = 0; & \text{on } \partial\left(B\backslash B_{\rho_k}\right), \end{cases}$$

normalized so that $\|\phi_k\|_{L^\infty(B)} = 1$. Multiplying (3.7) by ϕ_k and integrating in $B\backslash B_{\rho_k}$, we get

$$C_0 e^{M_k} \int_{B\backslash B_{\rho_k}} e^{u_k}\phi_k\, dx \leq \int_{\partial(B\backslash B_{\rho_k})} \frac{\partial\phi_k}{\partial\nu} u_k\, d\sigma + \lambda_{1,k}\int_{B\backslash B_{\rho_k}} u_k\phi_k\, dx.$$

But $u_k \geq 0$ and $\frac{\partial\phi_k}{\partial\nu} \leq 0$ on $\partial(B\backslash B_{\rho_k})$ so that

$$C_0 e^{M_k} \int_{B\backslash B_{\rho_k}} e^{u_k}\phi_k\, dx \leq \lambda_{1,k}\int_{B\backslash B_{\rho_k}} u_k\phi_k\, dx.$$

Now using the inequality $e^u \geq u$, it yields that

$$C_0 e^{M_k} \leq \lambda_{1,k}.$$

However, since the annulus $B\backslash B_{\rho_k}$ has a width that does not converge to zero, $\lambda_{1,k}$ remains uniformly bounded. It follows that M_k is bounded as $k \to \infty$, which is a contradiction. □

Lemma 3.3. *For $i = 0, 1, 2$, we have*

$$|w^{(i)}(t)| \leq C(1 + |t|) \quad \textit{for all } t \leq 0, \tag{3.8}$$

and for all $i = 1, 2, 3$

$$|v_i(t)| \leq C(1 + |t|) \quad \textit{for all } t \leq 0. \tag{3.9}$$

Proof. Since $u \geq 0$ and w is bounded above, we have $|w(t)| \leq C(1 + |t|)$. Moreover, by Eq. (3.2), and interpolation inequalities such as in Chapter 6 of [24], we get that for any $t \leq -1$ and $i = 1, 2$

$$\begin{aligned} |w^{(i)}(t)| &\leq C \sup_{[t-1,t+1]} \left(|w| + 2(N-2)|e^w - e^{2t} - 1|\right) \\ &\leq C \sup_{[t-1,t+1]} \left(|w| + 2(N-2)|e^w - 1|\right). \end{aligned}$$

Since w is bounded above, the second term in the supremum is bounded. Then (3.8) and (3.9) follow from the bound of w. □

Lemma 3.4. *For $i = 1, 2, 3$*

$$|v_i(t)| \leq C \quad \textit{for all } t \leq 0, \tag{3.10}$$

for $i = 1, 2$

$$|w^{(i)}(t)| \leq C \quad \textit{for all } t \leq 0. \tag{3.11}$$

Proof. It is direct that v_3 is bounded for all $t \leq 0$. Since $v_1(t) = e^{w(t)}$ (recall the change of variables (2.3) and that we assume $\lambda = 2(N-2)$) and w is bounded above, we have $v_1(t)$ is bounded as $t \to -\infty$. Next we prove that v_2 is bounded for all $t \leq 0$.

Integrating the following equation

$$\frac{d}{ds}\left(v_2(s)e^{(N-2)s}\right) = \left[-2(N-2)v_1(s) + v_3(s)\right]e^{(N-2)s}$$

in $[t, t_0]$ with $t \leq t_0 \leq 0$, we get

$$v_2(t) = e^{-(N-2)t}\left(v_2(t_0)e^{(N-2)t_0} + 2(N-2)\int_t^{t_0} e^{(N-2)s}v_1(s)\, ds - \frac{2(N-2)}{N}(e^{Nt_0} - e^{Nt})\right).$$

Since v_1 is bounded, the integral $\int_{-\infty}^{t_0} e^{(N-2)s} v_1(s)\, ds$ exists. If

$$\frac{2(N-2)}{N} e^{Nt_0} - 2(N-2) \int_{-\infty}^{t_0} e^{(N-2)s} v_1(s)\, ds \neq v_2(t_0) e^{(N-2)t_0},$$

we deduce that $|v_2(t)|$ grows exponentially as $t \to -\infty$, which contradicts (3.9). Therefore we get

$$v_2(t_0) = -2(N-2) e^{-(N-2)t_0} \int_{-\infty}^{t_0} e^{(N-2)s} v_1(s)\, ds + \frac{2(N-2)}{N} e^{2t_0} \quad \forall\, t_0 \le 0. \tag{3.12}$$

It follows that $|v_2(t)| \le C$ for all $t \le 0$, because v_1 is bounded.

Finally, the relations

$$w'(t) = v_2 + 2, \qquad w''(t) = -2(N-2)v_1 + (2-N)v_2 + v_3,$$

imply (3.11). □

Proof of Theorem 1.4. The statements in the theorem are consequence of the following properties, that we will prove next.

(i) If $\liminf_{t\to-\infty} w(t) = -\infty$, then $w(t) \to -\infty$, $v_i(t) \to 0$ as $t \to -\infty$ for $i = 1, 2, 3$, and u is a regular solution.
(ii) If $\liminf_{t\to-\infty} w(t) > -\infty$, then $w(t) \to 0$, $(v_1, v_2, v_3) \to P_2$ as $t \to -\infty$, and u is a weakly singular solution.

To prove these claims it is useful to define

$$E(t) = \frac{1}{2}(w'(t))^2 + 2(N-2)(e^{w(t)} - w(t)) - (N-2)C_1 e^{2t},$$

where $C_1 > 0$ is a constant such that $|w'(t)| \le C_1$ for all $t \le 0$. This constant exists thanks to Lemma 3.4. Let us compute

$$E'(t) = (w(t)'' + 2(N-2)(e^{w(t)} - 1))w(t)' - 2(N-2)C_1 e^{2t}$$

for $t \le 0$. Using Eq. (3.2) we get

$$E'(t) = -(N-2)w'(t)^2 + 2(N-2)e^{2t}(w'(t) - C_1) \le 0. \tag{3.13}$$

Let us prove (i) and so we assume $\liminf_{t\to-\infty} w(t) = -\infty$. First, we show that $w(t) \to -\infty$ as $t \to -\infty$. By contradiction, we assume that $w(t)$ does not tend to $-\infty$ as $t \to -\infty$. Then we can find sequences $s_k \to -\infty$, $\tau_k \to -\infty$, such that $s_k > \tau_k$,

$$w(s_k) \to -\infty, \quad w(\tau_k) \text{ is bounded.}$$

But then $E(\tau_k)$ is bounded and $E(s_k) \to \infty$ as $k \to \infty$. However, by (3.13), $E(s_k) \le E(\tau_k)$, which is a contradiction.

Now, since $w(t) \to -\infty$ as $t \to -\infty$, we can easily deduce $v_1(t) \to 0$ as $t \to -\infty$. Using formula (3.12), we obtain $v_2(t) \to 0$ as $t \to -\infty$. Therefore $\lim_{t\to-\infty} V(t) = P_1$.

Since $v_2(t) \to 0$ as $t \to -\infty$, we have $\lim_{r\to 0} ru'(r) = 0$. Then for any $\epsilon > 0$, there exists $r_0 > 0$ such that for any $0 < r < r_0$, we have $|ru'(r)| < \epsilon$. Integrating from r to r_0 in this inequality, for any $0 < r < r_0$ we obtain

$$0 \le u(r) \le -\epsilon \ln r + C, \qquad e^{u(r)} \le Cr^{-\epsilon}, \tag{3.14}$$

for some $C > 0$.

We can then get that $u'(r)$ is bounded for $r > 0$ small enough. In fact, Eq. (1.1) can be written as

$$-\left(s^{N-1}u'(s)\right)' = \lambda s^{N-1}(e^{u(s)} - 1).$$

Integrating above equation from δ to r with $(\delta, r) \subset (0, r_0)$ and using (3.14), letting $\delta \to 0$, we have

$$|u'(r)| \le Cr^{1-N} \int_0^r s^{N-1}\left(s^{-\epsilon} - 1\right) ds \le C$$

for $0 < r < r_0$. From the boundedness of u' near $r = 0$ we also get that u is bounded near $r = 0$. This shows that u is regular.

We prove now (ii), so we assume that $\liminf_{t\to-\infty} w(t) > -\infty$. Since w is bounded above by Lemma 3.2, we have w is bounded. By Lemma 3.4, the derivatives of w are bounded, then we get that $E(t)$ is bounded as $t \to -\infty$. From the boundedness of E together with the boundedness of the derivatives of w and (3.13), we deduce that

$$\int_{-\infty}^{0} w'(t)^2\, dt < +\infty. \tag{3.15}$$

Set $\psi_T(t) = w'(t+T)$, then we get that

$$\psi_T \to 0 \quad \text{in } L^2(0,1) \text{ as } T \to -\infty.$$

Moreover, ψ_T satisfies the equation

$$-\psi_T''(t) + (2-N)\psi_T'(t) = 2(N-2)e^{w(T+t)}\psi_T(t) - 4(N-2)e^{2(T+t)}.$$

Using regularity theory, we have $\psi_T(\frac{1}{2}) \to 0$ and $\psi_T'(\frac{1}{2}) \to 0$ as $T \to -\infty$. Thus we obtain that $w'(t) \to 0$ as $t \to -\infty$ and similarly $w''(t) \to 0$ as $\to -\infty$. This implies that $\lim_{t\to-\infty} v'(t) = -2$. Since $v'(t) = u'(e^t)e^t$ we see that u is a weakly singular solution by the definition. We get in addition that $(v_1, v_2, v_3) \to (1, -2, 0)$ as $t \to -\infty$. That is, $\lim_{t\to-\infty} V(t) = P_2$. □

A direct corollary of the proof of Theorem 1.4 is the following.

Corollary 3.5. *Let u be a radial singular solution to (1.1) and let $V(t) = (v_1(t), v_2(t), v_3(t))$ be the corresponding trajectory to (2.4). Then* $\lim_{t\to-\infty} V(t) = P_2 = (1, -2, 0)$.

