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Mild solutions of non-autonomous second order problems with nonlocal initial conditions

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ABSTRACT

In this paper we establish the existence of mild solutions for a non-autonomous abstract semi-linear second order differential equation submitted to nonlocal initial conditions. Our approach relies on the existence of an evolution operator for the corresponding linear equation and the properties of the Hausdorff measure of non-compactness.

1. Introduction

This paper is devoted to study the existence of mild solutions to the problems described by the non-autonomous abstract semi-linear second order differential equation with nonlocal conditions

$$\left.\begin{aligned} &u''(t) = A(t)u(t) + f\big(t, N(t)(u)\big), \quad t \in J,\\ &u(0) = g(u),\\ &u'(0) = h(u). \end{aligned}\right\} \tag{1.1}$$

In this text, X is a Banach space endowed with a norm $\|\cdot\|$ and $J = [0, a]$ with $a > 0$. In problem (1.1) we assume that $A(t) : D(A(t)) \subseteq X \to X$ for $t \in J$ are closed linear operators with domain $D(A(t)) = D$ for all $t \in J$. Moreover, we denote by $C(J, X)$ the space consisting of continuous functions from J into X provided with the norm of uniform convergence. As general conditions, we always assume that $g, h, N(\cdot) : C(J; X) \to X$ are continuous maps, the function $t \mapsto N(t)(u)$ is continuous for each $u \in C(J; X)$, and $f : J \times X \to X$ is a function that satisfies Carathéodory type conditions, which will be defined later.

The concept of *nonlocal initial condition* was introduced to extend the classical theory of initial value problems. This notion is more appropriate than the classical to describe natural phenomena because it allows us to consider additional information. For the importance of nonlocal conditions in different fields of applied sciences see [12,15,49,50] and the references cited

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therein. For example, in [15] the author describes the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula $g(u) = \sum_{i=0}^{p} c_i u(t_i)$, where c_i, $i = 0, 1, \dots, p$, are given constants and $0 < t_0 < t_1 < \cdots < t_p < 1$.

Early work in this area was made by Byszewski in [6–9]. Thenceforth, the study of differential equations with nonlocal initial conditions has been an active topic of research. The interested reader can consult [1,2,10,11,14,16,21–23,26–28,31, 32,36–38,43–46,51–53,55] and the references therein for recent developments on issues similar to those addressed on this paper.

On the other hand, there exists an extensive literature concerning abstract second order problems. In the autonomous case, the existence of solutions to the second order abstract Cauchy problem is strongly related with the concept of cosine functions. We refer the reader to [17,39–42] for basic concepts about the theory of cosine functions. Similarly to what happens in the autonomous case, the existence of solutions to the non-autonomous second order abstract Cauchy problem corresponding to the family $\{A(t)\colon t \in J\}$ is directly related to the concept of evolution operator generated by the family $\{A(t)\colon t \in J\}$. Various techniques to establish the existence of an evolution operator $\{S(t,s)\colon t,s \in J\}$ generated by the family $\{A(t)\colon t \in J\}$ can be found in the literature. Our aim in this paper is to establish the existence of mild solutions of problem (1.1). The results are based on the properties of evolution operators and measure of non-compactness.

This paper is organized as follows. In Section 2, we collect the properties of evolution operators and measure of non-compactness that are needed to establish our results. In Section 3, we show existence of mild solutions of problem (1.1), and finally in Section 4, we include some applications.

The terminology and notations are those generally used in works about evolution equations. In particular, if $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, we indicate by $\mathcal{L}(Z,Y)$ the Banach space of bounded linear operators from Z into Y endowed with the uniform operator topology, and we abbreviate this notation to $\mathcal{L}(Z)$ whenever $Z = Y$. By $B_r[x, Z]$ we denote the closed ball with center x and radius r in Z. When the space Z is clearly determined from the context, we abbreviate this notation to $B_r[x]$.

2. Preliminaries

The non-autonomous second order abstract Cauchy problem has received some attention in recent years due to its applications in various fields. Specially, several authors have studied the abstract Cauchy problem

$$\left.\begin{aligned} &u''(t) = A(t)u(t) + f(t), \quad t \in J,\\ &u(s) = x,\\ &u'(s) = y. \end{aligned}\right\} \tag{2.1}$$

We refer the reader to [4,20,29,34,47] for information about this topic. In particular, as we have already mentioned, the existence of solutions of problem (2.1) is related with the existence of the evolution operator $\{S(t,s)\}_{t,s\in J}$ for the homogeneous equation

$$\left.\begin{aligned} &u''(t) = A(t)u(t), \quad t \in J,\\ &u(s) = x,\\ &u'(s) = y. \end{aligned}\right\} \tag{2.2}$$

In this paper, we will use the concept of evolution operator $\{S(t,s)\}_{t,s\in J}$ associated with problem (2.2) introduced by Kozak in [25]. With this purpose, we assume that the domain of $A(t)$ is a subspace D dense in X and independent of $t \in J$, and for each $x \in D$ the function $t \mapsto A(t)x$ is continuous.

Definition 2.1. Let $S : J \times J \to \mathcal{L}(X)$. The family $\{S(t,s)\}_{t,s\in J}$ is said to be an *evolution operator* generated by the family $\{A(t)\colon t \in J\}$ if the following conditions are fulfilled:

(D1) For each $x \in X$ the map $(t,s) \mapsto S(t,s)x$ is continuously differentiable, and
  (a) For each $t \in J$, $S(t,t) = 0$;
  (b) For all $t,s \in J$ and each $x \in X$, $\frac{\partial}{\partial t} S(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s} S(t,s)x|_{t=s} = -x$.

(D2) For all $t,s \in J$, if $x \in D$, then $S(t,s)x \in D$, the map $(t,s) \mapsto S(t,s)x$ is of class C^2, and
  (a) $\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$;
  (b) $\frac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x$;
  (c) $\frac{\partial^2}{\partial s \partial t} S(t,s)x|_{t=s} = 0$.

(D3) For all $s,t \in J$, if $x \in D$, then $\frac{\partial}{\partial t} S(t,s)x \in D$. Further, there exist $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x$ and $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x$, and
  (a) $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x = A(t)\frac{\partial}{\partial s} S(t,s)x$;
  (b) $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x = A(t)\frac{\partial}{\partial t} S(t,s)x$;
  and the mapping $(t,s) \mapsto A(t)\frac{\partial}{\partial s} S(t,s)x$ is continuous.

Assuming that $f : J \to X$ is an integrable function, Kozak [25] has proved that the function $u : J \to X$ given by

$$u(t) = -\frac{\partial}{\partial s} S(t,s)x + S(t,s)y + \int_s^t S(t,\xi) f(\xi)\, d\xi,$$

is the mild solution of problem (2.2). Motivated by this result, we establish the following notion.

Definition 2.2. A continuous function $u : J \to X$ is said to be a *mild solution* of problem (1.1) if the equation

$$u(t) = -\frac{\partial}{\partial s} S(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi) f\big(\xi, N(\xi)(u)\big)\, d\xi, \quad t \in J,$$

is satisfied.

Henceforth, we assume that there exists an evolution operator $\{S(t,s)\}_{t,s\in J}$ associated with the family $\{A(t)\colon\ t \in J\}$. To abbreviate the text, we introduce the operator $C(t,s) = -\frac{\partial S}{\partial s}(t,s)$. With this notation, a mild solution of problem (1.1) is a continuous function that satisfies the equation

$$u(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi) f\big(\xi, N(\xi)(u)\big)\, d\xi, \quad t \in J.$$

In addition, we set $K, K_1 > 0$ for constants such that

$$\big\|C(t,s)\big\| \leqslant K, \qquad \left\|\frac{\partial}{\partial t} S(t,s)\right\| \leqslant K_1, \tag{2.3}$$

for all $s, t \in J$. Since the operator-valued map $C(t,\cdot)$ is strongly continuous, for $x \in X$, we have

$$S(t,s)x = -\int_s^t \frac{\partial}{\partial \xi} S(t,\xi)x\, d\xi = \int_s^t C(t,\xi)x\, d\xi,$$

which implies that

$$\big\|S(t,s)\big\| \leqslant K|t-s|, \quad s,t \in J.$$

Moreover, it is clear that

$$\big\|S(t_2,s) - S(t_1,s)\big\| \leqslant K_1|t_2 - t_1|, \quad \text{for all } t_1, t_2, s \in J. \tag{2.4}$$

Most of our results are based on the concept of measure of non-compactness. For this reason, we next recall a few properties of this concept.

