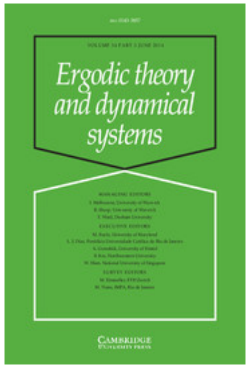

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Abstract. Let $(Y_t : t > 0)$ be a STIT tessellation process and $a > 1$. We prove that the random sequence $(a^n Y_{a^n} : n \in \mathbb{Z})$ is a non-anticipating factor of a Bernoulli shift. We deduce that the continuous time process $(a^t Y_{a^t} : t \in \mathbb{R})$ is a Bernoulli flow. We use the techniques and results in Martínez and Nagel [Ergodic description of STIT tessellations. *Stochastics* **84**(1) (2012), 113–134]. We also show that the filtration associated to the non-anticipating factor is standard in Vershik's sense.

1. *Introduction and main results*

1.1. *Introduction.* A STIT tessellation process $Y = (Y_t : t > 0)$ is a Markov process taking values in the space of tessellations on $\mathbb{R}^\ell$, for some $\ell \geq 1$. The process Y is assumed to be spatially stationary, and it was first constructed in [**9**]. In §1.5 we revisit this construction and recall some of its main properties. In [**6**] it was shown that Y is spatially mixing.

A polytope with non-empty interior W is called a window. The STIT process Y induces a tessellation process on W denoted by $Y \wedge W = (Y_t \wedge W : t > 0)$, which is a pure jump process.

Let $a > 1$. Define the renormalized process $\mathcal{Z} = (\mathcal{Z}_t := a^t Y_{a^t} : t \in \mathbb{R})$ and the discrete process along the integers $\mathcal{Z}^d = (\mathcal{Z}_n : n \in \mathbb{Z})$. In [**7**] it was shown that the renormalized process $\mathcal{Z}$ is time stationary and that for any window W the induced discrete process $\mathcal{Z}^d \wedge W = (\mathcal{Z}_n \wedge W : n \in \mathbb{Z})$ is a non-anticipating factor of a (generalized) Bernoulli shift. In [**7**] it was shown that $\mathcal{Z}^d \wedge W$ has infinite entropy. Ornstein theory implies that $\mathcal{Z}^d \wedge W$ is isomorphic to a Bernoulli shift of infinite entropy and that the time continuous process restricted to a window $\mathcal{Z} \wedge W = (\mathcal{Z}_t \wedge W : t \in \mathbb{R})$ is isomorphic to a Bernoulli flow.

Here we extend the above results from tessellations on windows to tessellations on the whole Euclidean space. We show in Theorem 1.2 that the discrete time process $\mathcal{Z}^d$ is a non-anticipating factor of a Bernoulli shift and that the time continuous $\mathcal{Z}$ is a Bernoulli flow. Let φ be the factor; the non-anticipating property implies that $(\varphi_n : n \leq m)$ is identically distributed to $(\mathcal{Z}_n : n \leq m)$ for all m. We complete our description of the renormalized discrete process by proving in Theorem 1.3 that the filtration $(\sigma(\varphi_n : n \le m) : m \le 0)$ is standard non-atomic in Vershik's sense; this means it is generated by a sequence of independent and identically distributed Uniform[0, 1] random variables.

1.2. *Notation and some basic facts.* We begin by fixing notation. $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \ge 0\}$ is the set of non-negative integers, $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$ the set of (strictly) positive integers and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N} = -\mathbb{Z}_+$ the set of non-positive integers. For a set $C \subseteq \mathbb{R}^\ell$ we denote by $\operatorname{int} C$ its interior, by $\operatorname{cl} C$ its closure and by $\partial C = \operatorname{cl} C \setminus \operatorname{int} C$ its boundary. When dealing with distributions or random elements we use $\sim$ to mean 'identically distributed as', or 'distributed as'.

A metric space $(\mathcal{X}, d)$ is Polish if it is complete and separable. A countable product space $\prod_{l \in \mathbb{N}} \mathcal{X}_l$ of Polish spaces is itself Polish with respect to some metric, for instance $d(x, y) = \sum_{l \in \mathbb{N}} 2^{-l} \min(d_l(x_l, y_l), 1)$, where d_l is the metric on $\mathcal{X}_l$.

We will always consider completed probability spaces $(\mathcal{X}, \mathcal{B}, \nu)$ even if we do not say so explicitly, and if $\mathcal{X}$ is a topological space we will reserve the notation $\mathcal{B}(\mathcal{X})$ for the Borel σ-field.

A Lebesgue probability space is isomorphic to the unit interval [0, 1] endowed with its Borel σ-field and a probability measure which is a convex combination of the Lebesgue measure and a pure atomic measure. Let $(\mathcal{X}, d)$ be a Polish space and ν be a probability measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$; then the probability space $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \nu)$ is Lebesgue (see [**2**]). Hence, if $\mathcal{X}' \in \mathcal{B}(\mathcal{X})$ is a non-empty Borel set and ν' is a probability measure on $(\mathcal{X}', \mathcal{B}(\mathcal{X}'))$ then $(\mathcal{X}', \mathcal{B}(\mathcal{X}'), \nu')$ is a Lebesgue probability space.

Let $(\mathcal{X}, d)$ be a metric space and $D_{\mathcal{X}}(\mathbb{R}_+)$ be the space of càdlàg (right continuous with left limits) trajectories taking values in $\mathcal{X}$, with time in $\mathbb{R}_+ = [0, \infty)$. The space $D_{\mathcal{X}}(\mathbb{R}_+)$ endowed with the Skorohod topology is metrizable (see [**4**, Corollary 5.5 in Ch. 3]). We denote by $d_{\text{Sk}}^{\mathcal{X}}$ the usual metric defining this topology. If $(\mathcal{X}, d)$ is separable then $(D_{\mathcal{X}}(\mathbb{R}_+), d_{\text{Sk}}^{\mathcal{X}})$ is also separable and if $(\mathcal{X}, d)$ is a Polish space then $(D_{\mathcal{X}}(\mathbb{R}_+), d_{\text{Sk}}^{\mathcal{X}})$ is also Polish (see [**4**, Theorem 5.6 in Ch. 3]). The Borel σ-field $\mathcal{B}(D_{\mathcal{X}})$ associated to $(D_{\mathcal{X}}(\mathbb{R}_+), d_{\text{Sk}}^{\mathcal{X}})$ is generated by the class of cylinders (see [**4**, Proposition 7.1]). We can replace the time set $\mathbb{R}_+$ by $\mathbb{R}$ in these definitions and properties.

1.3. *The space of tessellations.* Let us introduce the tessellations on $\mathbb{R}^\ell$, for some $\ell \ge 1$. A *polytope* is the compact convex hull of a finite point set, and we will always assume that it has non-empty interior. A *locally finite covering* of polytopes is a countable family of polytopes whose union is $\mathbb{R}^\ell$ and such that every bounded set can only intersect a finite number of them. The polytopes of a covering are called *cells*.

A *tessellation* T is a locally finite covering of $\mathbb{R}^\ell$ by cells having pairwise disjoint interiors. By $\mathbb{T}$ we mean the space of tessellations of $\mathbb{R}^\ell$. We enumerate the family of (infinite) countable cells of a tessellation $T \in \mathbb{T}$ as $T = \{C(T)^l : l \in \mathbb{N}\}$. If $T \in \mathbb{T}$ is such that the origin 0 belongs to the interior of its cell, the first cell $C(T)^1$ in the enumeration is the one containing 0. The boundary of a tessellation ∂T is the union of the boundaries of its cells, that is $\partial T = \bigcup_{l \in \mathbb{N}} \partial C(T)^l$. Every tessellation T is determined by ∂T, in fact $(\partial T)^c$ is the countable union of its open connected bounded components, and their closures define the family of cells of a unique tessellation.

Let $W \subset \mathbb{R}^\ell$ be a fixed polytope with non-empty interior, called a *window*. A *tessellation in* W is a locally finite countable covering of W by polytopes with disjoint interiors. Let $\mathbb{T}_W$ be the space of tessellations of W. By compactness, each $R \in \mathbb{T}_W$ has a finite number of cells and this number is denoted by $\#R$. The trivial tessellation is $R = \{W\} \in \mathbb{T}_W$, which has boundary ∂W. A cell of $R \in \mathbb{T}_W$ is said to be an *interior cell* (in W) if it does not intersect ∂W.

Let $T \in \mathbb{T}$. For every non-empty set $D \subseteq \mathbb{R}^\ell$ such that $D = \mathrm{cl}(\mathrm{int}\, D)$, we define $T \wedge D = \{C \cap D : C \in T, \mathrm{int}(C \cap D) \neq \emptyset\}$. When $\vec{T} = (T_l : l \in L)$ is a family of tessellations we put $\vec{T} \wedge D = (T_l \wedge D : l \in L)$. Analogously, if $\mathbf{T} = (\vec{T}^j : j \in J)$ is a class of families of tessellations we write $\mathbf{T} \wedge D = (\vec{T}^j \wedge D : j \in J)$.

Let W be a window and $T \in \mathbb{T}$. We have $T \wedge W \in \mathbb{T}_W$. Take a pair of windows W, W' such that $W' \subseteq W$. Every $Q \in \mathbb{T}_W$ defines in the same way as before a tessellation $Q \wedge W' \in \mathbb{T}_{W'}$. For each cell $C' \in Q \wedge W'$ there is a unique cell $C \in Q$ containing it, and C is called the extension of C' in Q (or in W). Each $a \in \mathbb{R}\backslash\{0\}$ defines the tessellation $aT = \{aC : C \in T\}$ where $aC = \{ax : x \in C\}$. Similarly, for a tessellation $Q \in \mathbb{T}_W$ we have $aQ = \{aC : C \in Q\} \in \mathbb{T}_{aW}$.

Let $\beta : \Omega \to \mathbb{T}$ be a function taking values in $\mathbb{T}$ and W be a window. We define the function $\beta \wedge W$ by $\beta \wedge W(\omega) = \beta(\omega) \wedge W$. In a similar way, for a function $\beta^J : \Omega \to \mathbb{T}^J$ we define $\beta^J \wedge W$ and for a function $\beta_W : \Omega \to \mathbb{T}_W$ we define $\beta_W \wedge W'$ for every window $W' \subseteq W$.

1.4. *Measurability considerations.* The family of closed sets $\mathbb{F}$ of $\mathbb{R}^\ell$ endowed with the Fell topology is a metrizable compact Hausdorff space; see [**14**, Ch. 12]. Let $\mathbb{F}' = \mathbb{F}\backslash\{\emptyset\}$ and $\mathbb{F}(\mathbb{F}')$ be the family of closed sets of $\mathbb{F}'$ endowed with the Fell topology and its associated Borel σ-field $\mathcal{B}(\mathbb{F}(\mathbb{F}'))$. Since a tessellation $T \in \mathbb{T}$ is a countable collection of polytopes, it is an element of $\mathbb{F}(\mathbb{F}')$. In [**14**, Lemma 10.1.2] it is shown that $\mathbb{T} \in \mathcal{B}(\mathbb{F}(\mathbb{F}'))$.