As a consequence of Theorem 1.4 and Corollary 3.5, we have the following.

Corollary 3.6. *For u a radial solution of (1.1) we have:*

(a) *u is regular if and only if* $\lim_{t\to-\infty} V(t) = P_1$;
(b) *u is singular if and only if* $\lim_{t\to-\infty} V(t) = P_2$.

4. The unstable manifold at P_2

In this section, we study the unstable manifold of P_2 and prove Theorem 1.5. First we have the following result.

Proposition 4.1. *Let $V(t) = (v_1(t), v_2(t), v_3(t)) : (-\infty, T) \to \mathbb{R}^3$ be the trajectory in $W^u(P_2)$ such that $v_3'(t) > 0$ as $t \to -\infty$, where T is the maximal time of existence. Then there exists some $t < T$ such that $v_3(t) \geq 2(N-2)v_1(t)$.*

Proof. First we observe that this trajectory satisfies

$$v_1'(t) > 0, \qquad v_2'(t) > 0, \qquad v_3'(t) > 0$$

for t close to $-\infty$ since the tangent vector to this trajectory becomes parallel to $(1, 2, 4(N-1))$ as it approaches P_2.

Let $z(t) = v_3(t) - 2(N-2)v_1(t)$ and by contradiction we assume that

$$z(t) < 0 \quad \text{for } \forall t \in (-\infty, T). \tag{4.1}$$

First, we remark that

$$v_2(t) < 0 \quad \text{for } \forall t \in (-\infty, T). \tag{4.2}$$

To prove this, let us suppose it fails, and so there is the first time $t_0 \in (-\infty, T)$, such that $v_2(t_0) = 0$. Since $\lim_{t\to-\infty} v_2(t) = -2$ we must have $v_2'(t_0) \geq 0$. But writing the second equation in (2.4) as

$$v_2'(t) = z(t) - (N-2)v_2(t)$$

we would get $z(t_0) \geq 0$, a contradiction with (4.1).

Using (2.4) and $v_2(t) < 0$ for all $t < T$ we can assert that the solution is defined for all t, that is $T = +\infty$. Indeed, the first equation in (2.4) yields

$$v_1(t) = v_1(t_0)e^{\int_{t_0}^{t}(2+v_2(s))\,ds}. \tag{4.3}$$

Since $v_2(t) < 0$ we see that $v_1(t)$ cannot blow up as $t \to T$, if T were finite. Also v_3 cannot blow up. This and the linearity of the second equation in (2.4) yield that $T = +\infty$.

Now, let us establish that

$$v_1(t) > 0 \quad \text{for } \forall t \in (-\infty, +\infty). \tag{4.4}$$

In fact, this is valid for t near $-\infty$ since $v_1(t) \to 1$ as $t \to -\infty$. If inequality (4.4) does not hold, then $v_1(t_0) = 0$ for some t_0, and it follows from (4.3) that $v_1(t) = 0$ for all t, a contradiction.

Next, we prove that

$$\limsup_{t\to+\infty} v_2(t) = 0. \tag{4.5}$$

Indeed, suppose not, we assume that there is a small number $\delta > 0$ such that $v_2(t) < -\delta < 0$ for all t. From the first equation in (2.4), we then get $v_1'(t) < (2-\delta)v_1(t)$, so we have $v_1(t) < v_1(0)e^{(2-\delta)t}$ for all $t > 0$. But by the third equation in (2.4), we have $v_3(t) = v_3(0)e^{2t}$. Hence $z(t) = v_3(0)e^{2t} - 2(N-2)v_1(0)e^{(2-\delta)t} \geq 0$ for some $t > 0$, which contradicts assumption (4.1).

From (4.2) and (4.5), there exists a sequence (t_k) with $t_k \to +\infty$ as $k \to +\infty$, such that

$$v_2'(t_k) > 0, \quad \text{and} \quad v_2(t_k) \to 0 \quad \text{as } k \to +\infty.$$

Moreover, by the second equation in (2.4) we have $0 > z(t_k) = v_2'(t_k) + (N-2)v_2(t_k) > (N-2)v_2(t_k)$. Therefore,

$$z(t_k) \to 0 \quad \text{as } k \to +\infty. \tag{4.6}$$

From (2.4), we have $z'(t) - 2z(t) = -2(N-2)v_1(t)v_2(t)$. Multiplying by e^{-2t} and integrating from t to t_k, we get

$$z(t_k) = e^{2(t_k-t)}\left(z(t) - 2(N-2)e^{2t}\int_t^{t_k} e^{-2s}v_1(s)v_2(s)ds\right). \tag{4.7}$$

From (4.2), (4.4), (4.6) and (4.7) we have that

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds < +\infty \quad \text{for any } t < +\infty. \tag{4.8}$$

Note that $v_1(t) = \frac{v_3(t)-z(t)}{2(N-2)}$ and hence

$$z'(t) - 2z(t) = (z(t) - v_3(t))v_2(t).$$

Multiplying by e^{-2t} and integrating from 0 to t_k, we find

$$z(t_k) = e^{2t_k}\left(z(0) + \int_0^{t_k} e^{-2s}z(s)v_2(s)ds - \int_0^{t_k} e^{-2s}v_2(s)v_3(s)ds\right).$$

Since $z(0) < 0$, $\int_0^{t_k} e^{-2s}z(s)v_2(s)ds$ and $-\int_0^{t_k} e^{-2s}v_2(s)v_3(s)ds$ are positive, we get

$$\int_0^{+\infty} e^{-2s}|v_2(s)|v_3(s)ds < +\infty. \tag{4.9}$$

Since $v_3(t) = v_3(0)e^{2t}$, (4.9) implies that

$$\int_0^{+\infty} |v_2(s)|ds < +\infty. \tag{4.10}$$

Since $z(t) < 0$ by assumption, we have $v_2(s) \le v_2(0)e^{-(N-2)s}$ for $s \ge 0$. Then for $t \ge 0$,

$$\begin{aligned}\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds &= -\int_t^{+\infty} e^{-2s}v_1(s)v_2(s)ds \\ &\ge -v_2(0)\int_t^{+\infty} e^{-Ns}v_1(s)ds.\end{aligned} \tag{4.11}$$

Integrating by parts and using (2.4) we get

$$\begin{aligned}\int_t^{\infty} e^{-Ns}v_1(s)\,ds &= \frac{1}{N}e^{-Nt}v_1(t) + \frac{1}{N}\int_t^{\infty} e^{-Ns}v_1'(s)\,ds \\ &= \frac{1}{N}e^{-Nt}v_1(t) + \frac{2}{N}\int_t^{\infty} e^{-Ns}v_1(s)\,ds + \frac{1}{N}\int_t^{\infty} e^{-Ns}v_1(s)v_2(s)\,ds\end{aligned}$$

and we deduce

$$\int_t^{\infty} e^{-Ns}v_1(s) = \frac{1}{N-2}e^{-Nt}v_1(t) + \frac{1}{N-2}\int_t^{\infty} e^{-Ns}v_1(s)v_2(s)\,ds.$$

Hence for $t > 0$, and since $v_2(s) < 0$

$$\int_t^{\infty} e^{-Ns}v_1(s) \ge \frac{1}{N-2}e^{-Nt}v_1(t) + \frac{1}{N-2}\int_t^{\infty} e^{-2s}v_1(s)v_2(s)\,ds. \tag{4.12}$$

From (4.11) and (4.12) we have

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \ge -\frac{v_2(0)}{N-2}v_1(t)e^{-Nt} + \frac{v_2(0)}{N-2}\int_t^{+\infty} v_1(s)|v_2(s)|e^{-2s}ds,$$

which implies that

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \ge \frac{-v_2(0)}{N-2-v_2(0)}v_1(t)e^{-Nt}. \tag{4.13}$$

Now, from (4.6) and (4.7) we have

$$-z(t) = 2(N-2)e^{2t}\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds. \tag{4.14}$$

From (4.14) and (4.13), we observe that

$$-z(t) \geq \frac{-2(N-2)v_2(0)}{N-2-v_2(0)}v_1(t)e^{(-N+2)t}. \tag{4.15}$$

Moreover, using (4.10)

$$v_1(t) = v_1(0)e^{2t}e^{\int_0^t v_2(s)ds} = v_1(0)e^{2t}e^{-\int_0^t |v_2(s)|ds} \geq v_1(0)e^{-C}e^{2t} \tag{4.16}$$

for some constant $C > 0$. Hence,

$$-z(t) \geq \frac{-2(N-2)v_1(0)v_2(0)}{N-2-v_2(0)}e^{-C}e^{(4-N)t} := C_1e^{(4-N)t}, \tag{4.17}$$

for $C_1 > 0$, which is a contradiction with (4.6) for $N = 3, 4$.

From now on we assume $N > 4$. By the second equation in (2.4) and $z(t) = v_3(t) - 2(N-2)v_1(t)$, we get that

$$-v_2(t) = -v_2(0)e^{(2-N)t} + e^{(2-N)t}\int_0^t (-z(s))e^{(N-2)s}ds.$$

By (4.17) we have

$$\begin{aligned} |v_2(t)| &= -v_2(t) \geq -v_2(0)e^{(2-N)t} + C_1e^{(2-N)t}\int_0^t e^{2s}ds \\ &\geq \frac{C_1}{2}e^{(2-N)t}(e^{2t}-1) \geq C_2e^{(4-N)t}, \end{aligned}$$

for $t > 1$ where C_2 is a positive constant. Therefore,

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \geq C_2\int_t^{+\infty} e^{(2-N)s}v_1(s)ds, \tag{4.18}$$

while, for $N > 4$ and $t > 0$

$$\begin{aligned} \int_t^{+\infty} e^{(2-N)s}v_1(s)ds &= \frac{1}{N-2}v_1(t)e^{(2-N)t} - \frac{1}{N-2}\int_t^{+\infty} e^{(2-N)s}v_1(s)|v_2(s)|ds + \frac{2}{N-2}\int_t^{+\infty} e^{(2-N)s}v_1(s)ds \\ &\geq \frac{1}{N-2}v_1(t)e^{(2-N)t} - \frac{1}{N-2}\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds + \frac{2}{N-2}\int_t^{+\infty} e^{(2-N)s}v_1(s)ds. \end{aligned}$$

So,

$$\int_t^{+\infty} e^{(2-N)s}v_1(s)ds \geq \frac{1}{N-4}v_1(t)e^{(2-N)t} - \frac{1}{N-4}\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds. \tag{4.19}$$

Combining (4.18) and (4.19), we get

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \geq \frac{C_2}{N-4+C_2}v_1(t)e^{(2-N)t}. \tag{4.20}$$

Then, from (4.14), (4.16) and (4.20) we obtain that

$$-z(t) \geq \frac{2(N-2)C_2v_1(0)e^{-C}}{N-4+C_2}e^{(6-N)t} := C_3e^{(6-N)t}, \tag{4.21}$$

for $C_3 > 0$, which is a contradiction with (4.6) for $N = 5, 6$.