For general information the reader can see [3]. In this paper, we use the notion of Hausdorff measure of non-compactness.

Definition 2.3. Let B be a bounded subset of a metric space Y. The *Hausdorff measure of non-compactness* of B is defined by

$$\eta(B) = \inf\{\varepsilon > 0\colon\ B \text{ has a finite cover by closed balls of radius } \varepsilon\}.$$

For a bounded set $B \subseteq X$, we next denote by $\overline{co}(B)$ the closed convex hull of the set B.

Remark 2.4. Let $B_1, B_2 \subseteq X$ be bounded sets. The Hausdorff measure of non-compactness has the following properties. For more details and the proof of the properties that follow, the reader can see [3].

(a) If $B_1 \subseteq B_2$, then $\eta(B_1) \leqslant \eta(B_2)$.
(b) $\eta(B) = \eta(\overline{B})$.
(c) $\eta(B) = 0$ if and only if B is totally bounded.
(d) For $\lambda \in \mathbb{R}$, $\eta(\lambda B) = |\lambda|\eta(B)$.
(e) $\eta(B_1 \cup B_2) = \max\{\eta(B_1), \eta(B_2)\}$.
(f) $\eta(B_1 + B_2) \leqslant \eta(B_1) + \eta(B_2)$, where $B_1 + B_2 = \{b_1 + b_2\colon\ b_1 \in B_1,\ b_2 \in B_2\}$.
(g) $\eta(B) = \eta(\overline{co}(B))$.

We next collect some specific properties of the Hausdorff measure of non-compactness which are needed to establish our results. Let X be a Banach space. In what follows, when we need to compare the measures of non-compactness in X and $C(J;X)$, we will use ζ to denote the Hausdorff measure of non-compactness defined in X, and γ to denote the Hausdorff measure of non-compactness on $C(J;X)$.

Lemma 2.5. *(See [3].) Let $W \subseteq C(J;X)$. If W is bounded and equicontinuous, then the set $\overline{co}(W)$ is also bounded and equicontinuous.*

For $W \subseteq C(J;X)$ and $t \in J$ fixed, we denote $W(t) = \{w(t)\colon\ w \in W\}$.

Lemma 2.6. *(See [3].) Let $W \subseteq C(J;X)$ be a bounded set. Then $\zeta(W(t)) \leqslant \gamma(W)$ for all $t \in J$. Furthermore, if W is equicontinuous on J, then $\zeta(W(t))$ is continuous on J, and*

$$\gamma(W) = \sup\{\zeta(W(t))\colon\ t \in J\}.$$

Definition 2.7. A set $W \subseteq L^1(J;X)$ is said to be *uniformly integrable* over J if there exists a positive function $k \in L^1(J;\mathbb{R}^+)$ such that $\|w(t)\| \leqslant k(t)$ a.e. for all $w \in W$.

Let $W \subseteq L^1(J;X)$ be a uniformly integrable set. In the following statements we denote by $F : L^1(J;X) \to X$ the map given by

$$F(u) = \int_0^a u(s)\,ds.$$

The next lemma was established in [19, Theorem 3.1].

Lemma 2.8. *Assume that X is a separable Banach space. If $W \subseteq L^1(J;X)$ is uniformly integrable, then $t \mapsto \zeta(W(t))$ is a measurable function and*

$$\zeta(F(W)) \leqslant \int_0^a \zeta(W(s))\,ds,$$

where $F(W) = \{F(w)\colon\ w \in W\}$.

The next property has been studied by several authors under different hypotheses, see [5,54] among others. We establish it here both for reference purposes and to unify the presentation and avoid some unnecessary hypotheses.

Lemma 2.9. *Let Y be a metric space and let $D \subseteq Y$ be a bounded set. Then there exists a countable set $D_0 \subseteq D$ such that $\eta(D) \leqslant \eta(D_0)$.*

Proof. We can assume that $\eta(D) > 0$. We fix $0 < \varepsilon < 1$ and $r = (1-\varepsilon)\eta(D) > 0$. Let $x_1 \in D$. Then there exists $x_2 \in D \setminus B_r[x_1]$. Applying repeatedly this argument, we can construct inductively a sequence $(x_n)_n$ in D so that $x_{k+1} \in D \setminus \bigcup_{i=1}^k B_r[x_i]$. Set $D_\varepsilon = \{x_n\colon\ n \in \mathbb{N}\}$. It is clear that $\eta(D_\varepsilon) \leqslant \eta(D)$. On the other hand, since $d(x_i,x_j) \geqslant r$ for all $i \neq j$, then $\eta(D_\varepsilon) > r$. We define $D_0 = \bigcup_{n=1}^\infty D_{1/n}$. It is clear that D_0 is a countable set. Moreover,

$$\eta(D_0) \geqslant \eta(D_{1/n}) \geqslant \left(1 - \frac{1}{n}\right)\eta(D),$$

and taking limit as $n \to \infty$, we infer that $\eta(D_0) \geqslant \eta(D)$. □

Corollary 2.10. *Let X be a Banach space, and $W \subseteq L^1(J;X)$ be a uniformly integrable set. Then there exists a countable set $W_0 \subseteq W$ such that*

$$\zeta(F(W)) = \zeta(F(W_0)) \leqslant 2\int_0^a \zeta(W_0(s))\,ds. \tag{2.5}$$

Definition 2.11. Let Y be a metric space. A continuous map $G : Y \to Y$ is said to be *η-condensing* if $\eta(G(B)) < \eta(B)$ for every bounded subset B of Y with $\eta(B) > 0$.

The following result was established by Darbo [13] in 1955 for η–k-set contractions, and for Sadovskiĭ [33] in 1967 for η-condensing maps.

Theorem 2.12. *Assume that B is a nonempty bounded closed and convex subset of a Banach space Y. Let $G : B \to B$ be an η-condensing map. Then G has a fixed point in B.*

The following result is a recent extension of Theorem 2.12 established in [30].

Theorem 2.13. *Let B be a closed and convex subset of a Banach space Y, let $G : B \to B$ be a continuous map such that $G(B)$ is bounded. For each bounded subset $D \subseteq B$, denote*

$$G^1(D) = G(D) \quad \text{and} \quad G^n(D) = G\big(\overline{co}\big(G^{n-1}(D)\big)\big), \quad n = 2, 3, \ldots.$$

If there exist $0 \leqslant r < 1$ and $n_0 \in \mathbb{N}$ such that

$$\eta\big(G^{n_0}(D)\big) \leqslant r\eta(D)$$

for every bounded set $D \subseteq B$, then G has a fixed point in B.

3. Existence results

In this section we will present our main results. As was explained in the introduction, in this paper we always assume that $g, h, N(\cdot) : C(J; X) \to X$ are continuous maps and the function $t \mapsto N(t)(u)$ is continuous for each $u \in C(J; X)$. Next we introduce some conditions related to function f.

(Cf1) The map $f : J \times X \to X$ satisfies the Carathéodory conditions, that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in J$.

(Cf2) There exist a function $m \in L^1(J; \mathbb{R}^+)$ and a non-decreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\big\| f(t, x)\big\| \leqslant m(t)\Phi\big(\|x\|\big)$$

for all $x \in X$ and almost all $t \in J$.

(Cf3) There exists a function $H \in L^1(J; \mathbb{R}^+)$ such that

$$\zeta\big(f(t, B)\big) \leqslant H(t)\zeta(B)$$

for almost all $t \in J$ and every bounded set $B \subseteq X$.

Before continuing our development, it is important to note that in the context of infinite dimensional spaces conditions (Cf2) and (Cf3) are different. We will justify our claim exhibiting a few elementary examples.

Example 3.1. Let $f : C([0, 1]; \mathbb{R}) \to C([0, 1]; \mathbb{R})$ given by

$$f(x)(\xi) = \sqrt{\big|x(\xi)\big|}, \quad \xi \in [0, 1].$$

It is easy to see that f is continuous. In fact, if the sequence $(x_n)_n$ converges to x for the norm of uniform convergence, then $\bigcup_{n=1}^{\infty} x_n([0, 1]) \cup x([0, 1])$ is a compact set. Since the function $\alpha(t) = \sqrt{|t|}$ is uniformly continuous on compact sets, then $f(x_n) = \alpha \circ x_n \to \alpha \circ x$ as $n \to \infty$ uniformly on $[0, 1]$. Moreover, $\|f(x)\| \leqslant \Phi(\|x\|)$, where $\Phi(t) = \sqrt{t}$ for $t \geqslant 0$. Hence, the function f verifies condition (Cf2).

