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SOME RESULTS FOR NONLOCAL ELLIPTIC AND PARABOLIC NONLINEAR EQUATIONS

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A Pepita, mi mujer.

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Abstract

This thesis is devoted to the study of qualitative properties of degenerate elliptic equations where the diffusion is purely nonlocal, and it is carried out in the framework of the theory of viscosity solutions.

The first part of the thesis is focused in the study of compactness properties of a family of *zero-th order nonlocal operators*, that is, elliptic nonlocal operators defined through a finite measure. We consider a one parameter family of zero-th order operators with the form

$$\mathcal{I}_\epsilon(u,x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)] K_\epsilon(z) dz,$$

where, for each $\epsilon \in (0,1)$, $K_\epsilon \in L^1(\mathbb{R}^N)$ is a radially symmetric, positive function. We set our problem in such a way $\mathcal{I}_\epsilon$ approaches the fractional Laplacian as $\epsilon \to 0^+$, implying that the L^1-norm of K_ϵ blows up as $\epsilon \to 0^+$. In the first result of this part we provide a common space-time modulus of continuity, independent of $\epsilon \in (0,1)$, for the family of bounded solutions of the nonlocal Heat equation in the plane associated to $\mathcal{I}_\epsilon$. The second result of this part considers a Dirichlet problem in a bounded domain $\Omega \subset \mathbb{R}^N$ associated to $\mathcal{I}_\epsilon$, and we conclude the compactness of the family of bounded solutions $\{u_\epsilon\}_\epsilon$ to these Dirichlet problems by finding a common modulus of continuity in $\bar{\Omega}$ for $\{u_\epsilon\}_\epsilon$, which is independent of ϵ.

The second part of the thesis is related to well-posedness, regularity and large time behavior for nonlocal equations with dominating gradient terms. We start with the well-posedness of a Hamilton-Jacobi equation with the form

$$\begin{aligned} \lambda u - \mathcal{I}(u) + H(x, Du) &= 0 \qquad \text{in } \Omega \\ u &= \varphi \qquad \text{in } \Omega^c, \end{aligned}$$

where the Hamiltonian H has a *Bellman form*. We set our problem in such a way the nonlocal operator $\mathcal{I}$ is of order less than 1 and therefore loss of the boundary condition may arise. In the second section of this part, we consider H coercive with a gradient growth stronger than the diffusive order of the nonlocal operator. The main result in this case is the Hölder continuity of *subsolutions* for this problem. Stability of the regularity estimates as $\lambda \to 0$ allows to conclude the asymptotic ergodic behavior as $t \to \infty$ for the associated parabolic problem in the torus. In this task, strong maximum principles are of main importance in the asymptotic analysis. Finally, we adapt the results obtained in the first two sections of this part of the thesis to obtain the large time behavior for the Cauchy-Dirichlet problem associated to H in both Bellman and coercive form. In this case, the influence of the exterior data in the equation through the nonlocal term makes the parabolic problem approaches the corresponding stationary problem as $t \to \infty$.

Resumen

Esta tesis está dedicada al estudio de propiedades cualitativas de ecuaciones elípticas degeneradas donde la difusión es puramente no local, y se lleva a cabo en el contexto de la teoría de soluciones viscosas.

La primera parte de la tesis trata el estudio de propiedades de compacidad de una familia de *operadores no locales de orden cero*, es decir, operadores elípticos no locales definidos a través de una medida finita. Consideramos un familia uni-paramétrica de operadores de orden cero de la forma

$$\mathcal{I}_\epsilon(u, x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)] K_\epsilon(z) dz,$$

donde, para cada $\epsilon \in (0,1)$, $K_\epsilon \in L^1(\mathbb{R}^N)$ es una función radialmente simétrica y positiva. Configuramos nuestro problema de manera que $\mathcal{I}_\epsilon$ aproxime el Laplaciano fraccionario cuando $\epsilon \to 0^+$, lo que implica que la norma L^1 de K_ϵ es no acotada a medida que $\epsilon \to 0^+$. Como primer resultado de esta parte obtenemos un módulo de continuidad en espacio-tiempo para la familia de soluciones acotadas de la ecuación del calor no local en el plano asociada a $\mathcal{I}_\epsilon$ que es independiente de $\epsilon \in (0,1)$. El segundo resultado de esta parte considera un problema de Dirichlet en un dominio acotado $\Omega \subset \mathbb{R}^N$ asociado a $\mathcal{I}_\epsilon$, y concluimos la compacidad de la familia de soluciones acotadas $\{u_\epsilon\}_\epsilon$ para estos problemas de Dirichlet encontrando un módulo de continuidad común en $\bar{\Omega}$ para $\{u_\epsilon\}_\epsilon$, que es independiente de ϵ.