Starting with (4.21) we can do the same process and obtain a contradiction for all $N \geq 3$. This ends the proof of the proposition. □

Proposition 4.2. *At any point of $W^u(P_2) \cap \{v_3 = 2(N-2)v_1\}$ the intersection is transversal.*

Proof. Let $V(t) = (v_1, v_2, v_3)$ be a trajectory in $W^u(P_2)$ with t in some interval $(-\infty, T)$ and $\lim_{t\to-\infty} V(t) = P_2$. Suppose that t_1 is such that $v_3(t_1) = 2(N-2)v_1(t_1)$. By contradiction, assume that $V'(t_1)$ is not transversal to the plane $\{v_3(t) = 2(N-2)v_1(t)\}$, that is,

$$V'(t_1) \in \{v_3 = 2(N-2)v_1\}.$$

Then, $v_3(t_1) = 2(N-2)v_1(t_1),\ v_3'(t_1) = 2(N-2)v_1'(t_1)$. From (2.4) we get $v_2(t_1) = 0$. Let $z(t) = v_3(t) - 2(N-2)v_1(t)$. The ODE (2.4) implies that

$$v_2' = z - (N-2)v_2, \qquad z' = 2z - 2(N-2)v_1v_2.$$

Treating v_1 as a given function, we see that v_2, z satisfy a first order non-autonomous linear ODE and the initial condition $v_2(t_1) = 0, z(t_1) = 0$. Since $v_2 = z = 0$ is a solution of the ODE with the same initial condition, by uniqueness we deduce $v_2(t) = 0$ for all t where it is defined. This contradicts $\lim_{t\to-\infty} v_2(t) = -2$. □

Proof of Theorem 1.5. The existence of some $\lambda_* > 0$ such that (1.1) has a singular solution is a consequence of Proposition 4.1. Indeed, let $V(t) = (v_1(t), v_2(t), v_3(t)) : (-\infty, T) \to \mathbb{R}^3$ be the trajectory in $W^u(P_2)$ such that $v_3'(t) > 0$ as $t \to -\infty$, where T is the maximal time of existence. Then there exists some $t < T$ such that $v_3(t) \geq 2(N-2)v_1(t)$. Let t_1 be the first time such that $v_3(t_1) = 2(N-2)v_1(t_1)$. Because the system (2.4) is autonomous, by shifting time, we can assume $t_1 = 0$. Let $P^* = V(0)$ be the point of intersection, and write $P^* = (P_1^*, P_2^*, P_3^*)$. Then

$$u(r) = -2\log(r) + \log\left(\frac{2(N-2)v_1(\log(r))}{\lambda_*}\right)$$

is a singular solution of (1.1) for $\lambda_* = P_3^*$.

The uniqueness of λ_* such that a singular solution of (1.1) exists is a consequence of Corollary 3.6, which says that singular solutions must be associated to trajectories in $W^u(P_2)$, and the trajectory in $W^u(P_2)$ with tangent vector close $(1, 2, 4(N-1))$ as it approaches P_2 is unique except a shift in time. This also yields the uniqueness of the singular solution. □

5. Multiplicity result: proof of Theorem 1.6

In this section, we assume that $3 \leq N \leq 9$ and prove multiplicity of solutions to problem (1.1). Let $P_1 = (0, 0, 0)$ and $P_2 = (1, -2, 0)$ be the stationary points of (2.4). We recall that P_1 has a 2-dimensional unstable manifold $W^u(P_1)$ and 1-dimensional stable manifold $W^s(P_1)$, while P_2 has a 1-dimensional unstable manifold $W^u(P_2)$ and a 2-dimensional stable manifold $W^s(P_2)$.

From Corollary 3.6 it follows that each regular radial solution of (1.1) corresponds to exactly one point in $W^u(P_1) \cap \{v_3 = 2(N-2)v_1\}$. By Proposition 4.2, we define λ_* to be the height $v_3 = \lambda_*$ where $W^u(P_2)$ first intersects the plane $\{v_3 = 2(N-2)v_1\}$, and we denote this intersection point by

$$P^* = (P_1^*, P_2^*, P_3^*) = \left(\frac{\lambda_*}{2(N-2)}, P_2^*, \lambda_*\right). \tag{5.1}$$

Let $V_0 : \mathbb{R} \to \mathbb{R}^3$ be the heteroclinic connection from P_1 to P_2 contained in $\{v_3 = 0\}$ as stated in Proposition 2.1 and let $\hat{V}_0 = V_0(-\infty, +\infty)$. Then $\hat{V}_0$ is contained in both $W^u(P_1)$ and $W^s(P_2)$.

Lemma 5.1. *$W^u(P_1)$ and $W^s(P_2)$ intersect transversally on points of $\hat{V}_0$. More precisely, for points $Q \in \hat{V}_0$ sufficiently close to P_2, there are directions in the tangent plane to $W^u(P_1)$ which are almost parallel to $v^{(1)}$, the tangent vector to $W^u(P_2)$ at P_2.*

Proof. Let u_β be the solution of the following initial value problem

$$\begin{cases} -\Delta u_\beta(r) = 2(N-2)e^{u_\beta(r)} - \beta & \text{for } 0 < r < R(\beta), \\ u_\beta(0) = 0, \qquad u_\beta'(0) = 0, \end{cases} \tag{5.2}$$

where $\beta \in \mathbb{R}$ is a parameter and $R(\beta) > 0$ is the maximal time of existence. We claim that $R(\beta) = +\infty$. Indeed, assume $R(\beta) < +\infty$ and fix $r_0 < R(\beta)$. Then for $r \in [r_0, R(\beta))$, from Eq. (5.2) we get

$$u_\beta'(r) = r_0^{N-1}u_\beta'(r_0)r^{1-N} - r^{1-N}\int_{r_0}^{r} t^{N-1}\left(2(N-2)e^{u_\beta(t)} - \beta\right)dt, \tag{5.3}$$

and this implies

$$u_\beta'(r) \leq r_0^{N-1}u_\beta'(r_0)r^{1-N} + \frac{|\beta|}{N}(r - r^{1-N}r_0^N) \quad \text{for } r_0 \leq r < R(\beta).$$

Integrating we see that

$$\limsup_{r\to R(\beta)} u_\beta(r) < +\infty.$$

Since u_β is bounded above in $[r_0, R(\beta))$, using again (5.3) we obtain

$$r_0^{N-1}u'_\beta(r_0)r^{1-N} - C(r - r^{1-N}r_0^N) \le u'_\beta(r) \quad \text{for } r_0 \le r < R(\beta),$$

and this shows that

$$\liminf_{r\to R(\beta)} u_\beta(r) > -\infty.$$

Control of u_β as $r \to R(\beta)$ also yields control of u'_β by (5.3) and this contradicts that $R(\beta)$ is the maximal time of existence. Therefore the solution $u_\beta(r)$ of (5.2) is defined for all $r > 0$.

Let $v_\beta(t) = u_\beta(r)$ with $r = e^t$ for $t \in (-\infty, +\infty)$ and set

$$v_{1,\beta}(t) = e^{v_\beta(t)+2t}, \qquad v_{2,\beta} = v'_\beta(t), \qquad v_{3,\beta}(t) = \beta e^{2t}.$$

Then $v_{1,\beta}, v_{2,\beta}, v_{3,\beta}$ satisfies system (2.4). Let $V_\beta = (v_{1,\beta}, v_{2,\beta}, v_{3,\beta})$. We have created in this way a family of trajectories in $W^u(P_1)$ with β as a parameter. Note that for $\beta = 0$, V_0 is just the heteroclinic connection of system (2.4) from P_1 to P_2 contained in the plane $\{v_3 = 0\}$ described in Proposition 2.1.

Define $X = \frac{\partial V}{\partial \beta}|_{\beta=0}$. Then X satisfies

$$X' = (M_2 + R(t))X \tag{5.4}$$

where M_2 is the matrix defined in (2.6) and

$$R(t) = \begin{bmatrix} v_{2,0}(t) + 2 & v_{1,0}(t) - 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that there exist $C, \alpha > 0$, such that $|V_0(t) - P_2| \le Ce^{-\alpha t}$ for all $t \ge 0$, which follows for example from Lemma 2.2. Therefore $|R(t)| \le Ce^{-\alpha t}$ for all $t \ge 0$. Recall that the eigenvalues of M_2 are $\nu_1 > 0$ and ν_2, ν_3, which are complex conjugates with negative real part. Let $v^{(k)} \in \mathbb{C}^3$ be the eigenvector associated to ν_k. By Theorem 8.1 of Chapter 3 in [25], there are solutions ψ_k to

$$\psi'_k = (M_2 + R(t))\psi_k, \quad \text{for } t > 0$$

such that $\lim_{t\to\infty} \psi_k(t)e^{-\nu_k t} = v^{(k)}$. Then

$$X(t) = \sum_{k=1}^{3} c_k\psi_k$$

for some constants $c_1, c_2, c_3 \in \mathbb{C}$. Since ν_2, ν_3 have negative real parts, $\psi_k(t) \to 0$ as $t \to \infty$, for $k = 2, 3$. If $c_1 = 0$ then $X(t) \to 0$ as $t \to \infty$ and this contradicts $\frac{\partial v_{3,\beta}}{\partial \beta}|_{\beta=0}(t) = e^{2t} > 0$ for all $t \ge 0$. So $c_1 \ne 0$ and therefore

$$X(t) = c_1 v^{(1)} e^{\nu_1 t} + o(e^{\nu_1 t}) \quad \text{as } t \to \infty.$$

This shows $X(t)$ is almost parallel to $v^{(1)}$ as $t \to \infty$. Since $v^{(1)}$ is the tangent vector to $W^u(P_2)$, then $X(t)$ is not tangent to $W^s(P_2)$ for t large. On the other hand, $X = \frac{\partial V}{\partial \beta}|_{\beta=0}$ is tangent to $W^u(P_1)$. This implies $W^s(P_2)$ and $W^u(P_1)$ intersect transversally on points of $\hat{V}_0$ close to P_2. Since the flow is invertible near $\hat{V}_0$, $W^u(P_1)$ and $W^s(P_2)$ intersect transversally at every point of $\hat{V}_0$. $\square$

We write (v_1, v_2, v_3) as points in the phase space $\mathbb{R}^3$ and let $\{e_1, e_2, e_3\}$ denote the canonical basis of $\mathbb{R}^3$.