On the other hand, assume that

$$\zeta\big(f(W)\big) \leqslant H\zeta(W), \tag{3.1}$$

for every bounded set $W \subseteq C([0, 1]; \mathbb{R})$ and certain constant $H > 0$. For each $n \in \mathbb{N}$, we take the constant function $x_n(t) = 1/n$ and the closed ball $W = B_{1/n^2}[x_n, C([0, 1]; \mathbb{R})]$. We know that $\eta(W) = 1/n^2$. Furthermore, it follows from (3.1) that there exist $\varphi, \psi \in B_{1/n^2}[0, C([0, 1]; \mathbb{R})]$ and $s \in [0, 1]$ such that $|\varphi(s) - \psi(s)| = 1/n^2$ and

$$\big\| f(x_n + \varphi) - f(x_n + \psi)\big\| \leqslant 2H\zeta(W) = \frac{2H}{n^2}.$$

Hence

$$\left|\sqrt{\frac{1}{n} + \varphi(s)} - \sqrt{\frac{1}{n} + \psi(s)}\right| = \frac{|\varphi(s) - \psi(s)|}{\sqrt{\frac{1}{n} + \varphi(s)} + \sqrt{\frac{1}{n} + \psi(s)}}$$
$$\leqslant \big\| f(x_n + \varphi) - f(x_n + \psi)\big\| \leqslant \frac{2H}{n^2}.$$

This implies that

$$1 \leqslant 2H\left(\sqrt{\frac{1}{n}+\varphi(s)}+\sqrt{\frac{1}{n}+\psi(s)}\right) \to 0, \quad n \to \infty,$$

which is a contradiction.

Example 3.2. If a function $f : X \to X$ satisfies (Cf3), then f also satisfies (Cf2). In fact, it follows from (Cf3) that f takes bounded sets into bounded sets. We define $\Psi : [0,\infty) \to [0,\infty)$ by $\Psi(\xi) = \sup_{\|x\| \leqslant \xi} \|f(x)\|$.

It is clear that Ψ is an increasing function and $\|f(x)\| \leqslant \Psi(\|x\|)$. It is also easy to see that Ψ is left continuous. Now, we define $\Phi : [0,\infty) \to [0,\infty)$ by

$$\Phi(t) = \begin{cases} \Psi(t+1) & \text{if } t \in \mathbb{N} \cup \{0\}, \\ \Psi(n+1) + [\Psi(n+2) - \Psi(n+1)](t-n) & \text{if } t \in [n, n+1],\ n \in \mathbb{N}. \end{cases}$$

Clearly Φ is a continuous and non-decreasing map, and $\|f(x)\| \leqslant \Psi(\|x\|) \leqslant \Phi(\|x\|)$. Hence f satisfies (Cf2).

However, for a function $f : J \times X \to X$ the assertion does not hold. In fact, let $f(t,x) = \alpha(t) f_0(x)$, where $f_0 : X \to X$ is a completely continuous function and $\alpha : J \to \mathbb{R}$ is a measurable function such that $\alpha \notin L^1(J;\mathbb{R}^+)$. In this case, clearly f verifies condition (Cf3) but not (Cf2).

Next, we consider the following condition for the family $\{N(t): t \in J\}$.

(CN1) There exists a constant $\nu > 0$ such that

$$\zeta\left(\left\{N(t)(u): u \in W\right\}\right) \leqslant \nu\gamma(W),$$

for all $t \in J$ and every bounded set $W \subseteq C(J;X)$.

We point out that condition (CN1) implies that, for all $t \in J$, $N(t)$ takes bounded sets into bounded sets. Thus, in this case, for $R \geqslant 0$ we denote

$$N_R = \sup\left\{\left\|N(t)(u)\right\|: t \in J,\ u \in C(J;X),\ \|u\|_\infty \leqslant R\right\}.$$

Note that, if f satisfies conditions (Cf1) and (Cf2), and $u \in C(J;X)$, then the function $t \mapsto f(t, N(t)(u))$ is integrable on J.

We are in a position to establish the following essential property.

Theorem 3.3. *Assume that $f : J \times X \to X$ satisfies* (Cf1), (Cf2), (Cf3), *and that N satisfies* (CN1). *Let $F : C(J;X) \to C(J;X)$ be the map given by*

$$Fu(t) = \int_0^t f\left(s, N(s)(u)\right) ds.$$

Let $W \subseteq C(J;X)$ be a bounded set. Then

$$\gamma\left(F(W)\right) \leqslant 2\nu\gamma(W) \int_0^a H(s)\,ds.$$

Proof. It is clear that the set of functions $\{f(\cdot, N(\cdot)(u)): u \in W\}$ is uniformly integrable. Therefore, according to Corollary 2.10, there exists a countable set $W_0 \subseteq W$ such that

$$\begin{aligned} \zeta\left(F(W)(t)\right) &\leqslant 2\int_0^t \zeta\left(f\left(s, N(s)\left(W_0(t)\right)\right)\right) ds \\ &\leqslant 2\nu \int_0^t H(s)\,ds\, \zeta\left(W_0(t)\right) \\ &\leqslant 2\nu \int_0^t H(s)\,ds\, \gamma(W). \end{aligned} \tag{3.2}$$

On the other hand, it follows from (Cf2) that $F(W)$ is equicontinuous. Consequently, using Lemma 2.6, we have that

$$\gamma\bigl(F(W)\bigr) \leqslant \sup_{t\in J} \zeta\bigl(F(W)(t)\bigr) \leqslant 2\nu \int_0^a H(s)\,ds\,\gamma(W),$$

which establishes the assertion. □

In what follows, we need a slightly extension of this result.

Corollary 3.4. *Assume that $p : J \times J \times X \to X$ is a function that satisfies the following conditions*:

(Cp1) *For each $t \in J$, the function $p(t,\cdot,\cdot)$ satisfies the Carathéodory conditions.*
(Cp2) *Let $B \subseteq X$ be a bounded set. The set $\{p(t,\cdot,x)\colon t \in J,\ x \in B\}$ is uniformly integrable.*
(Cp3) *Let $B \subseteq X$ be a bounded set. The set $\{p(\cdot,s,x)\colon s \in J,\ x \in B\}$ is equicontinuous.*
(Cp4) *There exists a positive function $\widetilde{H} : J \times J \to \mathbb{R}$ such that $\widetilde{H}(t,\cdot)$ is integrable for all $t \in J$, and*

$$\zeta\bigl(\bigl\{p(t,s,x)\colon x \in B\bigr\}\bigr) \leqslant \widetilde{H}(t,s)\zeta(B),$$

for each bounded set $B \subseteq X$.

Assume further that N satisfies condition (CN1)*. Let $F : C(J;X) \to C(J;X)$ be the map given by*

$$Fu(t) = \int_0^t p\bigl(t,s,N(s)(u)\bigr)\,ds.$$

Let $W \subseteq C(J;X)$ be a bounded set. Then

$$\gamma\bigl(F(W)\bigr) \leqslant 2\nu \sup_{t\in J} \int_0^t \widetilde{H}(t,s)\,ds\,\gamma(W).$$

Furthermore, if the function $t \mapsto \zeta(N(t)(W))$ is measurable, then

$$\zeta\bigl(F(W)(t)\bigr) \leqslant 2 \int_0^t \widetilde{H}(t,s)\zeta\bigl(N(s)(W)\bigr)\,ds, \tag{3.3}$$

for $t \in J$.