We note that the space of boundaries of tessellations is a subset of $\mathbb{F}'$ and is endowed with the trace of the Fell topology and the Borel σ-field. The topological and measurable structures are preserved when representing a tessellation by its boundary; in particular, all sequences $(T_n : n \in \mathbb{N}) \subset \mathbb{T}$ and $T \in \mathbb{T}$ satisfy $T_n \to T \Leftrightarrow \partial T_n \to \partial T$.

Let $\mathbb{F}_W$ be the family of closed subsets of W and $\mathbb{F}'_W = \mathbb{F}_W\backslash\{\emptyset\}$. The set $\mathbb{F}(\mathbb{F}'_W)$ is endowed with the Fell topology and its associated Borel σ-field. We have $\mathbb{T}_W \in \mathcal{B}(\mathbb{F}(\mathbb{F}'_W))$. The σ-field $\mathcal{B}(\mathbb{T}_W)$ will be identified with the sub-σ-field $\mathcal{B}(\mathbb{T}) \wedge W$ of $\mathcal{B}(\mathbb{T})$, defined by

$$\mathcal{B}(\mathbb{T}) \wedge W := \{B = \{T \in \mathbb{T} : T \wedge W \in B_W\} : B_W \in \mathcal{B}(\mathbb{T}_W)\}.$$

Since $\mathbb{T}$ is a Borel set in $\mathbb{F}(\mathbb{F}')$, for any probability measure ν defined on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ the completed probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu)$ is Lebesgue. An analogous statement can be made for the set of tessellations restricted to a window $\mathbb{T}_W$.

Let $(W_l : l \in \mathbb{N})$ be a strictly increasing sequence of windows such that $W_l \subset \mathrm{int}\, W_{l+1}$ and $W_l \nearrow \mathbb{R}^\ell$ as $l \nearrow \infty$. For $m, l \in \mathbb{N}$, $m > l$, define $\gamma_{m,l} : \mathbb{T}_{W_m} \to \mathbb{T}_{W_l}$, $T_m \to \gamma_{m,l}(T_m) = T_m \wedge W_l$. The projective space $\lim_{\leftarrow} \mathbb{T}_{W_l}$ is the set of points $(T_l : l \in \mathbb{N}) \in \prod_{l \in \mathbb{N}} \mathbb{T}_{W_l}$ that satisfy $\gamma_{m,l}(T_m) = T_l$ for $m > l$. We denote by $\gamma_m : \lim_{\leftarrow} \mathbb{T}_{W_l} \to \mathbb{T}_{W_m}$ the projection $\gamma_m(T_l : l \in \mathbb{N}) = T_m$.

Any tessellation $T \in \mathbb{T}$ defines a point $(T_l : l \in \mathbb{N}) \in \lim_{\leftarrow} \mathbb{T}_{W_l}$ by $T_l = T \wedge W_l$ for $l \in \mathbb{N}$. For the converse a certain condition must be satisfied. A sequence $(T_l : l \in \mathbb{N}) \in \lim_{\leftarrow} \mathbb{T}_{W_l}$ is said to satisfy the *finite extension* property if for all $l \in \mathbb{N}$ there exists $m > l$ such that the extensions in T_m of all the cells of T_l, are interior cells in T_m. This property implies for all $m' \geq m$ that the cells of $T_{m'}$ extending the cells of T_l are the same as the cells of T_m extending the cells of T_l. Note that the finite extension property is satisfied when $T_l = T \wedge W_l$ for all $l \in \mathbb{N}$, for some tessellation $T \in \mathbb{T}$. We claim that if $(T_l : l \in \mathbb{N}) \in \lim_{\leftarrow} \mathbb{T}_{W_l}$ satisfies the finite extension property then there is a unique $T \in \mathbb{T}$ such that $T_l = T \wedge W_l$ for all $l \in \mathbb{N}$. Let us show that the tessellation T is characterized by the set of cells $T = \{C(n_0, C_{n_0}) : n_0 \in \mathbb{N}, C_{n_0} \in T_{n_0}\}$, where $C(n_0, C_{n_0}) := \bigcup_{n \geq n_0} C_n$ and $(C_n : n > n_0)$ is uniquely defined by $C_n \in T_n$ and $C_{n_0} \subseteq C_n$ for all $n > n_0$. From the finite extension property there exists $n' \geq n_0$ such that $C_n = C_{n'}$ for all $n \geq n'$. Then every cell $C(n_0, C_{n_0})$ is bounded. Also two such cells $C(n_0, C_{n_0})$, $C(n^*, C_{n^*})$ are either equal or have disjoint interiors. Then $T = \{C(n_0, C_{n_0}) : n \in \mathbb{N}, C_{n_0} \in T_{n_0}\}$ is a locally finite covering of $\mathbb{R}^\ell$ whose cells have pairwise disjoint interiors. So, T is a tessellation and it is clear that $T_l = T \wedge W_l$ for all $l \in \mathbb{N}$. Thus, the claim is shown. Hence, the set of tessellations can be represented as a Borel part of the projective space.

Let us prove that the topology of the space of tessellations inherited from the projective space is the same as that already imposed on $\mathbb{T}$. Since these spaces are metric, it suffices to show that

$$\text{for all } (R_k : k \in \mathbb{N}) \subset \mathbb{T},\ R \in \mathbb{T} : R_k \to R \Leftrightarrow \text{for all } l \in \mathbb{N} : R_k \wedge W_l \to R \wedge W_l. \qquad (1)$$

In the proof we represent tessellations by their boundaries and use [**14**, Theorem 12.2.2]. This theorem implies that $\partial R_k \to \partial R$ if and only if the following two conditions are satisfied:

(a) for all $x \in \partial R$, for all k there exists $x_k \in \partial R_k$ such that $x_k \to x$;

(b) for all subsequences (k_j), if $x_{k_j} \in \partial R_{k_j}$ satisfies $x_{k_j} \to x$, then $x \in \partial R$.

The same holds for the convergence of closed sets restricted to a window. Now let us prove (1). The $\Rightarrow$ part is straightforward because $R \to R \wedge W_l$ is continuous. Let us prove the $\Leftarrow$ part. We must show that (a) and (b) are satisfied. Let $x \in \partial R$. Since $x \in \operatorname{int} W_l$ for some $l \in \mathbb{N}$ we deduce that $x \in \partial(R \wedge W_l)$. Then there exists a sequence (x_k) with $x_k \in \partial(R_k \wedge W_l)$ and such that $x_k \to x$. Since $x \notin \partial W_l$, property (a) follows because $x_k \in \partial R_k$ for all k except for a finite number. Let us prove (b). We can assume that $x \in \operatorname{int} W_l$ for some $l \in \mathbb{N}$, and that the subsequence (x_{k_j}) is in $\partial(R_{k_j} \wedge W_l)$. We get $x \in \partial(R \wedge W_l)$ and from $x \notin \partial W_l$ we obtain $x \in \partial R$. Hence the equivalence (1) is shown. From (1) we deduce the following property for the σ-fields:

$$\mathcal{B}(\mathbb{T}) \wedge W_l \nearrow \mathcal{B}(\mathbb{T}) \quad \text{as } l \nearrow \infty.$$

It is well known that any sequence of probability measures $(\mu_l : l \in \mathbb{N})$, such that μ_l is defined on $\mathbb{T}_{W_l}$ and $\mu_l = \mu_m \circ \gamma_{m,l}^{-1}$ for all $m > l$, defines a unique probability measure μ on $\lim_{\leftarrow} \mathbb{T}_{W_l}$ that satisfies $\mu_m = \mu \circ \gamma_m^{-1}$ for all $m \in \mathbb{N}$. This result can be also retrieved from [**14**, Theorem 2.3.1]. Note that $\mu(\mathbb{T}) = 1$ if and only if μ-almost surely (a.s.) the sequences $(T_n : n \in \mathbb{N}) \in \lim_{\leftarrow} \mathbb{T}_{W_l}$ satisfy the finite extension property.

1.5. *The STIT tessellation process.* Let us give some elements of the construction of the STIT tessellation process $Y = (Y_t : t > 0)$ from [**8, 9**]. Y is a Markov process whose marginals Y_t take values in $\mathbb{T}$. The law of the STIT process Y only depends on a (non-zero) locally finite and translation invariant measure Λ on the space of hyperplanes $\mathcal{H}$ in $\mathbb{R}^\ell$. The set $\mathcal{H}$ is endowed with the trace of the Fell topology and with the associated Borel σ-field.

Let $S^{\ell-1}$ be the set of unit vectors in $\mathbb{R}^{\ell-1}$ and $\widetilde{S}^{\ell-1} = S^{\ell-1}/\equiv$ be the set of equivalence classes for the relation $u \equiv -u$. A measure on $\widetilde{S}^{\ell-1}$ is defined by a measure on $S^{\ell-1}$ that is invariant under $u \to -u$. Each hyperplane $h \in \mathcal{H}$ can be represented by an element in $\mathbb{R} \times \widetilde{S}^{\ell-1}$. The image of Λ under this representation is $\lambda \otimes \kappa$, where λ is the Lebesgue measure on $\mathbb{R}$ and κ is a finite measure on $\widetilde{S}^{\ell-1}$ (see [**14**, Section 4.4 and Theorem 4.4.1]). From local finiteness it follows that

$$\Lambda([B]) < \infty \ \text{ for all } B \text{ bounded in } \mathcal{B}(\mathbb{R}^\ell) \quad \text{where } [B] = \{H \in \mathcal{H} : H \cap B \neq \emptyset\}. \tag{2}$$

It is assumed that the linear space generated by the support of κ is $\mathbb{R}^\ell$; this is written as

$$\langle \text{Support } \kappa \rangle = \mathbb{R}^\ell. \tag{3}$$

Let W be a window. From (2) we get $0 < \Lambda([W]) < \infty$. The translation invariance of Λ yields

$$\Lambda([cW]) = c\Lambda([W]) \quad \text{for all } c > 0 \tag{4}$$

(see, for example, [**14**, Theorem 4.4.1]). Denote by $\Lambda^W(\bullet) = \Lambda([W])^{-1}\Lambda(\bullet \cap [W])$ the normalized probability measure on the set of hyperplanes intersecting W.

