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# Antiferromagnetic Ising model in triangulations with applications to counting perfect matchings

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ABSTRACT

In this work we give a lower bound for the groundstate degeneracy of the antiferromagnetic Ising model in the class of stack triangulations, also known as planar 3-trees. The geometric dual graphs of stack triangulations form a class, say $\mathcal{C}$, of cubic bridgeless planar graphs, i.e. $G \in \mathcal{C}$ iff its geometric dual graph is a planar 3-tree. As a consequence, we show that every graph $G \in \mathcal{C}$ has at least $3 \cdot \varphi^{(|V(G)|+8)/30} \geq 3 \cdot 2^{(|V(G)|+8)/44}$ distinct perfect matchings, where $\varphi$ is the golden ratio. Our bound improves (slightly) upon the $3 \cdot 2^{(|V(G)|+12)/60}$ bound obtained by Cygan, Pilipczuk, and Škrekovski (2013) for the number of distinct perfect matchings also for graphs $G \in \mathcal{C}$ with at least 8 nodes.

Our work builds on an alternative perspective relating the number of perfect matchings of cubic bridgeless planar graphs and the number of so called groundstates of the widely studied Ising model from statistical physics. With hindsight, key steps of our work can be rephrased in terms of standard graph theoretic concepts, without resorting to terminology from statistical physics. Throughout, we draw parallels between the terminology we rely on and some of the concepts introduced/developed independently elsewhere.



## 1. Introduction

The Ising model is one of the most studied models in statistical physics. The study of the Ising model in a system (graph) helps to understand physical phenomena associated to its thermodynamic properties [14]. In particular, antiferromagnetic systems are interesting, in part because of their lack of order at zero temperature [12]. So far, the Ising model has been widely explored for large scale and mostly regular (lattice like) systems using various different methods [2]. In this context, our work contributes in two ways: (1) to the understanding of the Ising model's behavior for irregular systems and not necessarily of small size, and (2) to the development of an adaptation of the extensively employed transfer matrix method [13] used by statistical physicists, and thus widen the scope of its applicability.

Let us now precisely describe the antiferromagnetic Ising model. By a slight abuse of notation we say that a map in a 2-dimensional surface is a *triangulation* if each face is bounded by a cycle of length 3. In particular, the underlying graph has no loop and may have multiple edges. Given a triangulation $\Delta = (V, E)$ we associate the *coupling constant* $c(e) = -1$ with each edge $e \in E$. For any $W \subseteq V$, a *spin assignment* of $W$ is any function $\mathbb{s} : W \to \{1, -1\}$ and $1, -1$ are called *spins*. A *state* of $\Delta$ is any spin assignment of $V$. The energy of a state $\mathbb{s}$ is defined as $-\sum_{e=\{u,v\}\in E} c(e)\mathbb{s}(u)\mathbb{s}(v)$. The states of minimum energy are called *groundstates*. The number of groundstates is usually called the *groundstate degeneracy* of $\Delta$, denoted $g(\Delta)$.

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A graph is said to be *cubic* if each vertex has degree 3 and *bridgeless* if it contains no cutedges. In the mid-1970's, Lovász and Plummer asserted that *for every cubic bridgeless graph with n vertices, the number of perfect matchings is exponential in n*. For bipartite graphs, the assertion was positively solved by Voorhoeve [15]. Chudnovsky and Seymour showed that it holds for planar graphs [4]. Independently and after the announcement of our work, Esperet, Kardoš, King, Král' and Norine [6] announced the positive resolution of the full conjecture.

We address the problem of counting perfect matchings of cubic bridgeless graphs in the dual setting. This relates the problem of counting perfect matchings to the groundstate degeneracy of the antiferromagnetic Ising model on triangulations. In order to explain this, let us recall the directed cycle double cover conjecture of Jaeger (see [10]): *Every cubic bridgeless graph can be embedded in an orientable surface so that each face is homeomorphic to an open disk (i.e., the embedding defines a map) and the geometric dual has no loop.* A set $M$ of edges of a triangulation $\Delta$ is *intersecting* if $M$ contains exactly one edge of each face of $\Delta$. Let $G$ be a cubic bridgeless graph. Assuming the directed cycle double cover conjecture, let $G^*$ denote the geometric dual of an embedding of $G$ in an orientable surface without loops. Hence, $G^*$ is a triangulation, and $M$ is an intersecting set of $G^*$ if and only if $M$ is a perfect matching of $G$. We can now reformulate the theorem of Esperet et al. as follows: *Each triangulation has an exponential number of intersecting sets of edges*.

Given a state $\mathtt{s}$ of $\Delta$ we say that edge $\{u, v\}$ is *frustrated* by $\mathtt{s}$ or that $\mathtt{s}$ *frustrates* edge $\{u, v\}$ if $\mathtt{s}(u) = \mathtt{s}(v)$. Note that each state frustrates at least one edge of each face of $\Delta$. A state is a groundstate if it frustrates the smallest possible number of edges. Clearly, if there is a state which frustrates exactly one edge of each face of $\Delta$, it is a groundstate and the set of edges frustrated is an intersecting set and in this case, the number of groundstates is at most twice the number of intersecting sets of edges. Note that this always happens for planar triangulations and not necessarily in general. The converse also holds for planar triangulations: if we delete an intersecting set of edges from a planar triangulation, then we get a bipartite graph and its bipartition determines a groundstate. Hence, Chudnovsky and Seymour's result [4] can be reformulated as follows: *each planar triangulation has an exponential (in the number of vertices) groundstate degeneracy*.

The result we derive in this work, from the lower bound for the groundstate degeneracy of the antiferromagnetic Ising model on stack triangulations, applies to a sub-class of graphs for which both [4,6] already establish the validity of Lovász and Plummer's conjecture, albeit for a much smaller rate of exponential growth and arguably by more complicated and involved arguments.

During this article's review process the existence of the work of Cygan, Pilipczuk, and Škrekovski [5] was called to our attention. The main result of [5] is a lower bound of $3 \cdot 2^{(|V(G)|+12)/60}$ for the number of perfect matchings of $G$ when $G$ is a Klee-graph with at least 8 nodes. It is easily seen that Klee-graphs are exactly the class of geometric duals of stack triangulations, so [5] independently addressed the exact same problem this work is concerned with. Cygan et al. were motivated, as pointed out in [5, Section 1], by the important role Klee-graphs play in the theory of matchings and cubic graphs, as well as the expectation that Klee-graphs are the 3-edge-connected cubic graphs with fewest perfect matchings – seemingly justifying the interest in lower bounding the number of its perfect matchings and providing examples of an infinite family of Klee-graphs with as few of them as possible (furthermore, Klee-graphs arose naturally in works related to Lovász and Plummer's conjecture [8,7] – although they do not seem to play role in its final resolution [6]).

We believe that the relevance of our main result (Corollary 2) is, that it validates the feasibility of the alternative perspective proposed in [11], for addressing Lovász and Plummer's conjecture.

Specifically, letting $\varphi = (1 + \sqrt{5})/2 \approx 1.6180$ denote the golden ratio, we establish the following:

**Theorem 1.** *The groundstate degeneracy of the antiferromagnetic Ising model in a stack triangulation $\Delta$ with $|\Delta|$ vertices is at least* $6 \cdot \varphi^{\frac{1}{15}(|\Delta|+2)}$.

As a rather direct consequence of the preceding theorem we obtain the following result.

**Corollary 2.** *The number of perfect matchings of a cubic graph $G$ whose dual graph is a stack triangulation (alternatively, Klee-graphs) is at least* $3 \cdot \varphi^{(|V(G)|+8)/30} \geq 3 \cdot 2^{(|V(G)|+8)/44}$.

There are two substantial differences between this work and [11]. First, in [11] we directly work with so called transfer matrices. Thus, in essence [11] is closer to the type of applications of the transfer matrix method typical of statistical physics. In contrast, in this work we develop the technique further by considering transfer vectors instead of transfer matrices. Second, although in [11] a lower bound on the number of satisfying states of triangulations of a convex $n$-gon is given, the result has no direct implication in terms of Lovász and Plummer's conjecture—since the dual of a triangulation of a convex $n$-gon is not cubic (the outer face of the $n$-gon gives rise to a degree $n$ node in the dual). In contrast, this work does imply that certain bridgeless cubic graphs (those whose dual is a stack triangulation), admit exponentially many (in the number of its nodes) perfect matchings (and establishes a better exponential growth rate than those implied by previous work for these graphs).

Before concluding this section, we would like to stress that our approach uses and was inspired by (rather elementary) techniques and concepts from statistical physics, which we could have "disguised" by resorting to either known or purposefully defined purely graph theoretic terminology. However, it seems more natural to elicit the theoretical/statistical physics perspective that influenced our work and rely on the language/terminology commonly used by physicists. Throughout, we try to point out where we use a different language to speak about notions introduced and/or developed

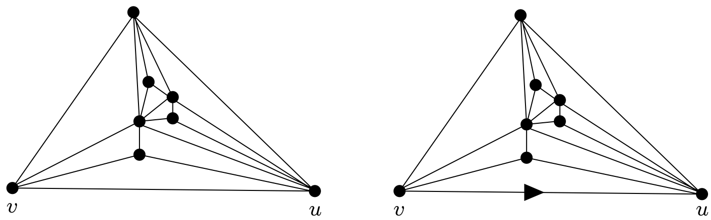


**Fig. 1.** A stack triangulation (left) and the rooted stack triangulation obtained by prescribing the counterclockwise orientation to the edge $\{v, u\}$ (right).

independently elsewhere. We also try to elicit where our viewpoint seems to be more convenient for proving stronger bounds and/or could potentially be applied in the derivation of similar bounds for other recursively defined structures.

### 1.1. Organization

The paper is organized as follows. We provide some mathematical background in Section 2. Then, in Section 3, we describe a bijection between rooted stack triangulations and colored rooted ternary trees—this bijection allows us to work with ternary trees instead of triangulations. In Section 4, we first introduce the concept of degeneracy vector in stack triangulations. This vector satisfies the condition that the sum of its coordinates is the number of (so called) pseudo-groundstates of the stack triangulation, from which a lower bound on the number of groundstates is immediately obtained. We also introduce the concept of root vector of a ternary tree and show that via the aforementioned bijection, the degeneracy vector of a stack triangulation is the same as the root vector of the associated colored rooted ternary tree. In Section 5, we adapt to our setting the transfer matrix method as used in statistical physics in the study of the Ising model. Some essential results are also established. In Section 6, we prove the main results of this work. In Section 7, we conclude with a brief discussion and comments about possible future research directions.

## 2. Preliminaries

We now introduce the main concepts and notation used throughout this work.

### 2.1. Stack triangulations

We start by formally defining the class of graphs called stack triangulations. The definition is recursive. Specifically, let $\Delta_0$ be a triangle. For $i \geq 1$, let $\Delta_i$ be the plane triangulation obtained by applying the following *growing rule* to $\Delta_{i-1}$.

**growing rule:** Given a plane triangulation $\Delta$,
**Step 1**: Choose an inner face $f$ from $\Delta$.
**Step 2**: Insert a new vertex $u$ at the interior of $f$.
**Step 3**: Connect the new vertex $u$ to each vertex of the boundary of $f$.

Clearly, the number of vertices of $\Delta_n$ is $n + 3$. A $\Delta_n$ thus obtained is called *stack triangulation*. Among others, the set of stack triangulations coincides with: (1) the set of plane triangulations having a unique Schnyder Wood (see [9]), (2) the collection of planar 3-trees (see [3, page 167]), and (3) the class of geometric duals of Klee-graphs. Also note that in the geometric dual, the previously defined growing rule, corresponds exactly to replacing a vertex of a (planar) cubic graph by a triangle.

It is clear from the definition of stack triangulations that they have a recursive structure. Since each application of a growing rule replaces an inner triangular face by three triangular faces, it is rather intuitive that there ought to be some association between stack triangulations and rooted trees (in fact, rooted ordered trees) of degree at most three. The associated tree structure will play a key role in our determination of the groundstate degeneracy of stack triangulations. For the moment, it should suffice to note that the associated tree structure depends on the choice of inner face during Step 1 at each application of the growing rule. This partly explains why we need to have a unique way of identifying/referring to each face of a stack triangulation. To achieve the latter, it will be handy to work with oriented versions of stack triangulations. We now introduce one basic such notion. Let $\Delta_n$ be a stack triangulation with $n \geq 0$ and $\Delta_0$ be the starting plane triangle in its construction. If we prescribe the counterclockwise orientation to any edge of the outer face of $\Delta_n$, that is, to any edge of $\Delta_0$, we say that $\Delta_n$ is a *rooted stack triangulation* (see Fig. 1).

A particular sub-class of stack triangulations will play an important role later on. We now formally specify this sub-class. Consider a plane triangle $\Delta_1$ and for $i \geq 2$, let $\Delta_i$ be the plane triangulation obtained by applying the growing rule to $\Delta_{i-1}$ restricting Step 1 so the face chosen is one of the three new faces obtained by the application of the growing rule to $\Delta_{i-2}$. For $n \geq 1$, we say that $\Delta_n$ is a *stack–strip triangulation* (for an example see Fig. 2). Clearly, stack–strip triangulations are a sub-class of stack triangulations.

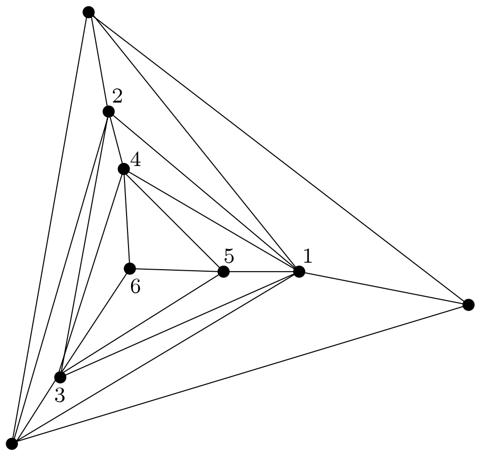


**Fig. 2.** Example of stack–strip triangulation (numbers correspond to the order in which nodes are added by the growing rule).