We call $\mathcal{S} \subset \mathbb{R}^3$ a spiral around P^* if there exist independent vectors $\sigma_1, \sigma_2 \in \mathbb{R}^3$, a continuous positive function $\rho : [0, \infty) \to \mathbb{R}$ with $\rho(t) \to 0$ as $t \to \infty$, and $\omega \in \mathbb{R}$ such that

$$\mathcal{S} = \{P^* + \rho(t)\cos(\omega t)\sigma_1 + \rho(t)\sin(\omega t)\sigma_2 + o(\rho(t)) : t \ge 0\}.$$

Lemma 5.2. *$W^u(P_1) \cap \{v_3 = 2(N-2)v_1\}$ contains a spiral $\mathcal{S}$ around the point P^*.*

Proof. The linearization of (2.4) at P_2 is given by the system

$$\begin{cases} \bar{v}'_1 = \bar{v}_2, \\ \bar{v}'_2 = -2(N-2)\bar{v}_1 + (2-N)\bar{v}_2 + \bar{v}_3, \\ \bar{v}'_3 = 2\bar{v}_3, \end{cases}$$

which is represented by the matrix M_2. Let $\bar{M}_2$ denote the matrix

$$\bar{M}_2 = \begin{bmatrix} \mathrm{Re}(\nu_2) & -\mathrm{Im}(\nu_2) & 0 \\ \mathrm{Im}(\nu_2) & \mathrm{Re}(\nu_2) & 0 \\ 0 & 0 & \nu_1 \end{bmatrix}$$

where ν_1, ν_2 are the eigenvalues (2.7). By Lemma 2.2, system (2.4) is C^1-conjugate in a neighborhood of P_2 to the flow generated by $\bar{M}_2$ around 0. More precisely, let X_t denote the flow generated by (2.4) and $Y_t = e^{\bar{M}_2 t}$. Then there are open neighborhoods $\mathcal{U}$ of P_2 and $\mathcal{V}$ of $\bar{O} = (0, 0, 0)$, and a C^1 diffeomorphism $\Phi : \mathcal{U} \to \mathcal{V}$ such that $Y_t(x) = \Phi \circ X_t \circ \Phi^{-1}(x)$ whenever $x \in \mathcal{V}$ and $\Phi^{-1}(x) \in \mathcal{U}$.

Let D be the 2-dimensional disk

$$D = \left\{ V = (v_1, v_2, v_3) : v_3 = 2(N-2)v_1,\ |V - P^*| < r_0 \right\},$$

where $r_0 > 0$ is fixed and small, so that $W^u(P_2) \cap \{v_3 = 2(N-2)v_1\}$ contains only the point P^*. This $r_0 > 0$ exists by Proposition 4.2. Also by this proposition, D is transversal to $W^u(P_2)$. Let $B^s \subset W^s(P_2) \cap \mathcal{U} \subset \{v_3 = 0\} \cap \mathcal{U}$ be an open neighborhood of P_2 relative to $W^s(P_2)$, which is diffeomorphic to a 2-dimensional disk. Define D_t as the connected component of $X_t(D) \cap \mathcal{U}$ that contains $X_t(P^*)$. We choose $\mathcal{U}$ smaller if necessary so that by the λ-Lemma of Palis [26], D_t is a C^1 manifold, which is C^1 close to B^s for t sufficiently negative. More precisely, let $\varepsilon > 0$ be small to be fixed later on. Then there exists $t_0 < 0$, $|t_0|$ large, such that for all $t \le t_0$, there is a diffeomorphism $\eta_t : D_t \to B^s$ such that $\|i' \circ \eta_t - i\|_{C^1(D_t)} \le \varepsilon$ where i, i' denote the inclusion maps. From now on we let $\mathcal{M} = D_{t_0}$.

We fix $Q \in \hat{V}_0$ such that $Q \in \mathcal{U}$ is sufficiently close to P_2. From Lemma 5.1, we can find a C^1 curve Γ contained in $W^u(P_1)$ of the form $\Gamma = \{\gamma(s) : |s| < \delta_0\}$ with $\gamma : (-\delta_0, \delta_0) \to \mathbb{R}^3$ a C^1 function such that $\gamma(0) = Q$ and $\gamma'(0)$ not tangent to $W^s(P_2)$ at Q. We can also assume that Γ is contained in $\mathcal{U}$ by taking δ_0 small. Choosing $\varepsilon > 0$ smaller if necessary we can assume that Γ intersects $\mathcal{M}$.

We want to prove that for $t > 0$ large, there is a point $P_t \in X_t(\Gamma) \cap \mathcal{M}$ and that the collection of points P_t describes a spiral around the point $X_{t_0}(P^*)$.

By the conjugation Φ, we will assume that P_2 is at the origin and near the origin the flow is given by $Y_t = e^{\bar{M}_2 t}$. Thus the image of $W^s(P_2) \cap \mathcal{U}$ through Φ is $\{(y_1, y_2, y_3) : y_3 = 0\}$, which is inside $\mathcal{V}$, and the image of B^s is $\{(y_1, y_2, y_3) : y_3 = 0, |y| < \delta\}$ for some $\delta > 0$.

Choosing ε small in the λ-Lemma, we can assume that the normal vector of $\widetilde{\mathcal{M}} := \Phi(\mathcal{M})$ near $\Phi(P^*)$ is almost parallel to $e_3 = (0, 0, 1)$. Thus by taking a subset of $\widetilde{\mathcal{M}}$, we may assume that $\widetilde{\mathcal{M}}$ is a C^1 graph with respect to the variables (y_1, y_2), that is, there exists a C^1 function $\varphi : \{\tilde{y} = (y_1, y_2) \in \mathbb{R}^2, |\tilde{y}| < \delta\} \to \mathbb{R}$ such that

$$\widetilde{\mathcal{M}} = \{(\tilde{y}, \varphi(\tilde{y})) : \tilde{y} \in \mathbb{R}^2, |\tilde{y}| < \delta\}.$$

Since $\gamma'(0)$ is not tangent to $W^s(P_2)$ at $\gamma(0)$, we have $\gamma_3'(0) \neq 0$. We may assume that $\varphi(\tilde{y}) > 0$ for $\tilde{y}$ near the origin and $\gamma_3'(0) > 0$.

We claim that for all $t > 0$ large there is a unique $s = s(t) > 0$ small so that $Y_t(\gamma(s)) \in \widetilde{\mathcal{M}}$. Indeed, this condition is equivalent to

$$e^{\nu_1 t}\gamma_3(s) = \varphi(e^{\nu_2 t}(\gamma_1(s) + i\gamma_2(s))). \tag{5.5}$$

Let $\tau = 1/t > 0$ and define, for $(\tau, s) \in (0, \delta_1) \times (-\delta_1, \delta_1)$ ($\delta_1 > 0$ a small fixed number)

$$F(\tau, s) = \gamma_3(s) - e^{-\nu_1/\tau}\varphi(e^{\nu_2/\tau}(\gamma_1(s) + i\gamma_2(s))).$$

Then, since $\nu_1 > \mathrm{Re}(\nu_2)$, F admits a C^1 extension to $\tau = 0$ and

$$F(0, s) = \gamma_3(s), \qquad \frac{\partial F}{\partial \tau}(0, s) = 0, \qquad \frac{\partial F}{\partial s}(0, s) = \gamma_3'(s).$$

Since $F(0, 0) = 0$ and $\frac{\partial F}{\partial s}(0, 0) > 0$, by the implicit function theorem, given $t > 0$ large there is a unique s small so that $F(1/t, s) = 0$. We obtain a C^1 function $s(t) > 0$ defined for all t large such that $Y_t(\gamma(s(t))) \in \widetilde{\mathcal{M}}$. Using (5.5) we see that

$$s(t) = \frac{e^{-\nu_1 t}}{\gamma_3'(0)}\varphi(0)(1 + o(1))$$

as $t \to \infty$. Writing $\nu_2 = \alpha + i\omega$, the point of intersection has the form

$$\tilde{P}_t = Y_t(\gamma(s(t))) = (0, 0, \varphi(0, 0)) + e^{\alpha t}\cos(\omega t)\tilde{\sigma}_1 + e^{\alpha t}\sin(\omega t)\tilde{\sigma}_2 + o(e^{\alpha t})$$

where

$$\tilde{\sigma}_1 = \left(\gamma_1(0), \gamma_2(0), \frac{\partial \varphi}{\partial y_1}(0, 0)\gamma_1(0) + \frac{\partial \varphi}{\partial y_2}(0, 0)\gamma_2(0)\right)$$

$$\tilde{\sigma}_2 = \left(-\gamma_2(0), \gamma_1(0), -\frac{\partial \varphi}{\partial y_1}(0, 0)\gamma_2(0) + \frac{\partial \varphi}{\partial y_2}(0, 0)\gamma_1(0)\right).$$