Let us construct $Y^W = (Y_t^W : t \geq 0)$, which is a pure jump Markov process whose marginals take values in $\mathbb{T}_W$. Let $D = (d_{n,m} : n \in \mathbb{Z}_+, m \in \mathbb{N})$ and $\tau = (\tau_{n,m} : n \in \mathbb{Z}_+, m \in \mathbb{N})$ be two independent families of independent random variables with distributions $d_{n,m} \sim \Lambda^W$ and $\tau_{n,m} \sim \text{Exponential}(1)$. We will define an increasing sequence of random times $(S_n : n \in \mathbb{Z}_+)$ and a sequence of random tessellations $(Y_{S_n}^W : n \in \mathbb{Z}_+)$ with starting points $S_0 = 0$ and $Y_0^W = \{W\}$. We will fix

$$Y_t^W = Y_{S_n}^W, \qquad t \in [S_n, S_{n+1}). \tag{5}$$

The sequences $(S_n : n \geq 0)$ and $(Y_{S_n}^W : n \geq 0)$ are defined by an inductive procedure. Let $\{C_n^1, \ldots, C_n^{l_n}\}$ be the cells of $Y_{S_n}^W$; we put

$$S_{n+1} = S_n + \tau(Y_{S_n}^W), \quad \text{where } \tau(Y_{S_n}^W) = \min\{\tau_{n,l}/\Lambda([C_n^l]) : l = 1, \ldots, l_n\}.$$

Let l^* be such that $\tau_{n,l^*}/\Lambda([C_n^{l^*}]) = \tau(Y_{S_n}^W)$ (it is a.s. uniquely defined). We take m as the first index such that $d_{n,m} \in [C_n^{l^*}]$, so $d_{n,m} \sim \Lambda^{C_n^{l^*}}$. The tessellation $Y_{S_{n+1}}^W$ is the one whose cells are $\{C_n^l : l \neq l^*\} \cup \{C_1', C_2'\}$, where C_1', C_2' is the partition of $C_n^{l^*}$ by the hyperplane $d_{n,m}$.

The process Y^W is a pure jump Markov process. This construction yields a law that is consistent with respect to W, which means that if W and W' are windows and $W' \subseteq W$, then $Y^W \wedge W' \sim Y^{W'}$. Moreover, as a consequence of (3), we get that a.s. for all W' there exists W (depending on the realization) containing W' such that the extensions in Y^W of all the cells of $Y^{W'}$ are interior cells in Y^W. This is the finite extension property.

From [**14**, Theorem 2.3.1] follows the existence of the marginals Y_t with values in $\mathbb{T}$ and such that $Y_t \wedge W \sim Y_t^W$ for every window W and all $t > 0$. In [**8**] it was shown that $Y = (Y_t : t > 0)$ is a well-defined Markov process, and this is a STIT tessellation process.

From this construction, and since S_1 is exponentially distributed with parameter $\Lambda([W])$, we get

$$\mathbb{P}(\partial(Y_t \wedge W) \cap \operatorname{int} W = \emptyset) = \mathbb{P}(Y_t \wedge W = \{W\}) = \mathbb{P}(Y_t \wedge W = Y_0 \wedge W) = e^{-t\Lambda([W])}.$$

For $t > 0$ define the marginal distributions

$$\xi^t(B) = \mathbb{P}(Y_t \in B) \text{ for all } B \in \mathcal{B}(\mathbb{T}) \quad \text{and} \quad \xi^t_W(D) = \mathbb{P}(Y_t \wedge W \in D) \text{ for all } D \in \mathcal{B}(\mathbb{T}_W).$$

Let us prove that

$$\xi^t \text{ is non-atomic and } \xi^t_W \text{ has a unique atom } \{W\}. \tag{6}$$

Let us first show the statement for ξ^t_W. Note that $\xi^t_W(\{W\}) = e^{-t\Lambda([W])} > 0$, so $\{W\}$ is an atom. Let us prove that it is unique. Assume that for some tessellation $Q_0 \in \mathbb{T}_W$, $Q_0 \neq \{W\}$, we have $\xi^t_W(\{Q_0\}) > 0$. From the construction there is a hyperface r of Q_0 contained in some hyperplane $H \in \mathcal{H}$. Since Λ is translation invariant and σ-finite, $\Lambda^W(\{H\}) = 0$ for all $H \in \mathcal{H}$. Consequently $\xi^t_W(\{Q_0\}) = 0$, so $\{W\}$ is the unique atom of ξ^t_W. (This argument was given in [**7**].) Now let us prove that ξ^t is non-atomic. If there exists T_0 such that $\xi^t(\{T_0\}) > 0$, we will necessarily have $\xi^t_W(\{T_0 \wedge W\}) > 0$ and so $T_0 \wedge W = \{W\}$ for all windows W. We will deduce $T_0 = \{\mathbb{R}^d\}$, which is in contradiction to $\mathbb{P}(Y_t = \{\mathbb{R}^d\}) = 0$.

From (5) it follows that $Y \wedge W$ is a pure jump Markov process with càdlàg trajectories. Then a.s. there exists $\lim_{h \searrow 0^+} Y_{t-h} \wedge W$ in $\mathbb{T}_W$ and $\lim_{h \searrow 0^+} Y_{t+h} \wedge W = Y_t \wedge W$. From (1) we deduce that also the STIT process Y has càdlàg trajectories. Then the trajectories of $Y \wedge W$ belong to $D_{\mathbb{T}_W}(\mathbb{R}_+)$ and the trajectories of Y are in $D_{\mathbb{T}}(\mathbb{R}_+)$.

Let $d^{\mathbb{T}}_{\mathrm{Sk}}$ be the usual metric defining the Skorohod topology on $D_{\mathbb{T}}(\mathbb{R}_+)$ and let $\mathcal{B}(D_{\mathbb{T}})$ be the Borel σ-field. The space $(D_{\mathbb{T}}(\mathbb{R}_+), d^{\mathbb{T}}_{\mathrm{Sk}})$ is separable. If the time set is $\mathbb{R}$ we write $D_{\mathbb{T}}(\mathbb{R})$, while $d^{\mathbb{T}}_{\mathrm{Sk}}$ and $\mathcal{B}(D_{\mathbb{T}})$ continue to denote the metric and the Borel σ-field, respectively. We use the same statements and notation for $d^{\mathbb{T}_W}_{\mathrm{Sk}}$, $\mathcal{B}(D_{\mathbb{T}_W})$ and $D_{\mathbb{T}_W}(\mathbb{R})$.

Let us consider the closure $\overline{\mathbb{T}}$ of $\mathbb{T}$ in $\mathbb{F}(\mathbb{F}')$. The space $D_{\overline{\mathbb{T}}}(\mathbb{R}_+)$ is endowed with the Skorohod topology generated by a metric $d^{\overline{\mathbb{T}}}_{\mathrm{Sk}}$. Since $(\overline{\mathbb{T}}, d^{\overline{\mathbb{T}}}_{\mathrm{Sk}})$ is a Polish space $(D_{\overline{\mathbb{T}}}(\mathbb{R}_+), d^{\overline{\mathbb{T}}}_{\mathrm{Sk}})$ is also Polish. Then we can consider that the trajectories of Y take values in the Polish space $(D_{\overline{\mathbb{T}}}(\mathbb{R}_+), d^{\overline{\mathbb{T}}}_{\mathrm{Sk}})$. For $D_{\overline{\mathbb{T}}}(\mathbb{R})$, $d^{\overline{\mathbb{T}}}_{\mathrm{Sk}}$ and $\mathcal{B}(D_{\overline{\mathbb{T}}})$ continue to denote the metric and the Borel σ-field, respectively. This is also done for $D_{\overline{\mathbb{T}_W}}(\mathbb{R})$.

The following scaling property was shown in [**9**, Lemma 5]. It is further used to state the renormalization in time and space:

$$\text{for all } t > 0, \quad tY_t \sim Y_1. \tag{7}$$

1.6. *Independent increments relation.* Let $T \in \mathbb{T}$ be a tessellation and $\vec{R} = (R^k : k \in \mathbb{N}) \in \mathbb{T}^{\mathbb{N}}$ be a sequence of tessellations. We define the tessellation $T \boxplus \vec{R}$ (also referred to as iteration or nesting) by its set of cells

$$T \boxplus \vec{R} = \{C(T)^k \cap C(R^k)^l : k \in \mathbb{N}, l \in \mathbb{N}, \operatorname{int}(C(T)^k \cap C(R^k)^l) \neq \emptyset\}.$$

Thus, we restrict the tessellation R^k to the cell $C(T)^k$, and this is done for all $k \in \mathbb{N}$. The same definition is made for tessellations and sequences of tessellations restricted to some window.

Let us fix a copy of the random process Y and let $\vec{Y}' = (Y'^m : m \in \mathbb{N})$ be a sequence of independent copies of Y, all of them being also independent of Y. In particular, $Y'^m \sim Y$. We put $\vec{Y}'_s = (Y'^m_s : m \in \mathbb{N})$ for $s > 0$. From the construction of Y the following property holds:

$$Y_{t+s} \sim Y_t \boxplus \vec{Y}'_s \quad \text{for all } t, s > 0.$$

This relation was first stated in [**9**, Lemma 2]. The construction made in [**9**] to prove this result also allows us to show the following relation stated in [**7**]. Let $\vec{Y}'^{(i)}$, $i = 1, \ldots, j$, be a sequence of j independent copies of $\vec{Y}'$ and also independent of Y. Then, for all $0 < s_1 < \cdots < s_j$ and all $t > 0$,

$$(Y_t, Y_{t+s_1}, \ldots, Y_{t+s_j}) \sim (Y_t, Y_t \boxplus \vec{Y}'^{(1)}_{s_1}, \ldots, (((Y_t \boxplus \vec{Y}'^{(1)}_{s_1}) \boxplus \cdots) \boxplus \vec{Y}'^{(j)}_{s_j - s_{j-1}})). \quad (8)$$

1.7. *Elements of ergodic theory.* An *abstract dynamical system* $(\Omega, \mathcal{B}(\Omega), \mu, \psi)$ is such that $(\Omega, \mathcal{B}(\Omega), \mu)$ is a Lebesgue probability space and $\psi : \Omega \to \Omega$ is a measure-preserving measurable transformation, that is, $\mu \circ \psi^{-1} = \mu$. To avoid overburdening our notation, instead of $(\Omega, \mathcal{B}(\Omega), \mu, \psi)$ we simply put (Ω, μ, ψ). For two dynamical systems (Ω, μ, ψ) and (Ω', μ', ψ'), the measurable map $\varphi : \Omega \to \Omega'$ is a *factor map* if $\varphi \circ \psi = \psi' \circ \varphi$ μ-a.s. and $\mu \circ \varphi^{-1} = \mu'$. If a factor map φ is a.s. a one-to-one surjection, then φ is an isomorphism.