### 2.2. Ternary trees

We define below a special class of trees, a sub-class of which we will end up placing in one-to-one correspondence with stack triangulations. A *rooted tree* is a tree $T$ with a special vertex $v \in V(T)$ designated to be the root. If $v$ is the root of $T$, we denote $T$ by $T_v$. A *rooted ternary tree* is a rooted tree $T_v$ such that all its vertices have at most three children. From now on, let $X$ be an arbitrary set with three elements. We say that a rooted ternary tree $T_v$ is *colored by X* (or simply *colored*) if; (1) each non-root vertex is labeled by an element of $X$, and (2) for every vertex of $V(T)$ all its children have different labels.

## 3. From stack triangulations to ternary trees

It is well known that stack triangulations are in bijection with ternary trees (see [1]). For our purposes, the usual bijection is not enough (we need the associated tree structure to more precisely reflect the way in which triangular faces touch each other). The main goal of this section is to precisely describe a one-to-one correspondence better suited for our purposes.

### 3.1. Bijection

Let $\Delta_n$ be a rooted stack triangulation with $n \geq 1$ and $\Delta_0$ be the starting plane triangle in its construction. We will show how to construct a colored rooted ternary tree $T(\Delta_n)$ which will be in bijective correspondence with $\Delta_n$.

Throughout this section, the following concept will be useful.

**Definition 1.** Let $\Delta$ be a rooted stack triangulation. Let $\tilde{\Delta}$ be a rooted stack triangulation obtained by prescribing the counterclockwise orientation to exactly one edge of each inner face of $\Delta$. We refer to $\tilde{\Delta}$ as an auxiliary stack triangulation of $\Delta$.

Note that in an auxiliary stack triangulation of $\Delta$, we allow inner faces of $\Delta$ to have edges oriented clockwise as long as exactly one of its edges is oriented counterclockwise. It is also allowed to have edges with both orientations.

We now, describe the key procedure in the construction of the colored rooted ternary tree $T(\Delta_n)$ in bijective correspondence with the rooted stack triangulation $\Delta_n$. The procedure starts by associating to $\Delta_n$ an auxiliary stack triangulation $\tilde{\Delta}_n$ in the most natural way. Indeed, let $\Delta_0, \ldots, \Delta_n$ be the stack triangulations obtained when applying the growing rule in the construction of $\Delta_n$. Note that the outer face of $\Delta_n$ is precisely the outer face of $\Delta_0$. Since $\Delta_n$ is rooted, one of the edges, say $e_0$, of the outer face of $\Delta_n$ is oriented counterclockwise. Fix the orientation of the edge of $\Delta_0$ corresponding to $e_0$ exactly as in $\Delta_n$, and thus determine an auxiliary stack triangulation of $\Delta_0$, say $\tilde{\Delta}_0$. Assume now that we have obtained auxiliary orientations $\tilde{\Delta}_0, \ldots, \tilde{\Delta}_i$ of $\Delta_0, \ldots, \Delta_i$, respectively. Since $\Delta_{i+1}$ is obtained from $\Delta_i$ by applying the growing rule to a specific face, say inserting a new vertex $u_i$ in the interior of a face $f_i$ of $\Delta_i$, we need only show how to pick the edges and their corresponding orientations for each of the three newly created faces of $\Delta_{i+1}$. In Fig. 3, we illustrate the mentioned choice of orientation and formally describe it in the following paragraph (also seizing the opportunity to introduce additional notation).

**Labeling/orientation procedure:**

**Step 1:** Let $\vec{e}_{f_i}$ be the counterclockwise oriented edge of $f_i$. The orientation of $\vec{e}_{f_i}$ induces a counterclockwise ordering of the three new faces around $u_i$ starting by the face that contains $\vec{e}_{f_i}$, say $f_i(1)$. Let $f_i(2)$ and $f_i(3)$ denote the second and third new faces according to the induced order. For each $j \in \{1, 2, 3\}$, we say that $f_i(j)$ is in position $j$ or that $j$ is the position of $f_i(j)$. (See Fig. 3.)

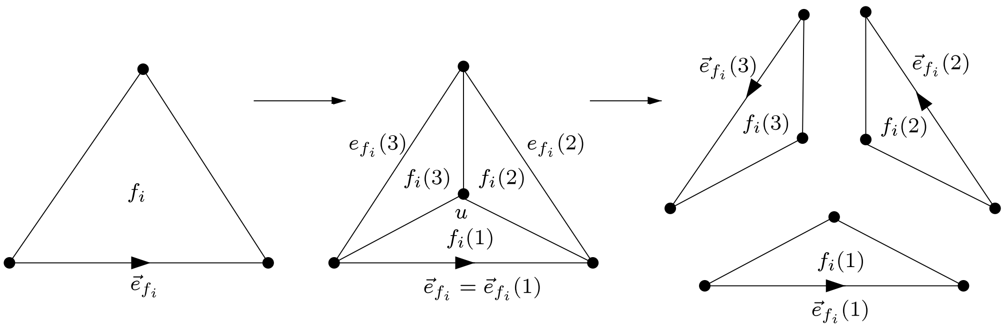


**Fig. 3.** Labeling/orientation procedure. Left to center illustrates Step 1. Center to right illustrates Step 2.

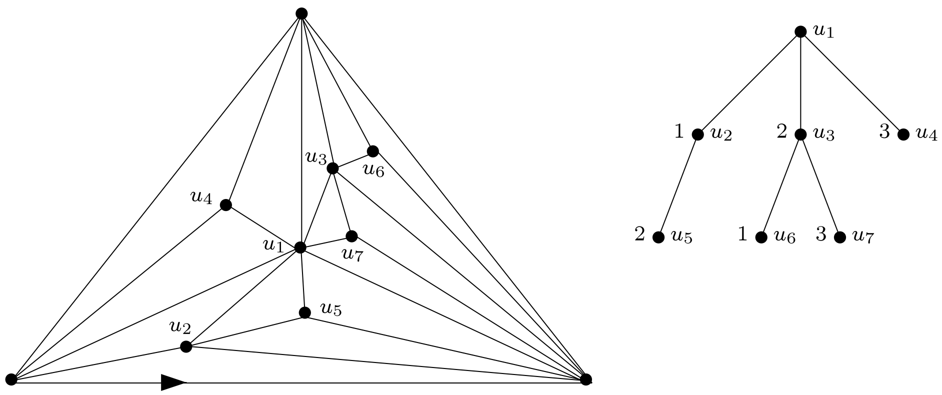


**Fig. 4.** Example of the bijection between a rooted stack triangulation $\Delta_n$ (left) and its associated colored rooted ternary tree $T(\Delta_n)$ (right). Node labels are shown to the left of each non-root node of $T(\Delta_n)$.

**Step 2**: For each $j \in \{2, 3\}$, take the unique edge $e_{f_i}(j)$ in $E(f_i) \cap E(f_i(j))$ and prescribe the counterclockwise orientation to this edge (see Fig. 3). For all other faces of $\Delta_i$ not contained in $f_i$, keep the same counterclockwise oriented edge. (Observe that for each $j \in \{1, 2, 3\}$, the triangle $f_i(j)$ has a prescribed counterclockwise orientation in one of its three edges. Moreover, note that $\vec{e}_{f_i} = \vec{e}_{f_i}(1)$.)

The set $\Theta_{\Delta_n} = \{(f_i, u_i, f_i(1), f_i(2), f_i(3))\}_{i\in\{1,\dots,n\}}$ will be henceforth referred to as the *growth history of* $\Delta_n$. Note that, for $j \in \{1, 2, 3\}$, each face $f_1(j)$ together with its oriented edge induce a rooted stack triangulation, henceforth denoted $\Delta_n^j$, on the vertices of $\Delta_n$ that lie on the boundary and interior of $f_1(j)$.

We can now describe, in terms of the growth history $\Theta_{\Delta_n}$ of $\Delta_n$, how the colored rooted ternary tree $T(\Delta_n)$ associated to $\Delta_n$ is constructed/defined. In essence, $T(\Delta_n)$'s nodes are in one-to-one correspondence with the internal nodes of $\Delta_n$. The parent of a node $u'$ is $u$, if the node $u'$ is inserted in an inner face $f$ (while applying the growing rule during the construction of $\Delta_n$) such that $f$ was created when $u$ was inserted. The label (alternatively, color) of $u'$ depends in the relative clockwise ordering of $f$ among the faces that were created together with $f$. Formally, (for an example see Fig. 4):

*Combinatorial description of* $T(\Delta_n)$: Let $X = \{1, 2, 3\}$. Let $V(T(\Delta_n)) = \{u_1, \dots, u_n\}$. Let $u_1$ be the root of $T(\Delta_n)$. For $i \in \{2, \dots, n\}$, $u_i$ is a child of vertex $u_j$ if there is a $k \in \{1, 2, 3\}$ such that $f_i = f_j(k)$. The label of $u_i$ is $k$.

For the sake of future reference we now state four easily verified properties regarding $T(\Delta_n)$. Each property is separately illustrated in Fig. 5.

**Remark 3.** Let $\Delta_n$ be a rooted stack triangulation. The colored ternary tree $T(\Delta_n)$ rooted on $v$, satisfies the following statements:

1. If $\Delta_n^i$ has 3 vertices for all $i \in \{1, 2, 3\}$, then $T(\Delta_n)$ has exactly one vertex $v$ (its root). (See Fig. 5(a).)
2. If there are $i, j \in \{1, 2, 3\}$ with $i \neq j$ such that $\Delta_n^i$ and $\Delta_n^j$ have 3 vertices and $\Delta_n^k$ with $k \in \{1, 2, 3\} \setminus \{i, j\}$ has at least 4 vertices, then the root $v$ has exactly one child $w$ labeled by $k$. Moreover, the root of $T(\Delta_n^k)$ is $w$, where $T(\Delta_n^k)$ is the colored sub-ternary tree of $T(\Delta_n)$ induced by $w$ and its descendants. (See Fig. 5(b).)
3. If there is an $i \in \{1, 2, 3\}$ such that $\Delta_n^i$ has 3 vertices and $j, k \in \{1, 2, 3\} \setminus \{i\}$ with $j \neq k$ such that $\Delta_n^j$ and $\Delta_n^k$ have at least 4 vertices, then the root $v$ has exactly two children $w_j$ and $w_k$ labeled by $j$ and $k$, respectively. Moreover, for every $t \in \{j, k\}$, the root of $T(\Delta_n^t)$ is $w_t$, where $T(\Delta_n^t)$ is the colored sub-ternary tree of $T(\Delta_n)$ induced by $w_t$ and its descendants. (See Fig. 5(c).)

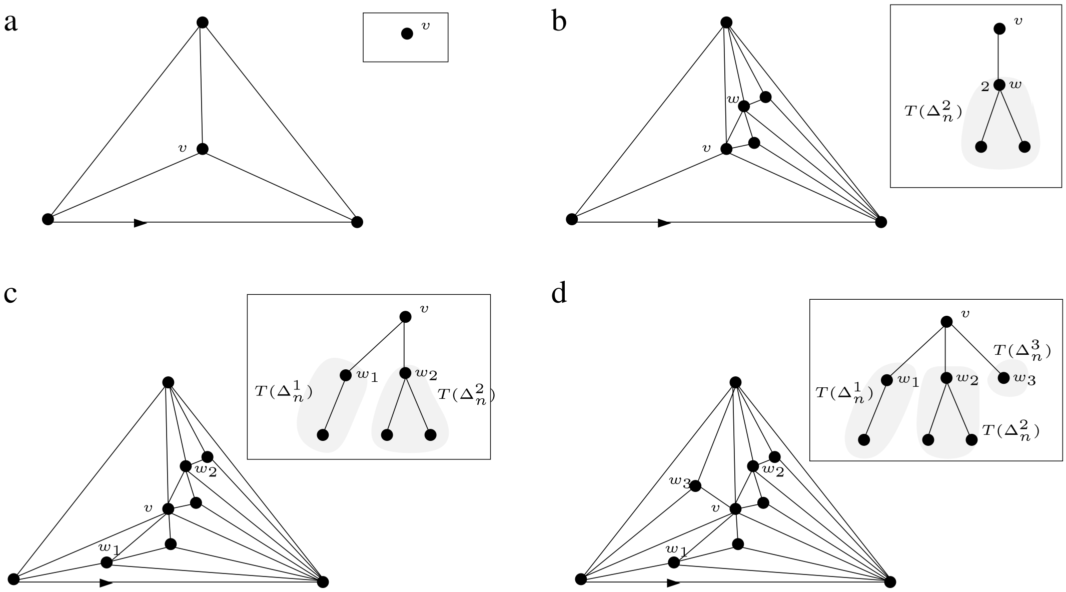


**Fig. 5.** Example of a rooted stack triangulations $\Delta_n$ and its corresponding associated colored rooted ternary trees $T(\Delta_n)$ (bounded by a rectangular box) for each of the four cases considered in Remark 3.

4. If $\Delta_n^i$, $i \in \{1, 2, 3\}$, has at least 4 vertices, then the root $v$ has three children $w_1$, $w_2$ and $w_3$ labeled by 1, 2 and 3, respectively. Moreover, for every $i \in \{1, 2, 3\}$, the root of $T(\Delta_n^i)$ is $w_i$, where $T(\Delta_n^i)$ is the colored sub-ternary tree of $T(\Delta_n)$ induced by $w_i$ and its descendants. (See Fig. 5(d).)