Therefore the curve $\{\tilde{P}_t, : t > t_1\}$, where $t_1 > 0$ is large, defines a spiral contained in $\widetilde{\mathcal{M}}$. Applying the conjugation Φ^{-1} we obtain a collection of points $P_t = \Phi^{-1}(\tilde{P}_t)$ in $\mathcal{M} \cap X_t(\Gamma)$ that forms a spiral around $X_{t_0}(P^*)$. Applying the flow X_{-t_0} we see that

$$\mathcal{S} = \{X_{t-t_0}(\gamma(s(t))) : t \geq t_1\}$$

with $t_1 > 0$ large has the structure of a spiral around P^*. By construction $\mathcal{S}$ is contained in $W^u(P_1) \cap \{v_3 = 2(N-2)v_1\}$. □

Proof of Theorem 1.6. Let us define λ_* to be the height $v_3 = \lambda_*$, where $W^u(P_2)$ first intersects the boundary plane $\{v_3 = 2(N-2)v_1\}$. Define $H_\lambda = \{v_3 = \lambda\}$. If $\lambda = \lambda_*$, we know that P^* lies on the line $\{v_3 = \lambda_*, v_3 = 2(N-2)v_1\}$. From Lemma 5.2, $W^u(P_1) \cap \{v_3 = 2(N-2)v_1\}$ contains a spiral $\mathcal{S}$ around the point P^*. Since the plane H_λ is transversal to $\{v_3 = 2(N-2)v_1\}$, it is possible to show that H_{λ_*} and $\mathcal{S}$ intersect an infinite number of times, which means that problem (1.1) has infinitely many radial regular solutions; see for example Lemma 4 in [14]. If $\lambda \neq \lambda_*$, but λ is close to λ_*, we have that $H_\lambda \cap \mathcal{S}$ contains a large number of points, which means that problem (1.1) has a large number of radial regular solutions. □

6. Proof of Theorem 1.7

In this section we always assume that $N \geq 10$ and prove Theorem 1.7.

First we give the asymptotic behavior of a radial singular solution to problem (1.1) near the origin.

Lemma 6.1. *Assume that (λ_*, u_*) is a radial singular solution of (1.1). Then*

$$u_*(r) = -2\log r + \log \frac{2(N-2)}{\lambda_*} + r^2 + o(r^2) \quad \text{as } r \to 0. \tag{6.1}$$

Proof. By Theorem 1.4, u_* is a weakly singular radial solution of (1.1). Define $v(t) = u_*(r)$ with $r = e^t$, and v_1, v_2, v_3 are given by (2.3). Therefore, from Corollary 3.6,

$$\lim_{t\to-\infty}(v_1, v_2, v_3) = (1, -2, 0).$$

By Lemmas 2.2 and 2.3, we have

$$(v_1, v_2, v_3) = (1, -2, 0) + (1, 2, 4(N-1))e^{2t}\left(1 + o(e^{\delta t})\right) \quad \text{as } t \to -\infty,$$

with $\delta > 0$ small. We then get

$$\begin{aligned} u_*(r) = v(t) &= -2t + \log \frac{2(N-2)v_1(t)}{\lambda_*} \\ &= -2\log r + \log \frac{2(N-2)\left(1 + e^{2t} + o(e^{(2+\delta)t})\right)}{\lambda_*} \\ &= -2\log r + \log \frac{2(N-2)}{\lambda_*} + \log(1 + r^2 + o(r^{2+\delta})) \\ &= -2\log r + \log \frac{2(N-2)}{\lambda_*} + r^2 + o(r^2) \quad \text{as } r \to 0. \quad \square \end{aligned}$$

For $\lambda > 0$, let us define

$$w(r) = -2\log r + \log \frac{2(N-2)}{\lambda} + \frac{\lambda}{2N} r^2. \tag{6.2}$$

Let $\rho > 0$ be a small number, which will be fixed later and let us write $c_\rho = w(\rho)$. Then w satisfies

$$\begin{cases} -\Delta w \leq \lambda(e^w - 1) & \text{in } B_\rho, \\ w(\rho) = c_\rho & \text{on } \partial B_\rho, \end{cases} \tag{6.3}$$

where B_ρ is a ball with radius ρ and center at the origin.

We have the following stability property of w.

Lemma 6.2. *Suppose $N \geq 10$ and let w be defined in (6.2). There exists $\rho \in (0, 1)$ small, such that w is stable in B_ρ, in the sense that*

$$\int_{B_\rho} |\nabla\varphi|^2 \geq \lambda \int_{B_\rho} e^w \varphi^2 \quad \text{for all } \varphi \in C_c^\infty(B_\rho). \tag{6.4}$$

Proof. Write $A = \frac{\lambda}{2N}$. Since $N \geq 10$,

$$\begin{aligned}\int_{B_\rho} |\nabla\varphi|^2 - \lambda e^w \varphi^2 &= \int_{B_\rho} |\nabla\varphi|^2 - 2(N-2)\frac{\varphi^2}{r^2} e^{Ar^2} \\ &= \int_{B_\rho} \left(|\nabla\varphi|^2 - 2(N-2)\frac{\varphi^2}{r^2}\right) - 2(N-2)(A+o(1)) \int_{B_\rho} \varphi^2 \\ &\geq \int_{B_\rho} \left(|\nabla\varphi|^2 - \frac{(N-2)^2}{4}\frac{\varphi^2}{r^2}\right) - 2(N-2)(A+o(1)) \int_{B_\rho} \varphi^2,\end{aligned}$$

where $o(1) \to 0$ as $\rho \to 0$. Let us recall the following improved Hardy inequality from [27]: for $\varphi \in C_c^\infty(B_\rho)$

$$\int_{B_\rho} \left(|\nabla\varphi|^2 - \frac{(N-2)^2}{4}\frac{\varphi^2}{r^2}\right) \geq H_2 \rho^{-2} \int_{B_\rho} \varphi^2,$$

where the constant H_2 is the first eigenvalue of the Laplacian in the unit ball in $N = 2$, hence it is positive and independent of N.

Choose $\rho > 0$ such that $2(N-2)(A+o(1)) \leq H_2\rho^{-2}$. Then (6.4) holds. □

Lemma 6.3. *Let $\rho \in (0, 1)$ be small and satisfy Lemma 6.2. Then for any radial regular solution u of (1.1) we have*

$$u(r) \leq \begin{cases} w(r) & \text{in } B_\rho \\ c_\rho & \text{in } B\backslash B_\rho, \end{cases} \tag{6.5}$$

where $w(r)$ is defined in (6.2).

Proof. Arguing by contradiction, suppose there exists $r_0 \in (0, \rho)$, such that $u(r_0) = w(r_0)$. Then

$$\begin{cases} -\Delta u = \lambda(e^u - 1) & \text{in } B_{r_0}; \\ -\Delta w \leq \lambda(e^w - 1) & \text{in } B_{r_0}; \\ u = w & \text{on } \partial B_{r_0}. \end{cases} \tag{6.6}$$

Therefore,

$$\begin{cases} -\Delta(w-u) \leq \lambda\left(e^w - e^u\right) & \text{in } B_{r_0}, \\ w - u = 0 & \text{on } \partial B_{r_0}. \end{cases} \tag{6.7}$$

Multiplying by $(w-u)^+$ and integrating in (6.7), we obtain

$$\int_{B_{r_0}} |\nabla(w-u)^+|^2 \leq \lambda \int_{B_{r_0}} (e^w - e^u)(w-u)^+. \tag{6.8}$$

From Lemma 6.2, w is stable in B_{r_0}, by taking $\varphi = (w-u)^+$ in (6.4), we then have

$$\int_{B_{r_0}} |\nabla(w-u)^+|^2 - \lambda e^w((w-u)^+)^2 \geq 0. \tag{6.9}$$

Combining (6.8) and (6.9), we get

$$\lambda \int_{B_{r_0}} e^w((w-u)^+)^2 \leq \lambda \int_{B_{r_0}} (e^w - e^u)(w-u)^+.$$

We rewrite it as

$$\int_{B_{r_0}} \left[(e^w - e^u)(w-u)^+ - e^w((w-u)^+)^2\right] \geq 0.$$

By convexity, the integrand is nonpositive, therefore,

$$(e^w - e^u)(w-u)^+ - e^w((w-u)^+)^2 = 0 \quad \text{a.e. in } B_{r_0},$$

then

$$(w-u)^+ = 0 \quad \text{a.e. in } B_{r_0}.$$

It implies that $w \leq u$ in B_{r_0}, which is impossible because u is a radial regular solution. Then $u(r) \leq w(r)$ for $r \in (0, \rho)$.

Since u is a radially decreasing regular solution, $u \leq c_\rho$ in $B\backslash B_\rho$. □

Now, let (λ, u_λ) be any radial solution to (1.1) (regular or singular), and define the operator L_γ as

$$L_\gamma(\phi) = -\Delta\phi - \lambda e^{u_\lambda}\phi + \gamma\phi$$

with $\gamma > 0$ large but fixed. We have the following lemma.

Lemma 6.4. *If $\gamma > 0$ is fixed large enough, we have:*

(a) *for $N \geq 11$, $\langle L_\gamma(\phi), \phi\rangle \geq C_1\|\phi\|^2_{H_0^1(B)}$ for all $\phi \in C_c^\infty(B)$;*

(b) *for $N = 10$, $\langle L_\gamma(\phi), \phi\rangle \geq C_2\|\phi\|^2_{L^2(B)}$ for all $\phi \in C_c^\infty(B)$,*
where C_1 and C_2 are positive constants.