Let $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ be a measurable space, where $\mathcal{S}$ is a Polish space (with respect to some metric) and $\mathcal{B}(\mathcal{S})$ is its Borel σ-field. Let $L = \mathbb{Z}_+$ or $L = \mathbb{Z}$. The shift transformation $\sigma_{\mathcal{S}} : \mathcal{S}^L \to \mathcal{S}^L$, $x \to \sigma_{\mathcal{S}}(x)$ with $\sigma_{\mathcal{S}}(x)_n = x_{n+1}$ for $n \in L$, is measurable. A dynamical system $(\mathcal{S}^L, \mu, \sigma_{\mathcal{S}})$ is called a *shift system*. Let $\mathcal{Y}^d = (\mathcal{Y}_n : n \in L)$ be a stationary sequence with state space $\mathcal{S}$ and $\mu^{\mathcal{Y}^d}$ be the distribution of $\mathcal{Y}^d$ on $\mathcal{S}^L$. The stationary property of $\mathcal{Y}^d$ means that $\mu^{\mathcal{Y}^d}$ is $\sigma_{\mathcal{S}}$-invariant and so $(\mathcal{S}^L, \mu^{\mathcal{Y}^d}, \sigma_{\mathcal{S}})$ is a shift system. Let $(\mathcal{S}, \mathcal{B}(\mathcal{S}), \nu_{\mathcal{S}})$ be a Lebesgue probability space; the shift $\sigma_{\mathcal{S}}$ preserves the product probability measure $\nu_{\mathcal{S}}^{\otimes L}$ and $(\mathcal{S}^L, \nu_{\mathcal{S}}^{\otimes L}, \sigma_{\mathcal{S}})$ is a Bernoulli shift. A dynamical system is said to be Bernoulli if it is isomorphic to a Bernoulli shift. The Ornstein isomorphism theorem states that two-sided Bernoulli shifts with the same entropy are isomorphic (see [**10, 11**]).

The inverse one-sided shift is given by $\sigma_{\mathcal{S}}^- : \mathcal{S}^{\mathbb{Z}_-} \to \mathcal{S}^{\mathbb{Z}_-}$, $(\sigma_{\mathcal{S}}^-(x))_n = x_{n-1}$ for $n \in \mathbb{Z}_-$. We set $\sigma_{\mathcal{S}}^{-n} := (\sigma_{\mathcal{S}}^-)^n$ for $n \in \mathbb{Z}_+$. The one-sided Bernoulli shifts $(\mathcal{S}^{\mathbb{Z}_+}, \nu^{\otimes \mathbb{Z}_+}, \sigma_{\mathcal{S}})$ and $(\mathcal{S}^{\mathbb{Z}_-}, \nu^{\otimes \mathbb{Z}_-}, \sigma_{\mathcal{S}}^-)$ are isomorphic.

Let $(\mathcal{S}^{\mathbb{Z}}, \mu, \sigma_{\mathcal{S}})$ and $(\mathcal{S}'^{\mathbb{Z}}, \mu', \sigma_{\mathcal{S}'})$ be two shift systems. A factor map $\varphi : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}'^{\mathbb{Z}}$ is *non-anticipating* if μ-a.s. in $x \in \mathcal{S}^{\mathbb{Z}}$ the coordinate $(\varphi(x))_n$ only depends on $(x_m : m \le n)$. By shift invariance it suffices that this condition is satisfied for some $n \in \mathbb{Z}$.

A flow (or continuous time dynamical system) $(\Omega, \mu, (\psi^t : t \in \mathbb{R}))$ is such that $(\Omega, \mathcal{B}(\Omega), \mu)$ is a Lebesgue probability space, $\mu \circ (\psi^t)^{-1} = \mu$ for $t \in \mathbb{R}$, $\psi^{t+s} = \psi^t \circ \psi^s$ μ-a.s. for $t, s \in \mathbb{R}$, and the joint application $[0, \infty) \times \Omega \to \Omega$, $(t, \omega) \to \psi^t(\omega)$ is measurable. The entropy of the flow is by definition the entropy of (Ω, μ, ψ^1). The shift flows are defined with respect to the shift transformations $\sigma^t(x_s : s \in \mathbb{R}) = (x_{s+t} : s \in \mathbb{R})$ for $t \in \mathbb{R}$. A stationary random process $\mathcal{Y} = (\mathcal{Y}_t : t \in \mathbb{R})$ with càdlàg trajectories, and whose marginals take values in a Polish space, defines a flow. In fact, for all $x \in D_{\mathcal{S}}(\mathbb{R})$ the trajectory $(\sigma^t(x) : t \in \mathbb{R})$ is right continuous in $t \in \mathbb{R}$ and so the joint application $[0, \infty) \times D_{\mathcal{S}}(\mathbb{R}) \to D_{\mathcal{S}}(\mathbb{R})$, $(t, \omega) \to \psi^t(\omega)$, is measurable (see, for instance, [**1**, Theorem 6.11 of Ch. I]).

A Bernoulli flow $(\Omega, \mu, (\psi^t : t \in \mathbb{R}))$ is a flow such that (Ω, μ, ψ^1) is isomorphic to a Bernoulli shift. The isomorphism theorem for Bernoulli flows (see [**13**]) states that two Bernoulli flows with the same entropy are isomorphic.

1.8. *Standardness.* Let us define standardness of filtrations in Vershik's sense, as introduced in [**15**]. We will restrict the discussion to Lebesgue probability spaces.

Let $(\Omega, \mathcal{B}, \nu)$ be a non-atomic Lebesgue probability space. A filtration of (complete) σ-fields $\mathcal{K} = (\mathcal{K}_n : n \in \mathbb{Z}_-)$ (contained in $\mathcal{B}$) is said to be *standard non-atomic* if $\mathcal{K}_n = \sigma(\theta_l : l \le n)$ for all $n \in \mathbb{Z}_-$, for some sequence $(\theta_n : n \in \mathbb{Z}_-)$ of independent and identically distributed Uniform[0, 1] random variables.

A filtration $\mathcal{G} = (\mathcal{G}_n : n \in \mathbb{Z}_-)$ (contained in $\mathcal{B}$) is *standard* if for some standard non-atomic filtration $\mathcal{K} = (\mathcal{K}_n : n \in \mathbb{Z}_-)$ the filtration $\mathcal{G}$ is *immersible* in $\mathcal{K}$. This means $\mathcal{G}_n \subseteq \mathcal{K}_n$ for all $n \in \mathbb{Z}_-$ and every $\mathcal{G}$-martingale is a $\mathcal{K}$-martingale.

The equivalence between condition (i) of Theorem 1 and condition (v) of Theorem 2 stated in [**3**] implies that: if a filtration $\mathcal{G} = (\mathcal{G}_n : n \in \mathbb{Z}_-)$ satisfies that $\mathcal{G}_0$ is separable, $\mathcal{G}_n = \sigma(\mathcal{G}_{n-1}, \alpha_n)$ with α_n independent from $\mathcal{G}_{n-1}$ for all $n \in \mathbb{Z}_-$ and $(\alpha_n : n \in \mathbb{Z}_-)$ is a sequence of independent and identically distributed Uniform[0, 1] random variables; then

$$\mathcal{G} \text{ is standard } \Leftrightarrow \mathcal{G} \text{ is standard non-atomic.} \tag{9}$$

In relation to this result, see the discussion following [**3**, Corollary 5]; other related results can be found in [**5, 15**].

1.9. *Renormalized process and previous results.* Let Y be a STIT tessellation process. Fix $a > 1$ and define the renormalized process $\mathcal{Z} = (\mathcal{Z}_s : s \in \mathbb{R})$ by

$$\mathcal{Z}_s = a^s Y_{a^s}, \quad s \in \mathbb{R}.$$

Note that $\mathcal{Z}_0 = Y_1$. Since $(Y_t : t > 0)$ is a Markov process, so is $(\mathcal{Z}_s : s \in \mathbb{R})$. From (7) all one-dimensional distributions of $(\mathcal{Z}_s : s \in \mathbb{R})$ are identical. In [**7**] it was shown that $\mathcal{Z}$ is a stationary Markov process, that is, $\mathcal{Z} \sim \sigma_{\mathbb{T}}^t \circ \mathcal{Z}$ for all $t \in \mathbb{R}$, where $(\sigma_{\mathbb{T}}^t \circ \mathcal{Z})_s = \mathcal{Z}_{s+t}$ for $s \in \mathbb{R}$.

The process $\mathcal{Z}$ inherits càdlàg trajectories from Y, so the trajectories of $\mathcal{Z}$ belong to $D_{\mathbb{T}}(\mathbb{R})$. Let $\mu^{\mathcal{Z}}$ be the law of $\mathcal{Z}$ on $D_{\mathbb{T}}(\mathbb{R})$. Since the process $\mathcal{Z}$ is stationary, $(D_{\mathbb{T}}(\mathbb{R}), \mu^{\mathcal{Z}}, (\sigma_{\mathbb{T}}^t : t \in \mathbb{R}))$ is a shift flow. Similarly for the process $\mathcal{Z} \wedge W = (\mathcal{Z}_s \wedge W : s \in \mathbb{R})$ restricted to a window W. Let $\mu_W^{\mathcal{Z}}$ be the law of this process on $D_{\mathbb{T}_W}(\mathbb{R})$. In [**7**] it was proved that $(D_{\mathbb{T}_W}(\mathbb{R}), \mu_W^{\mathcal{Z}}, (\sigma_{\mathbb{T}}^t : t \in \mathbb{R}))$ is a mixing shift flow.

Let $\mathcal{Z}^d = (\mathcal{Z}_n : n \in \mathbb{Z})$ be the restriction of $\mathcal{Z}$ to integer times. The law $\mu^{\mathcal{Z}^d}$ of $\mathcal{Z}^d$ on $\mathbb{T}^{\mathbb{Z}}$ is $\sigma_{\mathbb{T}}$-invariant. For a window W, the law $\mu_W^{\mathcal{Z}^d}$ of $\mathcal{Z}^d \wedge W = (\mathcal{Z}_n \wedge W : n \in \mathbb{Z})$ on $\mathbb{T}_W^{\mathbb{Z}}$ is $\sigma_{\mathbb{T}_W}$-invariant.

Let $\xi = \xi^1$ be the law of $Y_1 = \mathcal{Z}_0$ and $\xi_W = \xi_W^1$ be the law of $Y_1 \wedge W = \mathcal{Z}_0 \wedge W$. We write

$$\varrho = \xi^{\otimes \mathbb{N}} \quad \text{and} \quad \varrho_W = \xi_W^{\otimes \mathbb{N}}.$$

The following ergodic properties of Y were shown in [**7**].

THEOREM 1.1. *Let W be a window. The dynamical system $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}^d}, \sigma_{\mathbb{T}_W})$ is a factor of the Bernoulli shift $((\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}, \varrho_W^{\otimes\mathbb{Z}}, \sigma_{\mathbb{T}_W^{\mathbb{N}}})$, that is, there exists $\varphi_W : (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}} \to \mathbb{T}_W^{\mathbb{Z}}$, measurable and defined $\varrho_W^{\otimes\mathbb{Z}}$-a.s., satisfying*

$$\sigma_{\mathbb{T}_W} \circ \varphi_W = \varphi_W \circ \sigma_{\mathbb{T}_W^{\mathbb{N}}} \quad \varrho_W^{\otimes\mathbb{Z}}\text{-a.s.} \tag{10}$$

and

$$\varrho_W^{\otimes\mathbb{Z}} \circ \varphi_W^{-1} = \mu_W^{\mathcal{Z}^d}. \tag{11}$$

The factor map φ_W is non-anticipating. Moreover, for all $m \in \mathbb{Z}$, $\varrho_W^{\otimes\mathbb{Z}}$-a.s. in $\mathbf{R}^W = (\vec{R}_n^W : n \in \mathbb{Z}) \in (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}$, the coordinate $\varphi_W(\mathbf{R}^W)_m$ depends only on a finite set of coordinates $(\vec{R}_n^W : n \in [m-N, m])$ of the point $\mathbf{R}^W$.