La segunda parte de la tesis está relacionada con la existencia y unicidad, regularidad y comportamiento a grandes tiempos para ecuaciones no locales con términos de gradiente dominantes. Comenzamos con la existencia y unicidad de una ecuación de Hamilton-Jacobi de la forma

$$\begin{aligned} \lambda u - \mathcal{I}(u) + H(x, Du) &= 0 \qquad \text{en } \Omega \\ u &= \varphi \qquad \text{en } \Omega^c, \end{aligned}$$

donde el Hamiltoniano H tiene una *forma de Bellman*. Estructuramos el problema de manera que el operador no local $\mathcal{I}$ es de orden menor que 1 y por lo tanto puede aparecer una pérdida de la condición de borde. En la segunda sección de esta parte, consideramos H coercivo con un crecimiento en el gradiente más fuerte que el orden de la difusión del operador no local. El resultado principal en este caso es la continuidad Hölder para *subsoluciones* para este problema. Estabilidad de las estimaciones de regularidad cuando $\lambda \to 0$ permiten concluir el comportamiento asintótico ergódico cuando $t \to \infty$ para el problema parabólico asociado en el toro. En esta tarea, principios del máximo fuertes son de importancia mayor en el análisis asintótico. Finalmente, adaptamos los resultados obtenidos en las primeras dos secciones de esta parte de la tesis para obtener el comportamiento a grandes tiempos para el problema de Cauchy-Dirichlet asociado a H en las formas Bellman y coercivo. En este caso, la influencia del dato exterior en la ecuación a través del término no local hace que el problema parabólico aproxime al correspondiente problema estacionario cuando $t \to \infty$.

Resumé

Cette thése se consacre à l'étude des propriétés qualitatives d'équations elliptiques dégénérées où la diffusion est purement non locale, et s'est réalisée dans le cadre de la théorie des solutions visqueuses.

La première partie de la thèse traite de l'étude des propriétés de compacité d'une famille d'opérateurs non locaux d'ordre zéro. Ces opérateurs sont d'opérateurs elliptiques non locaux définis par le biais d'une mesure bornée. On considère une famille d'opérateurs uniparamétrique d'ordre zéro de la forme

$$\mathcal{I}_\epsilon(u,x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)] K_\epsilon(z) dz,$$

où, pour chaque $\epsilon \in (0,1)$, $K_\epsilon \in L^1(\mathbb{R}^N)$ est une fonction radialement symétrique et positive. On configure notre probléme de sorte que $\mathcal{I}_\epsilon$ tende vers du Laplacien fractionnaire quand $\epsilon \to 0^+$, ce qui implique que la norme L^1 des K_ϵ n'est pas bornée lorsque $\epsilon \to 0^+$. Un premier résultat de cette partie est un module de continuité dans l'espace-temps pour la famille des solutions bornées de l'équation de la chaleur non-locale dans le plan associé à $\mathcal{I}_\epsilon$, indépendante de $\epsilon \in (0,1)$. Le second résultat de cette partie considère le problème de Dirichlet sur un domaine borné $\Omega \subset \mathbb{R}^N$ associé à $\mathcal{I}_\epsilon$, et conclut à la compacité de la famille de solutions bornées $\{u_\epsilon\}_\epsilon$ pour ces problèmes de Dirichlet, en exhibant un module de continuité commun sur $\bar{\Omega}$ pour $\{u_\epsilon\}_\epsilon$, indépendant de ϵ.