Proof. For $\rho > 0$ small given in Lemma 6.2, from Lemmas 6.1 and 6.3, we have

$$\begin{aligned}
\langle L_\gamma(\phi), \phi\rangle &= \int_B L_\gamma(\phi)\phi = \int_B \left(|\nabla\phi|^2 - \lambda e^{u_\lambda}\phi^2 + \gamma\phi^2\right) \\
&= \int_B |\nabla\phi|^2 - \int_{B_\rho} \lambda e^{u_\lambda}\phi^2 - \int_{B\setminus B_\rho} \lambda e^{u_\lambda}\phi^2 + \int_B \gamma\phi^2 \\
&\geq \int_B |\nabla\phi|^2 - 2(N-2)\int_{B_\rho} \frac{\phi^2}{r^2}(1 + Ar^2 + o(r^2)) - C\int_{B\setminus B_\rho} \phi^2 + \int_B \gamma\phi^2 \\
&\geq \int_B \left(|\nabla\phi|^2 - 2(N-2)\frac{\phi^2}{r^2}\right) + [\gamma - \max\{2(N-2)(A + o(1)), C\}]\int_B \phi^2
\end{aligned}$$

where $A = \frac{\lambda}{2N}$ for a radial regular solution u_λ, $A = 1$ for a radial singular solution u_λ, and $o(1) \to 0$ as $\rho \to 0$. Choose γ large such that the second term of above is nonnegative, we then get the conclusion by Hardy's inequality. □

We now define

$$\|\phi\|_H^2 := \int_B \left(|\nabla\phi|^2 - \lambda e^{u_\lambda}\phi^2 + \gamma\phi^2\right)$$

which is a norm on $C_c^\infty(B)$ with associated inner product

$$(\phi, \varphi)_H = \int_B \left(\nabla\phi\nabla\varphi - \lambda e^{u_\lambda}\phi\varphi + \gamma\phi\varphi\right).$$

Completing $C_c^\infty(B)$ with respect to this norm we obtain a Hilbert space H. We denote by H^* the dual of H. We have $H_0^1(B) \subset H \subset L^2(B)$ and therefore $L^2(B) \subset H^* \subset H^{-1}(B)$. Actually by Lemma 6.4, if $N \geq 11$, the space H is just $H_0^1(B)$.

Given $h \in L^2(B) \subset H^*$ we consider the following problem

$$L_\gamma\phi = h \quad \text{in } B, \quad \text{and} \quad \phi = 0 \quad \text{on } \partial B. \tag{6.10}$$

We say that $\phi \in H$ is a weak solution of problem (6.10) if

$$(\phi, \varphi)_H = \langle h, \varphi\rangle_{H^*,H} \quad \text{for all } \varphi \in H.$$

By the Lax–Milgram theorem, for $h \in L^2(B)$, problem (6.10) has a unique weak solution $\phi \in H$.

Lemma 6.5. *Let $T : L^2(B) \to L^2(B)$ be the operator defined by $Th = \phi$, where ϕ is the solution of (6.10). Then T is compact and the natural embedding $H \hookrightarrow L^2(B)$ is compact.*

Proof. For $N \geq 11$, both statements hold since $T : L^2(B) \to H = H_0^1(B)$ and $H_0^1(B) \hookrightarrow L^2(B)$ is compact, by the Rellich–Kondrachov theorem. For $N = 10$, we observe that L_γ satisfies

$$\langle L_\gamma(\phi), \phi\rangle \geq c_r\|\phi\|^2_{L^r(B)} \quad \forall\phi \in C_c^\infty(B)$$

for $2 \leq r < \frac{2N}{N-2}$ where $c_r > 0$, thanks to an improved Hardy inequality of Brezis and Vázquez [27]. Then the statements are proved in [28]. □

Proposition 6.6. *The radial singular solution (λ_*, u_*) of (1.1) has a finite Morse index.*

Proof. By Lemma 6.5, if $\gamma > 0$ is large, $(-\Delta - \lambda_* e^{u_*} + \gamma)^{-1}$ is well defined and compact from $L^2(B)$ into itself, and hence its spectrum except 0 consists of eigenvalues, and these eigenvalues form a sequence that converges to 0. Hence $-\Delta - \lambda_* e^{u_*}$ is negative definite on a finite dimensional space only. □

Next we prove a bound for the Morse index of any radial regular solution of (1.1).

Proposition 6.7. *There is an integer $K \geq 1$ independent of λ, such that for any radial regular solution u_λ of (1.1) we have*

$$1 \leq m(u_\lambda) \leq K, \tag{6.11}$$

where $m(u_\lambda)$ denotes the Morse index of u_λ.

Proof. From (1.1) we get

$$\int_B |\nabla u_\lambda|^2 = \lambda \int_B (e^{u_\lambda} - 1)u_\lambda.$$

Therefore,

$$\int_B \left(|\nabla u_\lambda|^2 - \lambda e^{u_\lambda} u_\lambda^2\right) = \lambda \int_B \left(e^{u_\lambda} - 1 - e^{u_\lambda} u_\lambda\right) u_\lambda < 0,$$

so $m(u_\lambda) \geq 1$.

We prove the proposition by contradiction. Suppose that $\{(\lambda_n, u_n)\}$ is a sequence of radial regular solutions of problem (1.1) and assume that $m(u_n) \to \infty$ as $n \to \infty$. Let us write $m(u_n) = m_n$ and

$$L_n = -\Delta - \lambda_n e^{u_n}.$$

Let

$$E_n = \text{span}\left\{\varphi \in L^2(B) : \varphi \text{ is eigenvector of } L_n \text{ with negative eigenvalue}\right\}$$

so that $\dim(E_n) = m_n$. Since L_n is symmetric there exist eigenfunctions $\varphi_{1,n}, \ldots, \varphi_{m_n,n} \in E_n$, namely

$$\begin{cases} L_n\varphi_{i,n} = \mu_{i,n}\varphi_{i,n} & \text{in } B, \\ \varphi_{i,n} = 0 & \text{on } \partial B, \end{cases}$$

with $\mu_{i,n} < 0$, that form an orthonormal basis of E_n in $L^2(B)$ sense, that is

$$\int_B \varphi_{i,n}\varphi_{j,n} = \delta_{ij} \quad \text{for } i, j \in \{1, 2, \ldots, m_n\}, \tag{6.12}$$

where δ_{ij} is Kronecker's delta.

Multiplying by $\varphi_{i,n}$ and integrating on B, we find

$$\int_B \left(|\nabla \varphi_{i,n}|^2 - \lambda_n e^{u_n} \varphi_{i,n}^2\right) = \mu_{i,n} \int_B \varphi_{i,n}^2 < 0.$$

Then

$$\begin{aligned} \int_B |\nabla\varphi_{i,n}|^2 &< \int_B \lambda_n e^{u_n}\varphi_{i,n}^2 = \int_{B_\rho} \lambda_n e^{u_n}\varphi_{i,n}^2 + \int_{B\setminus B_\rho} \lambda_n e^{u_n}\varphi_{i,n}^2 \\ &\leq \int_{B_\rho} \lambda_n e^{-2\log r + \log\frac{2(N-2)}{\lambda_n} + A_n r^2}\varphi_{i,n}^2 + C\int_{B\setminus B_\rho}\varphi_{i,n}^2 \\ &= 2(N-2)\int_{B_\rho} \frac{\varphi_{i,n}^2}{r^2}(1 + A_n r^2 + o(r^2)) + C\int_{B\setminus B_\rho}\varphi_{i,n}^2 \\ &\leq \frac{8}{N-2}\int_B |\nabla\varphi_{i,n}|^2 + \max\{2(N-2)(A_n + o(1)), C\}\int_B \varphi_{i,n}^2. \end{aligned}$$

If $N \geq 11$ we deduce

$$\int_B |\nabla\varphi_{i,n}|^2 \leq \frac{N-2}{N-10}\max\{2(N-2)(A_n + o(1)), C\},$$

where $A_n = \frac{\lambda_n}{2N}$. Let us assume $N \geq 11$ and leave the case $N = 10$ for later. Thus $(\varphi_{i,n})_n$ is bounded in $H_0^1(B)$. By a diagonal argument, there is a subsequence (which we write the same), such that for each $i \in \{1, 2, \ldots\}$, $\varphi_{i,n} \rightharpoonup \varphi_i$ weakly in $H_0^1(B)$, $\varphi_{i,n} \to \varphi_i$ strongly in $L^2(B)$ and almost everywhere in B as $n \to +\infty$. Therefore for all $i \geq 1$,

$$\|\varphi_i\|_{H_0^1(B)} \leq \liminf_{n\to+\infty} \|\varphi_{i,n}\|_{H_0^1(B)} \leq C, \qquad \|\varphi_i\|_{L^2(B)} = 1.$$

Moreover, taking $n \to \infty$ in (6.12)

$$\int_B \varphi_i\varphi_j = \delta_{ij} \quad \text{for } i, j \geq 1. \tag{6.13}$$

Since $(\varphi_i)_{i\geq 1}$ is bounded in $H_0^1(B)$, there is a subsequence $(\varphi_{i_j})_j$ of (φ_i) such that $\varphi_{i_j} \to \varphi$ in $L^2(B)$ as $j \to +\infty$, and $\|\varphi\|_{L^2(B)} = 1$. But from (6.13) we get

$$\int_B \varphi_{i_j}\varphi_{i_m} = 0 \quad \text{for } j \neq m.$$

Taking the limit, as $j \to +\infty$ and $m \to +\infty$, we have

$$\int_B \varphi^2 = 0,$$

which is a contradiction.