Furthermore, the shift system $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}^d}, \sigma_{\mathbb{T}_W})$ is isomorphic to a Bernoulli shift of infinite entropy and $(D_{\overline{\mathbb{T}_W}}(\mathbb{R}), \mu_W^{\mathcal{Z}}, (\sigma_{\overline{\mathbb{T}_W}}^t : t \in \mathbb{R}))$ is a Bernoulli flow of infinite entropy.

The last part of this theorem follows from Ornstein theory (see [**12, 13**]).

1.10. *Main results.* Let us extend the above result from the process on tessellations on windows to the process on tessellations on the whole Euclidean space.

Let W be a window. The mapping $\Phi_W : D_{\mathbb{T}}(\mathbb{R}) \to D_{\mathbb{T}_W}(\mathbb{R})$, $\mathcal{Z} \to \Phi_W(\mathcal{Z}) = \mathcal{Z} \wedge W$, is a factor from $(D_{\mathbb{T}}(\mathbb{R}), \mu^{\mathcal{Z}}, (\sigma_{\mathbb{T}}^t : t \in \mathbb{R}))$ onto $(D_{\mathbb{T}_W}(\mathbb{R}), \mu_W^{\mathcal{Z}}, (\sigma_{\mathbb{T}_W}^t : t \in \mathbb{R}))$. This factor is obviously non-anticipating. Analogously in the discrete case, $\Phi_W^d : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}_W^{\mathbb{Z}}$, $\Phi_W(\mathcal{Z}^d) = \mathcal{Z}^d \wedge W$, is a factor from $(\mathbb{T}^{\mathbb{Z}}, \mu^{\mathcal{Z}}, \sigma_T)$ onto $(\mathbb{T}_W^{\mathbb{Z}}, \mu_W^{\mathcal{Z}}, \sigma_{T_W})$. From Theorem 1.1 we deduce that $(\mathbb{T}^{\mathbb{Z}}, \mu^{\mathcal{Z}}, \sigma_T)$ has a Bernoulli factor with infinite entropy, so it has infinite entropy. Hence, $(D_{\mathbb{T}}(\mathbb{R}), \mu^{\mathcal{Z}}, (\sigma_{\mathbb{T}}^t : t \in \mathbb{R}))$ also has infinite entropy.

Let $\mathbf{R} = (\vec{R}_n : n \in \mathbb{Z})$ be a random element distributed as $\mathbf{R} \sim \varrho^{\mathbb{Z}}$. Then the components $\vec{R}_n = (R_n^m : m \in \mathbb{N})$, $n \in \mathbb{N}$, are independent sequences of tessellations and the distribution of each one of them is ϱ. We also use $\mathbf{R} \in (\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}}$ to denote a realization of the random vector. We have $\mathbf{R} \wedge W \sim \varrho_W^{\mathbb{Z}}$, where $\mathbf{R} \wedge W := (\vec{R}_n \wedge W : n \in \mathbb{Z}) \in (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}$. By using this representation we have $(\mathbf{R} \wedge W) \wedge W' \sim \varrho_{W'}^{\mathbb{Z}}$ for every pair of windows W, W' such that $W' \subseteq W$.

In the next result φ_W refers to the factor in Theorem 1.1. We point out that this map is $\varrho^{\otimes\mathbb{Z}}$-a.s. defined in the following sense: $\varphi_W(\mathbf{R} \wedge W)$ is well defined for points $\mathbf{R} \in (\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}}$ in a set of full $\varrho^{\otimes\mathbb{Z}}$-measure. In the following we shall identify the σ-field $\mathcal{B}((\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}})$ with the following sub-σ-field of $\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}})$:

$$\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}}) \wedge W := \{B = \{\mathbf{R} \in (\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}} : \mathbf{R} \wedge W \in B_W\} : B_W \in \mathcal{B}((\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}})\}. \tag{12}$$

In a similar way, for a pair of windows W, W' such that $W' \subseteq W$, we identify $\mathcal{B}((\mathbb{T}_{W'}^{\mathbb{N}})^{\mathbb{Z}})$ with

$$\mathcal{B}((\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}) \wedge W' := \{B = \{\mathbf{R}_W \in (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}} : \mathbf{R}_W \wedge W' \in B_{W'}\} : B_{W'} \in \mathcal{B}((\mathbb{T}_{W'}^{\mathbb{N}})^{\mathbb{Z}})\},$$

which is a sub-σ-field of $\mathcal{B}((\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}})$.

THEOREM 1.2. *The shift system* $(\mathbb{T}^{\mathbb{Z}}, \mu^{\mathcal{Z}^d}, \sigma_{\mathbb{T}})$ *is a factor of the Bernoulli shift* $((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}}, \varrho^{\otimes\mathbb{Z}}, \sigma_{\mathbb{T}^{\mathbb{N}}})$*, that is, there exists* $\varphi : (\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$*, measurable and defined* $\varrho^{\otimes\mathbb{Z}}$*-a.s., satisfying*

$$\sigma_{\mathbb{T}} \circ \varphi = \varphi \circ \sigma_{\mathbb{T}^{\mathbb{N}}} \quad \varrho^{\otimes\mathbb{Z}}\text{-a.s.}$$

and

$$\varrho^{\otimes\mathbb{Z}} \circ \varphi^{-1} = \mu^{\mathcal{Z}^d}.$$

The factor map φ *is non-anticipating: for all* $m \in \mathbb{Z}$*,* $\varrho^{\otimes\mathbb{Z}}$*-a.s. in* $\mathbf{R} = (\vec{R}_n : n \in \mathbb{Z}) \in (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}}$*, the coordinate* $\varphi(\mathbf{R})_m$ *only depends on* $(\vec{R}_n : n \le m)$*.*

Besides, $\varphi \wedge W = \varphi_W$ $\varrho^{\otimes\mathbb{Z}}$*-a.s. and* $\varphi_W \wedge W' = \varphi_{W'}$ $\varrho^{\otimes\mathbb{Z}}$*-a.s. for each pair of windows* W, W' *such that* $W' \subseteq W$*.*

Finally, $(D_{\overline{\mathbb{T}}}(\mathbb{R}), \mu^{\mathcal{Z}}, (\sigma_{\mathbb{T}}^{\underline{t}} : t \in \mathbb{R}))$ *is a Bernoulli flow of infinite entropy.*

Since φ is non-anticipating, a consequence of this theorem is that $(\varphi_n : n \le m) \sim (\mathcal{Z}_n : n \le m)$ for all $m \in \mathbb{Z}$. Concerning standardness we will prove the following result.

THEOREM 1.3. *Let* φ *be the non-anticipating factor of Theorem 1.2, and define* $\mathcal{G}_n = \sigma(\varphi_k : k \le n)$*. Then the filtration* $\mathcal{G} = (\mathcal{G}_n : n \in \mathbb{Z}_-)$ *is standard non-atomic.*

2. *Proof of Theorem 1.2*

We can assume that throughout the evolution of the STIT tessellation process Y, the origin 0 belongs to the interior of the cell containing it a.s. As stated, the first cell $C(T)^1$ of a tessellation T is the one containing 0. For $b > 1$, $T \in \mathbb{T}$, we have already defined $bT \in \mathbb{T}$. When $\vec{R} = (R^m : m \in \mathbb{N}) \in \mathbb{T}^{\mathbb{N}}$ we put $b\vec{R} = (bR^m : m \in \mathbb{N})$. We recall that $a > 1$.

Let $(\vec{Y}_1'^{(i)} : i \in \mathbb{N})$ be independent copies of $\vec{Y}_1'$. By using (8) we get

$$(\mathcal{Z}_{n+k} : k \in \mathbb{Z}_+) \sim \left(a^k \mathcal{Z}_n \boxplus_{i=1}^{k} \frac{a^{k+1-i}}{a-1} \vec{Y}_1'^{(i)} : k \in \mathbb{Z}_+\right),$$

where $M \boxplus_{i=1}^{k} \vec{M}'^{(i)} = (\cdots(M \boxplus \vec{M}'^{(1)}) \boxplus \cdots) \boxplus \vec{M}'^{(k)}$, for a tessellation M and a family of sequences of tessellations $(\vec{M}'^{(i)} : i \in \mathbb{N})$.

Let W be a window. Since the process is translation invariant we can assume that W contains the origin in its interior. Let $\vec{R}^- = (R^k : k \in \mathbb{Z}_-)$ be a random sequence of independent copies of Y_1, so $\vec{R}^- \sim \xi^{\otimes\mathbb{Z}_-}$. In [**7**, Lemma 2.1], equation (4) is used to show that

$$\mathbb{P}(\forall k \in \mathbb{Z}_- : \partial R^k \cap \operatorname{int}(a^k W) = \emptyset) = \mathbb{P}(\partial R^0 \cap \operatorname{int}(W) = \emptyset)^{a/(a-1)} > 0,$$

and from the ergodic theorem it is deduced that

$$\mathbb{P}(\exists (n_i \ge 1 : i \in \mathbb{Z}_+) \nearrow, \forall i \in \mathbb{Z}_+ \, \forall k \in \mathbb{Z}_- : \partial R^{-n_i+k} \cap \operatorname{int}(a^k W) = \emptyset) = 1. \tag{13}$$

Below we shall encounter the following evolution equation: for $T^0, R^0 \in \mathbb{T}$ and $(\vec{Q}_n : n \in \mathbb{N})$ a family of sequences of tessellations,

$$\text{for all } n \in \mathbb{N}, \quad T^{n+1} = \left(aT^n \boxplus \frac{a}{a-1} \vec{Q}_{n+1}\right) \wedge W,$$
$$R^{n+1} = \left(aR^n \boxplus \frac{a}{a-1} \vec{Q}_{n+1}\right) \wedge W. \tag{14}$$

It is easily shown that these types of sequences satisfy

$$T^0 \wedge a^{-n} W = R^0 \wedge a^{-n} W \Rightarrow T^n \wedge W = R^n \wedge W \tag{15}$$

(see [**7**, Lemma 2.2]). Let us prove that the functions φ_W in Theorem 1.1 can be constructed in such a way that they satisfy the following projective property.