La deuxième partie de la thèse est liée à l'existence et l'unicité, la régularité et le comportement pour les temps grands d'équations non locales comportant un terme de gradient dominant. On commence par l'existence et l'unicité d'une équation de Hamilton-Jacobi de la forme

$$\begin{aligned} \lambda u - \mathcal{I}(u) + H(x, Du) &= 0 \qquad \text{sur } \Omega \\ u &= \varphi \qquad \text{sur } \Omega^c, \end{aligned}$$

où l'Hamiltonien H a une forme de Bellman. Nous structurons le problème afin que l'ordre de l'opérateur non local $\mathcal{I}$ soit inférieur à 1 et que par consequent, la condition de bord se perde. Dans la deuxième section de cette partie, nous considérons H coercitive avec une croissance plus forte que l'ordre de la diffusion de l'opérateur de gradient non local. Le résultat principal dans ce cas est la continuité Hölder des sous solutions de ce probleme. La stabilité de régularité lorsque $\lambda \to 0$permet de conclure à un comportement asymptotique ergodique pour $t \to \infty$ pour le probléme parabolique associé sur le tore. Les principes de maximum fort sont ici d'une importance capitale ppour l'analyse asymptotique.

Enfin, nous adaptons les résultats des deux premières sections de cette partie de la thèse pour le comportement aux temps grand pour le problème de Cauchy-Dirichlet associé à H sous les formes Bellman et coercitive. Dans ce cas, l'influence des conditions exterieures sur l'équation à travers son terme non local, fait que le probléme parabolique tend vers le probléme stationnaire équivalent lorsque $t \to \infty$.

Contents

Chapter 1

Introduction

1.1 Preliminaries.

In recent years, the study of nonlocal elliptic problems have attracted the attention of the mathematical community by its theoretical wealth, as for its broad range of applications. The distinctive element on these problems is the presence of a nonlocal elliptic operator playing the diffusive role, being the *fractional Laplacian* the most important example of such an operator. For $\alpha \in (0, 2)$, the fractional Laplacian of order α is defined by the expresion

$$(-\Delta)^{\alpha/2}\phi(x) = -C_{N,\alpha} \text{ P.V.} \int_{\mathbb{R}^N} [\phi(x+z) - \phi(x)]|z|^{-(N+\alpha)}dz,$$

where $x \in \mathbb{R}^N$ and ϕ is a suitable function. Here, $C_{N,\alpha} > 0$ is a well-known normalizing constant and P.V. stands for the Cauchy principal value (see [69]).

The study of elliptic nonlocal operators has a starting point in the late 20's by the seminal works of Lévy, Kintchine, Kolmogorov and many others, related with infinitely divisible distributions and Lévy processes. The heart of the matter is the fact that these operators are the *infinitesimal generator* of an associated pure jump Lévy process (see [38, 102]). The increasing interest on these operators is explained through this probabilistic perspective, since Lévy processes consistently describe the reality of a broad range of diffusion models for which the Brownian motion approach was not totally precise. For example, asset price models driven by Lévy processes were very important to understand the behavior of the financial markets that could not be explained by the classical Black-Scholes model based in the Brownian motion (see [56]). From this point, this topic has deepen its research in probability [97] and, moreover, it has enlarged its horizons because of its strong connections with other branches of mathematics and sciences like abstract potential theory [40, 92], analysis of PDE's [111, 46, 106, 45], front propagation in reactive systems [96], asset pricing in finance [98, 56] and many others.

It is the purpose of this thesis to be a contribution on the study of nonlocal elliptic problems in the PDE framework using the theory of viscosity solutions. It is remarkable that the viscosity machinery originally introduced to deal with first and second-order equations (see [11, 14, 66]) has shown to be effective in the study of nonlocal problems, mostly because of the natural comparison properties enjoyed by elliptic nonlocal operators. The basic tools of the nonlocal viscosity theory can be found in [3, 23, 37, 103, 104] and references therein. From this point, further very interesting results for nonlocal equations in the viscosity approach have been obtained in recent years. For instance, we can mention here well-posedness for Dirichlet problems [85, 21], strong maximum principles [55], systems of integro-differential equations [39], regularity [22, 47, 48, 87, 107, 108], large solutions [71, 54], homogenization [105], and large time behavior [19], among many others.