## 4. Transfer method

The main tool we use to carry out our work, is an adaptation of a method (well known among physicist) called the transfer matrix method. In this section, we adapt the method to our context.

The proofs of the results claimed in this section are not immediate, however the results themselves are not that surprising. Hence, in order to arrive sooner to the more substantial part of this work, we have preferred to move this section's proofs to Appendix.

### 4.1. Methodology

We start by introducing some additional concepts. We say that a state $\mathtt{s}$ *is satisfying for* a face $f$ of a planar triangulation $\Delta$, if there is exactly one edge in the boundary of $f$ that is frustrated by $\mathtt{s}$. Moreover, we say that $\mathtt{s}$ is a *satisfying state* of $\Delta$ if $\mathtt{s}$ is satisfying for every inner face. Clearly, a satisfying state of $\Delta$ is satisfying for its outer face if and only if it is a groundstate. Not only do groundstates of a planar triangulation $\Delta$ correspond to perfect matchings in its geometric dual (planar) cubic graph $G$, more generally, each state $\mathtt{s}$ of $\Delta$ corresponds to a $T$-join in $G$ with $T = V(G)$ — the $T$-join consisting of edges of $G$ which are associated (by duality) to the edges of $\Delta$ frustrated by $\mathtt{s}$ (an edge set is called a *T-join* if in the induced subgraph of this edge set, the collection of all the odd-degree vertices is $T$).

In general terms, our aim is to obtain for each stack triangulation $\Delta$ a vector $\mathbf{v}_\Delta$ in $\mathbb{R}^4$ such that the sum of its coordinates equals twice the number of satisfying states of $\Delta$. We now elaborate on this. Let $n \geq 1$ and $\Delta_n$ be a rooted stack triangulation. Let $\Delta_0 = (v_1, v_2, v_3)$ denote the starting triangle in the construction of $\Delta_n$ such that $\{v_1, v_2\}$ is the oriented edge with $v_1$ the tail and $v_2$ the head. We wish to construct a vector $\mathbf{v}_{\Delta_n} \in \mathbb{R}^4$ such that its coordinates are indexed by the ordered set $I = \{+++, ++-, +-+, -++\}$. For every $\phi \in I$, the $\phi$th coordinate of $\mathbf{v}_{\Delta_n}$, denoted $\Delta_n[\phi]$, is defined as the number of satisfying states of $\Delta_n$ when the spin assignment of $(v_1, v_2, v_3)$ is equal to $\phi$. The vector $\mathbf{v}_{\Delta_n}$ will be called the *degeneracy vector* of $\Delta_n$. In particular, $\mathbf{v}_{\Delta_0} = (0, 1, 1, 1)^t$ is the degeneracy vector of a triangle. Clearly, for every $\phi \in I$ we have the relation

$$\Delta_n[\phi] = \Delta_n[-\phi]. \tag{1}$$

Let $\Theta_{\Delta_n} = \{(f_i, u_i, f_i(1), f_i(2), f_i(3))\}_{i\in\{1,\ldots,n\}}$ be the growth history of $\Delta_n$. Let $v$ denote $u_1$. Recall that $f_1(j)$ induces a rooted stack triangulation $\Delta_n^j$ according to the growth history of $\Delta_n$ (see Section 3.1), and that the oriented edge of $\Delta_n^1$ is $\{v_1, v_2\}$ with $v_1$ its tail and $v_2$ its head; the oriented edge of $\Delta_n^2$ is $\{v_2, v_3\}$ with $v_2$ its tail and $v_3$ its head; and the oriented edge of $\Delta_n^3$ is $\{v_3, v_1\}$ with $v_3$ its tail and $v_1$ its head. The following result shows how to express the degeneracy vector of $\Delta_n$ in terms of the degeneracy vectors $\mathbf{v}_{\Delta_n^1}$, $\mathbf{v}_{\Delta_n^2}$, and $\mathbf{v}_{\Delta_n^3}$. Note that the existence of such a relation is not really surprising, given the recursive nature of the definition we have given for stack triangulations. However, what is remarkable is the elegance albeit non-trivial form taken by the relation.

**Proposition 4.** *For each $j \in \{1, 2, 3\}$, let $\mathbf{v}_{\Delta_n^j} = (v_j^k)_{k\in\{0,1,2,3\}}$. Then,*

$$\mathbf{v}_{\Delta_n} = \begin{pmatrix} v_1^0 v_2^0 v_3^0 + v_1^1 v_2^1 v_3^1 \\ v_1^0 v_2^2 v_3^3 + v_1^1 v_2^3 v_3^2 \\ v_1^2 v_2^3 v_3^0 + v_1^3 v_2^2 v_3^1 \\ v_1^3 v_2^0 v_3^2 + v_1^2 v_2^1 v_3^3 \end{pmatrix}.$$

We point out that the above recursive formula is identical to the one derived by Cygan et al. in [5, Lemma 4.2], the explanation for this will be clarified and explained in the following section.

### *4.2. Root vectors of ternary trees*

We will now introduce the concept of *root vector* of a colored rooted ternary tree. Then, we will see that $\mathbf{v}_\Delta$ is the degeneracy vector of the rooted stack triangulation $\Delta$ if and only if $\mathbf{v}_\Delta$ is the root vector of the colored rooted ternary tree $T(\Delta)$.

**Definition 2.** Let $T$ be a colored ternary tree rooted at $v$. For a node $u$ of $T \setminus \{v\}$, denote by $l_u \in \{1, 2, 3\}$ its label. We recursively define the root vector $\mathbf{v} \in \mathbb{R}^4$ of $T$ associated to $v$ according to the following rules.
**Rule 0:** $\mathbf{v} = (1, 1, 1, 1)^t$ when $v$ does not have any children.
**Rule 1:** If $v$ has exactly one child $u$ with $\mathbf{u} = (u_s)_{s=0,\dots,3}$, then

$$\mathbf{v} \in \left\{(u_1, u_0 + u_1, u_3, u_2)^t, (u_1, u_3, u_2, u_0 + u_1)^t, (u_1, u_2, u_0 + u_1, u_3)^t\right\}.$$

The choice of $\mathbf{v}$ depends on the label of $u$; if $l_u = i$, $\mathbf{v}$ is the $i$th vector in $[\mathbf{u}]$.
**Rule 2:** If $v$ has two children $u$ and $w$ with $\mathbf{u} = (u_s)_{s=0,\dots,3}$, $\mathbf{w} = (w_s)_{s=0,\dots,3}$, and $(l_u, l_w) \in \{(1, 2), (2, 3), (3, 1)\}$, then

$$\mathbf{v} \in \left\{ \begin{pmatrix} u_1 w_1 \\ u_0 w_2 + u_1 w_3 \\ u_3 w_2 \\ u_3 w_0 + u_2 w_1 \end{pmatrix}, \begin{pmatrix} u_1 w_1 \\ u_3 w_2 \\ u_3 w_0 + u_2 w_1 \\ u_0 w_2 + u_1 w_3 \end{pmatrix}, \begin{pmatrix} u_1 w_1 \\ u_3 w_0 + u_2 w_1 \\ u_0 w_2 + u_1 w_3 \\ u_3 w_2 \end{pmatrix} \right\}.$$

The choice of $\mathbf{v}$ depends on $(l_u, l_w)$; if $l_u = i$, $\mathbf{v}$ is the $i$th vector in this set.
**Rule 3:** If $v$ has three children $u$, $w$ and $z$ with $\mathbf{u} = (u_s)_{s=0,\dots,3}$, $\mathbf{w} = (w_s)_{s=0,\dots,3}$, $\mathbf{z} = (z_s)_{s=0,\dots,3}$, and $(l_u, l_w, l_z) = (1, 2, 3)$, then

$$\mathbf{v} = \begin{pmatrix} u_0 w_0 z_0 + u_1 w_1 z_1 \\ u_0 w_2 z_3 + u_1 w_3 z_2 \\ u_2 w_3 z_0 + u_3 w_2 z_1 \\ u_3 w_0 z_2 + u_2 w_1 z_3 \end{pmatrix}.$$

**Remark 5.** The preceding definition actually associates a vector in $\mathbb{R}^4$ to each node of a colored rooted ternary tree. We shall henceforth adopt the convention of denoting the vector associated to a node (e.g. $v$) by the same symbol but in roman type (e.g. $\mathbf{v}$).

The following result, which is central to our approach, establishes that determining the degeneracy vector of rooted stack triangulations is equivalent to determining the root vector of colored rooted ternary trees.

**Lemma 6.** *Let $n \geq 1$ and $\Delta_n$ be a rooted stack triangulation. Then, the root vector of the colored ternary tree $T(\Delta_n)$ in bijection with $\Delta_n$ equals the degeneracy vector of $\Delta_n$.*

The last stated lemma tells us that a component wise lower bound on the coordinates of the root vector of the colored ternary tree $T(\Delta_n)$ suffices to bound the coordinates of the degeneracy vector of $\Delta_n$, thence also the number groundstates of $\Delta_n$. Obtaining such a component wise lower bound is not simple. In fact, in order to succeed, we show in the next section how to first "prune" the tree $T(\Delta_n)$ and leave it in a more "manageable" form.

One can also try to interpret degeneracy vectors of a triangulation $\Delta_n$ (and through Lemma 6 also the root vector of the colored ternary tree $T(\Delta_n)$) in terms of the geometric dual (planar) cubic graph $G$ of $\Delta_n$. Recalling the duality between states of $\Delta_n$ and $T$-joins of $G$, $T = V(G)$, there are two interpretations possible: (1) the first component of the root vector is the number of $T$-joins of $G$ where the only vertex of degree 3 is the one representing the outer face; the other three components of the vector are the numbers of perfect matchings of the graph containing the first (second, third) edge incident to that vertex, or (2) if we ignore the outer face of $\Delta_n$ (and the corresponding vertex of $G$), we can speak about a cubic graph with three pending edges (semi-edges). Then, the first component of the vector is the number of perfect matchings containing all the three pending edges, and the other three components are the numbers of perfect matchings containing each one of the three edges. The latter of these interpretations is precisely the viewpoint adopted by Cygan et al. [5] where the cubic graphs with pending edges mentioned above were named *tripods*, and the geometric dual edges associated to the edges of the outer face of $\Delta_n$ where named *legs* (of the tripod). Thus, it is not surprising that the recursive formula obtained in [5, Lemma 4.2] for the number of perfect matchings in tripods coincides exactly with the one of our Proposition 4 for counting satisfying states. As far as we can tell, the main advantage of working with transfer vectors, as we do, is that they take into account the number of perfect matchings of the geometric dual of $\Delta_n$ that do not contain those edges associated to $\Delta_n$'s outer face (in the language of [5], excluding the legs of the tripod). Via Proposition 4, it is then possible to recursively propagate bounds maintaining a better overall control of the terms that arise.

## 5. Colored rooted ternary trees

The goal of this section is to prove a result that we should refer to as the Main lemma which shows that the groundstate degeneracy of stack triangulations is exponential in the number of its nodes. First, we introduce notation that will be useful when dealing with rooted ternary trees. We denote by $|T|$ the number of vertices of the ternary tree $T$. For any node $u$ of $T$, we denote by $T_u$ the colored rooted sub-ternary tree of $T$ rooted at $u$ and induced by $u$ and its descendants. If $v$ is the father of $u$ in $T$, we say that $T_u$ is a *component* of $v$.

Also, we denote by $P_{\tilde{w},w}$ any path with end nodes $\tilde{w}$ and $w$. Moreover, $\|P_{\tilde{w},w}\| = |P_{\tilde{w},w}| - 1$ denotes the length of $P_{\tilde{w},w}$.

### 5.1. Remainders

In this subsection we introduce the concept of *remainder* of a rooted ternary tree and prove some useful and fundamental claims related to this concept. We will show that it is possible to remove some remainders and some *potential remainders* from a rooted ternary tree in such a way that we are left with a remainder-free rooted ternary tree of size at least $\frac{2}{5}$ of the original one (Lemma 10). The root vector of the derived remainder-free tree will provide a coordinate wise lower bound on the coordinates of the root vector of the original rooted ternary tree. The underlying motivation for this section is that lower bounding the coordinates of a root vector is significantly easier for remainder-free colored rooted ternary trees.

**Definition 3.** Let $v$ be a leaf of $T$ and $w$ be its father. Consider the following cases:

(I) If $u \neq v$ is a child of $w$, then $|T_u| \geq 3$. In addition, if $w$ has three children, say $v, u, z$, then there are nodes $\tilde{u}, \tilde{z}$ such that neither $\tilde{u}$, nor $\tilde{z}$ has exactly one child, $T_u = P_{u,\tilde{u}} \cup T_{\tilde{u}}$, $T_z = P_{z,\tilde{z}} \cup T_{\tilde{z}}$ and $\|P_{u,\tilde{u}}\| = \|P_{z,\tilde{z}}\| = 3$.

(II) If $T_w$ is just the edge $wv$, then the father of $w$, say $y$, has two children $w$ and $u$, where $|T_u| \geq 3$. In addition, there exists a node $\tilde{u}$ without exactly one child such that $T_u = P_{u,\tilde{u}} \cup T_{\tilde{u}}$ and $\|P_{u,\tilde{u}}\| \in \{2, 3, 4\}$.

