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STEADY STATE ANALYSIS FOR A RELAXED CROSS DIFFUSION MODEL

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Abstract. In this article we study the existence the existence of nonconstant steady state solutions for the following relaxed cross-diffusion system

$$\begin{cases} \partial_t u - \Delta[a(\tilde{v})u] = 0, & \text{in } (0,\infty)\times\Omega, \\ \partial_t v - \Delta[b(\tilde{u})v] = 0, & \text{in } (0,\infty)\times\Omega, \\ -\delta\Delta\tilde{u} + \tilde{u} = u, & \text{in } \Omega, \\ -\delta\Delta\tilde{v} + \tilde{v} = v, & \text{in } \Omega, \\ \partial_n u = \partial_n v = \partial\tilde{u} = \partial_n\tilde{u} = 0, & \text{on } (0,\infty)\times\partial\Omega, \end{cases}$$

with Ω a bounded smooth domain, n the outer unit normal to $\partial\Omega$, $\delta > 0$ denotes the relaxation parameter. The functions $a(\tilde{v})$, $b(\tilde{u})$ account for nonlinear cross-diffusion, being $a(\tilde{v}) = 1 + \tilde{v}^\gamma$, $b(\tilde{u}) = 1 + \tilde{u}^\eta$ with $\gamma, \eta > 1$ a model example. We give conditions for the stability of constant steady state solutions and we prove that under suitable conditions Turing patterns arise considering δ as a bifurcation parameter.

1. Introduction.

1.1. Cross diffusion models and segregation patterns.

The mechanism of cross diffusion has been introduced by Shigesada Kawasaki and Terramoto in [13] to model the trend of a species to avoid another one and thereby, possibly segregate. In this pioneer paper, cross diffusion depends linearly on population density. For instance, if we consider a two species system it may take the following form:

$$\begin{cases} \partial_t u - \Delta[(d_1 + a_{11}u + a_{12}v)u] = r_1(u,v), & \text{in } Q_T, \\ \partial_t v - \Delta[(d_2 + a_{21}u + a_{22}v)v] = r_2(u,v), & \text{in } Q_T, \\ +\text{boundary conditions.} \end{cases} \quad (1)$$

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where $Q_T = (0,T) \times \Omega$ with Ω a bounded smooth open set of $\mathbb{R}^d$, $0 < T \leqslant \infty$, and $r_1(u,v), r_2(u,v)$ are the reaction terms, In this model diffusion pressure acts on three different levels: constant diffusion $d_1,\ d_2$, self diffusion $a_{11},\ a_{22}$ and cross diffusion a_{12}, a_{21}. System (1) and its steady states have been widely studied throughout the literature. Local existence theorems can be found in [1], global existence can be found for instance in [7, 14] for classical solutions with conditions on the coefficients and in [3] for weak solutions. A derivation of a triangular version of this system, in which only one of the species is subjected to cross diffusion pressure, from (fast) reaction-diffusion systems can be found in [8, 5]. Investigation regarding the segregating effect of cross diffusion using the Turing approach has been done in many works, for instance in [10]. In the latter, it is shown that cross diffusion may drive instability of constant steady states which cannot happen through only diffusion.

In [2] the first author and collaborators studied a cross diffusion system in absence of reaction. Particularly, the question was to investigate the possibility of segregating behavior in absence of reaction competition. For the case of (1), with $r_1(u,v), r_2(u,v) \equiv 0$, a negative answer has already been given in [3]. Thus, to generate segregating behavior, one needs to complexify system (1) incorporating nonlinear cross diffusion. In [2], the following system was introduced:

$$\begin{cases} \partial_t u - \Delta[a(\tilde{v})u] = 0, & \text{in } Q_T, \\ \partial_t v - \Delta[b(\tilde{u})v] = 0, & \text{in } Q_T, \\ -\delta\Delta\tilde{u} + \tilde{u} = u, & \text{in } \Omega, \\ -\delta\Delta\tilde{v} + \tilde{v} = v, & \text{in } \Omega, \\ \partial_n u = \partial_n v = \partial\tilde{u} = \partial_n\tilde{u} = 0, & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{2}$$

with $u(\cdot,0) = u_0 \geq 0$, $v(\cdot,0) = v_0 \geq 0$ in Ω. In this system $\delta > 0$ is a relaxation parameter, being $\tilde{u}$, $\tilde{v}$ regularizations (local averages) of u and v. Thus, cross-diffusion consider averages, at δ space scale, of the population densities instead of their local values. A key feature of the system (2) is that the total population is conserved throughout time, and that $u, \tilde{u}$ and $v, \tilde{v}$ have the same average respectively, that is

$$\int_\Omega u = \int_\Omega \tilde{u} = \int_\Omega u^0 \quad \text{and} \quad \int_\Omega v = \int_\Omega \tilde{v} = \int_\Omega v^0. \tag{3}$$

since the system has no reaction term.

We consider the following hypothesis on a and b.

(H0) $a,\ b \in C^1(\mathbb{R}_+)$.

(H1) There exist $\nu > 0,$ such that $\nu^2 \leqslant a(u),\ \nu^2 \leqslant b(u)$.

Existence of solutions for system (2) was studied in[2], particularly a priori bounds and global existence of solutions where established under assumptions (H0), (H1) and

(H2) There exist $\eta < 1,\ K > 0,$ such that $|a'(u)| \leqslant Ka^\eta(u)$ and $|b'(u)| \leqslant Kb^\eta(u)$, for $d = 1, 2$.

We observe that (H0), (H1) and (H2) imply that there exists $C > 0$ and $p > 1$ such that $a(u), b(u) \leqslant C(1 + u^p)$. More recently a more general global well posedness result was obtained by the first author and collaborators in any dimensions [9].

To study the existence of nonconstant steady states of (2) we may start considering whether Turing patterns arise. In [2] such type of patterns are studied and the spatially inhomogeneous steady state solutions of (2) are characterized using numerical computations.

In this article we show that nonconstant steady states bifurcating from constant solutions exists using standard techniques and also we show examples that illustrate the different structure that this branches may have. To study the Turing stability analysis of constant steady states we have to consider the following eigenvalue problem associated to the linearization the system (2) around the constant steady state $(u, v, \tilde{u}, \tilde{v}) = (\bar{u}, \bar{v}, \bar{u}, \bar{v})$:

$$\begin{cases} \Delta(a(\bar{v})\varphi_1 + a'(\bar{v})\bar{u}\tilde{\varphi}_2) = \mu\varphi_1, & \text{in } \Omega, \\ \Delta(b(\bar{u})\varphi_2 + b'(\bar{u})\bar{v}\tilde{\varphi}_1) = \mu\varphi_2, & \text{in } \Omega, \\ -\delta\Delta\tilde{\varphi}_1 + \tilde{\varphi}_1 = \varphi_1 & \text{in } \Omega, \\ -\delta\Delta\tilde{\varphi}_2 + \tilde{\varphi}_2 = \varphi_2 & \text{in } \Omega, \\ \partial_n\varphi_1 = \partial_n\varphi_2 = \partial_n\tilde{\varphi}_1 = \partial_n\tilde{\varphi}_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

It can be easily checked that μ is an eigenvalue of (4) if and only if for some $k \geq 1$ we have

$$(\varphi_1, \varphi_2, \tilde{\varphi}_1, \tilde{\varphi}_2) = e_k \left(\alpha_1^k, \alpha_2^k, \frac{\alpha_1^k}{1 + \delta\lambda_k}, \frac{\alpha_2^k}{1 + \delta\lambda_k} \right),$$

with λ_k, e_k eigenpair of the Laplacian with Neumann boundary condition

$$\Delta e_k + \lambda_k e_k = 0 \text{ in } \Omega, \ \partial_n e_k = 0 \ \text{ on } \partial\Omega. \tag{5}$$

and (α_1^k, α_2^k) is a nontrivial solution of

$$\mu\alpha_1^k + \lambda_k a(\bar{v})\alpha_1^k + \lambda_k a'(\bar{v})\bar{u}\tfrac{\alpha_2^k}{1+\delta\lambda_k} = 0,$$

$$\mu\alpha_2^k + \lambda_k b(\bar{u})\alpha_2^k + \lambda_k b'(\bar{u})\bar{v}\tfrac{\alpha_1^k}{1+\delta\lambda_k} = 0.$$

That is, $-\frac{\mu}{\lambda_k}$ has to be an eigenvalue of

$$M(\delta, \lambda_k) = \begin{pmatrix} a(\bar{v}) & \frac{a'(\bar{v})\bar{u}}{1+\delta\lambda_k} \\ \frac{b'(\bar{u})\bar{v}}{1+\delta\lambda_k} & b(\bar{u}) \end{pmatrix}. \tag{6}$$

Clearly the eigenvalues of $M(\delta, \lambda_k)$ have positive real part, if and only if

$$a(\bar{v})b(\bar{u}) - \frac{a'(\bar{v})b'(\bar{u})\bar{u}\bar{v}}{(1 + \delta\lambda_k)^2} > 0.$$

We observe that this condition holds independently on δ if $a'(\bar{v})b'(\bar{u}) \leqslant 0$, thus all the eigenvalues of (4) have negative real parts. Now, when $a'(\bar{v})b'(\bar{u}) > 0$ the quantity $\det(M(\delta, \lambda_k))$ is a nondecreasing function of k. Therefore, the condition

$$\det(M(\delta, \lambda_1)) > 0$$

is a sufficient condition for the eigenvalues μ of (4) to have negative real part for all $k \geqslant 1$. Under this condition, system (2) is *linearly* stable around $(\bar{u}, \bar{v}, \bar{u}, \bar{v})$. Moreover, the quantity $\det(M(\delta, \lambda_k))$ is also a nonincreasing function of δ. Therefore, if

$$\det M(0, \lambda_1) = a(\bar{v})b(\bar{u}) - a'(\bar{v})b'(\bar{u})\bar{u}\bar{v} \geq 0,$$

then the steady state is always linearly stable. Thus, to have Turing induced unstability one must firstly have

$$a(\bar{v})b(\bar{u}) - a'(\bar{v})b'(\bar{u})\bar{u}\bar{v} < 0. \tag{7}$$

In this case, the function $\det(M(\delta,\lambda_1))$ is increasing from the latter negative quantity to $a(\bar v)b(\bar u)$ and there exists a unique δ_0 such that $\det(M(\delta_0,\lambda_1)) = 0$ that is

$$a(\bar v)b(\bar u) - \frac{\bar u\bar v a'(\bar v)b'(\bar u)}{(1+\delta_0\lambda_1)^2} = 0. \tag{8}$$

Hence, the steady state $(\bar u,\bar v,\bar u,\bar v)$ is linearly unstable for $\delta < \delta_0$, because

$$a(\bar v)b(\bar u) - \frac{a'(\bar v)b'(\bar u)\bar u\bar v}{(1+\delta\lambda_1)^2} < 0, \tag{9}$$

and linearly stable for $\delta > \delta_0$. We will show in Section 2 that indeed linear stability implies nonlinear stability for system (2), when the dimension d is less than 2. The proof will use arguments used for the proof of existence in [2].