1.2 Thesis Plan and Main Results

This thesis may be divided in two main parts. The first part is focused on compactness results of a family of zero-th order nonlocal problems approaching the fractional Laplacian. The second part is devoted to the existence, uniqueness, regularity and asymptotic behavior of nonlocal problems in presence of gradient terms.

1.2.1 Compactness Results for Zeroth-Order Nonlocal Operators Approaching the Fractional Laplacian

Compactness Results for a Class of Zero-th Order Heat Equations

In chapter 2, we study various convergence results for a class of nonlinear fractional heat equations of the form

$$\begin{cases} u_t(t,x) - \mathcal{I}[u(t,\cdot)](x) = f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $\mathcal{I}$ is a nonlocal nonlinear operator of Isaacs type. Our aim is to study the convergence of solutions when the order of the operator changes in various ways. In particular we consider zero order operators approaching fractional operators through scaling and fractional operators of decreasing order approaching zero order operators. We further give rate of convergence in cases when the solution of the limiting equation has appropriate regularity assumptions.

Compactness Results for a Class of Zero-th Order Dirichlet Problems in Bounded Domains

In chapter 3, we consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and a parametric family of radially symmetric kernels $K_\epsilon : \mathbb{R}^N \to \mathbb{R}_+$ such that, for each $\epsilon \in (0,1)$, its L^1-norm is finite but it blows up as $\epsilon \to 0$. Our aim is to establish an ϵ independent modulus of continuity in Ω, for the solution u_ϵ of the homogeneous Dirichlet problem

$$\begin{cases} -\mathcal{I}_\epsilon[u] &= f & \text{in } \Omega. \\ u &= 0 & \text{in } \Omega^c, \end{cases}$$

where $f \in C(\bar{\Omega})$ and the operator $\mathcal{I}_\epsilon$ has the form

$$\mathcal{I}_\epsilon[u](x) = \frac{1}{2} \int_{\mathbb{R}^N} [u(x+z) + u(x-z) - 2u(x)] K_\epsilon(z) dz$$

and it approaches the fractional Laplacian as $\epsilon \to 0$. The modulus of continuity is obtained combining the comparison principle with the translation invariance of $\mathcal{I}_\epsilon$, constructing suitable barriers that allow to manage the discontinuities that the solution u_ϵ may have on $\partial\Omega$. Extensions of this result to fully non-linear elliptic and parabolic operators are also discussed.

1.2.2 Nonlocal Elliptic Problems with Dominant Gradient Terms

Existence and Uniqueness for Nonlocal Equations with Dominant Gradient Terms in Bellman Form

In chapter 4, we are interested on the well-posedness of Dirichlet problems associated to integro-differential elliptic operators of order $\alpha < 1$ in a bounded smooth domain Ω . The main difficulty arises because of losses of the boundary condition for sub and supersolutions due to the lower diffusive effect of the elliptic operator compared with the drift term. We consider the notion of viscosity solution with generalized boundary conditions, concluding strong comparison principles in $\bar{\Omega}$ under rather general assumptions over the drift term. As a consequence, existence and uniqueness of solutions in $C(\bar{\Omega})$ is obtained via Perron's method.

Regularity Results and Large Time Behavior for Integro-Differential Equations with Coercive Hamiltonians

In chapter 5, we obtain regularity results for elliptic integro-differential equations driven by the stronger effect of coercive gradient terms. This feature allows us to construct suitable strict supersolutions from which we conclude Hölder estimates for bounded subsolutions. In many interesting situations, this gives way to a priori estimates for subsolutions. We apply this regularity results to obtain the ergodic asymptotic behavior of the associated evolution problem in the case of superlinear equations. One of the surprising features in our proof is that it avoids the key ingredient which are usually necessary to use the Strong Maximum Principle: linearization based on the Lipschitz regularity of the solution of the ergodic problem. The proof entirely relies on the Hölder regularity.