If Case I holds, we say that $\{v\}$ is a *remainder* of $T$ and that $w$ is the *generator* of $\{v\}$. Moreover, if $w$ has exactly $i \in \{2, 3\}$ children, then we say that $\{v\}$ is $i$-big. If Case II holds we say that $\{v, w\}$ is a *remainder* of $T$ and that $y$ is its *generator*. We say that $T$ is remainder-free if it does not contain any remainder. We denote the set of remainders of $T$ by $R(T)$. We also denote by $B(T)$ the subset of $R(T)$ consisting of the 2-big remainders. Furthermore, we say that $\{v\}$ ($\{v, w\}$, respectively) is a potential remainder, if the father of $v$ has exactly 3 children and the first part of Case I (Case II, respectively) holds. The generator of a potential remainder is the same as the generator of a remainder. We denote the set of potential remainders of $T$ by $P(T)$. Finally, for any subset $R'$ of $R(T) \cup P(T)$, we denote the set of generators of $R'$ by $G(R')$.

See Fig. 6 for an illustration of the distinct situations encompassed by each of the preceding definition's cases.

We note that all remainders, except 2-big remainders, are also potential remainders. In other words, $P(T) \cup R(T) = P(T) \cup B(T)$. The next proposition claims that the generator of a 2-big remainder or a potential remainder is the generator of either exactly one 2-big remainder, or exactly one potential remainder.

**Proposition 7.** *Let $T$ be a rooted ternary tree. If $w \in G(P(T)) \cup G(B(T))$, then $w$ is the generator of exactly one element from $P(T) \cup B(T)$.*

**Proof.** For the sake of contradiction, suppose that $w$ is the generator of at least two elements in $P(T) \cup B(T)$, say $S_1$ and $S_2$. We consider three possible cases which cover all possible scenarios: (i) $S_1 = \{v\}$ and $S_2 = \{u\}$, (ii) $S_1 = \{v, \tilde{v}\}$ and $S_2 = \{u, \tilde{u}\}$, and (iii) $S_1 = \{v\}$, $S_2 = \{u, \tilde{u}\}$. If $S_1 = \{v\}$ and $S_2 = \{u\}$, then by Case I of Definition 3, we get that $|T_v| \geq 3$. If $S_1 = \{v, \tilde{v}\}$ and $S_2 = \{u, \tilde{u}\}$, then by Case II of Definition 3, we get that $|T_{\tilde{v}}| \geq 3$. If $S_1 = \{v\}$ and $S_2 = \{u, \tilde{u}\}$, then by Case II of Definition 3, we have that $|T_v| \geq 3$. Hence, all feasible cases lead to contradictions. ∎

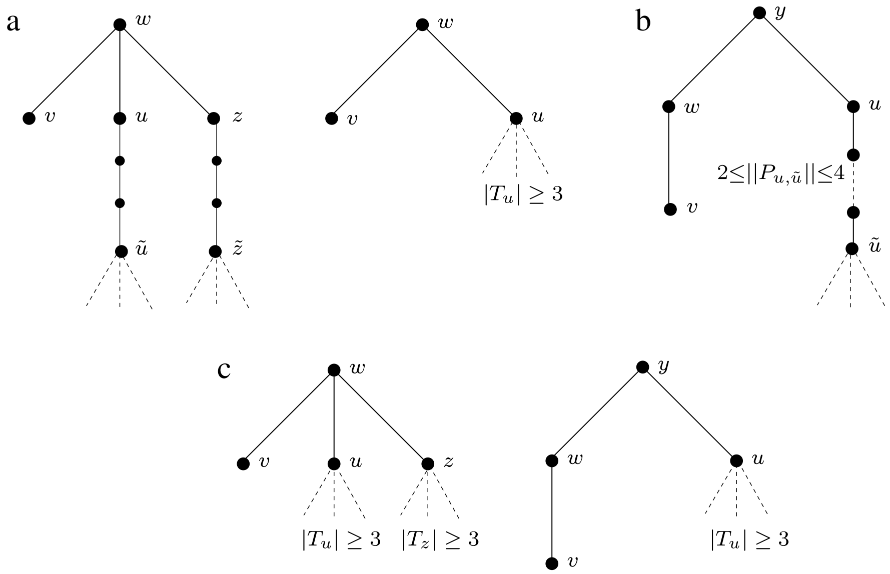


**Fig. 6.** In subfigure (a): structure of $T_w \subseteq T$ having a remainder $v$ of $T$ with generator $w$. Case where $\{v\}$ is 3-big (left) and 2-big (right). In subfigure (b): structure of $T_y \subseteq T$ having a remainder $\{v, w\}$ with generator $y$. In subfigure (c): potential remainder $\{v\}$ with generator $w$ (left) and potential remainder $\{v, w\}$ with generator $y$ (right).

Let $R' \subseteq P(T) \cup R(T)$, we denote by $V_{R'}$ the subset of vertices of $T$ which belong to the elements of $R'$, i.e. $V_{R'} = \cup_{S\in R'}\{v : v \in S\}$. The following lemma states that the deletion of the set of all 2-big remainders and an arbitrary subset (possibly empty) of potential remainders from a rooted ternary tree $T$ produces a rooted ternary tree with no 2-big remainders and such that each of its remainders or potential remainders is a potential remainder of $T$.

**Lemma 8.** *Let $T$ be a rooted ternary tree, $P'$ be a subset of $P(T)$ not necessarily non-empty and $\tilde{T} = T \setminus (V_{B(T)} \cup V_{P'})$. Then $\tilde{T}$ does not have 2-big remainders and $R(\tilde{T})$, $P(\tilde{T})$ are subsets of $P(T) \setminus P'$.*

**Proof.** We first prove that $B(\tilde{T}) = \emptyset$. For the sake of contradiction, assume $\{v\}$ is a 2-big remainder of $\tilde{T}$. Then, the father of $v$ in $\tilde{T}$, say $w$, has two children $v$, $u$ with $|\tilde{T}_u| \geq 3$. Since $v$ is a leaf of $\tilde{T}$, it must also hold that $v$ is a leaf of $T$ (otherwise, all of $v$'s children in $T$ must belong to some potential remainder or to some 2-big remainder, a situation that is not possible). Since $w$ has two children in $T$, it follows that $\{v\} \in B(T)$, a contradiction.

For proving that $R(\tilde{T})$ and $P(\tilde{T})$ are subsets of $P(T)\setminus P'$, it is enough to show that $P(\tilde{T}) \subseteq P(T)\setminus P'$, since $R(\tilde{T})\setminus B(\tilde{T}) \subseteq P(\tilde{T})$ and $B(\tilde{T}) = \emptyset$. Assume $S$ is a potential remainder of $\tilde{T}$. We first consider the case $S = \{v\}$. Then, the father of $v$ in $\tilde{T}$ has tree children $v$, $u$, $z$ with $|\tilde{T}_u|, |\tilde{T}_z| \geq 3$ (first part of Case I of Definition 3). Clearly, $|T_u| \geq |\tilde{T}_u|$ and $|T_z| \geq |\tilde{T}_z|$. Since $v$ has no children in $\tilde{T}$, it follows that $v \notin G(R(T))$. Thus, $v$ is a leaf of $T$ and $\{v\} \in P(T)$.

Assume now that $S = \{v, w\}$ and therefore, $S$ satisfies the first part of Case II of Definition 3. Let $y$ be the generator of $S$ and the father of $w$ in $\tilde{T}$. Then, $y$ has two children $w$, $u$ in $\tilde{T}$ with $|\tilde{T}_u| \geq 3$. We again have that $|T_u| \geq |\tilde{T}_u| \geq 3$ and that $v$ is a leaf of $T$. Assume $y$ has three children in $T$, say $w$, $u$, $z$. Then, $\{z\} \in R(T)$, implies that $|T_w| \geq 3$, and hence $w \in G(R(T))$. Therefore, $|T_v| \geq 3$, but this cannot happen because $v$ is a leaf of $T$. Thus, $y$ must have only two children in $T$. If $w$ has exactly two children in $T$, then $w \in G(R(T))$ and $|T_v| \geq 3$, contradicting again the fact that $v$ is a leaf. Hence, $w$ has only one child in $T$ and then, $\{v, w\} \in P(T)$. This finishes the proof of the lemma. ■

Let $T$ be a rooted ternary tree. We denote by $R_d(T)$ the set of remainders of $T$ with the deepest generators in $T$. The following statement claims that from a rooted ternary tree $T$ with no 2-big remainders (we want $T$ to be obtained by using Lemma 8) the deletion of a remainder that belongs to $R_d(T)$ results in a rooted ternary tree $\tilde{T}$ such that the depth of the elements in $G(R_d(\tilde{T}))$ is at most the depth of the elements in $G(R_d(T))$.

**Lemma 9.** *Let $T$ be a rooted ternary tree such that $B(T) = \emptyset$. Let $S \in R_d(T)$, $\tilde{T} = T \setminus V(S)$ and $x$ be the generator of $S$. Then, no descendant of $x$ in $\tilde{T}$ is the generator of a remainder of $\tilde{T}$.*

**Proof.** We first make the following straightforward observation: if $x'$ is a descendant of $x$ in $\tilde{T}$, then $x' \in V(T)$ and the number of children of $x'$ in $\tilde{T}$ and in $T$ is equal. Now, for the sake of contradiction, we assume that $S' \in R(\tilde{T})$ and that its generator, say $x'$, is a descendant of $x$. By Lemma 8, we know that $S' \in P(T)$ and that $S'$ is either a 3-big remainder, or a remainder of size 2. If we assume that $S' = \{v\}$ is a 3-big remainder of $\tilde{T}$, then there are paths $P_{u,\tilde{u}}$ and $P_{z,\tilde{z}}$ in $\tilde{T}$ satisfying Case II of Definition 3. By the aforementioned observation, we have that $P_{u,\tilde{u}}$ and $P_{z,\tilde{z}}$ are paths in $T$ satisfying

Case II of Definition 3 in $T$, and hence, $S' \in R(T)$. Analogously, the statement follows in the case that $S'$ is a remainder of size 2. ■

Our next claim is the main result of this subsection. Basically, by using the previously established results, we show that after deleting all 2-big remainders from a rooted ternary tree $T$, it is possible to sequentially delete the remainders with the deepest generators, in such a way that the resulting tree is remainder-free and has size at least $\frac{2}{5}|T|$.

**Lemma 10.** *Let $T$ be a rooted ternary tree. There exist $P' \subseteq P(T)$ such that $\tilde{T} = T \setminus (V_{B(T)} \cup V_{P'})$ is remainder-free and $|\tilde{T}| \geq \frac{2}{5}|T|$.*

**Proof.** Let $T^0, T^1, \ldots, T^p$ be the sequence of distinct rooted ternary trees such that $T^0 = T \setminus V_{B(T)}$, for each $i \in \{1, \ldots, p\}$, $T^i$ is obtained from $T^{i-1}$ by deleting a remainder, say $S_i$, from $R_d(T^{i-1})$ and $R(T^p) = \emptyset$. By Lemma 8, for each $i \in \{1, \ldots, p\}$ we have $R(T^i) \subseteq P(T)$ and hence, $p \leq |P(T)|$.

We claim that $\tilde{T} = T^p$ satisfies the statement of Lemma 10. We first note that $T^p = T \setminus (V_{B(T)} \cup V_{P'})$, where $P' = \{S_i : i \in \{1, \ldots, p\}\} \subseteq P(T)$. Secondly, in order to prove that $|T^p| \geq \frac{2}{5}|T|$, we show that there exist $W \subseteq V(T^p)$ such that $|W| \geq \frac{2}{5}(|W| + |V_{B(T)}| + |V_{P'}|)$.

By Lemma 8 and Remark 3, we have $G(B(T)) \cup G(P') \subseteq V(T^p)$. Let $U \subseteq V(T^p)$ be the union of $G(B(T)) \cup G(P')$ and the set of nodes of $T^p$ that have exactly one child in $T^p$.

By Definition 3, the subset of nodes $U$ induces a set $\Omega$ of disjoint rooted sub-ternary trees of $T^p$ such that for every $Q \in \Omega$ the following statements hold:

(i) Each node of $Q$ has at most 2 children.

(ii) Let $w$ be a node of $Q$ with exactly 2 children, say $u, z$. Then there are paths $(u, \hat{u}, u')$, $(z, \hat{z}, z')$ in $Q$ such that every node from $\{u, \hat{u}, z, \hat{z}\}$ has exactly one child in $Q$ and for each $x \in \{u', z'\}$, either $x$ is a leaf of $Q$, or $x$ has exactly one child in $Q$ that has exactly two children in $Q$. Let $\alpha_w = \{w, u, \hat{u}, u', z, \hat{z}, z'\}$. Moreover, if $x \in V_w$ is the generator of a potential remainder of size 2 in $P'$, then $x \in \{u, z\}$ and if $w'$ is another node of $Q$ with exactly two children, then $\alpha_w \cap \alpha_{w'} = \emptyset$. Let $R(\alpha_w)$ denote the subset of elements in $P' \cup B(T)$ with generator in $\alpha_w$.

(iii) If $v$ is the root of $Q$, then there exists a node $v'$ such that $Q = P_{v,v'} \cup T_{v'}$, $\|P_{v,v'}\| \geq 0$, and $v'$ is either a leaf of $Q$, or a node with exactly one child in $Q$ that has exactly two children in $Q$. Moreover, by Lemma 9, if $P_{v,v'} = (v_1 = v, \ldots, v_l = v')$ and $x \in V(P_{v,v'})$ is the generator of a potential remainder of size 2 in $P'$, then $x \in \{v_{l-2}, v_{l-3}, v_{l-4}\}$. Let $R_{v,v'}$ denote the subset of elements in $P' \cup B(T)$ with generator in $V(P_{v,v'})$.