As we pointed out above, for (9) to hold we need $a'(\bar v)b'(\bar u) > 0$, which holds when $a'(\bar v), b'(\bar u) < 0$ or $a'(\bar v), b'(\bar u) > 0$. But, from the modeling point of view it makes more sense to consider the latter case where a and b are increasing, that is cross-diffusion pressures increases with population density.

Regarding non constant steady states, we will prove in Section 4 that under suitable conditions a branch of nonconstant steady states bifurcate from $(\bar u,\bar v,\bar u,\bar v)$ when $\delta = \delta_0$. We study the shape of the bifurcation branches in Subsection 4.2 showing examples where the bifurcation is transcritical, subcritical or supercritical. It will be established that for δ large, all the steady states with fixed average are constants. Then, using Rabinowitz's global bifurcation theorem, we have particular situations where system (2) admits at least three nonconstant steady states with the same average.

1.2. **Main results.** We first state that in our model how Turing stability in system (2) leads to nonlinear stability for the constant steady state. This is stated in the following result.

Theorem 1.1. *Suppose the coefficients satisfy (H0)-(H2) and that dimension is* 1 *or* 2*. Suppose* δ*,* $\bar u, \bar v$ *satisfy*

$$a(\bar v)b(\bar u) - \frac{a'(\bar v)b'(\bar u)\bar u\bar v}{(1+\delta\lambda_1)^2} > 0 \tag{10}$$

and $\|u^0-\bar u\|_{L^2(\Omega)}, \|v^0-\bar v\|_{L^2(\Omega)}$ *are small enough, then we have*

$$\|u-\bar u\|_{L^2(\Omega)}, \|v-\bar v\|_{L^2(\Omega)} \xrightarrow[t\to+\infty]{} 0.$$

We observe that (10) holds for δ sufficiently large, whenever $a'(\bar v), b'(\bar u) \neq 0$. Indeed, in this case the constant steady states are always locally asymptotically stable for large values of δ. In Section 3 we will show the next result, which establishes that in this case the system (2) does not admit nonconstant steady states.

Theorem 1.2. *Suppose that the coefficients satisfy (H0)-(H1). Let* $(\bar u,\bar v)$ *be fixed. Then for* δ *large enough, the only equilibrium of* (2) *with average* $(\bar u,\bar v)$ *is the constant solution* $(u\equiv\bar u, v\equiv\bar v)$*.*

As we vary δ from ∞ to 0 the constant equilibrium $(\bar u,\bar v)$ changes its linear stability properties (from stable when δ is large, to unstable when δ is small). At the value δ_0, where the stability changes, a branch of nonconstant steady states arises.

Theorem 1.3. *Suppose that the first nonzero eigenvalue λ_1 of the Neumann Laplacian is simple. Suppose also that, $a,\ b \in C^2(\mathbb{R})$, satisfy (H1) and (7) and let $\delta_0 > 0$ be characterized by (8). Then there exists a family of nonconstant positive stationary solutions of (2) bifurcating from $(\bar{u}, \bar{v})$.*

This result will be proved in Subsection 4.1 using standard bifurcation techniques. In Subsection 4.2 we study the behavior of the bifurcating solutions. The next result establishes situations where the bifurcation is transcritical.

Proposition 1. *Suppose that a, b are $C^3(\mathbb{R})$, that the hypothesis (H0)-(H1) hold and*

$$\int_\Omega e_1^3\, dx \neq 0.$$

Then there exists a nontrivial polynomial $p(a', b', a'', b'', \bar{u}, \bar{v})$, such that if

$$p(a'(\bar{v}), b'(\bar{v}), a''(\bar{v}), b''(\bar{v}), \bar{u}, \bar{v}) \neq 0$$

Then, there exists $\hat{\delta} > 0$ such that the equation (2) admits non trivial steady state solutions for all $\delta \in (-\hat{\delta} + \delta_0, \hat{\delta} + \delta_0)$, where δ_0 is characterized by (8).

The hypothesis of the proposition above do not hold in the case that $\Omega = (0, \pi)$. In this case we show that depending on the parameters, the bifurcation can be either subcritical or supercritical. Moreover in the case that the bifurcation is supercritical (that is the branches exists for values of $\delta > \delta_0$), we have that at least three nonconstant solutions with fixed average exists (Proposition 3).

2. **Stability of constant solutions.**

The goal of this section is to prove Theorem 1.1. We start by proving several lemmas related to the local stability of a constant steady state $(\bar{u}, \bar{v})$ of (2). We will mainly follow the structure of the proof of Proposition 4.2 from [2] and adapt it to the specific question of stability. Particularly, one of the tools is a version of duality estimates introduced in [12, 6] adapted to the conservative case:

Lemma 2.1. *Let Ω be an open smooth subset of $\mathbb{R}^d$. Suppose that $t, x \mapsto a(t, x) > 0$ satisfies*

$$\partial_n a(t, x) = 0,\ \text{for } x \in \partial\Omega,\ t \in (0, T), \tag{11}$$

then, solutions to

$$\begin{cases} \partial_t u(t, x) - \Delta(a(t, x)u(t, x)) = 0, & x \in Q_T, \\ \partial_n (a(t, x)u(t, x)) = 0,\ \text{for } x \in (0, T) \times \partial\Omega, \\ u(0, \cdot) = u^0\ \text{in } \Omega, \end{cases}$$

satisfy for any constant C the following estimate,

$$\left\| \sqrt{a}(u - \bar{u}) \right\|_{L^2(Q_T)} \leqslant \left\| (u^0 - \bar{u}) \right\|_{H^{-1}(\Omega)} + \bar{u} \left\| \frac{a - C}{\sqrt{a}} \right\|_{L^2(Q_T)}. \tag{12}$$

The proof of this lemma is given in the appendix.

2.1. **Weak stability.** From now on we will use the following notations:

$$(w, \tilde{w}, z, \tilde{z}) := (u - \bar{u}, \tilde{u} - \bar{u}, v - \bar{v}, \tilde{v} - \bar{v}).$$

We also introduce the quantities

$$A_r(T) = \sup_{t \leqslant T, x \in \Omega} \left| \frac{a(\tilde{v}) - a(\bar{v})}{\sqrt{a(\tilde{v})}(\tilde{v} - \bar{v})} \right|, \qquad A_l(T) = \inf_{t \leqslant T, x \in \Omega} \sqrt{a(\tilde{v})}, . \tag{13}$$

Similarly, with obvious notations we introduce the quantities B_r, B_l. The utility of these functions is summarized in the following lemma:

Lemma 2.2. *We have the following inequalities:*

$$\begin{aligned} \|w\|_{L^2(Q_T)} \leqslant \quad & \tfrac{1}{\nu}\|w^0\|_{H^{-1}(\Omega)} + \frac{\bar{u}A_r(T)}{\nu A_l(T)(1+\delta_0\lambda_1)}\|z^0\|_{H^{-1}(\Omega)} \\ & + \frac{\bar{u}\bar{v}A_r(T)B_r(T)}{A_l(T)B_l(T)(1+\delta_0\lambda_1)^2}\|w\|_{L^2(Q_T)}, \end{aligned} \tag{14}$$

$$\begin{aligned} \|z\|_{L^2(Q_T)} \leqslant \quad & \tfrac{1}{\nu}\|z^0\|_{H^{-1}(\Omega)} + \frac{\bar{v}B_r(T)}{\nu B_l(T)(1+\delta_0\lambda_1)}\|w^0\|_{H^{-1}(\Omega)} \\ & + \frac{\bar{u}\bar{v}A_r(T)B_r(T)}{A_l(T)B_l(T)(1+\delta_0\lambda_1)^2}\|z\|_{L^2(Q_T)}. \end{aligned} \tag{15}$$

Proof of Lemma 2.2. This is mainly a consequence of the lemma 2.1. Indeed, if we choose in (12) $C = a(\bar{v})$, then, as a consequence of the mean value theorem, we have

$$A_l(T)\|w\|_{L^2(Q_T)} \leqslant \|w^0\|_{H^{-1}} + \bar{u}\left\| A_r(T)|\tilde{z}| \right\|_{L^2(Q_T)},$$

$$\|w\|_{L^2(Q_T)} \leqslant \frac{1}{\nu}\|w^0\|_{H^{-1}} + \bar{u}\frac{A_r(T)}{A_l(T)}\|\tilde{z}\|_{L^2(Q_T)}.$$