Existence, Uniqueness and Asymptotic Behavior for Nonlocal Parabolic Problems with Dominant Gradient Terms

In chapter 6, we deal with the well-posedness of Dirichlet problems associated to nonlocal Hamilton-Jacobi parabolic equations in a bounded, smooth domain Ω, in the case when the classical boundary condition may be lost. We address the problem for both coercive and noncoercive Hamiltonians: for coercive Hamiltonians, our results rely more on the regularity properties of the solutions, while noncoercive case are related to optimal control problems and the arguments are based on a careful study of the dynamics near the boundary of the domain. Comparison principles for bounded sub and supersolutions are obtained in the context of viscosity solutions with generalized boundary conditions, and consequently we obtain the existence and uniqueness of solutions in $C(\bar{\Omega} \times [0, +\infty))$ by the application of Perron's method. Finally, we prove that the solution of these problems converges to the solutions of the associated stationary problem as $t \to +\infty$ under suitable assumptions on the data.

Part I

Compactness Results for Zeroth-Order Nonlocal Operators Approaching the Fractional Laplacian

Chapter 2

Compactness Results for a Class of Zero-th Order Heat Equations

This chapter is based in the joint work with Patricio Felmer [72].

2.1 Introduction.

During the last years there have been an increasing interest in the study of diffusion equations involving various type of nonlocal operators, both of fractional and zero order. For the basic Cauchy problem

$$\begin{cases} u_t(t,x) - \mathcal{I}[u(t,\cdot)](x) = f(t,x) & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

authors consider diffusion operator $\mathcal{I}$ that may range from the Laplacian

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \tag{2.2}$$

passing through the fractional Laplacian

$$(-\Delta)^\sigma u = -C_{\sigma,n} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2\sigma}} dy, \tag{2.3}$$

with $\sigma \in (0,1)$ (with $C_{\sigma,n}$ a normalizing constant), to zero order convolution operators of the form

$$\mathcal{J}(u) = \int_{\mathbb{R}^n} J(x-y)u(y)dy - u(x), \tag{2.4}$$

where J is a compactly supported non-negative function whose integral is one. These linear operators are particular cases of a very large class of nonlinear operators including Bellman and Isaacs type operators.

It is the purpose of this article to analyze problem (2.1) when the order of the diffusion operator varies towards limiting values. We present various approximation schemes and corresponding convergence theorems in cases of interest by themselves and we propose a general theorem that allows to analyze the proof in each case through a unified approach.

Our analysis of the equations is made in the framework of viscosity solutions and consequently, stability properties and comparison principle are key analytical tools.

The operator $\mathcal{I}$ represents a nonlocal, nonlinear operator which is defined through a family of kernels, except in two well distinguished cases as we see below. More precisely, we consider a family of kernels $\mathcal{K}$, where each $K \in \mathcal{K}$ is a function $K : \mathbb{R}^n \to \mathbb{R}_+$, locally bounded in $\mathbb{R}^n \setminus \{0\}$, that satisfies:

(K1) *Uniform integrability at infinity:* There is a constant $C_r > 0$ depending on r, such that

$$\sup_{K\in\mathcal{K}} \int\limits_{B_r(0)^c} K(z)dz \leq C_r.$$

(K2) *Uniform weighted integrability at zero:* There is a constant $A > 0$ such that

$$\sup_{K\in\mathcal{K}} \int\limits_{B_1(0)} |z|^2 K(z)dz \leq A < +\infty.$$

For each $K \in \mathcal{K}$ and $u : \mathbb{R}^n \to \mathbb{R}$ we define the linear nonlocal operator

$$L_K[u](x) = \int\limits_{\mathbb{R}^n} \delta(u, x, z)K(z)dz, \tag{2.5}$$

where $\delta(u, x, z) = u(x + z) + u(x - z) - 2u(x)$. In order to have L_K well defined, we need some regularity and asymptotic behavior assumptions on u as we see in Section §2.

Given a two parameter family of kernels $K_{\alpha,\beta} \in \mathcal{K}$, we define the operator

$$\mathcal{I}[u](x) \;=\; \inf_{\alpha} \sup_{\beta} L_{\alpha,\beta}[u](x), \tag{2.6}$$

known as Isaacs type operator. Here $L_{\alpha,\beta} = L_{K_{\alpha,\beta}}$. In case of a single parameter, we consider the Bellman type operator

$$\mathcal{I}[u](x) = \sup_{\alpha} L_{\alpha}[u](x),$$

or with the infimum instead of the supremum. These operators are nonlinear, positively homogeneous and, for Bellman type, convex (concave).