LEMMA 2.1. *Let* $(W_l : l \in \mathbb{N})$ *be a strictly increasing sequence of windows such that* $W_l \subset \operatorname{int} W_{l+1}$ *for all* l *and* $W_l \nearrow \mathbb{R}^\ell$. *Then there exists a sequence of functions* $(\varphi_{W_l} : l \in \mathbb{N})$ *which satisfies (10) and (11) and the other properties mentioned in Theorem 1.1, and*

$$\text{for all } l \geq k, \quad \varphi_{W_l} \wedge W_k = \varphi_{W_k} \quad \varrho^{\mathbb{Z}}\text{-a.s.} \tag{16}$$

Proof. Let us give an equivalent but slightly different construction of the factor map $\varphi_W : (\mathbb{T}_W^{\mathbb{N}})^{\mathbb{Z}} \to \mathbb{T}_W^{\mathbb{Z}}$, with respect to the one supplied in [**7**]. Property (13) applied to the sequences $(R_n^1 \wedge W : n \in \mathbb{Z}_-)$ shows that $\varrho^{\otimes \mathbb{Z}}$-a.s. the set of $n \in \mathbb{N}$ that satisfy $\partial(R_{k-n}^1 \wedge W) \cap \operatorname{int}(a^k W) = \emptyset$ for all $k \in \mathbb{Z}_-$ is infinite. Then we can order them as an increasing sequence $N_i^W = N_i^W(\mathbf{R} \wedge W)$, $i \in \mathbb{N}$, and we get

$$\text{for all } i \in \mathbb{N},\ N_i^W \geq 1,\ N_i^W \nearrow \infty \quad \text{and} \quad \partial(R^1_{k-N_i^W} \wedge W) \cap \operatorname{int}(a^k W) = \emptyset \text{ for all } k \in \mathbb{Z}_-. \tag{17}$$

For $i \in \mathbb{N}$ we define the function $\varphi_W^i(\mathbf{R} \wedge W)$ by

$$\begin{aligned} &\text{for all } n \leq -N_i^W, \quad (\varphi_W^i(\mathbf{R} \wedge W))_n = \{W\}, \\ &\text{for all } n \geq -N_i^W, \quad (\varphi_W^i(\mathbf{R} \wedge W))_{n+1} = \left(a(\varphi_W^i(\mathbf{R} \wedge W))_n \boxplus \frac{a}{a-1} \vec{R}_n \right) \wedge W. \end{aligned} \tag{18}$$

This construction is done for all $i \in \mathbb{N}$. Since the operation $\boxplus$ depends on the enumeration of the cells of the tessellations, we need to point out two facts. First, from (17) and since the evolution equation (18) satisfies (14), we use (15) to obtain

$$\text{for all } j \geq i \geq 1, \quad (\varphi_W^j(\mathbf{R} \wedge W))_{N_i^W} = \{W\}. \tag{19}$$

This first property guarantees the following one. We can perform the construction in (18) such that for $j \geq i$ the cells in the tessellations $(\varphi_W^j(\mathbf{R} \wedge W))_n$ are enumerated in the same way as in $(\varphi_W^i(\mathbf{R} \wedge W))_n$, for the coordinates $n \geq -N_i^W$. This ensures that

$$\text{for all } j \geq i, \ \text{for all } n \geq -N_i^W, \quad (\varphi_W^j(\mathbf{R} \wedge W))_n = (\varphi_W^i(\mathbf{R} \wedge W))_n.$$

Therefore, there exists

$$\varphi_W(\mathbf{R} \wedge W) = \lim_{j \to \infty} \varphi_W^j(\mathbf{R} \wedge W) \quad \varrho^{\otimes \mathbb{Z}}\text{-a.s.}$$

which satisfies

$$\text{for all } i \in \mathbb{N}, \ \text{for all } n \geq -N_i^W, \quad (\varphi_W(\mathbf{R} \wedge W))_n = (\varphi_W^i(\mathbf{R} \wedge W))_n. \tag{20}$$

We will simply write $\varphi_W = \varphi_W(\mathbf{R} \wedge W)$, $\varphi_W^i = \varphi_W^i(\mathbf{R} \wedge W)$ and $(\varphi_W^i)_n = (\varphi_W^i(\mathbf{R} \wedge W))_n$. It is straightforward to show that φ_W satisfies the commuting property (10). Equalities (20) and (18) also prove that the factor map φ_W is non-anticipating. The relation (11), $\varrho_W^{\otimes \mathbb{Z}} \circ \varphi_W^{-1} = \mu_W^{\mathcal{Z}^d}$ was shown in [**7**] by a coupling argument. Hence, $\varphi_W = \lim_{N \to \infty} \varphi_W^i$

is pointwise defined $\varrho^{\otimes\mathbb{Z}}$-a.s. and is a non-anticipating factor satisfying the properties given in Theorem 1.1.

The sequence $(W_l : l \in \mathbb{N})$ is strictly increasing and satisfies $W_l \nearrow \mathbb{R}^\ell$. We can assume that $0 \in \operatorname{int} W_1$. To simplify the notation, we put $R^m_{n;l} = R^m_n \wedge W_l$, $\vec{R}_{n;l} = \vec{R}_n \wedge W_l$, $\mathbf{R}_l = \mathbf{R} \wedge W_l$, $N_i^{W_l} = N_i^l$, $\varphi_l = \varphi_{W_l}$, $\varphi_{n;l} = (\varphi_{W_l})_n$ and $\varphi^i_{n;l} = (\varphi^i_{W_l})_n$.

We apply property (13) to the sequence of independent random tessellations $(R^1_{n;l} : n \in \mathbb{Z}_-)$ to get that $\varrho^{\otimes\mathbb{Z}}$-a.s. for every $l \in \mathbb{N}$ there exists a sequence of points $N_i^l = N_i^l(\mathbf{R}_l)$, $i \in \mathbb{N}$, that satisfies property (17),

$$\text{for all } i \in \mathbb{N},\ N_i^l \geq 1,\ N_i^l \nearrow \infty \quad \text{and} \quad \partial R^1_{m-N_i^l;l} \cap \operatorname{int}(a^m W_l) = \emptyset \ \text{ for all } m \in \mathbb{Z}_-.$$

For $l \geq k$ we have $W_k \subset W_l$ and $R^1_{n;k} = R^1_{n;l} \wedge W_l$. Then, for all $s \geq 1$,

$$\partial R^1_{m-s;l} \cap \operatorname{int}(a^m W_l) = \emptyset \text{ implies } \partial R^1_{m-s;k} \cap \operatorname{int}(a^m W_k) = \emptyset.$$

Hence,

$$l \geq k \text{ implies } \{N_i^l : i \in \mathbb{N}\} \subseteq \{N_i^k : i \in \mathbb{N}\}.$$

From (18) we deduce that the set of equalities

$$\text{for all } l \geq k,\ \text{ for all } n \geq -N_i^k, \quad \varphi_{n;k} = \varphi_{n;l} \wedge W_k \quad \varrho^{\otimes\mathbb{Z}}\text{-a.s.,} \tag{21}$$

will be satisfied once the enumerations of the cells in the tessellations $\varphi_{n;l}(\mathbf{R}_l) \wedge W_k$ and $\varphi_{n;k}(\mathbf{R}_k)$ use the same order. In fact, if this property is satisfied then

$$\text{for all } n \geq -N_i^k, \quad \left(a\varphi_{n;l}(\mathbf{R}_l) \boxplus \frac{a}{a-1}\vec{R}_{n;l}\right) \wedge W_k = \left(a\varphi_{n;k}(\mathbf{R}_k) \boxplus \frac{a}{a-1}\vec{R}_{n;k}\right).$$

So, we will be able to deduce that $\varphi_{n+1;l}(\mathbf{R}_l) \wedge W_k = \varphi_{n+1;k}(\mathbf{R}_k)$ for all $n \geq -N_i^k$, and then equality (21) will be satisfied. The same enumeration of $\varphi_{n;l}(\mathbf{R}_l) \wedge W_k$ and $\varphi_{n;k}(\mathbf{R}_k)$, for $l \geq k$, can be achieved as described below.

Note that (19) implies that for all $n \geq -N_i^W$ the enumerations of the cells in $\varphi^j_{n;k}$ and $\varphi^i_{n;k}$ can be chosen to be the same for all $j \geq i$. This will fix the order for $\varphi_{n;k}$.

Let us now introduce some useful notions for the proposed ordering. First, let $\prec$ be the lexicographical order on $\mathbb{R}^\ell$ given by $(a^1, \ldots, a^\ell) \prec (b^1, \ldots, b^\ell)$ if and only if there exists some $j \in \{1, \ldots, \ell\}$ such that $a^i = b^i$ for $i < j$ and $a^j < b^j$. Now, let $A_1 = \{x_1, \ldots, x_r\}$ and $A_2 = \{y_1, \ldots, y_s\}$ be two non-empty finite subsets of $\mathbb{R}^\ell$, with no inclusion relation (neither $A_1 \subseteq A_2$ nor $A_2 \subseteq A_1$ is satisfied). We enumerate its elements by using the lexicographical order $\prec$, so $x_1 \prec \cdots \prec x_r$ and $y_1 \prec \cdots \prec y_s$. We put $A_1 \sqsubset A_2$ if for some $t < \min\{r, s\}$ we have $x_i = y_i$ for $i = 1, \ldots, t$ and $x_{t+1} \prec y_{t+1}$. Note that $\sqsubset$ is a total order relation in any family of non-empty finite sets of $\mathbb{R}^\ell$ such that for all pairs of them there is no inclusion relation.

Let $E(C)$ be the (finite) set of extremal points of a cell C of a tessellation. Note that if C and C' are two different cells of the same tessellation then there is no inclusion relation between $E(C)$ and $E(C')$, and so either $E(C) \sqsubset E(C')$ or $E(C') \sqsubset E(C)$ is satisfied.

Let us enumerate the cells in a recursive way in k. Let $k = 1$. Take $n \leq 0$. Let $i \in \mathbb{N}$ be the first index such that $-N_i^1 \leq n$. The cells of the tessellations $\varphi^i_{n;1}(\mathbf{R}_1)$ are enumerated as follows. The first is the cell containing 0, after we put the other cells enumerated by $\sqsubset$. This is the order chosen for $\varphi_{n;1}$.