For $N = 10$, we define the Hilbert space H as the completion of $C_c^\infty(B)$ with respect to the norm

$$\|\phi\|_H^2 := \int_B \left(|\nabla\phi|^2 - \lambda_* e^{u_*}\phi^2 + \gamma\phi^2\right)$$

with $\gamma > 0$ large but fixed and u_* the radial singular solution of (1.1) with $\lambda = \lambda_*$. Then

$$\begin{aligned}\|\varphi_{i,n}\|_H^2 &= \int_B \left(|\nabla\varphi_{i,n}|^2 - \lambda_* e^{u_*}\varphi_{i,n}^2\right) + \gamma\int_B \varphi_{i,n}^2\\ &= \mu_{i,n}\int_B \varphi_{i,n}^2 + \int_B \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right)\varphi_{i,n}^2 + \gamma\int_B \varphi_{i,n}^2\\ &< \int_B \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right)\varphi_{i,n}^2 + \gamma\int_B \varphi_{i,n}^2.\end{aligned}$$

Let $\rho > 0$ be as in Lemma 6.2. Let $A_n = \frac{\lambda_n}{2N}$. From Lemmas 6.1 and 6.3, we find

$$\begin{aligned}\int_B \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right)\varphi_{i,n}^2 &= \int_{B_\rho} \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right)\varphi_{i,n}^2 + \int_{B\setminus B_\rho} \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right)\varphi_{i,n}^2\\ &\leq \int_{B_\rho} \left(\lambda_n e^{-2\log r + \log\frac{2(N-2)}{\lambda_n} + A_n r^2} - \lambda_* e^{-2\log r + \log\frac{2(N-2)}{\lambda_*} + r^2 + o(r^2)}\right)\varphi_{i,n}^2 + C\int_{B\setminus B_\rho} \varphi_{i,n}^2\\ &\leq C\int_B \varphi_{i,n}^2.\end{aligned}$$

Thus we get

$$\|\varphi_{i,n}\|_H^2 \leq (C+\gamma)\int_B \varphi_{i,n}^2 \leq C.$$

That is, $(\varphi_{i,n})_n$ is bounded in H. By Lemma 6.5, the natural embedding $H \hookrightarrow L^2(B)$ is compact, so using the same argument as the case $N \geq 11$ we obtain a contradiction. This ends the proof of Proposition 6.7. □

Lemma 6.8. *Suppose that u_1, u_2 are radial regular solutions of (1.1) associated to the same parameter $\lambda > 0$. Then the graph of u_1 must intersect with the graph of u_2.*

Proof. By contradiction, assume that $u_1(r) > u_2(r)$ for any $r \in (0,1)$, and set $v = u_1 - u_2$. By Eq. (1.1) we have

$$\begin{cases} -\Delta v = \lambda(e^{u_1} - e^{u_2}) > \lambda e^{u_2} v & \text{in } B;\\ v > 0 & \text{in } B;\\ v = 0 & \text{on } \partial B.\end{cases} \tag{6.14}$$

We consider the following eigenvalue problem

$$\begin{cases} -\Delta\psi = \lambda e^{u_2}\psi + \mu\psi & \text{in } B;\\ \psi > 0 & \text{in } B;\\ \psi = 0 & \text{on } \partial B.\end{cases} \tag{6.15}$$

Multiplying by ψ and v in (6.14) and (6.15) respectively, and then integrating on B, we get

$$\lambda\int_B e^{u_2}\psi v + \mu\int_B \psi v > \lambda\int_B e^{u_2}\psi v,$$

so $\mu > 0$, that is u_2 is a stable radial regular solution. Then $m(u_2) = 0$ and this contradicts Proposition 6.7. □

Proof of Theorem 1.7. The first part follows from Propositions 6.6 and 6.7.

Let K be an integer such that $m(u_\lambda) \le K$ for any radial regular solution u_λ of (1.1) and $m(u_*) \le K$. This integer exists by Propositions 6.6 and 6.7. Next we prove that the graph of any radial regular solution u_λ of (1.1) intersects with that of the radial singular solution u_* at most $2K + 1$ times in $(0, 1)$. We follow the idea of Theorem 1.2 in [10].

By contradiction, suppose that the graph of u_λ intersects with the graph of u_* at least $2K + 2$ times in $(0, 1)$. There are two cases: $\lambda < \lambda_*$ and $\lambda \ge \lambda_*$.

For $\lambda < \lambda_*$, we can show $m(u_\lambda) \ge K + 1$, contradicting Proposition 6.7. Indeed, since the graph of (λ, u_λ) intersects with that of (λ_*, u_*) at least $2K + 2$ times in $(0, 1)$, there are at least $K + 1$ intervals $J_i \subset (0, 1) (i = 1, 2, \dots, K + 1)$ such that $u_\lambda > u_*$ in J_i. Let

$$h_i = \begin{cases} u_\lambda - u_* & \text{in } J_i; \\ 0 & \text{in } (0, 1)\setminus J_i. \end{cases}$$

Since u_λ and u_* satisfy Eq. (1.1), we have

$$\begin{aligned} -\Delta(u_\lambda - u_*) &= \lambda(e^{u_\lambda} - 1) - \lambda_*(e^{u_*} - 1) \\ &< \lambda(e^{u_\lambda} - e^{u_*}) \le \lambda e^{u_\lambda}(u_\lambda - u_*). \end{aligned}$$

Therefore

$$Q_{u_\lambda}(h_i) = \int_B [|\nabla h_i|^2 - \lambda e^{u_\lambda} h_i^2] dx < 0.$$

Since the functions $h_i,\ i = 1, \dots, K + 1$ are linearly independent, we conclude that $m(u_\lambda) \ge K + 1$.

For $\lambda \ge \lambda_*$, similarly we can obtain that $m(u_*) \ge K + 1$. This contradicts Proposition 6.6. In fact, because the graph of u_λ intersects with that of u_* at least $2K + 2$ times in $(0, 1)$, there are at least $K + 1$ intervals $J_k \subset (0, 1)$ $(k = 1, 2, \dots, K + 1)$ such that $u_* > u_\lambda$ in J_k. Let

$$h_k = \begin{cases} u_* - u_\lambda & \text{in } J_k; \\ 0 & \text{in } (0, 1)\setminus J_k. \end{cases}$$

Note that

$$-\Delta h_k < \lambda_* e^{u_*} h_k \quad \text{in } J_k$$

and this implies

$$Q_{u_*}(h_k) = \int_B [|\nabla h_k|^2 - \lambda_* e^{u_*} h_k^2] dx < 0.$$

Therefore $m(u_*) \ge K + 1$.

Next we prove that the number of regular solutions to (1.1) is bounded by $(K + 1)^2$ for each $\lambda \in (\lambda_0, \mu_1)$.

By contradiction, for each fixed $\lambda \in (\lambda_0, \mu_1)$, we suppose that there are at least $(K + 1)^2 + 1$ radial regular solutions to (1.1), denoted by u_i $(i = 0, 1, \dots, (K + 1)^2)$. Without loss of generality, assume $u_0(0) > u_1(0) > \cdots > u_{(K+1)^2}(0)$. By Lemma 6.8, the graph of $u_i,\ i = 1, \dots, (K + 1)^2$, must intersect with that of u_0. Let a_i be the first point such that $u_i(a_i) = u_0(a_i)$ for $i = 1, \dots, (K + 1)^2$. Then there are the following two cases.

Case 1: There are at least $(K + 1)$ different points a_i such that $u_0 - u_i > 0$ in $(0, a_i)$ and $u_i(a_i) = u_0(a_i)$.

Case 2: There exists some point $a_{i_0} \in (0, 1)$, such that there are at least $(K + 1)$ regular solutions that first intersect u_0 at a_{i_0}.

Case 1. We rearrange the indices so that $a_1 < \cdots < a_{K+1}$. Now $u_1(0), \dots, u_{K+1}(0)$ are not necessarily ordered. Let $\varphi_i = (u_0 - u_i)\chi_{(0,a_i)}$. We claim that $\{\varphi_i : i = 1, 2, \dots, (K + 1)\}$ is linearly independent. Indeed, suppose that

$$\sum_{i=1}^{K+1} c_i \varphi_i = 0.$$

Since $a_{i-1} < a_i$, there exists $r_{i-1} \in (a_{i-1}, a_i)$, such that $\varphi_1(r_{i-1}) = 0,\ \ \varphi_2(r_{i-1}) = 0, \dots, \varphi_{i-1}(r_{i-1}) = 0, \varphi_i(r_{i-1}) \ne 0$, then we can get $c_i = 0$, for $i = 1, 2, \dots, (K + 1)$. Then

$$\begin{aligned} Q_{u_0}(\varphi_i) &= \int_{\{|x|<a_i\}} [|\nabla \varphi_i|^2 - \lambda e^{u_0} \varphi_i^2] dx \\ &= \lambda \int_{\{|x|<a_i\}} [e^{u_0} - e^{u_i} - e^{u_0}(u_0 - u_i)](u_0 - u_i) dx < 0 \end{aligned}$$

by strict convexity and $u_0 - u_i > 0$ in $\{|x| < a_i\}$. This implies that $m(u_0) \ge K + 1$, contradicting Proposition 6.7.

Case 2. Rearranging indices, there are at least $K+1$ solutions $u_1, \ldots, u_{K+1}$ that satisfy $(u_0(r) - u_j(r)) > 0$ for $r \in (0, a_{i_0})$ and $u_j(a_{i_0}) = u_0(a_{i_0}), j = 1, \ldots, K+1$. Set $\varphi_j = (u_0 - u_j)\chi_{(0,a_{i_0})}$, we claim that

$$\{\varphi_j : j = 1, \ldots, K+1\} \text{ is linearly independent.} \tag{6.16}$$

Claim (6.16) together with $Q_{u_0}(\varphi_j) < 0$ yields that $m(u_0) \ge K+1$, contradicting $1 \le m(u_0) \le K$.

Let us show that the claim (6.16) holds. From now on, we write $r_0 = a_{i_0}$. We assume that there exist $c_j, \ j = 1, \ldots, K+1$, such that

$$\sum_{j=1}^{K+1} c_j\varphi_j(r) = 0 \quad \text{for all } r \in (0, r_0],$$

that is,

$$\sum_{j=1}^{K+1} c_j u_j(r) = \left(\sum_{j=1}^{K+1} c_j\right) u_0(r) \quad \text{for all } r \in (0, r_0]. \tag{6.17}$$

We will deduce $c_1 = \cdots = c_{K+1} = 0$ from the following assertion:

$$\sum_{j=1}^{K+1} c_j(u_j'(r_0))^n = \left(\sum_{j=1}^{K+1} c_j\right)(u_0'(r_0))^n, \quad \text{for all integers } n \ge 0. \tag{6.18}$$

In the following we will establish (6.18). We denote $g^{(n)}$ the n-th derivative of g and set

$$f(u) := -\lambda(e^u - 1), \quad \forall u \in \mathbb{R}; \qquad b = u_0(r_0).$$

Then $f^{(n)}(u_j(r_0)) = -\lambda e^b$ for any integer $n \ge 1$.