From the equation for $\tilde{z}$ we have

$$\|\tilde{z}\|_{L^2(Q_T)} \leqslant \frac{1}{1+\delta\lambda_1}\|z\|_{L^2(Q_T)},$$

where λ_1 denotes the first nonzero eigenvalue of (5). Therefore,

$$\|w\|_{L^2(Q_T)} \leqslant \frac{1}{\nu}\|w^0\|_{H^{-1}} + \frac{\bar{u}}{1+\delta\lambda_1}\frac{A_r(T)}{A_l(T)}\|z\|_{L^2(Q_T)}.$$

Similarly, we prove that

$$\|z\|_{L^2(Q_T)} \leqslant \frac{1}{\nu}\|z^0\|_{H^{-1}} + \frac{\bar{v}}{1+\delta\lambda_1}\frac{B_r(T)}{B_l(T)}\|w\|_{L^2(Q_T)},$$

and combining this two inequalities we obtain the inequalities 14, 15 □

As a first consequence, we have the following result

Lemma 2.3 (Global weak stability)**.** *Suppose that (H0) and (H1) hold, and that there exists C independent of T such that*

$$A_r(T), B_r(T) \leqslant C,$$

with A_r, B_r defined as in (13). Then for $\delta > 0$ big enough, we have that there exists a constant $\bar{C}$ depending on $\bar{u}$, $\bar{v}$ and δ such that

$$\|(w,z)\|_{L^2(Q_T)} \leqslant \bar{C}\|(w^0,z^0)\|_{H^{-1}}.$$

Especially, the constant $\bar{C}$ does not depend on T.

Proof. The proof is a direct application of Lemma 2.1. We observe that by (14) and the hypothesis we have that

$$\|w\|_{L^2(Q_T)} \leqslant \frac{\|w^0\|_{H^{-1}}}{\nu} + \frac{\bar{u}}{1+\delta\lambda_1}\frac{C}{\nu}\|z^0\|_{H^{-1}} + \frac{\bar{u}\bar{v}}{(1+\delta\lambda_1)^2}\frac{A_r B_r}{A_l B_l}\|w\|_{L^2(Q_T)}.$$

Then choosing δ such that

$$\frac{\bar{u}\bar{v}}{(1+\delta\lambda_1)^2}\frac{A_r B_r}{A_l B_l} \leqslant \frac{\bar{u}\bar{v}}{(1+\delta\lambda_1)^2}\frac{C^2}{\nu^2} < 1,$$

we can conclude. □

It is worth giving here a simple example. The lower bound ν^2 on a, b is an hypothesis we always need. The second constraint is satisfied on the following case

$$a = b, \quad a(\tilde{v}) = 1 + \tilde{v}^2, \qquad A_r = \sup \frac{\tilde{v} + \bar{v}}{\sqrt{1+\tilde{v}^2}} \leqslant 1 + \bar{v}, \quad B_r \leqslant 1 + \bar{u}.$$

Now we will study the local stability. The idea is driven by the following argument: if $\tilde{w}, \tilde{z}$ is uniformly close to 0, then A_r, B_r, A_l, B_l are uniformly close respectively to $\frac{a'(\bar{v})}{\sqrt{a(\bar{v})}}, \frac{b'(\bar{u})}{\sqrt{b(\bar{u})}}, \sqrt{a(\bar{v})}, \sqrt{b(\bar{u})}$. If furthermore condition (9) is satisfied, we have then

$$\frac{\bar{u}\bar{v}A_r B_r}{A_l B_l (1+\delta_0\lambda_1)^2} \leqslant \frac{\bar{u}\bar{v}a'(\bar{v})b'(\bar{u})}{a(\bar{v})b(\bar{u})(1+\delta_0\lambda_1)^2} + \text{uniformly small} < 1.$$

This idea will drive the derivation of a time independent bound of $\|(w, z)\|_{L^2(Q_T)}$ as stated in the following lemma.

Lemma 2.4 (Local weak stability)**.** *Suppose that (H0) and (H1) hold and that* $\bar{u}, \bar{v}, \delta$ *satisfy the Turing stability condition*

$$a(\bar{v})b(\bar{u}) - \frac{\bar{u}\bar{v}a'(\bar{v})b'(\bar{u})}{(1+\delta\lambda)^2} > 0 \tag{16}$$

Then there exists $\varepsilon_0 > 0$ *depending on* $\bar{u}, \bar{v}, \delta$ *such that if for all* $t \leqslant T$ *we have*

$$\|(\tilde{w}, \tilde{z})(t)\|_{L^\infty(\Omega)} \leqslant \varepsilon, \tag{17}$$

then

$$\|\sqrt{a}w\|_{L^2(Q_t)}, \|\sqrt{b}z\|_{L^2(Q_t)}, \|(w, z)\|_{L^2(Q_t)} \leqslant C(\delta, \varepsilon, \bar{u}, \bar{v})\|(w^0, z^0)\|_{H^{-1}}, \tag{18}$$

for all $t \leqslant T$*. In particular, the bound does not depend on* T*.*

Proof. The proof is very similar to the one of Lemma 2.3 since if a is C^1,

$$A_r(T) \leqslant \frac{a'(\bar{v})}{\sqrt{a(\bar{v})}} + o(1),$$

$$\frac{1}{A_l(T)} \leqslant \frac{1}{\sqrt{a(\bar{v})}} + o(1),$$

and similarly for B_l, B_r. We have then

$$\frac{\bar{u}\bar{v}}{(1+\delta\lambda)^2}\frac{A_r B_r}{A_l B_l} \leqslant \frac{\bar{u}\bar{v}a'(\bar{v})b'(\bar{u})}{a(\bar{v})b(\bar{u})(1+\delta\lambda)^2} + o(1) < 1,$$

for $\varepsilon > 0$ small enough thanks to (16). Thereby we obtain the bound on w, z. □

2.2. **Bootstraping the weak stability.** In this subsection we will prove Theorem 1.1. Note that the previous lemma do not need any dimension assumption. The following computations and lemma however need the assumption $d \leqslant 2$ for the use Sobolev embeddings. We start with the following

Lemma 2.5. *Suppose $d = 1, 2$ and $u^0, v^0 \in L^2(\Omega)$, and condition (17) is fullfilled then, for any $1 < q < 2$, there exists a constant K independent of T such that for $t \leqslant T$,*

$$\int |w|^q + |z|^q(t) \leqslant K \int_\Omega |w^0|^q + |z^0|^q.$$

Proof. We start with the following formal computation:

$$\frac{d}{dt}\int_\Omega |w|^q + q(q-1)\int_\Omega |w|^{q-2} a(\tilde v)|\nabla w|^2 = -q(q-1)\int_\Omega a'(\tilde v)|w|^{q-2} w\nabla w \nabla \tilde z.$$

We notice then that

$$-\int_\Omega a'(\tilde v)|w|^{q-2} w\nabla w\nabla \tilde z \leqslant \frac{1}{2}\int_\Omega \frac{a'(\tilde v)^2}{a(\tilde v)}|w|^q|\nabla \tilde z|^2 + \frac{1}{2}\int_\Omega a(\tilde v)|w|^{q-2}|\nabla w|^2$$

Combining that and Holder's inequality, we obtain

$$\frac{d}{dt}\int_\Omega |w|^q \leqslant C(\varepsilon)q(q-1)\int_\Omega |w|^q|\nabla \tilde z|^2 \leqslant C(\varepsilon)q(q-1)\|w\|_2^q\|\nabla \tilde z\|^2_{\frac{4}{2-q}}.$$

Then, by elliptic regularity and Sobolev embbeddings, there exists a constant depending only on δ, q such that

$$\|\nabla \tilde z\|_{\frac{4}{2-q}} \leqslant C(\delta, q)\|z\|_{\frac{4}{4-q}}.$$

And since $q < \frac{4}{4-q} < 2$ we can interpolate

$$\|z\|_{\frac{4}{4-q}} \leqslant \|z\|_q^{\frac{q}{2}}\|z\|_2^{1-\frac{q}{2}}.$$

Finally, we have

$$\frac{d}{dt}\int_\Omega |w|^q \leqslant C(\delta, q, \varepsilon)\|z\|_q^q\|w\|_2^q\|z\|_2^{2-q}.$$

Using Young's inequality,

$$\|w\|_2^q\|z\|_2^{2-q} \leqslant \frac{q}{2}\|w\|_2^2 + (1-\frac{q}{2})\|z\|_2^2 \leqslant \int_\Omega w^2 + z^2,$$

this leads to

$$\frac{d}{dt}\int_\Omega |w|^q \leqslant C(\delta, q, \varepsilon)\left(\int_\Omega w^2 + z^2\right)\int_\Omega |z|^q.$$

Summing up, we obtain

$$\frac{d}{dt}\int_\Omega |w|^q + |z|^q \leqslant C(\delta, q, \varepsilon)\left(\int_\Omega w^2 + z^2\right)\int_\Omega |w|^q + |z|^q.$$

Using Gronwall's lemma, we arrive at

$$\int_\Omega (|w(t,x)|^q + |z(t,x)|^q)dx \leqslant \exp\left(C\int_{Q_t} w^2 + z^2\right)\int_\Omega (|w^0|^q + |z^0|^q).$$

Then, using lemma 2.4, we conclude conclude for $t \leqslant T$

$$\textstyle\int_\Omega |w|^q + |z|^q(t) \leqslant \exp\left(C\int_{Q_T} w^2 + z^2\right)\int_\Omega |w^0|^q + |z^0|^q \quad \leqslant K\int_\Omega |w^0|^q + |z^0|^q.$$

Where K does not depend on T (but depends on δ, q, ε). ∎

Remark 1. To be more rigorous, one should apply this to $(\alpha + |w|^2)^{q/2}$ and then let $\alpha \to 0$ in the integral version of Gronwall lemma.