Let $k > 1$ and assume that the order has been defined up to $k-1$. The cells of the tessellations $\varphi_{n;k}(\mathbf{R}_k)$ in the kth window are enumerated as follows: the cells of the tessellation $\varphi_{n;k}(\mathbf{R}_k)$ in W_k that are extensions of cells of $\varphi_{n;k-1}(\mathbf{R}_{k-1})$ preserve their previous order; the remaining cells of the tessellation $\varphi_{n;k}(\mathbf{R}_k)$ in W_k are enumerated according to $\sqsubset$ and are put immediately after the previous ones. This enumeration gives the same order to the cells in $\varphi_{n;l}(\mathbf{R}_l) \wedge W_k$ and in $\varphi_{n;k}(\mathbf{R}_k)$, for all $l \geq k$. Then equality (21) is satisfied. This shows relation (16). □

Let us finish the proof of Theorem 1.2. From (16) and [**14**, Theorem 2.3.1] we get that there exists a function φ taking values in $\mathbb{T}^{\mathbb{Z}}$, defined $\varrho^{\mathbb{Z}}$-a.s. and such that $\varphi \wedge W_k = \varphi_k$ for all $k \in \mathbb{N}$ $\varrho^{\mathbb{Z}}$-a.s. Let us verify that it is a non-anticipating factor. Let us first show that it is a factor. From (10) and (16),

$$\sigma_{\mathbb{T}_{W_k}} \circ (\varphi \wedge W_k) = (\varphi \wedge W_k) \circ \sigma_{\mathbb{T}^{\mathbb{N}}_{W_k}} \quad \varrho^{\mathbb{Z}}\text{-a.s.}$$

From $\mathbb{T}_{W_k} = \mathbb{T} \wedge W_k$ we find that $(\sigma_{\mathbb{T}} \circ \varphi) \wedge W_k = \sigma_{\mathbb{T}_{W_k}} \circ (\varphi \wedge W_k)$ $\varrho^{\mathbb{Z}}$-a.s. and $(\varphi \circ \sigma_{\mathbb{T}^{\mathbb{N}}}) \wedge W_k = (\varphi \wedge W_k) \circ \sigma_{\mathbb{T}^{\mathbb{N}}_{W_k}}$ $\varrho^{\mathbb{Z}}$-a.s. Then $(\sigma_{\mathbb{T}} \circ \varphi) \wedge W_k = (\varphi \circ \sigma_{\mathbb{T}^{\mathbb{N}}}) \wedge W_k$ $\varrho^{\mathbb{Z}}$-a.s., and by taking $k \to \infty$ we get $\sigma_{\mathbb{T}} \circ \varphi = \varphi \circ \sigma_{\mathbb{T}^{\mathbb{N}}}$ $\varrho^{\mathbb{Z}}$-a.s.

On the other hand, equality (11) states that

$$\text{for all } k \in \mathbb{N}, \quad \varrho^{\otimes\mathbb{Z}}_{W_k} \circ \varphi^{-1}_{W_k} = \mu^{\mathcal{Z}^d}_{W_k}. \tag{22}$$

By using the identification between the σ-field $\mathcal{B}((\mathbb{T}^{\mathbb{N}}_W)^{\mathbb{Z}})$ and $\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}}) \wedge W$ defined in (12), we can write

$$\varphi^{-1}_{W_k}|_{\mathcal{B}((\mathbb{T}^{\mathbb{N}}_{W_k})^{\mathbb{Z}})} = \varphi^{-1}|_{\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}})\wedge W_k} \text{ and } \mu^{\mathcal{Z}^d}_{W_k}|_{\mathcal{B}((\mathbb{T}^{\mathbb{N}}_{W_k})^{\mathbb{Z}})} = \mu^{\mathcal{Z}^d}|_{\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}})\wedge W_k}. \tag{23}$$

Hence

$$\varrho^{\otimes\mathbb{Z}}_{W_k} \circ \varphi^{-1}_{W_k}|_{\mathcal{B}((\mathbb{T}^{\mathbb{N}}_{W_k})^{\mathbb{Z}})} = \varrho^{\otimes\mathbb{Z}} \circ \varphi^{-1}|_{\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}})\wedge W_k)}. \tag{24}$$

From (22)–(24) we get the equality

$$\text{for all } k \in \mathbb{N}, \quad \varrho^{\otimes\mathbb{Z}} \circ \varphi^{-1}|_{\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}})\wedge W_k} = \mu^{\mathcal{Z}^d}|_{\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}})\wedge W_k}.$$

Since $\mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}}) \wedge W_k \nearrow \mathcal{B}((\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}})$ we deduce the measure equality $\varrho^{\otimes\mathbb{Z}} \circ \varphi^{-1} = \mu^{\mathcal{Z}^d}$. Then φ is a factor.

By definition $(\varphi(\mathbf{R}))_n$ only depends on $(\mathbf{R}_j : j \leq n)$, then φ is non-anticipating. The proof of Theorem 1.2 for the discrete sequence $\mathcal{Z}^d$ is complete.

Let us show the last part of Theorem 1.2 for the time continuous process $\mathcal{Z}$. Since $(\mathbb{T}^{\mathbb{Z}}, \mu^{\mathcal{Z}^d}, \sigma_{\mathbb{T}})$ is a factor of a Bernoulli shift we get from Ornstein theory that it is also Bernoulli (see [**12, 13**]). Since $(D_{\mathbb{T}}(\mathbb{R}), \mathcal{B}(D_{\mathbb{T}}), \mu^{\mathcal{Z}})$ is a Lebesgue space and $(\mathbb{T}^{\mathbb{Z}}, \mu^{\mathcal{Z}^d}, \sigma_{\mathbb{T}})$ is a Bernoulli shift of infinite entropy, we can apply [**13**, Theorem 4 in §12, part 2] to get that $(D_{\mathbb{T}}(\mathbb{R}), \mu^{\mathcal{Z}}, (\sigma^t_{\mathbb{T}} : t \in \mathbb{R}))$ is a Bernoulli flow of infinite entropy. The proof of Theorem 1.2 is complete.

3. *Proof of Theorem 1.3*

We shall first state a general result on non-anticipating functions.

3.1. *Immersions and non-anticipating functions.* Let $(A, \mathcal{A})$ be a measurable space, with A a Polish space (with respect to some metric) endowed with its Borel σ-field $\mathcal{A}$. We set $A_l = A$ and $\mathcal{A}_l = \mathcal{A}$ for all $l \in \mathbb{Z}_-$. For $n \in \mathbb{Z}_-$ we denote by $A^{\times n} = \prod_{l \le n} A_l$, $\Xi_n^A : A^{\times 0} \to A$ the nth coordinate function and by $\mathcal{A}^{\otimes n} = \sigma(\Xi_l^A : l \le n)$ the σ-field in $A^{\times 0}$ generated by all the coordinates up to the nth.

Let ν be a non-atomic probability measure on $(A, \mathcal{A})$, so $(A, \mathcal{A}, \nu)$ is a non-atomic Lebesgue space. Let $\nu^{\otimes 0}$ be the product probability measure on $(A^{\times 0}, \mathcal{A}^{\otimes 0})$. There exists a family of one-to-one surjective measurable functions $\theta_l : A_l \to [0, 1]$ for $l \in \mathbb{Z}_-$, such that $\nu \circ \theta_l^{-1}$ is the Lebesgue measure. We have $\mathcal{A}_l = \sigma(\theta_l)$, $\mathcal{A}^{\otimes n} = \sigma(\theta_l : l \le n)$, and then the filtration $(\mathcal{A}^{\otimes n} : n \in \mathbb{Z}_-)$ is standard non-atomic.

The probability measure space $(A^{\times 0}, \mathcal{A}^{\otimes n}, \nu^{\otimes 0}|_{\mathcal{A}^{\otimes n}})$ is identified with the product measure space $(A^{\times n}, \bigotimes_{l \le n} \mathcal{A}_l, \nu^{\otimes n})$, where $\nu^{\otimes n}$ is the product measure on $(A^{\times n}, \bigotimes_{l \le n} \mathcal{A}_l)$. We put $x^n = (x_l : l \le n) \in A^{\times n}$ for all $x \in A^{\times 0}$ and $n \in \mathbb{Z}_-$. We will also denote by x^n a generic element of $A^{\times n}$. We will identify (a.s.) a function $f : A^{\times 0} \to \mathbb{R}$ $\mathcal{A}^{\otimes n}$-measurable with a function $f : A^{\times n} \to \mathbb{R}$ $\bigotimes_{l \le n} \mathcal{A}_l$-measurable. Then, when f is also bounded, we write $\int_{A^{\times 0}} f \, d\nu^{\otimes 0} = \int_{A^{\times n}} f \, d\nu^{\otimes n}$.

Let $(B, \mathcal{B})$ be another measurable space, B a Polish space and $\mathcal{B}$ its Borel σ-field. We consider the notions $B^{\times n}$, Ξ_n^B, $\mathcal{B}^{\otimes n}$ defined on it. Let $\eta : A^{\times 0} \to B^{\times 0}$ be a non-anticipating shift invariant measurable function. This means that for all $n \in \mathbb{Z}_-$ the function $\eta_n = \Xi_n^B \circ \eta$ is such that $\eta_n(x)$ only depends on $x^n \in A^{\times n}$ and $\eta_n(x) = \eta_0(\sigma^n x)$, where $\sigma^n : A^{\times 0} \to A^{\times 0}$ is given by $(\sigma^n x)_l = x_{l+n}$ for all $l \in \mathbb{Z}_-$. We put $\eta^n := (\eta_m : m \le n)$.

The measurable space $(B^{\times 0}, \mathcal{B}^{\otimes 0})$ is endowed with the probability measure $\nu^{\otimes 0} \circ \eta^{-1}$. Since $\bigotimes_{l \le n} \mathcal{B}_l$ and $\mathcal{B}^{\otimes n}$ are identified, we can also identify $(\eta^n)^{-1}|_{\bigotimes_{l \le n} \mathcal{B}_l}$ and $\eta^{-1}|_{\mathcal{B}^{\otimes n}}$. Hence,

$$\text{for all } n \in \mathbb{Z}_-, \quad \eta^{-1}(\mathcal{B}^{\otimes n}) = \sigma(\eta^n).$$

LEMMA 3.1. *The filtration of σ-fields $(\sigma(\eta^n) : n \in \mathbb{Z}_-)$ is immersed in the filtration $(\mathcal{A}^{\otimes n} : n \in \mathbb{Z}_-)$, with respect to $\nu^{\otimes 0}$.*

Proof. Since η is a non-anticipating shift invariant function, $x^n = y^n$ implies that $\eta^n(x) = \eta^n(y)$. Then the fibers of $\mathcal{A}^{\otimes n}$ are contained in the fibers of $\sigma(\eta^n)$, so $\sigma(\eta^n) \subseteq \mathcal{A}^{\otimes n}$.

Now let $(f_n : n \in \mathbb{Z}_-)$ be a $(\sigma(\eta^n) : n \in \mathbb{Z}_-)$-martingale with respect to $\nu^{\otimes 0}$. We must show that it is a $(\mathcal{A}^{\otimes n} : n \in \mathbb{Z}_-)$-martingale.

Recall that a bounded measurable function $f : \mathcal{A}^{\otimes 0} \to \mathbb{R}$ is $\eta^{-1}(\mathcal{B}^{\otimes n})$-measurable $\nu^{\otimes 0}$-a.s. if and only if there exists a $\mathcal{B}^{\otimes n}$-measurable function $f' : B^{\otimes 0} \to \mathbb{R}$ such that $f = f' \circ \eta$. This function f' is $\nu^{\otimes 0} \circ \eta^{-1}$-a.s. defined. It is straightforward to show that $(f_n : n \in \mathbb{Z}_-)$ is a $(\eta^{-1}(\mathcal{B}^{\otimes n}) : n \in \mathbb{Z}_-)$-martingale with respect to $\nu^{\otimes 0}$ if and only if $(f'_n \circ \eta : n \in \mathbb{Z}_-)$ is a $(\mathcal{B}^{\otimes n} : n \in \mathbb{Z}_-)$-martingale with respect to $\nu^{\otimes 0} \circ \eta^{-1}$. Therefore, this last property is our hypothesis.