Proof. For fixed $t \in J$ and $u \in W$, we define $v(s) = p(t,s,N(s)(u))$ and $V = \{v\colon u \in W\}$. It follows from (Cp1) and (Cp2) that $V \subseteq L^1(J;X)$ is a uniformly integrable set. Applying Corollary 2.10, there exist countable sets $V_0 = \{v_n\colon n \in \mathbb{N}\}$ and $W_0 = \{u_n\colon n \in \mathbb{N}\} \subseteq W$ such that $v_n(\cdot) = p(t,\cdot,N(\cdot)(u_n))$, and

$$\begin{aligned}
\zeta\bigl(F(W)(t)\bigr) &= 2\zeta\bigl(F(W_0)(t)\bigr) \\
&= 2\zeta\left(\left\{\int_0^t p\bigl(t,s,N(s)(u_n)\bigr)\,ds\colon n \in \mathbb{N}\right\}\right) \\
&\leqslant 2\int_0^t \zeta\bigl(\bigl\{p\bigl(t,s,N(s)(u_n)\bigr)\colon n \in \mathbb{N}\bigr\}\bigr)\,ds \\
&\leqslant 2\int_0^t \widetilde{H}(t,s)\zeta\bigl(N(s)(W)\bigr)\,ds,
\end{aligned}$$

which shows that the inequality (3.3) holds. Now, by using condition (CN1), we have

$$\zeta\bigl(F(W)(t)\bigr) \leqslant 2\nu \int_0^t \widetilde{H}(t,s)\,ds\,\gamma(W),$$

for $t \in J$. In addition, combining conditions (Cp2), (Cp3) and (CN1) with the equality

$$\int_0^{t+s} p\big(t+s,\xi,N(\xi)(u)\big)\,d\xi - \int_0^t p\big(t,\xi,N(\xi)(u)\big)\,d\xi$$
$$= \int_0^t \big[p\big(t+s,\xi,N(\xi)(u)\big) - p\big(t,\xi,N(\xi)(u)\big)\big]\,d\xi + \int_t^{t+s} p\big(t+s,\xi,N(\xi)(u)\big)\,d\xi,$$

we deduce that $F(W)$ is an equicontinuous subset of $C(J;X)$. The assertion is now a consequence of Lemma 2.6. □

In order to show the generality of our presentation, we exhibit below a pair of simple examples of maps that verify the condition (CN1).

Example 3.5. Let $Q : J \to \mathcal{L}(X)$ be a strongly continuous operator-valued map. Then

$$N(t)(u) = Q(t)u(t), \quad t \in J,$$

satisfies the condition (CN1). In particular, this occurs for $Q(t) = I$. In this case, the differential equation (1.1) is reduced to the usual second order equation

$$u''(t) = A(t)u(t) + f\big(t,u(t)\big).$$

Example 3.6. Let $k : J \times J \times X \to X$ be a continuous function. Assume that k takes bounded sets into bounded sets, and that there exists a positive function $\mu \in L^1(J;\mathbb{R}^+)$ such that

$$\zeta\big(\big\{k(s,t,x)\colon x \in B\big\}\big) \leqslant \mu(s)\zeta(B),$$

for every bounded set $B \subseteq X$. Then

$$N(t)(u) = \int_0^a k\big(s,t,u(s)\big)\,ds, \quad t \in J,$$

satisfies condition (CN1). In fact, it is clear that $N(\cdot)(u)$ is continuous for each $u \in C(J;X)$. Moreover, applying again (3.2), we have

$$\zeta\big(N(t)(W)\big) \leqslant \int_0^a \mu(s)\,ds\,\gamma(W),$$

for every bounded set $W \subseteq C(J;X)$. In this case, the differential equation (1.1) is reduced to the integro-differential equation

$$u''(t) = A(t)u(t) + f\left(t, \int_0^a k\big(s,t,u(s)\big)\,ds\right).$$

In the statements that follow, the functions g and h take bounded sets into bounded sets. To represent this property, we will use the notation

$$g_R = \sup\big\{\big\|g(u)\big\|\colon \|u\| \leqslant R\big\} < \infty,$$
$$h_R = \sup\big\{\big\|h(u)\big\|\colon \|u\| \leqslant R\big\} < \infty,$$

for $R \geqslant 0$.

Lemma 3.7. *Assume that* (Cf1), (Cf2), (CN1) *are satisfied, and let K be the constant involved in* (2.3)*. Assume further that there exists a constant $M \geqslant 0$ such that*

$$K\left[g_M + ah_M + \Phi(N_M)\int_0^a (a-s)m(s)\,ds\right] \leqslant M. \tag{3.4}$$

Then the function $F : C(J;X) \to C(J;X)$ given by

$$(Fu)(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,s)f\big(s, N(s)(u)\big)\,ds, \quad t \in J, \tag{3.5}$$

is continuous and maps $B_M[0]$ to $B_M[0]$.

Proof. Since the function $s \mapsto f(s, N(s)(u))$ is integrable on J, we infer that F is well defined. We next show that F is a continuous map. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C(J;X)$ such that $u_n \to u$, $n \to \infty$, for the norm of uniform convergence. Since g and h are continuous maps,

$$C(t,0)g(u_n) + S(t,0)h(u_n) \to C(t,0)g(u) + S(t,0)h(u), \quad n \to \infty,$$

uniformly for $t \in J$. Similarly, since $N(s)(u_n) \to N(s)(u)$, $n \to \infty$, for each $s \in J$, it follows from (Cf1) that $f(s, N(s)(u_n)) \to f(s, N(s)(u))$, $n \to \infty$. Moreover, in view of

$$\big\| f\big(s, N(s)(u_n)\big)\big\| \leqslant m(s)\Phi\big(\big\|N(s)(u_n)\big\|\big) \leqslant m(s)\Phi(N_R),$$

where $R \geqslant 0$ is a constant such that $\|u_n\|_\infty \leqslant R$, applying the Lebesgue dominated convergence theorem we obtain that $F(u_n) \to F(u)$ as $n \to \infty$.

On the other hand, if $\|u\|_\infty \leqslant M$, it follows from (2.3) and (3.4) that

$$\begin{aligned}\big\|(Fu)(t)\big\| &\leqslant \big\|C(t,0)g(u)\big\| + \big\|S(t,0)h(u)\big\| + \left\|\int_0^t S(t,s)f\big(s, N(s)(u)\big)\,ds\right\| \\ &\leqslant K(g_M + ah_M) + K\Phi(N_M)\int_0^t (t-s)m(s)\,ds \\ &\leqslant K(g_M + ah_M) + K\Phi(N_M)\int_0^a (a-s)m(s)\,ds \\ &\leqslant M,\end{aligned}$$

which implies that $F(B_M[0]) \subseteq B_M[0]$. □

We next consider the following condition for functions g, h.

(Cgh) There exists $\beta > 0$ such that

$$\zeta\big(g(W)\big) + a\zeta\big(h(W)\big) \leqslant \beta\gamma(W),$$

for every bounded set $W \subseteq C(J;X)$.

We point out that if condition (Cgh) is fulfilled, then g and h take bounded sets into bounded sets.

Theorem 3.8. *Assume that* (Cf1), (Cf2), (Cf3), (CN1) *and* (Cgh) *are fulfilled. If*

$$K\left[\beta + \nu\int_0^a (a-s)H(s)\,ds\right] < 1, \tag{3.6}$$

and there exists a constant $M \geqslant 0$ such that (3.4) holds, then problem (1.1) has at least one mild solution.

Proof. It follows from Lemma 3.7 that $F : B_M[0] \to B_M[0]$ is continuous. Let now W be a bounded subset of $C(J;X)$ with $\gamma(W) > 0$. It follows directly from Definition 2.3 that

$$\gamma\big(\big\{C(\cdot,0)g(u)\colon u \in W\big\}\big) \leqslant K\zeta\big(g(W)\big),$$
$$\gamma\big(\big\{S(\cdot,0)h(u)\colon u \in W\big\}\big) \leqslant Ka\zeta\big(h(W)\big).$$

We define the map $F_1 : C(J;X) \to C(J;X)$ given by

$$F_1(u)(t) = \int_0^t S(t,s)f\big(s, N(s)(u)\big)\,ds. \tag{3.7}$$

Let $p(t,s,x) = S(t,s)f(s,x)$. It is easy to see that p satisfies the hypotheses of Corollary 3.4. Furthermore, the function $\widetilde{H}$ involved in the statement of Corollary 3.4 can be chosen as $\widetilde{H}(t,s) = K(t-s)H(s)$. Therefore, we get

$$\gamma\big(\{F_1(u)\colon u \in W\}\big) \leqslant 2K \sup_{t\in J} \int_0^t (t-s)H(s)\,ds\,\gamma(W)$$
$$= 2\nu K \int_0^a (a-s)H(s)\,ds\,\gamma(W),$$

and combining these estimates, we have

$$\gamma\big(F(W)\big) \leqslant K\left(\zeta\big(g(W)\big) + a\zeta\big(h(W)\big) + 2\nu \int_0^a (a-s)H(s)\,ds\,\gamma(W)\right)$$
$$< \gamma(W),$$

which implies that F is a condensing map. The assertion is a consequence of Theorem 2.12. □

Corollary 3.9. *Assume that* (Cf1), (Cf2), (Cf3) *and* (CN1) *are fulfilled and that $S(t,s)$ is compact for all $s,t \in J$. Assume further that the following conditions are satisfied*:

(a) *The map $g : C(J;X) \to X$ is continuous and satisfies*

$$\zeta\big(g(W)\big) < \frac{1}{K}\gamma(W),$$

for every bounded set $W \subseteq C(J;X)$ such that $\gamma(W) \neq 0$.