Equation (2.1) with an operator $\mathcal{I}$ defined as above, is related to *two player stochastic games with jumps*, in which the trajectory of the stochastic process governing the dynamics is modeled by a Lévy Process. In this context, the solution u of (2.1) represents the value function of the game, see for instance [43]. The jumps in a Lévy process are closely related with its *Lévy measure*, which in our case is given by $K(z)dz$. Usually, Lévy measures are singular and even non-integrable at zero, depicting the high intensity of small jumps. The structural conditions of these measures at zero and at infinity are determined, in our case, by conditions (K1) and (K2), respectively.

Given a family of kernels $\mathcal{K}$ we define the *extremal operators* as

$$\mathcal{M}^-[u](x) = \inf_{K\in\mathcal{K}} L_K[u](x), \quad \text{and} \quad \mathcal{M}^+[u](x) := \sup_{K\in\mathcal{K}} L_K[u](x)$$

whenever the expression is well defined. Then for every admissible pair of functions u, v and each $x \in \mathbb{R}^n$, the operator $\mathcal{I}$ satisfies

$$\mathcal{M}^-[u-v](x) \leq \mathcal{I}[u](x) - \mathcal{I}[v](x) \leq \mathcal{M}^+[u-v](x). \tag{2.7}$$

We say that $\mathcal{I}$ is elliptic, following the definition given by Caffarelli and Silvestre in [47] and [48].

Now we are prepared to describe in detail our main results. They are classified in two classes: the first class deals with two approximation schemes related to zero order operators approximating fractional operators, while the second class involves a family of operators having as an example $(-\Delta)^\sigma$, ranging from the local Laplacian to the identity, when σ moves from 1 to 0.

2.1.1 Approximating zero order and fractional operators.

Here we present two approximation schemes, one in which a fractional operator is approached by zero order operators and the other where, reversely, a zero order operator is approximated by fractional operators. Our motivation is a recent result by Elgueta, Cortázar and Rossi [59] and subsequent work in [5] and [60]. In [59], the authors approximate the heat equation in a bounded domain with Dirichlet boundary conditions, by a sequence of nonlocal equations involving the rescaling of an operator of the form (2.4). This result was proved assuming that the solution of the limiting equation satisfies some regularity assumption that in our case is not available in general, as for Isaacs type operators. We propose the study in the framework of viscosity solutions, which has the necessary flexibility for dealing with these nonlinear problems.

For the first scheme, we consider $\sigma \in (0,1)$ and $\mathcal{K}_0^1$ be the set of kernels

$$K(z) = \frac{a(\hat{z})}{|z|^{n+2\sigma}}, \quad \text{for } \hat{z} = \frac{z}{|z|} \text{ and } z \in \mathbb{R}^n \setminus \{0\}, \tag{2.8}$$

defined for a function $a \in \mathcal{L}_{\lambda,\Lambda}$, where $0 < \lambda \leq \Lambda$ are fixed parameters and

$$\mathcal{L}_{\lambda,\Lambda} = \{a \in L^\infty(S^{n-1}) \,/\, 0 < \lambda \leq a(\hat{z}) \leq \Lambda,\, \hat{z} \in S^{n-1}\}. \tag{2.9}$$

We let $\mathcal{K}_\epsilon^1$ be the set of nonsingular kernels K_ϵ associated to K given by

$$K_\epsilon(z) = \epsilon^{-(n+2\sigma)} \frac{a(\hat{z})}{1 + |z/\epsilon|^{n+2\sigma}}, \quad \epsilon \in (0,1]. \tag{2.10}$$

Then we define the set of kernels for the first scheme $\mathcal{K}^1 = \cup_{\epsilon \in [0,1]} \mathcal{K}_\epsilon^1$ and we consider a two parameter family of kernel $\mathcal{K}_{\alpha,\beta,\epsilon} \in \mathcal{K}_\epsilon^1$, with $\epsilon \in [0,1]$, to define the Isaacs type operators

$$\mathcal{I}_\epsilon[u](x) = \inf_\alpha \sup_\beta L_{\alpha,\beta,\epsilon}[u](x), \tag{2.11}$$

where $L_{\alpha,\beta,\epsilon}$ is the linear operator associated to $K_{\alpha,\beta,\epsilon}$. The idea is to approximate a fractional operator by zero order one, as $\varepsilon \to 0$.