Corollary 1. *For any $1 < q < 2$, there exists $\alpha = \alpha(q, \varepsilon)$ such that if ,*

$$\|w^0, z^0\|_q \leqslant \alpha,$$

then, for all $t > 0$ we have

$$\|\tilde{w}, \tilde{z}\|_\infty \leqslant \varepsilon.$$

Proof. This is mainly a consequence of elliptic regularity and Sobolev embeddings. We remind that, as long as $\|\tilde{w}, \tilde{z}\|_\infty \leqslant \varepsilon$ for $t \leqslant T$, we have by the previous lemma

$$\|w(t), z(t)\|_q \leqslant C\|w^0, z^0\|_q$$

with a constant independent on T. Since $d \leqslant 2$, by Sobolev embeddings and elliptic regularity, we have,

$$\|(\tilde{w}(t), \tilde{z}(t))\|_\infty \leqslant C'\|(w^0, z^0)\|_q.$$

The constant C' being still independent on T. We choose then α such that $C'\alpha < \varepsilon$. Finally, we use a standard bootstrap argument:

- as initially, we have $\|(\tilde{w}^0, \tilde{z}^0)\|_\infty < \varepsilon$, this remains true for $t < T*$ (and we choose $T*$ as the maximal time) by continuity of $\tilde{u}, \tilde{v}$ in L^q see [2] or [9] for more details,
- suppose $T* < \infty$, then we have $\|\tilde{w}(T*), \tilde{z}(T*)\|_\infty < \varepsilon$ which contradicts the maximality of $T*$,
- therefore, $T* = \infty$ and by lemma 2.4, $\|(w, z)\|_{L^2(\mathbb{R}_+ \times \Omega)}$ is bounded and $\|(w, z)\|_q$ is uniformly bounded (by $C'\|(w^0, z^0)\|_q$).

□

Lemma 2.6. *Suppose conditions (16) is fullfilled, then there exists $\alpha > 0$ such that if*

$$\|(w, z)\|_2 \leqslant \alpha,$$

then

$$\|(w, z)\|_2 \xrightarrow[t \to +\infty]{} 0$$

The proof of this lemma follows from the following result:

Lemma 2.7. *Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfy $(f')_+ \leqslant C$ and $\int_0^\infty f(t)dt < +\infty$, then $f(t) \to 0$ as $t \to +\infty$.*

Proof. Let $\varepsilon > 0$ and let T be large enough so that $T - \frac{\varepsilon}{C} > 0$ and $\int_{T-\frac{\varepsilon}{C}}^\infty f(t)dt < \frac{\varepsilon^2}{3C}$. Suppose there exists $t \geqslant T$ such that $f(t) \geqslant \varepsilon$, then for $s \in [t - \frac{\varepsilon}{C}, t]$, we have, by definition of $(f')_+$,

$$f(s) \geqslant \varepsilon - C(t - s).$$

So that

$$\int_{T-\frac{\varepsilon}{C}}^\infty f(t)dt \geqslant \int_{t-\frac{\varepsilon}{C}}^t f(s)ds \geqslant \int_{t-\frac{\varepsilon}{C}}^t \varepsilon - C(t-s)ds = \frac{\varepsilon^2}{2C} > \frac{\varepsilon^2}{3C},$$

which gives a contradiction. Thus, for any $\varepsilon > 0$ there exists T such that $t \geqslant T$ implies $f(t) \leqslant \varepsilon$. This concludes the proof of the lemma. □

Going back to the proof of the theorem, we first recall, that using interpolation inequality and Corollary 1 we can always choose α such that $\|(w^0, z^0)\|_q \leqslant \eta(q, \varepsilon)$ and thereby

$$\|(\tilde{w}, \tilde{z})\|_\infty \leqslant C(\varepsilon), \tag{19}$$

uniformly in time.

Now, multiplying the first equation of (2) by w and integrating by parts we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega w^2+\int_\Omega a(\tilde v)|\nabla w|^2 &= -\int ua'(\tilde v)\nabla w\nabla\tilde z,\\ &= -\int \bar u a'(\tilde v)\nabla w\nabla \tilde z-\int wa'(\tilde v)\nabla w\nabla\tilde z\end{aligned}$$

We use (19) and Sobolev embeddings to prove that

$$-\int \bar u a'(\tilde v)\nabla w\nabla\tilde z\leqslant \overline{C}(\varepsilon)\int_\Omega|\nabla\tilde z|^2+\frac{1}{4}\int_\Omega a(\tilde v)|\nabla w|^2\leqslant C(\varepsilon,q)\|z\|_q^2+\frac{1}{4}\int_\Omega a(\tilde v)|\nabla w|^2$$

for some contants $\overline{C}(\varepsilon), C(\varepsilon,q)$ depending also on $\bar u,\ \bar v$. Similarly, we have also

$$-\int wa'(\tilde v)\nabla w\nabla\tilde z\leqslant \hat C(\varepsilon)\int_\Omega w^2|\nabla\tilde z|^2+\frac{1}{4}\int_\Omega a(\tilde v)|\nabla w|^2.$$

Finally, we have

$$\int_\Omega w^2|\nabla\tilde z|^2\leqslant \|w\|_{2r}^2\|\nabla\tilde z\|_{2r'}^2\leqslant C(\delta)\|w\|_2^{2\theta}\|w\|_{2s}^{2-2\theta}\|z\|_q^2$$

where we choose r, r' such that

$$\frac{1}{r}+\frac{1}{r'}=1,\quad \frac{1}{2r'}=\frac{1}{q}-\frac{1}{2}\quad\text{and}\quad \frac{1}{2r}=\frac{\theta}{2}+\frac{1-\theta}{2s},$$

Thanks to Young's inequality and Sobolev embeddings, we can write, recalling that $\|z\|_q$ is uniformly bounded,

$$\int_\Omega w^2|\nabla\tilde z|^2\leqslant C\int w^2+\frac{\nu}{4}\int|\nabla w|^2$$

Therefore we end up with

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2+\frac{\nu}{4}\int_\Omega|\nabla w|^2\leqslant C\int w^2.$$

Integrating the inequality we obtain a uniform bound on $\int_\Omega w^2$. The inequality becomes

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2+\frac{\nu}{4}\int_\Omega|\nabla w|^2\leqslant C.$$

And we can conclude thanks to lemmas 2.7 and 2.2. □

3. **Uniqueness of steady states for large value of the relaxation parameter δ.** In this section we will show that given positive $\bar u,\bar v$ then for δ large the only equilibrium (u,v) of the system (2) with fixed averages $\frac{1}{|\Omega|}\int_\Omega u=\bar u$ and $\frac{1}{|\Omega|}\int_\Omega v=\bar v$ is $(\bar u,\bar v)$, thus proving Theorem 1.2.

Let (u,v) be a steady state of the system (2) satisfying

$$\frac{1}{|\Omega|}\int_\Omega u=\bar u,\ \frac{1}{|\Omega|}\int_\Omega v=\bar v,$$

The steady states of the system should satisfy $\Delta[a(\tilde v)u]=\Delta[b(\tilde u)v]=0$. Therefore there exist two constants C_1, C_2 depending on u, v, such that

$$u=\frac{C_1}{a(\tilde v)},\quad v=\frac{C_2}{b(\tilde v)}. \tag{20}$$

From Markov inequality, and (3) it follows that for $\alpha > 1$ we have

$$\frac{|\tilde{v} \geqslant \alpha\bar{v}|}{|\Omega|} \leqslant \frac{1}{\alpha}.$$

It follows that $|\tilde{v} \leqslant \alpha\bar{v}| \geqslant |\Omega|(1-1/\alpha)$. Using (20) and $\int_\Omega u = \bar{u}|\Omega|$ we obtain

$$\begin{aligned} |\Omega|\bar{u} &= \textstyle\int \frac{C_1}{a(\tilde{v})} \\ &\geqslant \int_{\{\tilde{v}\leqslant\alpha\bar{v}\}} \frac{C_1}{a(\tilde{v})} \\ &\geqslant |\Omega|(1-\tfrac{1}{\alpha}) \sup_{\tilde{v}\leqslant\alpha\bar{v}} \frac{C_1}{a(\tilde{v})}. \end{aligned}$$

and then,

$$C_1 \leqslant \frac{\bar{u}}{|\Omega|}\frac{\alpha}{\alpha-1}\inf_{\tilde{v}\leqslant\alpha\bar{v}} a(\tilde{v}).$$

Similarly, we can show that

$$C_2 \leqslant \frac{\bar{v}}{|\Omega|}\frac{\alpha}{\alpha-1}\inf_{\tilde{u}\leqslant\alpha\bar{u}} b(\tilde{u}).$$

This proves that there exists C_0 depending on α, $\bar{u},\bar{v}$, such that $C_1, C_2 \leqslant C_0$. From (H1) we have that $u \leqslant C_0/\nu^2$, $v \leqslant C_0/\nu^2$, and by the maximum principle we conclude that $\tilde{u} \leqslant C_0/\nu^2$, $\tilde{v} \leqslant C_0/\nu^2$ (note that this bound does not depend on δ). Now, the equation on $\tilde{u}$ can be written in the form

$$-\delta\Delta(\tilde{u}-\bar{u}) + (\tilde{u}-\bar{u}) = \frac{C_1}{a(\tilde{v})} - \bar{u}.$$