The previous statements imply that if $\{w_1, \ldots, w_q\}$ is the set of nodes of $Q$ with exactly two children in $Q$, then $V(P_{v,v'})$, $\alpha_{w_1}, \ldots, \alpha_{w_q}$ is a partition of $V(Q)$. Moreover, by Item (ii), for each $i \in \{1, \ldots, q\}$ it holds that $|\alpha_{w_i}| \geq \frac{7}{16}(|\alpha_{w_i}| + |V_{R(\alpha_w)}|) > \frac{2}{5}(|\alpha_{w_i}| + |V_{R(\alpha_w)}|)$. On the other hand, by Item (iii), if either $l \leq 4$ or $l \geq 6$, then it is routine to check that $|V(P_{v,v'})| \geq \frac{2}{5}(|V(P_{v,v'})| + |R_{v,v'}|)$. We are left with the case that $l = 5$. We suppose first that $v$ is the root of $T^p$. In this case we consider the set of nodes $V(P_{v,v'}) \cup \{z\}$, where $z$ is a leaf of $T^p$ and therefore $z \notin U$. By Item (iii), we obtain

$$|V(P_{v,v'}) \cup \{z\}| \geq \frac{6}{14}(|V(P_{v,v'}) \cup \{z\}| + |R_{v,v'}|) \geq \frac{2}{5}(|V(P_{v,v'}) \cup \{z\}| + |R_{v,v'}|). \tag{2}$$

Hence, we shall assume that $v$ is not the root of $T^p$. Then, there exist a node $y \notin U$ in $V(T^p)$ such that $y$ is the father of $v$. We have that $y$ is the father of at most 3 such nodes. In the case $y$ is the father of exactly one such node, we have, again by Item (iii), that (2) is valid replacing $z$ by $y$. If $y$ is the father of exactly two such nodes, say $v$ and $u$, then by Item (iii) we have

$$|V(P_{v,v'}) \cup V(P_{u,u'}) \cup \{y\}| \geq \frac{11}{27}(|V(P_{v,v'}) \cup V(P_{u,u'}) \cup \{y\}| + |R_{v,v'}| + |R_{u,u'}|),$$

where $\frac{11}{27} > \frac{2}{5}$. Finally, again by Item (iii), in the case that $y$ is the father of three such nodes, say $v$, $w$ and $u$, we obtain

$$|V(P_{v,v'}) \cup V(P_{w,w'}) \cup V(P_{u,u'}) \cup \{y\}| \geq \frac{16}{40}(|V(P_{v,v'}) \cup V(P_{w,w'}) \cup V(P_{u,u'}) \cup \{y\}| + |R_{v,v'}| + |R_{w,w'}| + |R_{u,u'}|),$$

where $\frac{16}{40} = \frac{2}{5}$. The statement of Lemma 10 follows by taking $W$ equal to the union of $U$ and the additional needed nodes in $V(T^p) \setminus U$. ■

### 5.2. *Counting satisfying states*

In this section, we establish properties of the root vectors of colored rooted ternary trees and relate them with the tree structure. Informally, for some special classes of colored rooted ternary trees, we obtain lower bounds for the sum of the coordinates of its associated root vector.

Recall that $\varphi = (1 + \sqrt{5})/2 \approx 1.6180$ denotes the golden ratio. For $s \in \{0, \ldots, 3\}$, let $e_s \in \mathbb{N}$ and $\mathbf{e} = (\varphi^{e_s})_{s=0,\ldots,3}$. Define

$$\Psi(\mathbf{e}) = 2\sum_{j=1}^{3} e_j, \quad \text{and} \quad \Phi(\mathbf{e}) = \Psi(\mathbf{e}) - |\{\, s \mid e_s > e_0 \,\}|\,.$$

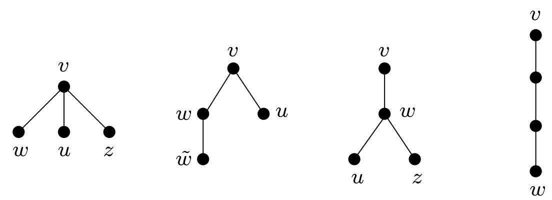


**Fig. 7.** All rooted ternary trees with 4 vertices.

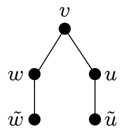


**Fig. 8.** The tree $T_v$ in the analysis of the case that $k = 5$.

Henceforth, for a vector $\mathbf{x}$ we let $[\![\mathbf{x}]\!]$ denote the collection of all vectors obtained by fixing the first coordinate of $\mathbf{x}$ and permuting the remaining coordinates in an arbitrary way. Note that if $\mathbf{e} = (\varphi^{e_s})_{s=0,\dots,3}$ with $e_0, e_1, e_2, e_3 \in \mathbb{N}$, then for all $\tilde{\mathbf{e}} \in [\![\mathbf{e}]\!]$ we have that $\Psi(\tilde{\mathbf{e}}) = \Psi(\mathbf{e})$ and $\Phi(\tilde{\mathbf{e}}) = \Phi(\mathbf{e})$. For a set $S$ of vectors, we let $[\![S]\!]$ denote the union of the sets $[\![\mathbf{x}]\!]$ where $\mathbf{x}$ varies over $S$.

Given vectors $\mathbf{x} = (x_s)_{s=0,\dots,3}$ and $\mathbf{y} = (y_s)_{s=0,\dots,3}$, we write $\mathbf{x} \geq \mathbf{y}$ if $x_s \geq y_s$ for all $s \in \{0, \dots, 3\}$.

The next two propositions, whose proofs we recommend skipping on a first reading, essentially establish lower bounds for the sum of the coordinates of the root vector $\mathbf{v}$ associated to "small size" colored rooted ternary trees.

**Proposition 11.** *Let $T_v$ be a colored rooted ternary tree with $|T_v| = k$, $k \in \{2, 3, 4, 5\}$. Then, there are $e_0, e_1, e_2, e_3 \in \mathbb{N}$ such that $\mathbf{v} \geq \mathbf{e} = (\varphi^{e_s})_{s=0,\dots,3}$ and $\Psi(\mathbf{e}) \geq 2k - 2$. Moreover, if $k \in \{2, 3\}$, then $\Psi(\mathbf{e}) = 2k - 2$.*

**Proof.** We first suppose that $k = 2$. Clearly $T_v$ is a rooted tree on $v$ with exactly one child $w$ which is a leaf of $T_v$. In other words, $T_v = P_{w,v}$ with $\|P_{w,v}\| = 1$. We observe that by applying Rules 0 and 1, we get that $\mathbf{w} = (1, 1, 1, 1)^t$ and $\mathbf{v} \in [\![(1, 2, 1, 1)^t]\!]$. Given that $1 = \varphi^0$ and $2 \geq \varphi^1$, it is easy to see that the desired vector $\mathbf{e}$ belongs to $[\![(\varphi^0, \varphi^1, \varphi^0, \varphi^0)]\!]$

Secondly, we assume that $k = 3$. Since $|T_v| = 3$, either $T_v = P_{w,v}$ with $\|P_{w,v}\| = 2$, or $v$ has exactly two children $w$ and $u$, which are leaves of $T_v$. In the first scenario, applying Rule 0 once and Rule 1 twice, we get that $\mathbf{v} \in [\![(2, 3, 1, 1)^t, (1, 2, 2, 1)^t]\!]$. Given that $1 = \varphi^0$, $2 \geq \varphi^1$ and $3 \geq \varphi^2$, we can take $\mathbf{e} \in [\![(\varphi^1, \varphi^2, \varphi^0, \varphi^0)^t, (\varphi^0, \varphi^1, \varphi^1, \varphi^0)^t]\!]$ satisfying the statement. In the second scenario, applying Rule 0, we get that $\mathbf{w}$ and $\mathbf{u}$ are vectors all of whose coordinates are 1. Applying Rule 2, we see that $\mathbf{v} \in [\![(1, 2, 1, 2)^t]\!]$. Given that $1 = \varphi^0$ and $2 \geq \varphi^1$, the desired vector $\mathbf{e}$ may be chosen from the set $[\![(\varphi^0, \varphi^1, \varphi^0, \varphi^1)^t]\!]$.

We now study the situation that $k = 4$. The tree $T_v$ may be one of the four trees depicted in Fig. 7. Each case is analyzed separately below (in the order in which they appear in Fig. 7).

For the first case, note that by Rule 0 we have that $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{z}$ are vectors all of whose coordinates are 1. Thus, by Rule 3, we get that $\mathbf{v} = (2, 2, 2, 2)^t$. Hence, $\mathbf{v} \geq \mathbf{e}$ where $\mathbf{e} = (\varphi^1, \varphi^1, \varphi^1, \varphi^1)^t$.

For the second case, by Rule 0 we have that all coordinates of $\mathbf{u}$ and $\tilde{\mathbf{w}}$ are 1. Thus, by Rule 1, $\mathbf{w} \in [\![(1, 2, 1, 1)^t]\!]$. Then, by Rule 2, we get that $\mathbf{v} \in [\![(2, 3, 1, 2)^t, (1, 2, 1, 3)^t, (1, 3, 2, 2)^t]\!]$. Given that $1 = \varphi^0$, $2 \geq \varphi^1$ and $3 \geq \varphi^2$ the result follows.

For the third case, note that $|T_w| = 3$ and that the structure of $T_w$ is the same as the second one considered in the proof of Proposition 11. Hence, we know that $\mathbf{w} \in [\![(1, 2, 1, 2)^t]\!]$. By Rule 1, we get that $\mathbf{v} \in [\![(2, 3, 2, 1)^t, (1, 2, 2, 2)^t]\!]$. Given that $1 = \varphi^0$, $2 \geq \varphi^1$ and $3 \geq \varphi^2$ the claimed result follows. We leave the last case to the interested reader.

Finally, we assume that $k = 5$. Suppose $T_v$ is as depicted in Fig. 8. Clearly, $\mathbf{w}, \mathbf{u} \in [\![(1, 2, 1, 1)^t]\!]$, so by Rule 2 we get that $\mathbf{v} \in [\![(4, 3, 1, 3)^t, (1, 3, 3, 2)^t, (2, 4, 2, 2)^t, (2, 2, 1, 5)^t, (1, 3, 1, 3)^t, (1, 3, 4, 3)^t]\!]$. The desired conclusion follows since $1 = \varphi^0$, $2 \geq \varphi^1$, $3 \geq \varphi^2$, $4 \geq \varphi^2$ and $5 \geq \varphi^3$. ■

**Proposition 12.** *Let $T_v$ be a colored rooted ternary tree such that $v$ has three children $u$, $w$ and $z$. Suppose that $1 \leq |T_u| \leq 3$ and $1 \leq |T_w| \leq 3$. Then,*

- *If $|T_z| = 2$, there are $e_0, e_1, e_2, e_3 \in \mathbb{N}$ such that $\mathbf{v} \geq \mathbf{e} = (\varphi^{e_s})_{s=0,\dots,3}$ and $\Psi(\mathbf{e}) \geq 8$.*
- *If $|T_z| = 3$, there are $e_0, e_1, e_2, e_3 \in \mathbb{N}$ such that $\mathbf{v} \geq \mathbf{e} = (\varphi^{e_s})_{s=0,\dots,3}$ and $\Psi(\mathbf{e}) \geq 10$.*

**Proof.** We first note that if $|T_x| \geq 2$, then $\mathbf{x} \geq (1, 1, 1, 1)^t$. This implies that it is enough to prove both statements for the case $|T_u| = 1$ and $|T_w| = 1$. Observe that by Rule 0, we have that $\mathbf{u} = \mathbf{w} = (1, 1, 1, 1)^t$.

For the first statement, assume $|T_z| = 2$. By Rule 1, we have that $\mathbf{z} \in [\![(1, 2, 1, 1)^t]\!]$. Then, by Rule 3 we have that $\mathbf{v} \in [\![(3, 3, 2, 2)^t, (2, 3, 3, 2)^t]\!]$. The result follows, since $2 \geq \varphi^1$ and $3 \geq \varphi^2$.