(b) *The map $h : C(J;X) \to X$ is continuous and takes bounded sets into bounded sets.*

If there exists a constant $M \geqslant 0$ such that (3.4) *holds, then problem* (1.1) *has at least one mild solution.*

Proof. We define F by (3.5). It follows from Lemma 3.7 that F is continuous and $F : B_M[0] \to B_M[0]$.

Let $W \subseteq B_M[0]$ with $\gamma(W) > 0$. It follows from (a) that

$$\gamma\big(\{C(\cdot,0)g(u)\colon u \in W\}\big) \leqslant K\zeta\big(\{g(u)\colon u \in W\}\big) < \gamma(W).$$

Moreover, since each operator $S(t,0)$ is compact and the operator-valued map $S(\cdot,0)$ is continuously differentiable, and h takes bounded sets into bounded sets, a direct application of the Arzelà–Ascoli theorem allows us to conclude that $\{S(\cdot,0)h(u)\colon u \in W\}$ is relatively compact in $C(J;X)$. This shows that condition (Cgh) holds with $\beta = 1/K$.

Let $p(t,s,x) = S(t,s)f(s,x)$. As was mentioned in the proof of Theorem 3.8 the function p satisfies the hypotheses of Corollary 3.4. In this case, the function $\widetilde{H}$ involved in the statement of Corollary 3.4 can be chosen as $\widetilde{H}(t,s) = 0$. Therefore, if F_1 is the map defined by (3.7), we have

$$\gamma\big(F_1(W)\big) = 0,$$

and

$$\gamma\big(F(W)\big) \leqslant \gamma\big(\{C(\cdot,0)g(u) + S(\cdot,0)h(u)\colon u \in W\}\big) + \gamma\big(F_1(W)\big) < \gamma(W).$$

This implies that (3.6) holds. The assertion is a consequence of Theorem 3.8. □

The condition (a) used in Corollary 3.9 is difficult to verify in concrete situations. For this reason, we modify slightly the statement of Corollary 3.9 to get the following property.

Corollary 3.10. *Assume that* (Cf1), (Cf2), (Cf3) *and* (CN1) *are fulfilled and that $S(t,s)$ is compact for all $s,t \in J$. Assume further that the following conditions are satisfied*:

(a) *The map $g : C(J;X) \to X$ satisfies the Lipschitz condition*

$$\big\|g(u_2) - g(u_1)\big\| \leqslant L_g\|u_2 - u_1\|,$$

for all $u_1, u_2 \in C(J;X)$.

(b) *The map $h : C(J; X) \to X$ is continuous and takes bounded sets into bounded sets.*

If $KL_g < 1$ and there is a constant $M \geqslant 0$ such that

$$K\left[L_g M + \|g(0)\| + ah_M + \Phi(N_M)\int_0^a (a-s)m(s)\,ds\right] \leqslant M, \tag{3.8}$$

then problem (1.1) has at least one mild solution.

Proof. It follows from the definition of the Hausdorff measure of non-compactness that

$$\zeta\big(g(W)\big) \leqslant L_g\gamma(W),$$

for every bounded set $W \subseteq C(J; X)$. Moreover, since

$$\big\|g(u)\big\| \leqslant \big\|g(u) - g(0)\big\| + \big\|g(0)\big\| \leqslant L_g\|u\| + \big\|g(0)\big\|,$$

we obtain that $g_M \leqslant L_g M + \|g(0)\|$. From Lemma 3.7 we have $F : B_M[0] \to B_M[0]$. The assertion is now a direct consequence of Corollary 3.9. □

Corollary 3.11. *Assume that* (Cf1), (Cf2), (CN1) *and* (Cgh) *are fulfilled, and that $\{f(s, N(s)(u))\colon u \in W\}$ is relatively compact for each bounded set $W \subseteq C(J; X)$. If $K\beta < 1$ and there exists a constant $M \geqslant 0$ such that (3.4) holds, then the problem (1.1) has at least one mild solution.*

Proof. Initially we argue as in the proof of Theorem 3.8 to obtain

$$\gamma\big(F(W)\big) \leqslant K\beta\gamma(W) + \gamma\big(F_1(W)\big).$$

Now, arguing as in the proof of Corollary 3.9, we define $p(t, s, x) = S(t, s)f(s, x)$. Since the set $\{S(t, s)f(s, N(s)(u))\colon u \in W\}$ is relatively compact, we can take $\widetilde{H}(t, s) = 0$ in the statement of Corollary 3.9 to conclude that $\gamma(F_1(W)) = 0$. Hence, we get

$$\gamma\big(F(W)\big) \leqslant K\beta\gamma(W) < \gamma(W),$$

and we complete the proof as in Theorem 3.8. □

For our next results we need to strengthen the condition (CN1) for the family of functions $\{N(t)\colon t \in J\}$. For a bounded set $W \subseteq C(J; X)$ and $t \in J$, we denote

$$\gamma\big(W, [0, t]\big) = \gamma\big(\{w|_{[0,t]}\colon w \in W\}\big).$$

(CN2) There exists a constant $\nu > 0$ such that

$$\zeta\big(N(t)(W)\big) \leqslant \nu\gamma\big(W, [0, t]\big)$$

for each bounded set $W \subseteq C(J; X)$.

Example 3.12. Let $N(t)(u) = u(t)$ be the map considered in Example 3.5. It is clear that the family $\{N(t)\colon t \in J\}$ satisfies condition (CN2).

Example 3.13. Let

$$N(t)(u) = \int_0^t k\big(t, s, u(s)\big)\,ds, \quad t \in J,$$

where $k : \{(t, s)\colon t \in J,\ 0 \leqslant s \leqslant t\} \times X \to X$ is a continuous function. Assume that k takes bounded sets into bounded sets, and there exists a positive function $\mu \in L^1(J; \mathbb{R}^+)$ such that

$$\zeta\big(\big\{k(t, s, x)\colon x \in B\big\}\big) \leqslant \mu(s)\zeta(B),$$

for every bounded set $B \subseteq X$. Then N satisfies condition (CN2). In fact, it is clear that $N(\cdot)(u)$ is continuous for each $u \in C(J; X)$. Moreover, proceeding as in Example 3.6,

$$\zeta\big(N(t)(W)\big) \leqslant \int_0^t \mu(s)\,ds\,\gamma\big(W,[0,t]\big),$$

for every bounded set $W \subseteq C(J;X)$.

Definition 3.14. (See [18].) Let X, Y be Banach spaces. We will say the map $h: X \to Y$ is *completely continuous* if h is continuous and $h(B)$ is relatively compact in Y for any bounded subset B of X.

Theorem 3.15. *Assume that* (Cf1), (Cf2), (Cf3) *and* (CN2) *are fulfilled, and that g, h are completely continuous maps. If the inequality* (3.4) *holds, then problem* (1.1) *has at least one mild solution.*

Proof. We define the map F by (3.5). It follows from Lemma 3.7 that F is continuous and $F(B_M[0]) \subseteq B_M[0]$. Moreover, $F(B_M[0])$ is an equicontinuous set of functions. In fact, since $g(B_M[0])$ is relatively compact and $C(\cdot,0)$ is strongly continuous, applying the Arzelà–Ascoli theorem, we infer that the set $\{C(\cdot,0)g(u):\ u \in B_M[0]\}$ is relatively compact in $C(J;X)$. Using the same argument we can establish that the set $\{S(\cdot,0)h(u):\ u \in B_M[0]\}$ is relatively compact in $C(J;X)$.