Multiplying by $(\tilde{u}-\bar{u})$ gives

$$\delta\int\left|\nabla(\tilde{u}-\bar{u})\right|^2 + \int\left|\tilde{u}-\bar{u}\right|^2 = \int\left(\frac{C_1}{a(\tilde{v})} - \bar{u}\right)(\tilde{u}-\bar{u}) = \int\left(\frac{C_1}{a(\tilde{v})} - \frac{C_1}{a(\bar{v})}\right)(\tilde{u}-\bar{u}),$$

since $\int \tilde{u}-\bar{u} = 0$. Because $\tilde{v} \leqslant C_0/\nu^2$ and $C_1 \leqslant C_0$ we have that there exists C_3 independent on δ such that

$$\left|\frac{C_1}{a(\tilde{v})} - \frac{C_1}{a(\bar{v})}\right| \leqslant C_3|\tilde{v}-\bar{v}|,$$

and then

$$\delta\int\left|\nabla(\tilde{u}-\bar{u})\right|^2 + \int\left|\tilde{u}-\bar{u}\right|^2 \leqslant C_3\int_\Omega |\tilde{v}-\bar{v}||\tilde{u}-\bar{u}|$$

Proceeding in the same way, choosing C_3 appropriately, we obtain

$$\delta\int\left|\nabla(\tilde{v}-\bar{v})\right|^2 + \int\left|\tilde{v}-\bar{v}\right|^2 \leqslant C_3\int_\Omega |\tilde{u}-\bar{u}||\tilde{v}-\bar{v}|.$$

Adding up these two inequalities and using Cauchy-Schwarz we get

$$\delta\|\nabla\tilde{u}\|_2^2 + \|\tilde{u}-\bar{u}\|_2^2 + \delta\|\nabla\tilde{v}\|_2^2 + \|\tilde{v}-\bar{v}\|_2^2 \leqslant C_3(\|\tilde{u}-\bar{u}\|_2^2 + \|\tilde{u}-\bar{u}\|_2^2). \tag{21}$$

On the other hand we have that

$\delta\|\nabla\tilde{u}\|_2^2 + \|\tilde{u}-\bar{u}\|_2^2 \geqslant (1+\delta\lambda_1)\|\tilde{u}-\bar{u}\|_2^2$ and $\delta\|\nabla\tilde{v}\|_2^2 + \|\tilde{v}-\bar{v}\|_2^2 \geqslant (1+\delta\lambda_1)\|\tilde{v}-\bar{v}\|_2^2$,

which together with (21) leads to the inequality

$$(1+\delta\lambda_1)(\|\tilde{u}-\bar{u}\|_2^2 + \|\tilde{v}-\bar{v}\|_2^2) \leqslant C_3(\|\tilde{u}-\bar{u}\|_2^2 + \|\tilde{v}-\bar{v}\|_2^2),$$

thus, if $1+\delta\lambda_1 > C_3$, we conclude that $u \equiv \bar{u}$ and $v \equiv \bar{v}$. □

4. From Turing instability to bifurcation.

4.1. Bifurcation at the critical value δ_0.

In this section, we prove that the critical value δ_0 characterizes the appearance of a new branch of equilibria. We suppose additional smoothness of the functions a and b:

$$a, b \in C^2(\mathbb{R}).$$

For two given positive constants $\bar{u}, \bar{v}$, we denote $W = (w, z)$ and F the function defined by

$$F(\delta, W) = \begin{pmatrix} \Delta\left[a(\bar{v} + \tilde{z})(\bar{u} + w)\right] \\ \Delta\left[b(\bar{u} + \tilde{w})(\bar{v} + z)\right] \end{pmatrix}.$$

We study this function on $(0, \infty) \times E^2$, where the space E is defined as

$$E = \{f \in W^{2,2}(\Omega), \int_\Omega f = 0, \ \partial_n f = 0 \text{ on } \partial\Omega\}.$$

In this formula $\tilde{w} = T_\delta w$, denotes the unique solution of

$$-\delta\Delta\tilde{w} + \tilde{w} = w \text{ in } \Omega, \ \partial_n \tilde{w} = 0 \ \text{ on } \partial\Omega, \tag{22}$$

and therefore the dependency on δ is hidden in $\tilde{w}, \tilde{z}$. We remark that for any δ, W satisfying $F(\delta, W) = 0$, if we set $u = \bar{u} + w$, $v = \bar{v} + z$ then (u, v) is a steady state of the system (2). The nonnegativity comes from the remark that $F(\delta, w) = 0$ implies

$$a(\tilde{v})u = C_1, \qquad b(\tilde{u})v = C_2,$$

as we assumed $a, b \geqslant \nu^2 > 0$ we have that u, v have a constant sign, since $\bar{u}$, $\bar{v} > 0$ u and v are positive.

We prove now the following existence result.

Theorem 4.1. *Assume the domain Ω is smooth and that the first nonzero eigenvalue λ_1 of (5) is simple. Suppose also that, a, $b \in C^2(\mathbb{R})$, satisfy (H0), (H1) and (7) and $\delta_0 > 0$ satisfy (8).*

Then the point $(\delta_0, 0)$ is a bifurcation point for F, that is, in some neighborhood of $(\delta_0, 0)$ the set $\{(\delta, W); F(\delta, W) = 0\}$ consist of of two branches, one a parameterized as a C^1 curve given by $(\delta(\varepsilon), W(\varepsilon))$, with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\delta(0) = \delta_0$ and $W(0) = 0$, and the other one is the trivial branch $(\delta, 0)$. These two branches of solutions only intersect at $(\delta_0, 0)$.

We observe that the above result proves Theorem 1.3 of the introduction. Indeed, $(\bar{u}, \bar{v}) + W$ is a steady state of (2) for if and only if $F(\delta, W) = 0$.

The proof is a direct application Crandall-Rabinowitz local bifurcation theorem (see [4, 11] for instance). We denote $F_W = \frac{\partial F}{\partial W}$, to prove Theorem 4.1 we have to show that the following conditions hold:

1) The kernel of $F_W(\delta_0, 0) = Span(\varphi)$.
2) The range of $F_W(\delta_0, 0)$ denoted by $R(F_W(\delta_0, 0))$ has codimension 1,
3) The cross derivative $F_{\delta,W} = \frac{\partial^2 F}{\partial_\delta \partial_W}$ satisfies $F_{\delta,W}(\delta, 0)(\varphi) \notin R(F_W(\delta_0, 0)$.

Moreover, the Crandall-Rabinowitz theorem states that if the above conditions hold we have that

$$W'(0) = \varphi. \tag{23}$$

We prove these points in three separate lemmas.

Lemma 4.2. *Under the hypothesis of Theorem 4.1 the following holds:*

$$\ker F_W(\delta_0, 0) = Span(\varphi), \;\; where\ \varphi = \begin{pmatrix} pe_1 \\ qe_1 \end{pmatrix} \tag{24}$$

and e_1 is an eigenfunction of (5) associated to λ_1 with $\int_\Omega e_1^2 = 1$, and (p, q) are given by

$$p = 1, \quad q = -\frac{a(\bar v)(1+\delta_0\lambda_1)}{a'(\bar v)\bar u} = -\frac{b'(\bar u)\bar v}{b(\bar u)(1+\delta_0\lambda_1)}. \tag{25}$$

Proof. Linearizing the system we obtain

$$F_W(\delta, 0)(\varphi) = \begin{pmatrix} \Delta[(a'(\bar v)\bar u\tilde\varphi_2 + a(\bar v)\varphi_1] \\ \\ \Delta[b'(\bar u)\bar v\tilde\varphi_1 + b(\bar u)\varphi_2] \end{pmatrix}, \tag{26}$$

where $\tilde\phi = T_\delta\phi$. We have that for $i = 1, 2$

$$\varphi_i = \sum_{k=1}^{\infty} \varphi_i(k)e_k,$$

with e_k eigenfunction of (5) associated to λ_k. Then

$$\tilde\varphi_i(k) = \frac{1}{1+\delta\lambda_k}\varphi_i(k),$$

and thereby,

$$F_W(\delta, 0)(\varphi)_1(k) = -\lambda_k\big(a'(\bar v)\bar u\tfrac{1}{1+\delta\lambda_k}\varphi_2(k) + a(\bar v)\varphi_1(k)\big)$$
$$F_W(\delta, 0)(\varphi)_2(k) = -\lambda_k\big(b'(\bar u)\bar v\tfrac{1}{1+\delta\lambda_k}\varphi_1(k) + b(\bar u)\varphi_2(k)\big).$$

It is here convenient to write it in a matrix way:

$$F_W(\delta, 0)(\varphi) = -\sum_{k=1}^{\infty} \lambda_k M(\delta, \lambda_k)\begin{pmatrix} \varphi_1(k) \\ \varphi_2(k) \end{pmatrix} e_k,$$

where $M(\delta, \lambda_k)$ is the matrix defined in (6).

Clearly

$$(\varphi_1, \varphi_2) \in \ker(F_W(\delta_0, 0)) \Leftrightarrow \forall k \geqslant 1, \begin{pmatrix} \varphi_1(k) \\ \varphi_2(k) \end{pmatrix} \in \ker M(\delta_0, \lambda_k). \tag{27}$$

From (8) and $\lambda_k > \lambda_1$ for $k > 1$, since λ_1 is simple, it follows that

$$\forall k \geqslant 2, \quad \det M(\delta_0, \lambda_k) > 0.$$

Therefore, $\ker M(\delta_0, \lambda_k) = \{0\}$ for $k \geqslant 2$ and

$$\ker M(\delta_0, \lambda_1) = \text{Span}\begin{pmatrix} p \\ q \end{pmatrix},$$

where p, q are given by (25). Hence, $\ker(F_W(\delta_0, 0)) = \text{Span}(\varphi)$ with $\varphi = \begin{pmatrix} p \\ q \end{pmatrix} e_1$. which ends the proof. □

Lemma 4.3. *The range of $F_W(\delta_0, 0)$ has codimension 1 and is characterized by*

$$(\phi_1, \phi_2) \in R(F_W(\delta_0, 0)) \Leftrightarrow \int_\Omega (p^*\phi_1 e_1 + q^*\phi_2 e_2)\, dx = 0, \tag{28}$$

where e_1 is as above and

$$p^* = \frac{b}{a}q, \quad q^* = p = 1, \tag{29}$$

with (p, q) *as in* (25).