In order to prove (6.18), we shall show that for each $j \in \{0, 1, 2, \ldots, K+1\}$,

$$u_j^{(n)}(r_0) = P_n(u_j'(r_0)) \quad \text{for any integer } n \ge 1, \tag{6.19}$$

where P_n is a polynomial of degree 1 for $n = 1, 2$, and of degree $n-2$ for $n \ge 3$, whose coefficients depend only on N, n, r_0, and b.

Indeed, for $n = 1$, (6.19) is direct and for $n = 2$ this follows from Eq. (1.1). By induction, assume that (6.19) holds for $n = k \ge 2$. From Eq. (1.1), we have

$$(\Delta u_j)^{(k-1)} = (f(u_j))^{(k-1)}. \tag{6.20}$$

We see that for $n \ge 0$,

$$\begin{aligned}(\Delta u_j)^{(n)} &= u_j^{(n+2)} + \frac{N-1}{r}u_j^{(n+1)} - n\frac{N-1}{r^2}u_j^{(n)} + n(n-1)\frac{N-1}{r^3}u_j^{(n-1)} - \cdots \\ &\quad + (-1)^{n-1}n!\frac{N-1}{r^n}u_j'' + (-1)^n n!\frac{N-1}{r^{n+1}}u_j',\end{aligned} \tag{6.21}$$

and by the formula for derivatives of a composition (e.g. Faa di Bruno [29]) we obtain

$$(f(u_j))^{(n)} = -\lambda e^{u_j}\sum_{\alpha_1,\ldots,\alpha_n}\frac{n!}{\alpha_1!(1!)^{\alpha_1}\alpha_2!(2!)^{\alpha_2}\cdots\alpha_n!(n!)^{\alpha_n}}\prod_{i=1}^{n}(u_j^{(i)})^{\alpha_i}, \tag{6.22}$$

where the sum ranges over integers $\alpha_1 \ge 0, \ldots, \alpha_n \ge 0$ with $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$. Using (6.20)–(6.22) with $n = k-1$ and $r = r_0$, we get

$$\begin{aligned}u_j^{(k+1)}(r_0) &= -\frac{N-1}{r_0}u_j^{(k)}(r_0) + (k-1)\frac{N-1}{r_0^2}u_j^{(k-1)}(r_0) - \cdots \\ &\quad - (-1)^{k-2}(k-1)!\frac{N-1}{r_0^{k-1}}u_j''(r_0) - (-1)^{k-1}(k-1)!\frac{N-1}{r_0^k}u_j'(r_0) \\ &\quad - \lambda e^b\sum_{\alpha_1,\ldots,\alpha_{k-1}}\frac{(k-1)!}{\alpha_1!(1!)^{\alpha_1}\alpha_2!(2!)^{\alpha_2}\cdots\alpha_{k-1}!((k-1)!)^{\alpha_{k-1}}}\prod_{i=1}^{k-1}(u_j^{(i)}(r_0))^{\alpha_i},\end{aligned}$$

where the sum ranges over integers $\alpha_1 \ge 0, \ldots, \alpha_{k-1} \ge 0$ with $\alpha_1 + 2\alpha_2 + \cdots + (k-1)\alpha_{k-1} = k-1$. By the induction assumption (6.19), we have $\prod_{i=1}^{k-1}(u_j^{(i)}(r_0))^{\alpha_i}$ is a polynomial in $u_j'(r_0)$ of degree at most $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5 + \cdots + (k-3)\alpha_{k-1} \le k-1$. Thus we see the validity of (6.19).

Next we prove that (6.18) holds, again by induction. From (6.17), we have

$$\sum_{j=1}^{K+1} c_j u_j^{(n)}(r_0) = \left(\sum_{j=1}^{K+1} c_j\right) u_0^{(n)}(r_0) \quad \text{for any integer } n \geq 0, \tag{6.23}$$

and so (6.18) holds for $n = 0, 1$. Suppose (6.18) holds for $n = k$. By Eq. (1.1), we get

$$(\Delta u_j)^{(n)} = (f(u_j))^{(n)}. \tag{6.24}$$

Since $u_j(r_0) = u_0(r_0)$ for $j = 1, 2, \ldots, K+1$, from (6.21)–(6.24), we obtain for any integer $n \geq 0$,

$$\sum_{j=1}^{K+1} c_j \left(\left(u_j'(r_0)\right)^n + A_{j,n}\right) = \left(\sum_{j=1}^{K+1} c_j\right)\left(\left(u_0'(r_0)\right)^n + A_{0,n}\right) \tag{6.25}$$

where

$$A_{j,n} = \sum_{\alpha_1,\ldots,\alpha_n} \frac{n!}{\alpha_1!(1!)^{\alpha_1}\alpha_2!(2!)^{\alpha_2}\cdots\alpha_n!(n!)^{\alpha_n}} \prod_{i=1}^{n} (u_j^{(i)}(r_0))^{\alpha_i}$$

and the sum ranges over integers $0 \leq \alpha_1 < n, \alpha_2 \geq 0, \ldots, \alpha_n \geq 0$ with $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$. In writing (6.25) we have used again the formula for the n-th order derivative of a composition, where we have isolated one term. Consider (6.25) for $n = k+1$. By (6.19) we know that $\prod_{i=1}^{k+1}(u_j^{(i)}(r_0))^{\alpha_i}$ is a polynomial in $u_j'(r_0)$ of degree at most

$$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5 + \cdots + (k-1)\alpha_{k+1}.$$

Since $0 \leq \alpha_1 < k+1$, we see that

$$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5 + \cdots + (k-1)\alpha_{k+1} < \alpha_1 + 2\alpha_2 + \cdots + (k+1)\alpha_{k+1} = k+1$$

and therefore $A_{j,n}$ can be expressed as a polynomial in $u_j'(r_0)$ of degree at most k. Thus by the induction assumption, we have

$$\sum_{j=1}^{K+1} c_j A_{j,n} = \left(\sum_{j=1}^{K+1} c_j\right) A_{0,n}$$

and so (6.18) holds for any integer $n \geq 0$.

Finally we turn to the proof of (6.16), namely the linear independence of $\varphi_j, j = 1, \ldots, K+1$. We denote $u_0'(r_0) = d_0$, $u_j'(r_0) = d_j$ for $j = 1, 2, \ldots, K+1$. For $n = 1, 2, \ldots, K+1$, we can rewrite (6.18) as

$$\begin{pmatrix} d_1 - d_0 & d_2 - d_0 & \cdots & d_{K+1} - d_0 \\ d_1^2 - d_0^2 & d_2^2 - d_0^2 & \cdots & d_{K+1}^2 - d_0^2 \\ d_1^3 - d_0^3 & d_2^3 - d_0^3 & \cdots & d_{K+1}^3 - d_0^3 \\ \vdots & \vdots & \ddots & \vdots \\ d_1^{K+1} - d_0^{K+1} & d_2^{K+1} - d_0^{K+1} & \cdots & d_{K+1}^{K+1} - d_0^{K+1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{K+1} \end{pmatrix} = 0. \tag{6.26}$$

A calculation shows that the determinant of the coefficient matrix of (6.26) is equal to a $(K+2) \times (K+2)$ Vandermonde determinant and the value is

$$\prod_{0 \leq j < i \leq K+1} (d_i - d_j) \neq 0.$$

Thus $c_1 = c_2 = \cdots = c_{K+1} = 0$ and this ends the proof of Theorem 1.7. □

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Appendix

Proof of Proposition 1.2. Suppose u is a classical solution of (1.1). Let $\phi_1 > 0$ be the first eigenfunction of $-\Delta$ corresponding to the first eigenvalue μ_1. Multiplying problem (1.1) by ϕ_1 and integrating over B, we find

$$\mu_1 \int_B u\phi_1 = \lambda \int_B (e^u - 1)\phi_1 > \lambda \int_B u\phi_1.$$

Thus $\lambda < \mu_1$.

Multiplying problem (1.1) by $x \cdot \nabla u$, and integrating over B, we have

$$-\int_B \Delta u(x \cdot \nabla u) = \lambda \int_B (e^u - 1)(x \cdot \nabla u). \tag{A.1}$$

But

$$\begin{aligned} -\int_B \Delta u(x \cdot \nabla u) &= -\frac{1}{2}\int_{\partial B} |\nabla u|^2 x \cdot \nu + \left(1 - \frac{N}{2}\right)\int_B |\nabla u|^2 \\ &\le \left(1 - \frac{N}{2}\right)\int_B |\nabla u|^2, \end{aligned} \tag{A.2}$$

since $x \cdot \nu \ge 0$ on ∂B. Moreover,

$$\lambda \int_B (e^u - 1)(x \cdot \nabla u) = -\lambda N \int_B (e^u - 1 - u). \tag{A.3}$$

From (A.1)–(A.3), we get

$$\left(\frac{N}{2} - 1\right)\int_B |\nabla u|^2 \le \lambda N \int_B (e^u - 1 - u).$$

We rewrite the above inequality as

$$\frac{N-2}{4}\int_B |\nabla u|^2 \le \lambda N \int_B (e^u - 1 - u) - \frac{N-2}{4}\int_B |\nabla u|^2.$$

Multiplying Eq. (1.1) by u and substituting we get

$$\frac{N-2}{4}\int_B |\nabla u|^2 \le \lambda \int_B \left[N(e^u - 1 - u) - \frac{N-2}{4}(e^u - 1)u\right].$$

The integrand on the right hand is negative for $u \ge C_0$, with C_0, a positive constant, so the integral can be restricted to the region $\{x : u(x) \le C_0\}$ and in this region

$$N(e^u - 1 - u) - \frac{N-2}{4}(e^u - 1)u \le C_1 u^2.$$

Thus

$$\frac{N-2}{4}\int_B |\nabla u|^2 \le \lambda C_1 \int_B u^2 \le \lambda C_2 \int_B |\nabla u|^2,$$

where $C_1 > 0$, $C_2 > 0$. This implies that $u = 0$ if $0 < \lambda < \frac{N-2}{4C_2}$. □

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