Let F_1 be the map given by (3.7). Proceeding as in the proof of Corollary 3.4 with $p(t,s,x) = S(t,s)f(s,x)$ we infer that the set $F_1(B_M[0])$ is equicontinuous.

We define $\mathfrak{B} = \overline{co}(F(B_M[0]))$. Since $\mathfrak{B} \subseteq B_M[0]$, then $F: \mathfrak{B} \to \mathfrak{B}$, and it follows from the previous assertions and Lemma 2.5 that the set $\mathfrak{B}$ is equicontinuous.

Let $D \subseteq \mathfrak{B}$. Since D is an equicontinuous set, it follows from Lemma 2.6 that

$$\gamma\big(D,[0,t]\big) = \sup_{0\leqslant s\leqslant t} \zeta\big(D(s)\big)$$

is a continuous function. Using the general properties of the Hausdorff measure of non-compactness and Corollary 3.4, we have that

$$\begin{aligned}
\zeta\big(F(D)(t)\big) &\leqslant \zeta\big(C(t,0)g(D)\big) + \zeta\big(S(t,0)h(D)\big) + \zeta\left(\int_0^t S(t,s)f\big(s,N(s)(D)\big)\,ds\right) \\
&= \zeta\left(\int_0^t S(t,s)f\big(s,N(s)(D)\big)\,ds\right) \\
&\leqslant 2\,\nu a K \int_0^t H(s)\gamma\big(D,[0,s]\big)\,ds \\
&\leqslant 2\,\nu a K \int_0^t H(s)\,ds\,\gamma\big(D,[0,t]\big)
\end{aligned}$$

and

$$\zeta\big(F\big(\overline{co}(D)\big)(t)\big) \leqslant 2\nu a K \int_0^t H(s)\gamma\big(\overline{co}(D),[0,s]\big)\,ds = 2\nu a K \int_0^t H(s)\gamma\big(D,[0,s]\big)\,ds.$$

Proceeding inductively, and arguing as above, we can show that

$$\begin{aligned}
\zeta\big(F^n(D)(t)\big) &= \zeta\big(F\big(F^{n-1}\big(\overline{co}(D)\big)\big)(t)\big) \\
&\leqslant (2\nu a K)^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} H(s_n)\cdots H(s_2)H(s_1)\,ds_n\cdots ds_1\,\gamma\big(D,[0,t]\big) \\
&= \frac{(2\nu a K)^n}{n!}\left(\int_0^t H(s)\,ds\right)^n \gamma\big(D,[0,t]\big).
\end{aligned}$$

Therefore,

$$\gamma\big(F^n(D)\big) = \sup_{t\in J} \zeta\big(F^n(D)(t)\big) \leqslant \frac{(2\nu aK)^n}{n!}\left(\int_0^t H(s)\,ds\right)^n \gamma(D).$$

Since $\frac{(2\nu aK)^n}{n!}(\int_0^t H(s)\,ds)^n \to 0$ as $n\to\infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{(2\nu aK)^{n_0}}{n_0!}\left(\int_0^t H(s)\,ds\right)^{n_0} = r < 1,$$

and applying Theorem 2.13 it follows that F has a fixed point in $\mathfrak{B}$. This fixed point is a mild solution of problem (1.1). □

The inequality (3.4) is somewhat difficult to verify. We next state a case where the verification of this hypothesis is immediate.

Corollary 3.16. *Assume that* (Cf1), (Cf2), (Cf3) *and* (CN2) *are fulfilled, and that g, h are bounded and completely continuous maps. If*

$$\int_0^a m(t)\sup\big\{\Phi\big(\big\|N(t)(u)\big\|\big)\colon u\in C(J;X)\big\}\,dt < \infty,$$

then problem (1.1) *has at least one mild solution.*

Proof. In this case the condition (3.4) is verified for any constant $M>0$ such that

$$K\left[\sup_{u\in C(J;X)}\big\|g(u)\big\| + a\sup_{u\in C(J;X)}\big\|h(u)\big\| + a\int_0^a m(t)\sup_{u\in C(J;X)}\Phi\big(\big\|N(t)(u)\big\|\big)\,dt\right] \leqslant M.$$

The assertion is an immediate consequence of Theorem 3.15. □

We turn to consider the case where the evolution operator $\{S(t,s)\}_{t,s\in J}$ is compact. We can use this property to avoid the condition (3.4).

Theorem 3.17. *Assume that* (Cf1), (Cf2) *and* (Cf3) *are fulfilled, and that $S(t,s)$ is a compact operator for all $0\leqslant s\leqslant t\leqslant a$. Suppose also that the following conditions are satisfied*:

(a) *The map $g: C(J;X)\to X$ is bounded and completely continuous.*
(b) *The map $h: C(J;X)\to X$ is bounded.*
(c) *There exists a constant $\nu>0$ such that $\|N(t)(u)\| \leqslant \nu\sup_{0\leqslant s\leqslant t}\|u(s)\|$, for all $t\in J$ and $u\in C(J;X)$.*

If, in further, $\int_c^\infty \frac{1}{\Phi(\xi)}\,d\xi = \infty$ for all $c>0$, then problem (1.1) *has at least one mild solution.*

Proof. We define F by (3.5). From Lemma 3.7 we have that F is continuous. Moreover, arguing as in the proof of Corollary 3.9, we can show that for all $R>0$ the set $F(B_R)$ is relatively compact in $C(J;X)$. Combining these assertions, we conclude that F is completely continuous. Let $u\in C(J;X)$ be a function such that $u=\lambda F(u)$, for some $0<\lambda<1$. Using (3.5), we can estimate

$$\begin{aligned}\big\|u(t)\big\| &\leqslant K\big\|g(u)\big\| + aK\big\|h(u)\big\| + aK\int_0^t m(s)\Phi\big(\big\|N(s)(u)\big\|\big)\,ds\\ &\leqslant c + aK\int_0^t m(s)\Phi\big(\big\|N(s)(u)\big\|\big)\,ds,\end{aligned}$$

where $c>0$ is a constant such that $K\|g(u)\| + aK\|h(u)\|\leqslant c$ for all $u\in C(J;X)$.

Let $\alpha(t) = c + aK\int_0^t m(s)\Phi(\|N(s)(u)\|)\,ds$. Then

$$\alpha'(t) = aKm(t)\Phi\big(\big\|N(t)(u)\big\|\big) \leqslant aKm(t)\Phi\big(\nu\alpha(t)\big),$$

which implies that

$$\frac{\alpha'(t)}{\Phi(\nu\alpha(t))} \leqslant aKm(t)$$

and, after integration on $[0, t]$, we obtain

$$\int_{\nu c}^{\nu\alpha(t)} \frac{d\xi}{\Phi(\xi)} \leqslant a\nu K \int_0^t m(s)\,ds < \infty.$$

It follows from this inequality that $\alpha(\cdot)$ is a bounded function, which implies the set $\{u \in C(J; X)\colon u = \lambda F(u),\ 0 < \lambda < 1\}$ is bounded. Applying the Leray–Schauder alternative theorem [18, Theorem II.5.4], we deduce that F has a fixed point. □

4. Applications

In this section we apply the theory developed in Section 3 to the study of the wave equation. Initially we will present a review of the basic properties of the second order abstract Cauchy problem.

Let $A_0 : D(A_0) \subseteq X \to X$ be the infinitesimal generator of a strongly continuous cosine family $\{C_0(t)\}_{t\in\mathbb{R}}$ of bounded linear operators on X, and let $\{S_0(t)\}_{t\in\mathbb{R}}$ be the sine family associated with $\{C_0(t)\}_{t\in\mathbb{R}}$, which is defined by

$$S_0(t)x = \int_0^t C_0(s)x\,ds,$$

for $x \in X$ and $t \in \mathbb{R}$. For the general properties of cosine families we refer the reader to [17,41,42]. It follows from this definition that

$$C_0(t)x - x = A\int_0^t S_0(s)x\,ds$$

for all $x \in X$ and $t \geqslant 0$.