Proof. Using the same decomposition as above we obtain

$$(\phi_1, \phi_2) \in R(F_W(\delta, 0)) \Leftrightarrow \forall k \geqslant 1, \begin{pmatrix} \phi_1(k) \\ \phi_2(k) \end{pmatrix} \in R(M(\delta_0, \lambda_k)).$$

Clearly, for all $k \geqslant 2$

$$\begin{pmatrix} \phi_1(k) \\ \phi_2(k) \end{pmatrix} \in R(M(\delta_0, \lambda_k)) = \mathbb{R}^2.$$

thus,

$$\phi \in R(F_w(\delta, 0)) \Leftrightarrow \begin{pmatrix} \phi_1(1) \\ \phi_2(1) \end{pmatrix} \in R(M(\delta_0, \lambda_1)).$$

Also, we have that $R(M(\delta_0, \lambda_1)) = (Span(p^*, q^*))^\perp$ where $(p^*, q^*) \in \ker M(\delta_0, \lambda_1)^T$. It is easy to check that thus (p^*, q^*) can be taken as in (29). We have then,

$$\phi \in R(F_W(\delta, 0)) \Leftrightarrow \begin{pmatrix} \phi_1(1) \\ \phi_2(1) \end{pmatrix} \in (Span(p^*, q^*))^\perp \Leftrightarrow \int_\Omega (p^* \phi_1 e_1 + q^* \phi_2 e_1)\, dx = 0,$$

which ends the proof. □

Lemma 4.4. *The following holds*

$$F_{\delta,W}(\delta_0, 0)(\varphi) \notin R(F_W(\delta, 0)). \tag{30}$$

Proof. Using the characterization of $R(F_w(\delta_0, 0))$ we have that (30) is equivalent to

$$\int_\Omega (p^* e_1, q^* e_1) \cdot F_{\delta,w}(\delta, 0)(\varphi) \neq 0.$$

Differentiating formula (26) with respect to δ in the spectral decomposition gives

$$F_{\delta,W}(\delta, 0)(\phi) = -\sum_{k=1}^{\infty} \lambda_k \frac{\partial}{\partial \delta} M(\delta, \lambda_k) \begin{pmatrix} \phi_1(k) \\ \phi_2(k) \end{pmatrix} e_k,$$

with

$$\frac{\partial}{\partial \delta} M(\delta, \lambda_k) = \begin{pmatrix} 0 & \dfrac{-\lambda_k}{(1+\delta\lambda_k)^2} a'(\bar{v})\bar{u} \\ \dfrac{-\lambda_k}{(1+\delta\lambda_k)^2} b'(\bar{u})\bar{v} & 0 \end{pmatrix}.$$

Therefore,

$$F_{\delta,W}(\delta_0, 0)(\varphi) = -\frac{\lambda_1^2}{(1+\delta_0\lambda_1)^2} \begin{pmatrix} 0 & a'(\bar{v})\bar{u} \\ b'(\bar{u})\bar{v} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},$$

$$(p^* e_1, q^* e_1).F_{\delta_0,W}(\delta, 0)(\varphi) = -\frac{\lambda_1^2}{(1+\delta\lambda_1)^2} \Big(a'(\bar{v})\bar{u} q p^* + b'(\bar{u})\bar{v} p q^* \Big).$$

Using the formulas (25,29), we have

$$pq^* > 0, \quad qp^* > 0.$$

On the other hand by (8) we have $a'(\bar{v}), \quad b'(\bar{u}) > 0$ or $a'(\bar{v}), \quad b'(\bar{u}) < 0$ from where we obtain $a'(\bar{v})\bar{u}qp^* + b'(\bar{u})\bar{v}pq^* \neq 0$ which concludes the proof. □

Proof of Theorem 4.1. The proof of 1), 2) and 3) follow directly from Lemmas 4.2, 4.3 and 4.4. □

4.2. **Bifurcation branches.** In this section we will characterize the shape of the bifurcating branch of solutions given by Theorem 4.1. According to this result there exists a unique family of nonzero solutions of $F(\delta, W) = 0$ in a neighborhood of $(\delta_0, 0)$ given by $(\delta(\varepsilon), W(\varepsilon))$ with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$. In this section we will study the expansion of $\delta(\varepsilon)$ which characterizes the direction of the bifurcation branch.

Throughout this section we denote $W = (w, z)$ that is

$$w = u - \bar{u},\ z = v - \bar{v},$$

and we will assume that $a,\ b$ are C^4 functions. The system (2) for stationary solutions can be rewritten in the new variables as

$$\begin{cases} \Delta[a(\bar{v} + \tilde{z})w] = 0 \ \text{ in } \Omega, \\ \Delta[b(\bar{u} + \tilde{w})z] = 0 \ \text{ in } \Omega, \\ -\delta\Delta\tilde{w} + \tilde{w} = w, \ \text{ in } \Omega, \\ -\delta\Delta\tilde{z} + \tilde{z} = z, \ \text{ in } \Omega, \\ \partial_n w = \partial_n z = \partial_n \tilde{w} = \partial_n \tilde{z} = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{31}$$

We expand

$$\begin{aligned} &w(\varepsilon) = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + o(\varepsilon^3), \quad z(\varepsilon) = \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + o(\varepsilon^3), \\ &\delta(\varepsilon) = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + o(\varepsilon^2), \end{aligned} \tag{32}$$

and by (23) we have $w_1 = pe_1 = e_1$ and $z_1 = qe_1$ with (p, q) as in (25). For $\delta > 0$ we denote $T_\delta(f)$ as in (22). Using this notation we set

$$\begin{aligned} \tilde{w}(\varepsilon) &= T_{\delta(\varepsilon)}(w(\varepsilon)) = \varepsilon\tilde{w}_1 + \varepsilon^2\tilde{w}_2 + \varepsilon^3\tilde{w}_3 + o(\varepsilon^3), \\ \tilde{z}(\varepsilon) &= T_{\delta(\varepsilon)}(z(\varepsilon)) = \varepsilon\tilde{z}_1 + \varepsilon^2\tilde{z}_2 + \varepsilon^3\tilde{z}_3 + o(\varepsilon^3). \end{aligned} \tag{33}$$

After performing an expansion in powers of ε we observe that

$$\begin{aligned} \tilde{w}_1 &= \frac{1}{1 + \delta_0\lambda_1} e_1, \\ \tilde{w}_2 &= T_{\delta_0}(w_2) + \delta_1 T_{\delta_0}(\Delta\tilde{w}_1) = T_{\delta_0}(w_2) - \delta_1 \frac{\lambda_1 e_1}{(1 + \delta_0\lambda_1)^2}, \\ \tilde{w}_3 &= T_{\delta_0}(w_3) + \delta_1 T_{\delta_0}(\Delta\tilde{w}_2) + \delta_2 T_{\delta_0}(\Delta\tilde{w}_1) \\ &= T_{\delta_0}(w_3) + \delta_1 T_{\delta_0}(\Delta\tilde{w}_2) - \delta_2 \frac{\lambda_1 e_1}{(1 + \delta_0\lambda_1)^2}, \end{aligned} \tag{34}$$

and similarly

$$\begin{aligned} &\tilde{z}_1 = \frac{q}{1 + \delta_0\lambda_1} e_1, \\ &\tilde{z}_2 = T_{\delta_0}(z_2) + \delta_1 T_{\delta_0}(\Delta\tilde{z}_1) = T_{\delta_0}(z_2) - \delta_1 \frac{\lambda_1 q e_1}{(1 + \delta_0\lambda_1)^2}, \\ &\tilde{z}_3 = T_{\delta_0}(z_3) + \delta_1 T_{\delta_0}(\Delta\tilde{z}_2) + \delta_2 T_{\delta_0}(\Delta\tilde{z}_1) = T_{\delta_0}(z_3) + \delta_1 T_{\delta_0}(\Delta\tilde{z}_2) - \delta_2 \frac{\lambda_1 q e_1}{(1 + \delta_0\lambda_1)^2}. \end{aligned} \tag{35}$$

Our first goal is to give an expression for δ_1 which will be a consequence of matching powers of ε in the expression for $F(\delta(\varepsilon), (w(\varepsilon), z(\varepsilon))) = 0$ and using Lemma 4.3. In order to keep notation simple, we will denote

$$a = a(\bar{v}),\ a' = a'(\bar{v}),\ a'' = a''(\bar{v}),$$

$$b = b(\bar{u}),\ b' = b'(\bar{v}),\ b'' = b''(\bar{u}).$$

Computing the terms of ε^2 in $F(\delta(\varepsilon),(w(\varepsilon),z(\varepsilon)))=0$, we obtain

$$\Delta(aw_2+a'\bar{u}\tilde{z}_2+a'\tilde{z}_1w_1+\frac{a''}{2}\bar{u}\tilde{z}_1^2)=0,$$

$$\Delta(bz_2+b'\bar{v}\tilde{w}_2+b'\tilde{w}_1z_1+\frac{b''}{2}\bar{v}\tilde{w}_1^2)=0.$$