The probability measure $\nu^{\otimes n} \circ \eta^{-1}$ is well defined on $(B^{\times 0}, \mathcal{B}^{\otimes n})$. Since $(\eta^n)^{-1}|_{\bigotimes_{l \le n} \mathcal{B}_l} = \eta^{-1}|_{\mathcal{B}^{\otimes n}}$, this probability measure is also well defined on $(B^{\times n}, \bigotimes_{l \le n} \mathcal{B}_l)$.

The Lebesgue probability space $(A^{\times n}, \bigotimes_{l\le n} \mathcal{A}, \nu^{\otimes n})$ is partitioned by $(\eta^{-1}z^n : z^n \in B^{\times n})$. Then $\nu^{\otimes n} \circ \eta^{-1}$-a.s. on $z^n \in B^{\times n}$ there exists a probability measure denoted by $\nu_{z,n}$ and defined on the measurable subset $\eta^{-1}\{z^n\} \subseteq A^{\times n}$ that satisfies the following property: for all bounded $\mathcal{A}^{\otimes n}$-measurable functions $g_n : A^{\times 0} \to \mathbb{R}$, the function given by

$$\widehat{g_n} : B^{\times 0} \to \mathbb{R}, \quad \widehat{g_n}(z) = \widehat{g_n}(z^n) = \int_{\eta^{-1}\{z^n\}} g_n \, d\nu_{z,n},$$

is $\mathcal{B}^{\otimes n}$-measurable and $\nu^{\otimes 0} \circ \eta^{-1}$-a.s. defined. Moreover, it satisfies

$$\int_{A^{\times 0}} g_n \, d\nu^{\otimes 0} = \int_{B^{\times n}} \widehat{g_n}(z^n) \, d\nu^{\otimes n} \circ \eta^{-1}(z^n). \tag{25}$$

Since $f'_n : B^{\times 0} \to \mathbb{R}$ is $\mathcal{B}^{\otimes n}$-measurable, $f'_n \circ \eta(x)$ only depends on $\eta^n(x)$. But $\eta^n(x)$ only depends on x^n, so $f'_n \circ \eta(x)$ only depends on $x^n \in A^{\times n}$. Let $h_{n-1} : A^{\times 0} \to \mathbb{R}$ be a bounded $\mathcal{A}^{\otimes(n-1)}$-measurable function, so $h_{n-1}(x)$ only depends on x^{n-1}. By using (25) we get

$$\begin{aligned}\int_{A^{\times 0}} f'_{n-1} \circ \eta h_{n-1} \, d\nu^{\otimes 0} &= \int_{B^{\times(n-1)}} f'_{n-1}(z^{n-1})\widehat{h}_{n-1}(z^{n-1}) \, d\nu^{\otimes(n-1)} \circ \eta^{-1}(z^{n-1}) \\ &= \int_{B^{\times 0}} f'_{n-1}\widehat{h}_{n-1} d\nu^{\otimes 0} \circ \eta^{-1} = \int_{B^{\times 0}} f'_n \widehat{h}_{n-1} \, d\nu^{\otimes 0} \circ \eta^{-1},\end{aligned}$$

where in the last equality we used the fact that $(f'_n : n \in \mathbb{Z}_-)$ is a $(\mathcal{B}^{\otimes n} : n \in \mathbb{Z}_-)$-martingale and that $\widehat{h}_{n-1}$ is a bounded $\mathcal{B}^{\otimes(n-1)}$-measurable function. Now, since $\widehat{h}_{n-1}$ is $\mathcal{B}^{\otimes(n-1)}$-measurable we get that $\widehat{h}_{n-1}(\eta^n(x))$ only depends on $\eta^{n-1}(x)$ and by using an argument entirely similar to that above we find that

$$\begin{aligned}\int_{B^{\times 0}} f'_n \widehat{h}_{n-1} \, d\nu^{\otimes 0} \circ \eta^{-1} &= \int_{B^{\times n}} f'_n(z^n)\widehat{h}_{n-1}(z^n) \, d\nu^{\otimes n} \circ \eta^{-1}(z^n) \\ &= \int_{A^{\times n}} f'_n \circ \eta h_{n-1} \, d\nu^{\otimes n} = \int_{A^{\times 0}} f'_n \circ \eta h_{n-1} \, d\nu^{\otimes 0}.\end{aligned}$$

Then $\int_{A^{\times 0}} f'_{n-1} \circ \eta h_{n-1} \, d\nu^{\otimes 0} = \int_{A^{\times 0}} f'_n \circ \eta h_{n-1} \, d\nu^{\otimes 0}$. Since $f_n = f'_n \circ \eta$, we have proved that $(f_n : n \in \mathbb{Z}_-)$ is a $(\mathcal{A}^{\otimes n} : n \in \mathbb{Z}_-)$-martingale. □

Since $\nu^{\otimes 0}$ is non-atomic, from the definitions made in §1.8, Lemma 3.1 can be written as follows.

COROLLARY 3.2. *The filtration* $(\sigma(\eta^n) : n \in \mathbb{Z}_-) = (\eta^{-1}\mathcal{B}^{\otimes n} : n \in \mathbb{Z}_-)$ *is standard (with respect to* $\nu^{\otimes 0}$*).*

3.2. *The proof.* We will apply the construction described in the previous section to the following setting: $A = \mathbb{T}^{\mathbb{N}}$, $\nu = \varrho$, $B = \mathbb{T}$ and $\eta : A^{\mathbb{Z}_-} \to B^{\mathbb{Z}_-}$ coincides with $\varphi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ defined in §2 but restricted to the set of coordinates in $\mathbb{Z}_-$. That is,

$$\text{for all } z \in A^{\mathbb{Z}}, \quad \eta(z^0) = \varphi^0(z) \quad \text{with } \varphi^0(z) = (\varphi_n(z) : n \in \mathbb{Z}_-) \text{ and } z^0 = (z_n : n \in \mathbb{Z}_-).$$

This definition is valid because the factor map φ is non-anticipating. Then $\eta : (\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}_-} \to \mathbb{T}^{\mathbb{Z}_-}$ satisfies

$$\text{for all } n \le -1, \quad \eta_{n+1} = \left(a\eta_n \boxplus \frac{a}{a-1}\vec{R}_n\right).$$

The mapping η inherits shift invariance and non-anticipation from φ. Then: for all $m \in \mathbb{Z}_-$, $\varrho^{\otimes \mathbb{Z}_-}$-a.s. in $\mathbf{R} = (\vec{R}_n : n \in \mathbb{Z}_-) \in (\mathbb{T}^{\mathbb{N}})^{\mathbb{Z}_-}$, $\eta(\mathbf{R})_m$ depends only on $(\vec{R}_n : n \le m)$. Therefore, we are in the framework of the previous section and from Lemma 3.1 we get that the filtration $(\sigma(\eta^n) : n \in \mathbb{Z}_-)$ is standard. But since φ is non-anticipating we have $\sigma(\eta^n) = \sigma(\varphi^n)$ and, from Corollary 3.2, the filtration $(\sigma(\varphi^n) : n \in \mathbb{Z}_-)$ is standard.

Let us show this filtration is standard non-atomic. This will be a straightforward consequence of the following result.

LEMMA 3.3. *There exists a sequence* $(\alpha_n : n \in \mathbb{Z}_-)$ *of independent and identically distributed Uniform*[0, 1] *random variables that satisfies* $\sigma(\varphi^n) = \sigma(\sigma(\varphi^{n-1}), \alpha_n)$ *and* α_n *is independent of* $\sigma(\varphi^{n-1})$ *for all* $n \in \mathbb{Z}_-$.

Proof. We claim that

$$\text{for all } n \in \mathbb{Z}_-, \text{ there exists } \alpha_n \sim \text{Uniform}[0, 1] \text{ such that}$$
$$\sigma(\varphi^n) = \sigma(\sigma(\varphi^{n-1}), \alpha_n) \text{ and } \alpha_n \text{ is independent of } \sigma(\varphi^{n-1}). \tag{26}$$

Note that these relations imply that the random variables $(\alpha_n : n \in \mathbb{Z}_-)$ are independent.

In showing the claim we use [**3**, Proposition 5] together with the separability of $\sigma(\varphi^n)$ for all $n \in \mathbb{Z}_-$. The equivalence between properties (iii) and (iv) in the aforementioned proposition implies that relation (26) is equivalent to

$$\text{for all } n \in \mathbb{Z}_-, \text{ every random variable } V_n \text{ that satisfies}$$
$$\sigma(\varphi^{n-1}, V_n) = \sigma(\varphi^n) \text{ has a diffusive law.}$$

Let us show that the latter property holds. We can assume that V_n is a random variable that takes values in $\mathbb{T}$. We will prove this property by contradiction, so assume that for some $n \in \mathbb{Z}_-$, $\sigma(\varphi^{n-1}, V_n) = \sigma(\varphi^n)$ and V_n has an atomic part. Then there exists $A \in \mathcal{B}(\mathbb{T})$ such that $a := \mathbb{P}(V_n \in A) > 0$ and, for all $C \in \mathcal{B}(\mathbb{T})$ with $C \subseteq A$, $\mathbb{P}(V_n \in C) = 0$ or $\mathbb{P}(V_n \in C) = a$ is satisfied. Let $(W_l : l \in \mathbb{Z}_+)$ be a sequence of windows such that $W_l \subset \text{int } W_{l+1}$ for all l and $W_l \nearrow \mathbb{R}^\ell$. Since a.s. in the tessellation φ^n there exist cells of φ^{n-1} that have been broken into more than one piece and $\sigma(\varphi^n) = \sigma(\varphi^{n-1}, V_n)$, we get the existence of some l that satisfies

$$\mathbb{P}(V_n \in A_l) > 0 \quad \text{where } A_l = \{T \in A : T \wedge W_l \neq \{W_l\}\}.$$

Since $\mathbb{P}(V_n \in A_l) > 0$, $A_l \subseteq A$ and A is an atom of V_n, we get that $\mathbb{P}(V_n \in A_l) = a$. An analogous argument shows that there exists a fixed tessellation $Q_0 \in \mathbb{T}_{W_l}$ with $Q_0 \neq \{W_l\}$ such that $\mathbb{P}(V_n \wedge W_l = Q_0) \ge a > 0$. This implies that $\mathbb{P}(\varphi_n \wedge W_l = Q_0) > 0$ and so $\mathbb{P}(a^n Y_{a^n} \wedge W_l = Q_0) > 0$. This is in contradiction to the fact that $\{W\}$ is the unique atom of $Y_{a^n} \wedge W$ for every window W (see (6)).

We conclude that the random variable V_n cannot have an atomic part. □

Since $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \varrho)$ is non-atomic, from Corollary 3.2 we get that the filtration $(\sigma(\varphi^n) : n \in \mathbb{Z}_-)$ is standard. From Lemma 3.3 the hypotheses for the equivalence (9) are satisfied. This equivalence shows that $(\sigma(\varphi^n) : n \in \mathbb{Z}_-)$ is standard non-atomic. Then Theorem 1.3 is satisfied.

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