The notation E stands for the space consisting of vectors $x \in X$ such that the function $C_0(\cdot)x$ is of class C^1. Kisyński in [24] has proved that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\leqslant t\leqslant 1} \left\|A_0 S_0(t)x\right\|, \quad x \in E,$$

is a Banach space. Moreover, the operator-valued function $G(t) = \left[\begin{smallmatrix} C_0(t) & S_0(t) \\ A_0S_0(t) & C_0(t) \end{smallmatrix}\right]$ is a strongly continuous group of bounded linear operators on the space $E \times X$, generated by the operator $\mathcal{A} = \left[\begin{smallmatrix} 0 & I \\ A_0 & 0 \end{smallmatrix}\right]$ defined on $D(A_0) \times E$. It follows from this property that $S_0(t) : X \to E$ is a bounded linear operator such that the operator-valued map $S_0(\cdot)$ is strongly continuous and $A_0S_0(t) : E \to X$ is a bounded linear operator such that $A_0S_0(t)x \to 0$ as $t \to 0$ for all $x \in E$. Furthermore, if $f : [0, \infty) \to X$ is a locally integrable function, then

$$v(t) = \int_0^t S_0(t-s)f(s)\,ds$$

defines an E-valued continuous function.

Finally, we mention that the function $u(\cdot)$ given by

$$u(t) = C_0(t-s)x + S_0(t-s)y + \int_s^t S_0(t-\xi)f(\xi)\,d\xi, \quad t \in J, \tag{4.1}$$

is called mild solution of the problem

$$\left.\begin{aligned} & u''(t) = A_0u(t) + f(t), \quad t \geqslant 0, \\ & u(s) = x, \\ & u'(s) = y. \end{aligned}\right\} \tag{4.2}$$

When $x \in E$, the function $u(\cdot)$ given by (4.1) is continuously differentiable, and

$$u'(t) = A_0S_0(t-s)x + C_0(t-s)y + \int_s^t C_0(t-\xi)f(\xi)\,d\xi, \quad t \in J.$$

Moreover, if $x \in D(A)$, $y \in E$ and f is a continuously differentiable function, then the function $u(\cdot)$ is a classical solution of problem (4.2).

It is well known that, except in the case $dim(X) < \infty$, a cosine function $C_0(t)$ cannot be compact for all $t \in [t_1, t_2]$, with $t_2 - t_1 > 0$ (see [39]). On the contrary, for the cosine functions that arise in specific applications, the sine function $S_0(t)$ is very often a compact operator for all $t \in \mathbb{R}$. A similar situation occurs for the evolution operator $S(t,s)$ generated by a family $\{A(t): t \in J\}$. We next consider a particular situation.

Assume that A_0 is the infinitesimal generator of a cosine function $C_0(t)$. Let $A(t) = A_0 + P(t)$ for all $t \in J$, where $P(\cdot): J \to \mathcal{L}(E;X)$ is a map such that the function $t \mapsto P(t)x$ is continuously differentiable in X for each $x \in E$. It has been established by Serizawa and Watanabe [35] that for each $(y,z) \in D(A_0) \times E$ the non-autonomous abstract Cauchy problem

$$\left.\begin{aligned} &u''(t) = \big(A_0 + P(t)\big)u(t), \quad t \in J,\\ &u(0) = y,\\ &u'(0) = z \end{aligned}\right\}$$

has a unique solution $u(\cdot)$ such that the function $t \mapsto u(t)$ is continuously differentiable in E. It is clear that the same argument allows us to conclude that equation

$$\left.\begin{aligned} &u''(t) = \big(A_0 + P(t)\big)u(t), \quad t \in J,\\ &u(s) = y,\\ &u'(s) = z, \end{aligned}\right\}$$

has a unique solution $u(\cdot,s)$ such that the function $t \mapsto u(t,s)$ is continuously differentiable in E. It follows from (4.1) that

$$u(t,s) = C_0(t-s)y + S_0(t-s)z + \int_s^t S_0(t-\xi)P(\xi)u(\xi,s)\,d\xi.$$

In particular, for $y = 0$, we have

$$u(t,s) = S_0(t-s)z + \int_s^t S_0(t-\xi)P(\xi)u(\xi,s)\,d\xi. \tag{4.3}$$

Consequently,

$$\big\|u(t,s)\big\|_1 \leqslant \big\|S_0(t-s)\big\|_{\mathcal{L}(X,E)}\|z\| + \int_s^t \big\|S_0(t-\xi)\big\|_{\mathcal{L}(X,E)}\big\|P(\xi)\big\|_{\mathcal{L}(E,X)}\big\|u(\xi,s)\big\|_1\,d\xi.$$

Applying the Gronwall–Bellman lemma, there exists $\widetilde{M} \geqslant 0$ such that $\|u(t,s)\|_1 \leqslant \widetilde{M}\|z\|$, for $s,t \in J$.

We define the operator $S(t,s)z = u(t,s)$. It follows from the previous estimate that $S(t,s)$ is a bounded linear map on E for the norm in X. Since E is dense in X, we can extend $S(t,s)$ to X. We keep the notation $S(t,s)$ for this extension.

This motivates the following result established by Henríquez in [20].

Lemma 4.1. *Under the preceding conditions, $\{S(t,s)\}_{t,s\in J}$ is the evolution operator generated by the family $\{A(t): t \in J\}$. Moreover, if $S_0(t)$ is compact for all $t \in \mathbb{R}$, then $S(t,s)$ is also compact for all $s,t \in J$.*

The one-dimensional wave equation modeled as an abstract Cauchy problem has been studied extensively. See for example [48]. In this section, we apply the results established in the preceding section to study the existence of solutions of the non-autonomous wave equation with nonlocal initial conditions. Initially, we will study the following problem

$$\left.\begin{aligned} &\frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi} + \tilde{f}\big(t, w(t,\xi)\big), \quad t \in J,\\ &w(t,0) = w(t,2\pi), \qquad \frac{\partial w}{\partial \xi}(t,0) = \frac{\partial w}{\partial \xi}(t,2\pi), \quad t \in J,\\ &w(0,\xi) = \sum_{i=0}^m g_i w(t_i,\xi),\\ &\frac{\partial w(0,\xi)}{\partial t} = \sum_{i=0}^m h_i w(t_i,\xi), \end{aligned}\right\} \tag{4.4}$$

for $0 \leqslant \xi \leqslant 2\pi$. Here $b : J \to \mathbb{R}$ is a continuous function, $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ satisfies appropriate conditions which will be specified later, $0 < t_0 < \cdots < t_m \leqslant a$, and $g_i, h_i \in \mathbb{R}$, for $i = 0, 1, \ldots, m$.

We model this problem in the space $X = L^2(\mathbb{T}, \mathbb{C})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. Specifically, in what follows we denote by $L^2(\mathbb{T}, \mathbb{C})$ the space of 2π-periodic 2-integrable functions from $\mathbb{R}$ into $\mathbb{C}$. Similarly, $H^2(\mathbb{T}, \mathbb{C})$ denotes the Sobolev space of 2π-periodic functions $u : \mathbb{R} \to \mathbb{R}$ such that $u'' \in L^2(\mathbb{T}, \mathbb{C})$.

We consider the operator A_0 defined by

$$A_0 z = \frac{d^2 z(\xi)}{d\xi^2} \quad \text{with domain } D(A_0) = H^2(\mathbb{T}, \mathbb{C}).$$

It is well known that A_0 is the infinitesimal generator of a strongly continuous cosine function $C_0(t)$ in X. Moreover, A_0 has discrete spectrum, the spectrum of A_0 consists of eigenvalues $-n^2$ for $n \in \mathbb{Z}$ with associated eigenvectors

$$z_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}, \quad n \in \mathbb{Z}.$$

Furthermore, the set $\{z_n \colon n \in \mathbb{Z}\}$ is an orthonormal basis of X. In particular,

$$A_0 z = \sum_{n \in \mathbb{Z}} -n^2 \langle z, z_n \rangle z_n,$$

for $z \in D(A_0)$. The cosine function $C_0(t)$ is given by

$$C_0(t) z = \sum_{n \in \mathbb{Z}} \cos(nt) \langle z, z_n \rangle z_n, \quad t \in \mathbb{R},$$

with associated sine function

$$S_0(t) z = t \langle z, z_0 \rangle z_0 + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\sin(nt)}{n} \langle z, z_n \rangle z_n, \quad t \in \mathbb{R}.$$

It is clear that $\|C_0(t)\| \leqslant 1$ for all $t \in \mathbb{R}$. Thus, $C_0(\cdot)$ is uniformly bounded on $\mathbb{R}$. Hence, $\|S_0(t)\| \leqslant |t|$, for all $t \in \mathbb{R}$. Moreover, $S_0(t)$ is a compact operator.