Assume now that $|T_z| = 3$. From the proof of Proposition 11 we know that $\mathbf{z} \in [\![(2, 3, 1, 1)^t, (1, 2, 2, 1)^t]\!]$. Then, by Rule 3 we have that $\mathbf{v} \in [\![(5, 2, 5, 2)^t, (3, 4, 3, 4)^t, (3, 3, 3, 3)^t, (2, 4, 2, 4)^t]\!]$. The desired conclusion follows since $2 \geq \varphi^1$, $3 \geq \varphi^2$, $4 \geq \varphi^2$ and $5 \geq \varphi^3$. ■

The next result elicits a key fact. It considers a colored rooted ternary tree $T$ rooted at $v$, consisting of a "long" path $P_{\tilde{v},v}$ starting at $v$, at whose other extreme $\tilde{v}$ another colored ternary tree $\tilde{T}$ is attached. If $\mathbf{v}$ (respectively $\tilde{\mathbf{v}}$) is the root vector associated to $T$ (respectively $\tilde{T}$), then the result essentially establishes that the coordinates of $\mathbf{v}$ are lower bounded by the coordinates of $\tilde{\mathbf{v}}$ plus a term that depends on the length of $P_{\tilde{v},v}$. Hence, if the coordinates of the root vector $\tilde{\mathbf{v}}$ are large and the path $P_{\tilde{v},v}$ is relatively long, the coordinates of the root vector $\mathbf{v}$ must be large. Formally, we have the following:

**Lemma 13.** *Let $T = T_v$ be a colored rooted ternary tree, such that $T_v = T_{\tilde{v}} \cup P_{\tilde{v},v}$ where $P_{\tilde{v},v}$ is non-trivial. If $\tilde{\mathbf{v}} \geq \tilde{\mathbf{e}} = (\varphi^{\tilde{e}_s})_{s=0,\ldots,3}$ with $\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \in \mathbb{N}$, then there are $e_0, e_1, e_2, e_3 \in \mathbb{N}$ such that $\mathbf{v} \geq \mathbf{e} = (\varphi^{e_s})_{s=0,\ldots,3}$ and $\Phi(\mathbf{e}) \geq \Phi(\tilde{\mathbf{e}}) + \|P_{\tilde{v},v}\|$.*

**Proof.** It is enough to prove the result for $P_{\tilde{v},v}$ of length 1. By Rule 1, we get that $\mathbf{v} \in [\tilde{\mathbf{v}}]$ where $\tilde{\mathbf{v}} \geq \tilde{\mathbf{e}}$ for some $\tilde{\mathbf{e}} \in [\![(\varphi^{\tilde{e}_1}, \varphi^{\tilde{e}_1} + \varphi^{\tilde{e}_0}, \varphi^{\tilde{e}_3}, \varphi^{\tilde{e}_2})^t]\!]$. Assume $\tilde{\mathbf{e}}$ is the vector within the double brackets (the other cases are similar). We now consider several scenarios:

- **Case** $\tilde{e}_1 > \tilde{e}_0 + 1$: Clearly, $\mathbf{v} \geq \mathbf{e} = (\varphi^{\tilde{e}_1}, \varphi^{\tilde{e}_1}, \varphi^{\tilde{e}_3}, \varphi^{\tilde{e}_2})^t$. Moreover, $\Psi(\mathbf{e}) = \Psi(\tilde{\mathbf{e}})$ and $|\{s \mid e_s > e_0\}| \leq \left|\{s \mid \tilde{e}_s > \tilde{e}_0\}\right| - 1$. Hence, $\Phi(\mathbf{e}) \geq \Phi(\tilde{\mathbf{e}}) + 1$.
- **Case** $\tilde{e}_1 \in \{\tilde{e}_0, \tilde{e}_0 + 1\}$: If $\tilde{e}_1 = \tilde{e}_0$, then $\varphi^{\tilde{e}_1} + \varphi^{\tilde{e}_0} = 2\varphi^{\tilde{e}_1} \geq \varphi^{\tilde{e}_1+1}$. Since $1 + \varphi = \varphi^2$, if $\tilde{e}_1 = \tilde{e}_0 + 1$, then $\varphi^{\tilde{e}_1} + \varphi^{\tilde{e}_0} = \varphi^{\tilde{e}_1+1}$. Hence, $\mathbf{v} \geq \mathbf{e} = (\varphi^{\tilde{e}_1}, \varphi^{\tilde{e}_1+1}, \varphi^{\tilde{e}_3}, \varphi^{\tilde{e}_2})^t$. Moreover, $\Psi(\mathbf{e}) = \Psi(\tilde{\mathbf{e}}) + 2$ and $|\{s \mid e_s > e_0\}| \leq \left|\{s \mid \tilde{e}_s > \tilde{e}_0\}\right| + 1$. Hence, $\Phi(\mathbf{e}) \geq \Phi(\tilde{\mathbf{e}}) + 1$.
- **Case** $\tilde{e}_1 \leq \tilde{e}_0 - 1$: Since $1 + \varphi = \varphi^2$, if $\tilde{e}_1 = \tilde{e}_0 - 1$, then $\varphi^{\tilde{e}_1} + \varphi^{\tilde{e}_0} = \varphi^{\tilde{e}_1+2}$. If $\tilde{e}_1 \leq \tilde{e}_0 - 2$, then $\varphi^{\tilde{e}_1} + \varphi^{\tilde{e}_0} \geq \varphi^{\tilde{e}_1+2}$. Hence, $\mathbf{v} \geq \mathbf{e} = (\varphi^{\tilde{e}_1}, \varphi^{\tilde{e}_1+2}, \varphi^{\tilde{e}_3}, \varphi^{\tilde{e}_2})^t$. Moreover, $\Psi(\mathbf{e}) = \Psi(\tilde{\mathbf{e}}) + 4$ and $|\{s \mid e_s > e_0\}| \leq \left|\{s \mid \tilde{e}_s > \tilde{e}_0\}\right| + 3$. Hence, $\Phi(\mathbf{e}) \geq \Phi(\tilde{\mathbf{e}}) + 1$. ■

The following result is an immediate consequence of Lemma 13.

**Corollary 14.** *Let $T = T_v$ be a colored rooted ternary tree, such that $T_v = T_{\tilde{v}} \cup P_{\tilde{v},v}$ where $P_{\tilde{v},v}$ is non-trivial. If $\tilde{\mathbf{v}} \geq \tilde{\mathbf{e}}$, then an $\mathbf{e}$ exists such that $\mathbf{v} \geq \mathbf{e}$ and*

$$\Psi(\mathbf{e}) \geq \Psi(\tilde{\mathbf{e}}) + \max\{\|P_{\tilde{v},v}\| - 3, 0\}.$$

The next two results will be helpful for handling the situation where a colored ternary tree $T$ rooted at $v$ has two or three children, where in addition colored ternary trees are rooted at each of $v$'s children. The results essentially show how to lower bound the coordinates of the root vector of $T$ in terms of the root vectors of the sub-ternary trees attached to each of $v$'s children. The lower bounds are relatively strong, except for some situations where the sub-trees rooted at $v$'s children are of very different sizes.

**Lemma 15.** *Let $T_v$ be a colored rooted ternary tree, such that $v$ has two children $w$ and $u$. If $\mathbf{w} \geq \mathbf{e}^w = (\varphi^{e^w_s})_{s=0,\ldots,3}$, with $e^w_0, e^w_1, e^w_2, e^w_3 \in \mathbb{N}$ and $\mathbf{u} \geq \mathbf{e}^u = (\varphi^{e^u_s})_{s=0,\ldots,3}$, with $e^u_0, e^u_1, e^u_2, e^u_3 \in \mathbb{N}$, then there are $e_0, e_1, e_2, e_3 \in \mathbb{N}$ such that $\mathbf{v} \geq \mathbf{e} = (\varphi^{e_s})_{s=0,\ldots,3}$ and $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u)$.*

**Proof.** Since $\mathbf{w} \geq \mathbf{e}^w$ and $\mathbf{u} \geq \mathbf{e}^u$, by Rule 2 we have that $\mathbf{v} \geq \tilde{\mathbf{v}}$ where

$$\tilde{\mathbf{v}} \in \left\{ \begin{pmatrix} \varphi^{e^w_1+e^u_1} \\ \varphi^{e^w_0+e^u_2} + \varphi^{e^w_1+e^u_3} \\ \varphi^{e^w_3+e^u_2} \\ \varphi^{e^w_3+e^u_0} + \varphi^{e^w_2+e^u_1} \end{pmatrix}, \begin{pmatrix} \varphi^{e^w_1+e^u_1} \\ \varphi^{e^w_3+e^u_2} \\ \varphi^{e^w_2+e^u_1} + \varphi^{e^w_3+e^u_0} \\ \varphi^{e^w_0+e^u_2} + \varphi^{e^w_1+e^u_3} \end{pmatrix}, \begin{pmatrix} \varphi^{e^w_1+e^u_1} \\ \varphi^{e^w_3+e^u_0} + \varphi^{e^w_2+e^u_1} \\ \varphi^{e^w_0+e^u_2} + \varphi^{e^w_1+e^u_3} \\ \varphi^{e^w_3+e^u_2} \end{pmatrix} \right\}.$$

Moreover, $\varphi^{e^w_0+e^u_2} + \varphi^{e^w_1+e^u_3} \geq \varphi^{e^w_1+e^u_3}$ and $\varphi^{e^w_2+e^u_1} + \varphi^{e^w_3+e^u_0} \geq \varphi^{e^w_2+e^u_1}$, so depending on the value of $\tilde{\mathbf{v}}$ we can take

$$\mathbf{e} \in \left\{ \begin{pmatrix} \varphi^{e^w_1+e^u_1} \\ \varphi^{e^w_1+e^u_3} \\ \varphi^{e^w_3+e^u_2} \\ \varphi^{e^w_2+e^u_1} \end{pmatrix}, \begin{pmatrix} \varphi^{e^w_1+e^u_1} \\ \varphi^{e^w_3+e^u_2} \\ \varphi^{e^w_2+e^u_1} \\ \varphi^{e^w_1+e^u_3} \end{pmatrix}, \begin{pmatrix} \varphi^{e^w_1+e^u_1} \\ \varphi^{e^w_2+e^u_1} \\ \varphi^{e^w_1+e^u_3} \\ \varphi^{e^w_3+e^u_2} \end{pmatrix} \right\},$$

and obtain that $\mathbf{v} \geq \mathbf{e}$ and $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u)$. ■

**Lemma 16.** *Let $T_v$ be a colored rooted ternary tree, such that $v$ has three children $w$, $u$ and $z$. If $\mathbf{w} \geq \mathbf{e}^w = (\varphi^{e^w_s})_{s=0,\ldots,3}$ with $e^w_0, e^w_1, e^w_2, e^w_3 \in \mathbb{N}$, $\mathbf{u} \geq \mathbf{e}^u = (\varphi^{e^u_s})_{s=0,\ldots,3}$ with $e^u_0, e^u_1, e^u_2, e^u_3 \in \mathbb{N}$ and $\mathbf{z} \geq \mathbf{e}^z = (\varphi^{e^z_s})_{s=0,\ldots,3}$ with $e^z_0, e^z_1, e^z_2, e^z_3 \in \mathbb{N}$, then there are $e_0, e_1, e_2, e_3 \in \mathbb{N}$ such that $\mathbf{v} \geq \mathbf{e} = (\varphi^{e_s})_{s=0,\ldots,3}$ and $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u) + \Psi(\mathbf{e}^z)$.*

**Proof.** By Rule 3 we have that

$$\mathbf{v} = \begin{pmatrix} \varphi^{e_0^w+e_0^u+e_0^z} + \varphi^{e_1^w+e_1^u+e_1^z} \\ \varphi^{e_0^w+e_3^u+e_2^z} + \varphi^{e_1^w+e_3^u+e_2^z} \\ \varphi^{e_2^w+e_3^u+e_0^z} + \varphi^{e_3^w+e_2^u+e_1^z} \\ \varphi^{e_3^w+e_0^u+e_2^z} + \varphi^{e_2^w+e_1^u+e_3^z} \end{pmatrix} \geq \begin{pmatrix} \varphi^{e_1^w+e_1^u+e_1^z} \\ \varphi^{e_1^w+e_3^u+e_2^z} \\ \varphi^{e_3^w+e_2^u+e_1^z} \\ \varphi^{e_2^w+e_1^u+e_3^z} \end{pmatrix}.$$

Let $\mathbf{e}$ be the last vector in the preceding expression and note that $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u) + \Psi(\mathbf{e}^z)$. ∎

Informally, this section's result show that the coordinates of the root vector of a colored rooted ternary tree $T$ associated to a stack triangulation is large (relative to the size of the tree), provided: (i) the tree is relatively balanced, or (ii) it contains many large paths. Unfortunately, the two latter situations do not cover all possible cases. Indeed, $T$ might possibly be large, relatively imbalanced, and not contain large paths. This situation is where the notion of remainder free becomes handy. In the next section we make precise how the notion is useful.

### *5.3. Main lemma*

The main result of this work, i.e. Theorem 1, will follow almost directly from the results established in the preceding section and the next key claim which roughly says that the root vector $\mathbf{v}$ of a colored rooted ternary remainder-free tree $T = T_v$ either has large coordinates relative to the size of $T_v$, or $T_v$ corresponds to a *short* path $P_{\tilde{v},v}$ and a tree $T_{\tilde{v}}$ whose root vertex $\tilde{v}$ has large coordinates relative to the size of $T_{\tilde{v}}$.

**Lemma 17.** *Let $T = T_v$ be a colored rooted ternary remainder-free tree such that $|T| \geq 4$. Then, there is a path $P_{\tilde{v},v}$ such that $T_v = T_{\tilde{v}} \cup P_{\tilde{v},v}$ with $0 \leq \|P_{\tilde{v},v}\| \leq 5$ (if $\|P_{\tilde{v},v}\| = 0$, then $\tilde{v} = v$ and $T_{\tilde{v}} = T_v$) and there are $e_0^{\tilde{v}}, e_1^{\tilde{v}}, e_2^{\tilde{v}}, e_3^{\tilde{v}} \in \mathbb{N}$ such that*

$$\tilde{\mathbf{v}} \geq \mathbf{e}^{\tilde{v}} = (\varphi^{e_s^{\tilde{v}}})_{s=0,\ldots,3}, \quad \textit{and} \quad \Psi(\mathbf{e}^{\tilde{v}}) \geq \frac{|T_{\tilde{v}}|+7}{2}. \tag{3}$$

**Proof.** We proceed by induction on $|T|$. For the base case $|T_v| = 4$, by Proposition 11, there exists an $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) \geq 6 > (|T_v|+7)/2$. Let $T_v$ be a colored rooted ternary tree remainder-free with $|T_v| \geq 5$. We separate the proof in three cases depending on the number of children of the root $v$. It is clear that for any node $u$ of $T_v$, the tree $T_u$ is a colored rooted ternary remainder-free tree.