Replacing the expressions for $\tilde{w}_2,\tilde{w}_1$ in the above equations we obtain

$$\Delta(aw_2+a'\bar{u}T_{\delta_0}(z_2)) = -\Delta(a'\tilde{z}_1w_1+\frac{a''}{2}\bar{u}\tilde{z}_1^2)-\delta_1\lambda_1^2a'\bar{u}\frac{qe_1}{(1+\delta_0\lambda_1)^2}, \quad (36)$$

$$\Delta(bz_2+b'\bar{v}T_{\delta_0}(w_2)) = -\Delta(b'\tilde{w}_1z_1+\frac{b''}{2}\bar{v}\tilde{w}_1^2)-\delta_1\lambda_1^2b'\bar{v}\frac{e_1}{(1+\delta_0\lambda_1)^2}, \quad (37)$$

and using Lemma 4.3 we obtain that

$$\left(f_1-\delta_1\lambda_1^2a'\bar{u}\frac{qe_1}{(1+\delta_0\lambda_1)^2}, f_2-\delta_1\lambda_1^2b'\bar{v}\frac{e_1}{(1+\delta_0\lambda_1)^2}\right)\perp(p^*,q^*)e_1, \quad (38)$$

with (p^*,q^*) given by (29) and

$$f_1=-\Delta(a'\tilde{z}_1w_1+\tfrac{a''}{2}\bar{u}\tilde{z}_1^2)$$

$$f_2=-\Delta(b'\tilde{w}_1z_1+\tfrac{b''}{2}\bar{v}\tilde{w}_1^2).$$

After doing some computations we obtain that (38) is equivalent to the following:

$$C_0\delta_1=-\frac{\lambda_1}{a(1+\delta_0\lambda_1)}\int_\Omega e_1^3\,dx\left(a'q^2b+\frac{a''}{2}\bar{u}b\frac{q^3}{1+\delta_0\lambda_1}+b'qa+\frac{b''}{2}\bar{v}a\frac{1}{1+\delta_0\lambda_1}\right), \quad (39)$$

with

$$C_0=\frac{\lambda_1^2}{(1+\delta_0\lambda_1)^2}\left(a'\bar{u}\frac{b}{a}q^2+b'\bar{v}\right)\neq 0. \quad (40)$$

Then, as a consequence of (39), (40) we have the following result:

Proposition 2. *Suppose that the hypothesis of Theorem 4.1, a,b are $C^3(\mathbb{R})$ and*

$$\int_\Omega e_1^3\,dx\neq 0,\quad a'bq^2+\frac{a''}{2}\bar{u}b\frac{q^3}{1+\delta_0\lambda_1}+b'qa+\frac{b''}{2}\bar{v}a\frac{1}{1+\delta_0\lambda_1}\neq 0.$$

Then, there exists $\hat{\delta}>0$ such that the equation $F(\delta,(w,z))=0$ admits non trivial solutions for all $\delta\in(\delta_0-\hat{\delta},\delta_0+\hat{\delta})$, with δ_0 as in (8).

Example. We consider the case $\bar{u}=\bar{v}=1$, $a(v)=1+v^\gamma$, $b(u)=1+u^\tau$. In this case we have that

$$\gamma\tau=4(1+\delta_0\lambda_1)^2,$$

and after doing some calculations we obtain:

$$a'bq^2+\frac{a''}{2}\bar{u}b\frac{q^3}{1+\delta_0\lambda_1}+b'qa+\frac{b''}{2}\bar{v}a\frac{1}{1+\delta_0\lambda_1}=\frac{\tau}{1+\delta_0\lambda_1}\left(-1+\frac{\tau}{2(1+\delta_0\lambda_1)}\right).$$

Therefore if $\int_\Omega e_1^3\neq 0$ and $\tau\neq 2(1+\delta_0\lambda_1)$ the conditions of Proposition 2 are satisfied.

Observe that if we have that $\int_\Omega e_1^3\,dx=0$ therefore $\delta_1=0$, for instance this happens in the 1-D case. In this situation in order to characterize the bifurcation branch we need to compute δ_2.

The expression for δ_2 is obtained in a similar way as done for δ_1, computing instead the terms of order ε^3 in the expansion of $F(\delta(\varepsilon),(w(\varepsilon),z(\varepsilon)))=0$ and using Lemma 4.3. Indeed, if $\delta_1=0$ we have that

$$\begin{aligned}\delta_2\frac{C_0}{\lambda_1} = -&\int_\Omega\left[a'\tilde{z}_1w_2p^*e_1+a'\tilde{z}_2w_1p^*e_1+\frac{a''}{2}\tilde{z}_1^2w_1p^*e_1+\frac{a'''}{6}\bar{u}\tilde{z}_1^3p^*e_1\right]dx\\ +&\int_\Omega\left[b'\tilde{w}_1z_2q^*e_1+b'\tilde{w}_2z_1q^*e_1+\frac{b''}{2}\tilde{w}_1^2z_1q^*e_1+\frac{b'''}{6}\bar{v}\tilde{w}_1^3q^*e_1\right]dx,\end{aligned} \tag{41}$$

where C_0 given in (40).

Example. Suppose that $\Omega=(0,\pi)$, $\bar{u}=\bar{v}=1$, $a(v)=1+v^\gamma$, $b(u)=1+u^\tau$. In this situation we have that $\lambda_k=k^2$, with $k\in\mathbb{N}$, $e_1=\sqrt{\frac{2}{\pi}}\cos x$ and

$$w_2=\frac{2}{\pi}p_2\cos 2x,\;\; z_2=\frac{2}{\pi}q_2\cos 2x,\;\; \tilde{w}_2=\frac{1}{1+4\delta_0}w_2,\;\; \tilde{z}_2=\frac{1}{1+4\delta_0}z_2,$$

with (p_2,q_2) solution of

$$M(\delta_0,4)\begin{pmatrix}p_2\\ q_2\end{pmatrix}=\begin{pmatrix}c_1\\ c_2\end{pmatrix},$$

with $M(\delta_0,4)$ defined in (6) is given in this case by

$$M(\delta_0,4)=\begin{pmatrix}2 & \frac{\gamma}{4\delta_0+1}\\ \frac{\tau}{4\delta_0+1} & 2\end{pmatrix},$$

and

$$\begin{aligned}c_1&=-\left(\tfrac{\gamma}{2(1+\delta_0)}+\tfrac{\gamma(\gamma-1)q^2}{4(1+\delta_0)^2}\right),\\ c_2&=-\left(\tfrac{\tau q}{2(1+\delta_0)}+\tfrac{\tau(\tau-1)}{4(1+\delta_0)^2}\right).\end{aligned}$$

After doing some long computations, the following expression for δ_2 is obtained

$$\begin{aligned}-\pi C_0\delta_2 = \;&\frac{\gamma q^2p_2}{1+\delta_0}+\frac{\gamma q_2 q}{1+4\delta_0}+\frac{3\gamma(\gamma-1)q^3}{4(1+\delta_0)^2}+\frac{\gamma(\gamma-1)(\gamma-2)q^4}{4(1+\delta_0)^3}\\ &+\frac{\tau q_2}{1+\delta_0}+\frac{\tau p_2 q}{1+4\delta_0}+\frac{3\tau(\tau-1)q}{4(1+\delta_0)^2}+\frac{\tau(\tau-1)(\tau-2)}{4(1+\delta_0)^3},\end{aligned}$$

with $C_0>0$ given in (40), and in this case $\gamma\tau=4(1+\delta_0)^2$, $q=-\frac{2(1+\delta_0)}{\gamma}=-\frac{\tau}{2(1+\delta_0)}$.

We can use MAPLE to evaluate the expresion for δ_2. For example, if we replace the value $\tau=4$, we obtain that

$$\delta_2=\frac{1}{3\pi C_0}\left[\frac{-18\gamma-72-304\gamma^{3/2}+24\gamma^3+256\gamma^2-99\gamma^{5/2}+17\gamma^{7/2}+192\sqrt{\gamma}}{(5\gamma-8\sqrt{\gamma}+3)\gamma^{5/2}}\right]. \tag{42}$$

We note that if $\tau=4$ we must have that $\gamma>1$ since $\gamma=(1+\delta_0)^2$. Evaluating the expression for $\gamma=(1+0.01)^2$ we have that $\delta_2<0$ hence the bifurcation is subcritical, thus there exists $0<\hat{\delta}<\delta_0$ such that system (2) admits at least two nonconstant steady states with average $(1,1)$ for $\hat{\delta}<\delta<\delta_0$. In this situation, by the principle of exchanged stability we expect that these steady states are stable.

Now, from the expression for δ_2 we have that if $\tau=4$, for all γ large enough the bifurcation is supercritical, hence there exists $\hat{\delta}>\delta_0$ such that system (2) admits at least two nonconstant steady states with average $(1,1)$ for $\delta_0<\delta<\hat{\delta}$. In this case, we expect the steady states to be unstable.

We will prove in the next proposition that if $\delta_2 > 0$, there exists $\delta_0 < \bar{\delta} < \hat{\delta}$ such that for $\delta_0 < \delta < \bar{\delta}$ there are at least three nonconstant steady states of (2) with average $(1,1)$. This result will be a consequence of Rabinowitz's global bifurcation theorem ([11]).

Proposition 3. *Assume that $\Omega = (0,\pi)$, $a(v) = 1 + v^\gamma$, $b(u) = 1 + u^\tau$, that δ_2 given in (42) is positive, and that $\gamma, \tau > 2$. Then, there exists a $\bar{\delta} > \delta_0$ such that for $\delta_0 < \delta < \bar{\delta}$ the system (2) has at least three nonconstant steady states of (2) with average $(1,1)$.*

Proof. We observe that by (42) when $\tau = 4$ and γ is large enough, then the hypothesis of the proposition hold.