For $t \in J$ the operators $P(t)$ are defined by

$$P(t) z = b(t) \frac{dz(\xi)}{d\xi} \quad \text{with domain } D\big(P(t)\big) = H^1(\mathbb{T}, \mathbb{C}).$$

Let $A(t) = A_0 + P(t)$, $t \in J$. It has been proved by Henríquez in [20] that the family $\{A(t) \colon t \in J\}$ generates an evolution operator $\{S(t,s)\}_{t,s \in J}$. From Lemma 4.1 we have that the operators $S(t,s)$ are compact. We now estimate the constant K involved in our statements. For $z \in E$, we abbreviate $x(t,s) = S(t,s)z$. We decompose $x(t,s) = \sum_{n \in \mathbb{Z}} x_n(t,s) z_n$, where $x_n(t,s) = \langle x(t,s), z_n \rangle$. It follows from (4.3) that

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} x_n(t,s) z_n &= S_0(t-s) z + \int_s^t S_0(t-\tau) P(\tau) \sum_{n \in \mathbb{Z}} x_n(\tau,s) z_n \, d\tau \\
&= (t-s)\langle z, z_0 \rangle z_0 + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\sin n(t-s)}{n} \langle z, z_n \rangle z_n + \int_s^t S_0(t-\tau) b(\tau) \sum_{n \in \mathbb{Z}} x_n(\tau,s) z_n' \, d\tau \\
&= (t-s)\langle z, z_0 \rangle z_0 + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\sin n(t-s)}{n} \langle z, z_n \rangle z_n + i \sum_{n \in \mathbb{Z}} n \int_s^t b(\tau) x_n(\tau,s) S_0(t-\tau) z_n \, d\tau \\
&= (t-s)\langle z, z_0 \rangle z_0 + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\sin n(t-s)}{n} \langle z, z_n \rangle z_n + i \sum_{n \in \mathbb{Z}} \int_s^t b(\tau) x_n(\tau,s) \sin n(t-\tau) z_n \, d\tau,
\end{aligned}$$

which implies that

$$\begin{aligned}
x_0(t,s) &= (t-s)\langle z, z_0 \rangle, \\
x_n(t,s) &= \frac{\sin n(t-s)}{n} \langle z, z_n \rangle + i \int_s^t b(\tau) \sin n(t-\tau) x_n(\tau,s) \, d\tau,
\end{aligned}$$

for $n \in \mathbb{Z}$, $n \neq 0$. We introduce the functions $v_n(t,s) = -\frac{\partial x_n(t,s)}{\partial s}$ for $n \in \mathbb{Z}$. It follows from the above expressions that

$$v_0(t,s) = \langle z, z_0 \rangle,$$
$$v_n(t,s) = \cos n(t-s)\langle z, z_n \rangle + i \int_s^t b(\tau) \sin n(t-\tau) v_n(\tau,s)\, d\tau, \quad n \neq 0.$$

Hence we obtain that

$$\left|v_n(t,s)\right| \leqslant \left|\langle z, z_n \rangle\right| + \int_s^t \left|b(\tau)\right| \left|v_n(\tau,s)\right| d\tau, \quad 0 \leqslant s \leqslant t,\ n \neq 0.$$

Applying the Gronwall–Bellman lemma, we obtain

$$\left|v_n(t,s)\right| \leqslant e^{\int_s^t |b(\tau)|\, d\tau} \left|\langle z, z_n \rangle\right|.$$

Since $C(t,s)z = -\frac{\partial S(t,s)z}{\partial s}$, it follows that

$$\left\|C(t,s)z\right\| \leqslant e^{\int_s^t |b(\tau)|\, d\tau} \|z\|.$$

Therefore, since $t \in J$, we can take $K = e^{\int_0^a |b(\tau)|\, d\tau}$.

We assume that $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$\left|\tilde{f}(t,r)\right| \leqslant m(t)|r|, \quad t \in J,\ r \in \mathbb{R},$$

where $m \in L^1(J : \mathbb{R}^+)$.

To complete our construction we define the functions f, N, g and h by

$$f(t,w)(\xi) = \tilde{f}\big(t, w(t,\xi)\big),$$
$$N(t)(w)(\xi) = w(t,\xi),$$
$$g(w)(\xi) = \sum_{i=0}^m g_i w(t_i,\xi),$$
$$h(w)(\xi) = \sum_{i=0}^m h_i w(t_i,\xi).$$

Using this construction, and defining $u(t) = w(t,\cdot) \in X$, the problem (4.4) is modeled in the abstract form of problem (1.1). It is clear that f satisfies conditions (Cf1) and (Cf2), with $\Phi(r) = r$; N satisfies the condition (CN1), with $\nu = 1$ and $N_R = R$, and g, h are bounded linear maps with $\|g\| = \sum_{i=0}^m |g_i|$ and $\|h\| = \sum_{i=0}^m |h_i|$. Therefore, the following result is an easy consequence of Corollary 3.9.

Corollary 4.2. *Under the above conditions, assume further that*

$$K\left[\sum_{i=0}^m \big(|g_i| + a|h_i|\big) + \int_0^a (a-s)m(s)\, ds\right] < 1, \tag{4.5}$$

then problem (4.4) has at least one mild solution.

Proof. It follows from our preceding considerations and Lemma 4.1 that $S(t,s)$ is compact. Moreover, condition (3.4) is an immediate consequence of (4.5). Since g is a bounded linear map,

$$\zeta\big(g(W)\big) \leqslant \|g\|\gamma(W) \leqslant \sum_{i=0}^m |g_i|\gamma(W) < \frac{1}{K}\gamma(W),$$

for every bounded set $W \subseteq C(J;X)$. Therefore, the hypotheses of Corollary 3.9 are fulfilled. □

We now are concerned with the problem

$$\left.\begin{aligned}
&\frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi} + \tilde{f}\left(t, \int_0^t p(s)w(s,\xi)\,ds\right), \quad t \in J,\\
&w(t,0) = w(t,2\pi), \qquad \frac{\partial w}{\partial \xi}(t,0) = \frac{\partial w}{\partial \xi}(t,2\pi), \quad t \in J,\\
&w(0,\xi) = \int_0^a \int_0^\xi q_0(s,\xi)w(s,r)\,dr\,ds,\\
&\frac{\partial w(0,\xi)}{\partial t} = \int_0^a q_1(s)w(s,\xi)\,ds,
\end{aligned}\right\} \tag{4.6}$$

for $0 \leqslant \xi \leqslant 2\pi$. To study this problem we keep notation as introduced in the analysis of problem (4.4). Additionally, we assume that $p, q_1 : J \to \mathbb{R}$ and $q_0 : J \times [0, 2\pi] \to \mathbb{R}$ are continuous functions, and that $q_0(t, 2\pi) = 0$ for all $t \in J$.

On the other hand, in this case, we define

$$N(t)(w)(\xi) = \int_0^t p(s)w(s,\xi)\,ds,$$

$$g(w)(\xi) = \int_0^\xi \int_0^a q_0(s,\xi)w(s,r)\,ds\,dr,$$

$$h(w)(\xi) = \int_0^a q_1(s)w(s,\xi)\,ds.$$

It is clear that $N(t)$, g, h are bounded linear maps with

$$\|N(t)\| = \int_0^t |p(s)|\,ds,$$

$$\|g\| \leqslant (2\pi a)^{1/2}\left(\int_0^{2\pi}\int_0^a q_0(s,\xi)^2\,ds\,d\xi\right)^{1/2},$$

$$\|h\| = \int_0^a |q_1(s)|\,ds.$$

Moreover, the map g is completely continuous. Therefore, using again Corollary 3.9, and arguing as above, we can state the following result.

Corollary 4.3. *Under the above conditions, assume further that*

$$K\left(\|g\| + a\|h\| + \nu\int_0^a (a-s)m(s)\,ds\right) \leqslant 1,$$

where $\nu = \int_0^a |p(s)|\,ds$. Then problem (4.6) has at least one mild solution.

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