**Case 1** ($v$ has one child $w$): We have $|T_w| = |T_v| - 1 \geq 4$. By induction $T_w = T_{\tilde{w}} \cup P_{\tilde{w},w}$ with $0 \leq \|P_{\tilde{w},w}\| \leq 5$ and $\tilde{\mathbf{w}}$ satisfying (3). If $\|P_{w,\tilde{w}}\| < 5$, then $T_v = T_{\tilde{w}} \cup P_{\tilde{w},v}$ and thus it satisfies the desired property. By Corollary 14, we know that there is and $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) \geq \Psi(\mathbf{e}^{\tilde{w}}) + \|P_{\tilde{w},v}\| - 3$. Given that $|T_v| = |T_{\tilde{w}}| + \|P_{w,\tilde{w}}\| + 1$, if $\|P_{w,\tilde{w}}\| = 5$, then there is an $\mathbf{e} \leq \mathbf{v}$ such that

$$\Psi(\mathbf{e}) \geq \Psi(\mathbf{e}^{\tilde{w}}) + \|P_{\tilde{w},v}\| - 3 \geq \frac{|T_{\tilde{w}}|+7}{2} + \|P_{\tilde{w},v}\| - 3 = \frac{|T_v|+7}{2}.$$

Therefore, $T_v$ satisfies the desired property.

**Case 2** ($v$ *has two children* $w$ *and* $u$): First, note that $T_w$ and $T_u$ have size at least 2 (otherwise we would have, say $|T_w| = 1$ and $|T_u| = |T_v| - |T_w| - 1 \geq 3$, implying that $w$ is a remainder of $T$, a contradiction). If $|T_w| = 2$, then either $|T_u| = 2$ or $|T_u| \geq 3$. If $|T_w| = |T_u| = 2$, then by Proposition 11 we have that there is an $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) = 8 > (|T_v|+7)/2$.

In the case that $|T_w| = 2$ and $|T_u| \geq 3$, we have that $S = V(T_w)$ is a potential remainder of $T$ (otherwise, $S$ would be a remainder of $T$, a contradiction). Let $P_{u,\tilde{u}}$ be the path of Case II of Definition 6 associated to the potential remainder $S$. Then, $T_u = T_{\tilde{u}} \cup P_{u,\tilde{u}}$ and $\|P_{u,\tilde{u}}\| \notin \{2, 3, 4\}$, where the node $\tilde{u}$ does not have exactly one child. By induction, $\tilde{\mathbf{u}}$ satisfies (3), and then by Proposition 11, we have that there is an $\mathbf{e}^w \leq \mathbf{w}$ such that $\Psi(\mathbf{e}^w) = 2$. By Corollary 14 and Lemma 15, we know that there is an $\mathbf{e} \leq \mathbf{v}$ such that

$$\Psi(\mathbf{e}) \geq 2 + \Psi(\mathbf{e}^{\tilde{u}}) + \max\{\|P_{u,\tilde{u}}\| - 3, 0\} \geq 2 + \frac{|T_{\tilde{u}}|+7}{2} + \max\{\|P_{u,\tilde{u}}\| - 3, 0\}.$$

If $\|P_{u,\tilde{u}}\| \leq 1$, then $|T_v| \leq |T_{\tilde{u}}| + 4$ and hence, $\Psi(\mathbf{e}) \geq (|T_v|+7)/2$. If not, $\|P_{u,\tilde{u}}\| \geq 5$ and then $\max\{\|P_{u,\tilde{u}}\| - 3, 0\} = \|P_{u,\tilde{u}}\| - 3$. Since $|T_v| = |T_{\tilde{u}}| + 3 + \|P_{u,\tilde{u}}\|$, the desired result follows.

Hence, we assume that $|T_w|, |T_u| \geq 3$. If $|T_w| = |T_u| = 3$, by Proposition 11 and Lemma 15, we get that there is an $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) = 8 > (|T_v|+7)/2$.

We now assume that $|T_w| = 3$ and $|T_u| \geq 4$. By induction, $T_u = T_{\tilde{u}} \cup P_{\tilde{u},u}$ with $0 \leq \|P_{\tilde{u},u}\| \leq 5$ and $\tilde{\mathbf{u}}$ satisfying (3). By Lemma 15, there is an $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u)$. By Proposition 11, Corollary 14, and the fact that $|T_v| = |T_{\tilde{u}}| + \|P_{\tilde{u},u}\| + 4$,

$$\begin{aligned}\Psi(\mathbf{e}) &\geq \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^{\tilde{u}}) + \max\{\|P_{\tilde{u},u}\| - 3, 0\} \\ &\geq 4 + \frac{|T_{\tilde{u}}|+7}{2} + \max\{\|P_{\tilde{u},u}\| - 3, 0\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{|T_v|+7}{2} + \frac{4-\|P_{\tilde{u},u}\|}{2} + \max\{\|P_{\tilde{u},u}\|-3,0\} \\
&\geq \frac{|T_v|+7}{2} + \frac{1}{2}\max\{3-\|P_{\tilde{u},u}\|, \|P_{\tilde{u},u}\|-3,0\} \\
&\geq \frac{|T_v|+7}{2}.
\end{aligned}
$$

Hence, $T_v$ satisfies the desired property.

Finally, we assume that $|T_w|, |T_u| \geq 4$. By induction, $T_w = T_{\tilde{w}} \cup P_{\tilde{w},w}$ and $T_u = T_{\tilde{u}} \cup P_{\tilde{u},u}$ where $0 \leq \|P_{\tilde{w},w}\|, \|P_{\tilde{u},u}\| \leq 5$ and $\tilde{u}$, $\tilde{w}$ satisfying (3). By Lemma 15, there is an $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u)$. By Corollary 14 and given that $|T_v| = |T_{\tilde{w}}| + |T_{\tilde{u}}| + \|P_{\tilde{w},w}\| + \|P_{\tilde{u},u}\| + 1$,

$$
\begin{aligned}
\Psi(\mathbf{e}) &\geq \Psi(\mathbf{e}^{\tilde{w}}) + \Psi(\mathbf{e}^{\tilde{u}}) + \max\{\|P_{\tilde{w},w}\|-3,0\} + \max\{\|P_{\tilde{u},u}\|-3,0\} \\
&\geq \frac{|T_{\tilde{w}}|+7}{2} + \frac{|T_{\tilde{u}}|+7}{2} + \max\{\|P_{\tilde{w},w}\|-3,0\} + \max\{\|P_{\tilde{u},u}\|-3,0\} \\
&= \frac{|T_v|+7}{2} + \frac{1}{2}\max\{\|P_{\tilde{w},w}\|-3, 3-\|P_{\tilde{w},w}\|\} + \frac{1}{2}\max\{\|P_{\tilde{u},u}\|-3, 3-\|P_{\tilde{u},u}\|\} \\
&\geq \frac{|T_v|+7}{2}.
\end{aligned}
$$

Hence, $T_v$ satisfies the desired property.

**Case 3** (*$v$ has three children $w$, $u$ and $z$*): Since $|T_v| \geq 5$, it cannot happen that $|T_w| = |T_u| = |T_z| = 1$. If $1 \leq |T_w| \leq 3$, $1 \leq |T_u| \leq 3$ and $2 \leq |T_z| \leq 3$, we have that: if $|T_z| = 2$, then $|T_v| \leq 9$ and by the first statement of Proposition 12 there is a vector $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) \geq 8 = 16/2 \geq (|T_v|+7)/2$; if $|T_z| = 3$, then $|T_v| \leq 10$ and by the second statement of Proposition 12 there is a vector $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) \geq 10 > 17/2 \geq (|T_v|+7)/2$. Therefore, $T_v$ satisfies the desired property.

We now assume that at least one of the children of $v$ induces a sub-tree with at least 4 vertices.

- If $1 \leq |T_w|, |T_u| \leq 2$ and $|T_z| \geq 4$, then by Rules 0, 1 and 3, we have

$$
\mathbf{v} \geq \begin{pmatrix} \varphi^{e_0^z} + \varphi^{e_1^z} \\ \varphi^{e_3^z} + \varphi^{e_2^z} \\ \varphi^{e_0^z} + \varphi^{e_1^z} \\ \varphi^{e_2^z} + \varphi^{e_3^z} \end{pmatrix}, \quad \text{or} \quad \mathbf{v} \geq \begin{pmatrix} \varphi^{e_0^z} + \varphi^{e_1^z} \\ \varphi^{e_2^z} + \varphi^{e_3^z} \\ \varphi^{e_3^z} + \varphi^{e_2^z} \\ \varphi^{e_0^z} + \varphi^{e_1^z} \end{pmatrix}, \quad \text{or} \quad \mathbf{v} \geq \begin{pmatrix} \varphi^{e_0^z} + \varphi^{e_1^z} \\ \varphi^{e_0^z} + \varphi^{e_1^z} \\ \varphi^{e_2^z} + \varphi^{e_3^z} \\ \varphi^{e_3^z} + \varphi^{e_2^z} \end{pmatrix}.
$$

If $e_3^z = e_2^z$, given that $2 > \varphi$, we may choose the vector $\mathbf{e}$ from the set

$$
\left\{ \begin{pmatrix} \varphi^{e_1^z} \\ \varphi^{e_3^z+1} \\ \varphi^{e_1^z} \\ \varphi^{e_2^z+1} \end{pmatrix}, \begin{pmatrix} \varphi^{e_1^z} \\ \varphi^{e_3^z+1} \\ \varphi^{e_2^z+1} \\ \varphi^{e_1^z} \end{pmatrix}, \begin{pmatrix} \varphi^{e_1^z} \\ \varphi^{e_1^z} \\ \varphi^{e_2^z+1} \\ \varphi^{e_3^z+1} \end{pmatrix} \right\}.
$$

If not, we have $e_3^z \geq e_2^z + 1$ (analogously $e_2^z \geq e_3^z + 1$) and given that $\varphi + 1 = \varphi^2$, we may choose the vector $\mathbf{e}$ from the set

$$
\left\{ \begin{pmatrix} \varphi^{e_1^z} \\ \varphi^{e_2^z+2} \\ \varphi^{e_1^z} \\ \varphi^{e_3^z} \end{pmatrix}, \begin{pmatrix} \varphi^{e_1^z} \\ \varphi^{e_3^z} \\ \varphi^{e_2^z+2} \\ \varphi^{e_1^z} \end{pmatrix}, \begin{pmatrix} \varphi^{e_1^z} \\ \varphi^{e_1^z} \\ \varphi^{e_3^z} \\ \varphi^{e_2^z+2} \end{pmatrix} \right\}.
$$

Therefore, for any choice of $\mathbf{e}$ we get that $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^z) + 4$. By induction, $T_z = T_{\tilde{z}} \cup P_{\tilde{z},z}$ with $0 \leq \|P_{\tilde{z},z}\| \leq 5$ and $\tilde{\mathbf{z}}$ satisfying (3). Since $|T_v| \leq |T_{\tilde{z}}| + \|P_{\tilde{z},z}\| + 5$, by Corollary 14,

$$
\begin{aligned}
\Psi(\mathbf{e}) &\geq 4 + \Psi(\mathbf{e}^{\tilde{z}}) + \max\{\|P_{\tilde{z},z}\|-3,0\} \\
&\geq \frac{|T_{\tilde{z}}|+7}{2} + 4 + \max\{\|P_{\tilde{z},z}\|-3,0\} \\
&\geq \frac{|T_v|+7}{2} + \frac{3-\|P_{\tilde{z},z}\|}{2} + \max\{\|P_{\tilde{z},z}\|-3,0\} \\
&= \frac{|T_v|+7}{2} + \frac{1}{2}\max\{3-\|P_{\tilde{z},z}\|, \|P_{\tilde{z},z}\|-3\} \\
&\geq \frac{|T_v|+7}{2}.
\end{aligned}
$$

Hence, $T_v$ satisfies the desired property.