Denote by $\Delta^{-1} f$, the solution of $g'' = f$ with $g'(0) = g'(\pi) = 0$ and $\int_0^\pi g(x)\,dx = 0$. Clearly $\Delta^{-1} : X \to X$ is a compact operator for $X = \{f \in C[0,\pi] \,/\, \int_0^\pi f(x)\,dx = 0\}$. Observe that if we write $u = 1 + w$, $v = 1 + z$ with $w,\ z$ with zero average, then $(1 + w, 1 + z)$ is a solution of (2) if and only if

$$(1 + (1 + \tilde{z})^\gamma)(1 + w) = C_1, \text{ and } (1 + (1 + \tilde{w})^\tau)(1 + z) = C_2,$$

with the constants $C_1,\ C_2$ given by

$$C_1 = \frac{\pi}{\int_0^\pi \frac{1}{1+(1+\tilde{z})^\gamma}}, \quad C_2 = \frac{\pi}{\int_0^\pi \frac{1}{1+(1+\tilde{w})^\tau}},$$

because $\int_0^\pi w\,dt = \int_0^\pi z\,dt = 0$. Therefore

$$w = \left(\frac{\pi}{\int_0^\pi \frac{1}{1+(1+\tilde{z})^\gamma}} \frac{1}{1 + (1 + \tilde{z})^\gamma} - 1 \right), \text{ and } z = \left(\frac{\pi}{\int_0^\pi \frac{1}{1+(1+\tilde{w})^\tau}} \frac{1}{1 + (1 + \tilde{w})^\tau} - 1 \right) \tag{43}$$

with $\tilde{w} = T_\delta w$ and $\tilde{z} = T_\delta z$. By doing some calculations, we have that $(w+1, z+1)$ is a solution of (31) if and only if $(\tilde{w}, \tilde{z})$ satisfy:

$$(I + \delta^{-1} K)(\tilde{w}, \tilde{z}) + \frac{1}{\delta} g((\tilde{w}, \tilde{z})) = 0, \tag{44}$$

with $K : X \to X$ compact given by

$$K(\tilde{w}, \tilde{z}) = \begin{pmatrix} -\Delta^{-1}\left(\tilde{w} + \frac{\gamma}{2}\tilde{z}\right) \\ -\Delta^{-1}\left(\tilde{z} + \frac{\tau}{2}\tilde{w}\right) \end{pmatrix}$$

and $g : X \to X$ smooth and $g(\tilde{w}, \tilde{z}) = o(\tilde{w}, \tilde{z})$ (to guarantee that $u = 1 + w$ and $v = 1 + z$ given by (43) are positive, it suffices to replace $(1 + \tilde{z})^\gamma$, $(1 + \tilde{w})^\tau$ by $|1 + \tilde{z}|^\gamma$, $|1 + \tilde{w}|^\tau$) in (43)).

We have that the set of solutions of (44) near $(\delta_0, 0)$ is given by $(\delta, 0)$ and

$$\Gamma = \{(\delta(\varepsilon), (\tilde{w}(\varepsilon)), \tilde{z}(\varepsilon)) \,/\, \varepsilon \in (-\varepsilon_0, \varepsilon_0)\}.$$

Set C the connected component of solutions of (44) that contains Γ. Observe that by Theorem Applying Rabinowitz's global bifurcation theorem we have that either:

1) C is not compact in $(0,\infty) \times X$ or

2) C contains points $(\delta_j, 0, 0)$ with δ_j a root of $\det M(\delta, j^2) = 0$, with $j \neq 1$.

Following the proof of Theorem 1.2, we have that any solution of (31) is bounded uniformly when δ belongs to a compact set of $(0,\infty)$, hence in this situation $\tilde{w}, \tilde{z}$ are uniformly bounded on X as well. Also, by Proposition 1.2 are no solutions of

(44) for δ large thus, if 1) holds we must have that C contains solutions of 44 with $\delta < \delta_0$.

Is easy to check that if δ_j a root of $\det M(\delta, j^2) = 0$, with $j \neq 1$ then $\delta_j < \delta_0$. Then if 2) holds C also contains solutions of 44 with $\delta < \delta_0$. Now, since by Theorem 4.1 there exists a neighborhood $(-\tilde{\delta}, \tilde{\delta}) \times V$ of $(\delta_0, 0)$ such that the only nontrivial solutions of (44) in this neighborhood are in Γ. Then, choosing $\tilde{\delta}$ close to δ_0, we must have that C contains solutions of (44) for all $\delta \in (-\tilde{\delta}, \tilde{\delta})$ which are not in $(-\tilde{\delta}, \tilde{\delta}) \times V$, hence we must have that for $\delta_0 < \delta < \tilde{\delta}$ there are at least three nonconstant solutions of (44). $\square$

5. **Illustrations.** We give here numerical illustrations on our model case on the interval $]0, 1[$.

$$a_1(v) = 1 + v^2, \quad b(u) = 1 + u^2.$$

The hypothesis of our theorem are then satisfied.

We choose for the numerical simulations $\bar{u} = 2, \bar{v} = 1$. As [2], we start from the initial conditions

$$u = 1.9 + 0.2\chi_{[0.1,0.6]}, \quad v = 1.$$

We plot the final steady state obtained (the smaller δ is, the further the curves are from constant steady state).

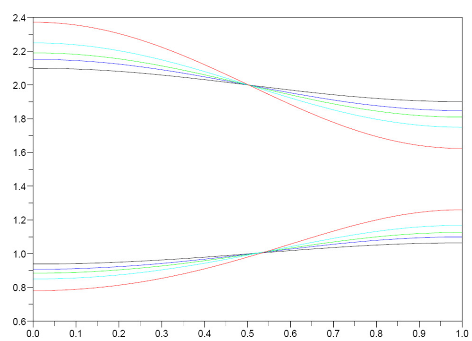

FIGURE 1. Steady states of the function u (upper curves) and v (lower curves). The parameter δ takes values $0.0255, 0.26, 0.261, 0.263, 0.264, 0.265$, the final steady state is closer to the constant steady state when δ is bigger.

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Appendix A. **Michel Pierre's estimate revisited.** We briefly remind the properties of the dual estimate introduced in [12, 6]: we consider the problems

$$\partial_t \psi + a\Delta\psi = F(t,x) \quad \text{ in } \Omega, \qquad \psi(T) = 0, \tag{45}$$

where the source term $F(t,x)$ is a smooth test function and

$$\partial_t u - \Delta(au) = 0, \quad \text{ in } \Omega, \quad u(0) = u^0, \tag{46}$$

together with Neumann boundary condition and the crucial assumption (11). Multiplying the equation (45) by $\Delta\psi$ and integrating over time and space, one obtains (see [2, 6])

$$\left\|\nabla\psi^0\right\|_{L^2(\Omega)}, \|\sqrt{a}\Delta\psi\|_{L^2(Q_T)} \leqslant \|F/\sqrt{a}\|_{L^2(Q_T)}.$$

We introduce the notation $w = u - \bar{u}$. The problem (46) can be read as:

$$\partial_t w - \Delta(aw) = \bar{u}\Delta a, \quad w(0) = w^0 \tag{47}$$

Multplying (45) by w and (47) by ψ, we obtain

$$-\int_\Omega w^0\psi^0 = \int_{Q_T} wF + \int_{Q_T} \bar{u}\psi\Delta a = \int_{Q_T} wF + \bar{u}\int_{Q_T} a\Delta\psi,$$

that is

$$\int_{Q_T} wF = -\int_\Omega w^0\psi^0 - \bar{u}\int_{Q_T} a\Delta\psi. \tag{48}$$

Since $\int_\Omega w^0 = 0$, we have

$$\int_\Omega w^0\psi^0 = \int_\Omega w^0(\psi^0 - \bar{\psi}^0).$$

And therefore,

$$\left|\int_\Omega w^0\psi^0\right| \leqslant \|\nabla\psi^0\|_2 \|w^0\|_{H^{-1}} \leqslant \|w^0\|_{H^{-1}} \|F/\sqrt{a}\|_{L^2(Q_T)}. \tag{49}$$

Using 48 we have

$$\left|\int_{Q_T} wF\right| \leqslant \left|\int_\Omega w^0\psi^0\right| + \left|\bar{u}\int_{Q_T} a\Delta\psi\right| \tag{50}$$

as $\int \Delta\psi = 0$, we can write for any constant C,

$$\int_\Omega a\Delta\psi = \int_\Omega (a - C)\Delta\psi.$$

Integrating over time, we have

$$\begin{aligned} \left|\int_{Q_T} a\Delta\psi\right| = \left|\int_{Q_T} (a-C)\Delta\psi\right| &\leqslant \left\|\tfrac{a-C}{\sqrt{a}}\right\|_{L^2(Q_T)} \|\sqrt{a}\Delta\psi\|_{L^2(Q_T)} \\ &\leqslant \left\|\tfrac{a-C}{\sqrt{a}}\right\|_{L^2(Q_T)} \|F/\sqrt{a}\|_{L^2(Q_T)}. \end{aligned} \tag{51}$$

Combining (49) and (51) with (50) we obtain (taking $F = \sqrt{a}w$),

$$\left|\int_{Q_T} wF\right| \leqslant \left(\|w^0\|_{H^{-1}} + \bar{u}\left\|\frac{(a-C)}{\sqrt{a}}\right\|_{L^2(Q_T)}\right) \|F/\sqrt{a}\|_{L^2(Q_T)}.$$

Since this is true for any smooth F, we can conclude

$$\|\sqrt{a}w\|_{L^2(Q_T)} \leqslant \|w^0\|_{H^{-1}} + \bar{u}\left\|\frac{(a-C)}{\sqrt{a}}\right\|_{L^2(Q_T)}.$$

We end this proof with a remark. Contrarily to the usual diffusion reaction term, we are not limited to consider only nonnegative solutions of (45). Therefore, this estimate cannot be extended to equations with reaction because in this case, we only have an inequality instead of an equality in (46).

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