For the second scheme we consider the approximation of zero order equations by fractional ones. We focus the statement of our result in the case of a zero order operator with an integrable kernel with singularity. See Section §8 for other possibilities. For $\rho > 1$ fixed and for every $\epsilon \in [0, 1]$ define the set $\mathcal{K}^2_\epsilon$ composed of all kernels of the form

$$K_\epsilon(z) = \frac{a(\hat{z})}{|z|^{n+\epsilon}}(1 + |\log|z||)^{-\rho}, \tag{2.12}$$

where $a \in L^\infty(S^{n-1})$ satisfies (2.9). We notice that for each $\epsilon > 0$ the kernel K_ϵ has a non integrable singularity of order $n + \epsilon$ at zero, so the corresponding integral operator is fractional with order $\epsilon/2$. The limiting kernel has still a singularity at zero, but the kernel is integrable. Then we consider $\mathcal{K}^2 = \cup_{\epsilon\in[0,1]}\mathcal{K}^2_\epsilon$ which is the set of kernels for the second scheme and we define operators like above using (2.11)

We state our convergence result for the first two approximation schemes. Here and in what follows we denote by $BUC(X)$ the set of bounded uniformly continuous functions defined on X.

Theorem 2.1. *Consider $f \in BUC((0, T) \times \mathbb{R}^n)$ and $u_0 \in BUC(\mathbb{R}^n)$. Let $\mathcal{I}_\epsilon, \mathcal{I}_0$ as in (2.11) using kernels in $\mathcal{K}^1$. Let u_ϵ be the bounded viscosity solution to the equation (2.1) associated to the operator $\mathcal{I}_\epsilon$. Then, $(u_\epsilon)_\epsilon$ converges uniformly over compact sets of $[0, T) \times \mathbb{R}^n$ to the unique bounded viscosity solution u of (2.1) associated to the operator $\mathcal{I}_0$.*

Similar statement holds if we consider $\mathcal{K}^2$ instead of $\mathcal{K}^1$.

There is an interesting interpretation of Theorem 2.1 in the context of stochastic games. In the first case it states that the value function of a game defined through controlled pure jump process of *infinite activity* (all paths have infinite jumps a.e.) can be approximated by a sequence of value functions of *finite activity* games (all paths have finite jumps a.e.), see [98] and [102] for expositions of Lévy processes and these ideas. In fact, by the form of the kernels in Theorem 2.1 we are approximating the value function of a game defined by a $2\sigma-$stable process by a sequence of value functions of a game defined by compound Poisson processes. The second part admits a similar interpretation, where the value function of a game defined by a finite activity process is approximated by a sequence of value functions of games defined by infinite activity processes.

When the limit function u in Theorem 2.1 satisfies an appropriate regularity assumption we can prove the global uniform convergence of u_ϵ providing an explicit convergence rate as it was the case in [59]. Precisely we have

Theorem 2.2. *Let u_ϵ and u be as in Theorem 2.1. Then we have*

1. If the operators are defined with kernels in $\mathcal{K}^1$ and if we assume $u \in C^1([0, T), C^{2\sigma+\gamma}(\mathbb{R}^n))$, for some $\gamma > 0$ and $2\sigma + \gamma < 1$, then

$$||u - u_\epsilon||_{L^\infty([0,T)\times\mathbb{R}^n)} \leq C\epsilon^\gamma. \tag{2.13}$$

2. If the operators are defined with kernels in $\mathcal{K}^2$ and if we assume $u \in C^1((0, T), C^\gamma(\mathbb{R}^n))$, for some $\gamma \in (0, 1)$, then

$$||u - u_\epsilon||_{L^\infty([0,T)\times\mathbb{R}^n)} \leq C\Big(\frac{1}{(\rho - 1)|\log(\epsilon)|^{\rho-1}} + \frac{\epsilon|\log(\epsilon)|}{\gamma^2}\Big). \tag{2.14}$$

The constants C depend on u and on the other parameters in the problem, but not on ϵ.