- If $2 \leq |T_w|, |T_u| \leq 3$ and $|T_z| \geq 4$. By Lemma 16, there is an $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u) + \Psi(\mathbf{e}^z)$. By Proposition 11, we have $\Psi(\mathbf{e}^w) = 2(|T_w| - 1)$ and $\Psi(\mathbf{e}^u) = 2(|T_u| - 1)$. By induction, $T_z = T_{\tilde{z}} \cup P_{\tilde{z},z}$ with $0 \leq \|P_{\tilde{z},z}\| \leq 5$ and $\tilde{\mathbf{z}}$ satisfying (3). Since $|T_v| = |T_w| + |T_u| + |T_{\tilde{z}}| + \|P_{\tilde{z},z}\| + 1$, by Corollary 14,
$$\begin{aligned}
\Psi(\mathbf{e}) &\geq 2(|T_w| - 1) + 2(|T_u| - 1) + \Psi(\mathbf{e}^{\tilde{z}}) + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \\
&\geq \frac{|T_{\tilde{z}}| + 7}{2} + 2(|T_w| + |T_u|) - 4 + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \\
&= \frac{|T_v| + 7}{2} + \frac{3}{2}(|T_w| + |T_u|) - \frac{\|P_{\tilde{z},z}\|}{2} - \frac{9}{2} + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \\
&\geq \frac{|T_v| + 7}{2} + \frac{3 - \|P_{\tilde{z},z}\|}{2} + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \\
&= \frac{|T_v| + 7}{2} + \frac{1}{2}\max\{3 - \|P_{\tilde{z},z}\|, \|P_{\tilde{z},z}\| - 3\} \\
&\geq \frac{|T_v| + 7}{2}.
\end{aligned}$$
Hence, $T_v$ satisfies the desired property.
- The case $|T_w| = 1, |T_u| = 3$, and $|T_z| \geq 4$. By Lemma 16, there is an $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u) + \Psi(\mathbf{e}^z)$. By Proposition 11, we have $\Psi(\mathbf{e}^u) = 4$. By induction, $T_z = T_{\tilde{z}} \cup P_{\tilde{z},z}$ with $0 \leq \|P_{\tilde{z},z}\| \leq 5$ and $\tilde{\mathbf{z}}$ satisfying (3). Since $|T_v| = |T_{\tilde{z}}| + \|P_{\tilde{u},u}\| + 5$, by Corollary 14,
$$\begin{aligned}
\Psi(\mathbf{e}) &\geq 4 + \frac{|T_{\tilde{z}}| + 7}{2} + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \\
&= \frac{|T_v| + 7}{2} + \frac{3 - \|P_{\tilde{u},u}\|}{2} + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \geq \frac{|T_v| + 7}{2}.
\end{aligned}$$
- If $1 \leq |T_w| \leq 3$ and $|T_u|, |T_z| \geq 4$, by Lemma 16, there is an $\mathbf{e} \leq \mathbf{v}$ such that $\Psi(\mathbf{e}) = \Psi(\mathbf{e}^w) + \Psi(\mathbf{e}^u) + \Psi(\mathbf{e}^z)$. By Proposition 11, we have $\Psi(\mathbf{e}^w) \geq 2(|T_w| - 1)$. By induction, $T_u = T_{\tilde{u}} \cup P_{\tilde{u},u}$ and $T_z = T_{\tilde{z}} \cup P_{\tilde{z},z}$ with $0 \leq \|P_{\tilde{u},u}\|, \|P_{\tilde{z},z}\| \leq 5$ and $\tilde{\mathbf{u}}, \tilde{\mathbf{z}}$ satisfying (3). Since $|T_v| = |T_w| + |T_{\tilde{u}}| + |T_{\tilde{z}}| + \|P_{\tilde{u},u}\| + \|P_{\tilde{z},z}\| + 1$, by Corollary 14,
$$\begin{aligned}
\Psi(\mathbf{e}) &\geq 2(|T_w| - 1) + \Psi(\mathbf{e}^{\tilde{u}}) + \Psi(\mathbf{e}^{\tilde{z}}) + \max\{\|P_{\tilde{u},u}\| - 3, 0\} + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \\
&\geq \frac{|T_{\tilde{u}}| + 7}{2} + \frac{|T_{\tilde{z}}| + 7}{2} + 2(|T_w| - 1) + \max\{\|P_{\tilde{u},u}\| - 3, 0\} + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \\
&= \frac{|T_v| + 7}{2} + \frac{3}{2}|T_w| - 2 + \frac{6 - \|P_{\tilde{u},u}\| - \|P_{\tilde{z},z}\|}{2} + \max\{\|P_{\tilde{u},u}\| - 3, 0\} + \max\{\|P_{\tilde{z},z}\| - 3, 0\} \\
&= \frac{|T_v| + 7}{2} + \frac{3}{2}|T_w| - 2 + \frac{1}{2}\max\{3 - \|P_{\tilde{u},u}\|, \|P_{\tilde{u},u}\| - 3\} + \frac{1}{2}\max\{3 - \|P_{\tilde{z},z}\|, \|P_{\tilde{z},z}\| - 3\}.
\end{aligned}$$
If $|T_w| \in \{2, 3\}$, then the last formula is greater than $\frac{|T_v|+7}{2}$. If $|T_w| = 1$, then we have that if $\|P_{\tilde{z},z}\| = 3$, then $\|P_{\tilde{u},u}\| \neq 3$, otherwise $\{w\}$ would be a remainder of $T$. Therefore, $T_v$ satisfies the desired property.
- If $|T_w|, |T_u|, |T_z| \geq 4$. Similar to the preceding case. ■

## 6. Proof of main results

**Proof of Theorem 1.** Recall that $T(\Delta_n)$ is a colored rooted ternary tree on $|\Delta_n| - 3$ nodes such that its root vector $\mathbf{v}$ is equal to the degeneracy vector of $\Delta_n$. By Lemma 10, there exist $P' \subseteq P(T(\Delta_n))$ such that the colored rooted ternary tree $\tilde{T}(\Delta_n) = T(\Delta_n) \setminus V_{P'}$ is remainder-free and $|\tilde{T}(\Delta_n)| \geq \frac{2}{5}|T(\Delta_n)|$. Clearly, the root vector $\tilde{\mathbf{v}}$ of $\tilde{T}(\Delta_n)$ is such that $\mathbf{v} \geq \tilde{\mathbf{v}}$. The Main lemma guarantees that there are $e_0, e_1, e_2, e_3 \in \mathbb{N}$ such that $\tilde{\mathbf{v}} \geq \mathbf{e} = (\varphi^{e_s})_{s=0,..,3}$ and
$$\Psi(\mathbf{e}) \geq \frac{(|\tilde{T}(\Delta_n)| - 5) + 7}{2} = \frac{|\tilde{T}(\Delta_n)| + 2}{2} \geq \frac{|T(\Delta_n)| + 5}{5} = \frac{|\Delta_n| + 2}{5}.$$
Moreover, since $\Delta_n[\phi] = \Delta_n[-\phi]$ for all $\phi \in \{+, -\}^3$. Hence, the groundstate degeneracy of $\Delta_n$ is at least $2\sum_{s=1}^{3} \varphi^{e_s} \geq 6\varphi^{\frac{1}{3}\Psi(\mathbf{e})} \geq 6\varphi^{(|\Delta_n|+2)/15}$. ■

**Proof of Corollary 2.** Let $G$ be a cubic planar graph such that its geometric dual graph is the stack triangulation $\Delta$. We know that the number of perfect matchings of $G$ is equal to half of the groundstate degeneracy of $\Delta$. From Euler's formula we get that $2|\Delta| = |G| + 4$. Therefore, by Theorem 1 we have that the number of perfect matchings of $G$ is at least $3\varphi^{(|G|+8)/30}$. ■

Regarding upper bounds, it is worth pointing out that Cygan et al. [5] built an infinite family of Klee-graphs with at most $O(1) \cdot 2^{|G|/17285} \approx O(1) \cdot \varphi^{|G|/12000}$ perfect matchings.

## 7. Final comments

The approach followed throughout this work seems to be specially well suited for calculating the groundstate degeneracy of triangulations that have some sort of recursive tree like construction, e.g. 3-trees. It would be interesting to identify other such families of triangulations where similar methods allowed to lower bound their groundstate degeneracy. One tempting possibility is to recall that all plane triangulations can be generated sequentially through a sequence of a constant set of rules, and derive an alternative (hopefully tighter) bound than the one of Chudnovsky and Seymour [4] for the class of planar bridgeless cubic graphs. Of particular interest would be to show that the approach we follow in this work can actually be successfully applied to obtain exponential lower bounds for non-trivial families of non-planar bridgeless cubic graphs.

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## Appendix. Proofs of results of Section 4

**Proof** (Proposition 4)**.** Let $\phi \in I$. Note that $\mathbf{v}_{\Delta_n}[\phi]$ equals the sum of the number of satisfying states of $\Delta_n$ when $(v_1, v_2, v_3, v)$ are assigned spins $(\phi, +)$ and $(\phi, -)$. For a given spin assignment to $(v_1, v_2, v_3, v)$, the number of satisfying states of $\Delta_n$, is obtained by multiplying the number of satisfying states of each $\Delta_n^i$ when the spin assignment of its outer faces agree with the fixed spins assigned to $(v_1, v_2, v_3, v)$.

First, consider the case where $\phi = +++$. If $v$'s spin is $+$, then

$$\Delta_n^1[+++] \cdot \Delta_n^2[+++] \cdot \Delta_n^3[+++] = v_1^0 v_2^0 v_3^0.$$

If $v$'s spin is $-$, then

$$\Delta_n^1[++-] \cdot \Delta_n^2[++-] \cdot \Delta_n^3[++-] = v_1^1 v_2^1 v_3^1.$$

Hence, $\Delta_n[\phi] = v_1^0 v_2^0 v_3^0 + v_1^1 v_2^1 v_3^1$.

Now, consider the case where $\phi = ++-$. If $v$'s spin is $+$, then

$$\Delta_n^1[+++] \cdot \Delta_n^2[+-+] \cdot \Delta_n^3[-++] = v_1^0 v_2^2 v_3^3.$$

Recalling that by identity (1) we have that $\Delta_n^2[+--] = \Delta_n^2[-++]$ and $\Delta_n^3[-+-] = \Delta_n^3[+-+]$, if $v$'s spin is $-$, then

$$\Delta_n^1[++-] \cdot \Delta_n^2[+--] \cdot \Delta_n^3[-+-] = v_1^1 v_2^3 v_3^2.$$

Hence, $\Delta_n[\phi] = v_1^0 v_2^2 v_3^3 + v_1^1 v_2^3 v_3^2$.

The other two remaining cases, where $\phi$ equals $+-+$ and $-++$, can be similarly dealt with and left to the interested reader. ∎

**Proof** (Lemma 6)**.** By induction on $n$. For the base case $n = 1$; the stack triangulation $\Delta_1$ is isomorphic to $K_4$ and $T(\Delta_1)$ is a vertex. It is clear that $\Delta_1[\phi] = 1$ for all $\phi \in I$, and the root vector of $T(\Delta_1)$ is obtained by Rule 0 in Definition 2.

Now, let $\Delta_n$ be a rooted stack triangulation with $n > 1$. We denote by $v$ the root of $T(\Delta_n)$. We separate the proof in cases according to the number of vertices of the rooted stack triangulations $\Delta_{n_i} = \Delta_n^i$ with $i \in \{1, 2, 3\}$. We note that if $n_i = 0$ for every $i \in \{1, 2, 3\}$, then $n = 1$. Thus, we can assume that $n_i \geq 1$ for at least one index $i \in \{1, 2, 3\}$. We now consider three possible situations.

First, assume there are $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$ such that $n_i = n_j = 0$ and $n_k \geq 1$. By definition of the degeneracy vector, we have that $\mathbf{v}_{\Delta_{n_i}} = \mathbf{v}_{\Delta_{n_j}} = (0, 1, 1, 1)^t$. Let $\mathbf{v}_{\Delta_{n_k}} = (v_k^t)_{t \in \{0,1,2,3\}}$. According to Proposition 4, we have that

$$\mathbf{v}_{\Delta_n} \in \left\{ \begin{pmatrix} v_1^1 \\ v_1^0 + v_1^1 \\ v_1^3 \\ v_1^2 \end{pmatrix}, \begin{pmatrix} v_2^1 \\ v_2^3 \\ v_2^2 \\ v_2^0 + v_2^1 \end{pmatrix}, \begin{pmatrix} v_3^1 \\ v_3^2 \\ v_3^0 + v_3^1 \\ v_3^3 \end{pmatrix} \right\},$$

where $\mathbf{v}_{\Delta_n}$ is the $k$th vector in the set above. Item 2 of Remark 3 says that $T(\Delta_{n_k})$ is labeled by $k$ and rooted on $w$, where $w$ is the unique child of $v$. Given that $1 \leq n_k < n$, by induction we get that $\mathbf{w} = \mathbf{v}_{\Delta_{n_k}}$. By Definition 2, we know that $\mathbf{v}$ is obtained from $\mathbf{w}$ by application of Rule 1. Hence, $\mathbf{v} = \mathbf{v}_{\Delta_n}$.

Assume now that there is an $i \in \{1, 2, 3\}$ such that $n_i = 0$ and $j, k \in \{1, 2, 3\} \setminus \{i\}$ with $j \neq k$ such that $n_j, n_k \geq 1$. We have that $\mathbf{v}_{\Delta_{n_i}} = (0, 1, 1, 1)^t$. Consider $\mathbf{v}_{\Delta_{n_j}} = (v_j^t)_{t\in\{0,1,2,3\}}$ and $\mathbf{v}_{\Delta_{n_k}} = (v_k^t)_{t\in\{0,1,2,3\}}$. Proposition 4 implies that

$$\mathbf{v}_{\Delta_n} \in \left\{ \begin{pmatrix} v_2^1 v_3^1 \\ v_2^3 v_3^2 \\ v_2^3 v_3^0 + v_2^2 v_3^1 \\ v_2^0 v_3^2 + v_2^1 v_3^3 \end{pmatrix}, \begin{pmatrix} v_1^1 v_3^1 \\ v_1^0 v_3^3 + v_1^1 v_3^2 \\ v_1^2 v_3^0 + v_1^3 v_3^1 \\ v_1^2 v_3^3 \end{pmatrix}, \begin{pmatrix} v_1^1 v_2^1 \\ v_1^0 v_2^2 + v_1^1 v_2^3 \\ v_1^3 v_2^2 \\ v_1^3 v_2^0 + v_1^2 v_2^1 \end{pmatrix} \right\},$$

where $\mathbf{v}_{\Delta_n}$ is the $i$th vector in the set above. Item 3 of Remark 3 guarantees that the root $v$ of $T(\Delta)$ has exactly two children $w$ and $u$ labeled $j$ and $k$, respectively. Moreover, $T(\Delta_n^j)$ and $T(\Delta_n^k)$ are rooted on $w$ and $u$, respectively. We know that $1 \leq n_j < n$ and $1 \leq n_k < n$, then by induction, $\mathbf{w} = \mathbf{v}_{\Delta_{n_j}}$ and $\mathbf{u} = \mathbf{v}_{\Delta_{n_k}}$. If we now apply Rule 2 of Definition 2, we get $\mathbf{v} = \mathbf{v}_{\Delta_n}$.

Finally, assume that $n > n_j \geq 1$ for every $j \in \{1, 2, 3\}$. Suppose that $\mathbf{v}_{\Delta_{n_j}} = (v_j^t)_{t\in\{0,1,2,3\}}$ for each $j \in \{1, 2, 3\}$. Item 4 of Remark 3 and the induction hypothesis imply that the root $v$ of $T(\Delta)$ has three children $w_1$, $w_2$ and $w_3$ such that $\mathbf{w}_j = \mathbf{v}_{\Delta_{n_j}}$ for each $j \in \{1, 2, 3\}$. By Proposition 4 and since $\mathbf{v}$ is derived by applying Rule 3 of Definition 2, the desired conclusion follows. ∎

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