We recall some regularity results for u when proper assumptions are made on the data by Silvestre in [108] and by Barles, Chasseigne and Imbert in [22], for the associated elliptic equation, and by Barles, Chasseigne, Ciomaga and Imbert in [18], Silvestre in [109] and Chang-Lara and Dávila in [52] for the parabolic case. However for a general nonlinear problem there are no regularity results available, so it would be interesting to obtain estimates like those above with weaker regularity assumptions

2.1.2 Approximating second order and zero order local operators with fractional operators.

The two approximation schemes that we consider next may be illustrated with the operator family $\sigma(1-\sigma)\Delta^\sigma$ as $\sigma \in (0,1)$. This family approaches the Laplacian through the concentration of the kernel, as $\sigma \to 1$ and the identity, through the flattening of the kernel, as $\sigma \to 0$. We describe these two schemes in a more general setting next.

We consider kernels defined as in (2.8) but with an extra factor

$$K_\sigma(z) = \sigma(1-\sigma)\frac{a(\hat{z})}{|z|^{n+2\sigma}},$$

with $\sigma \in (0,1)$. We define $\mathcal{K}_\sigma$ as the set of all kernels of this type with $\sigma \in (0,1)$ and then we define $\mathcal{K}^3 = \cup_{\sigma\in(1/2,1)}\mathcal{K}_\sigma$ and $\mathcal{K}^4 = \cup_{\sigma\in(0,1/2)}\mathcal{K}_\sigma$. From here we define the corresponding Isaacs type operators $\mathcal{I}_\sigma$, as in (2.6), with a two parameter family of kernels $K_{\alpha,\beta}$ in $\mathcal{K}_\sigma$. In these cases, σ is the index defining the sequence of operators.

Next we set the limit operators for the cases $\sigma = 1$ and $\sigma = 0$. In the first case, we define for each parameter pair (α, β) the matrix $C^{\alpha,\beta}$ as

$$c^{\alpha,\beta}_{k,l} = \frac{1}{2}\int\limits_{\mathbb{S}^{n-1}} \omega_k\omega_l a_{\alpha,\beta}(\omega)d\omega,$$

for $k, l \in \{1, ..., n\}$. Then we define

$$F(M) = \inf_\alpha \sup_\beta \operatorname{tr}(C^{\alpha,\beta}M), \quad \text{for } M \in \mathcal{S}^{n\times n}, \tag{2.15}$$

where $\mathcal{S}^{n\times n}$ is the set of symmetric $n \times n$ matrices. With these definitions the limit equation for $\sigma = 1$ becomes

$$\left\{\begin{array}{ll} u_t(t,x) - F(D^2_x u(t,x)) = f(t,x), & x \in \mathbb{R}^n, t \in (0,T), \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{array}\right. \tag{2.16}$$

In the case of $\sigma = 0$ we consider two parameter family of functions $a_{\alpha,\beta}$ and define the numbers $c_+ = \inf\limits_\alpha \sup\limits_\beta c_{\alpha,\beta}$ and $c_- = \sup\limits_\alpha \inf\limits_\beta c_{\alpha,\beta}$, where

$$c_{\alpha,\beta} = \int\limits_{\mathbb{S}^{n-1}} a_{\alpha,\beta}(\omega)d\omega.$$

Then we consider the homogeneous function

$$G(u) = \left\{\begin{array}{ll} c_+u, & u \geq 0 \\ c_-u, & u \leq 0 \end{array}\right. \tag{2.17}$$

for the limiting equation, that in this case becomes

$$\begin{cases} u_t(t,x) + G(u(t,x)) = f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.18}$$

Now we state our second main approximation theorem, when σ approaches the extreme values 1 and 0.

Theorem 2.3. *Let $f \in BUC((0,T) \times \mathbb{R}^n)$ and $u_0 \in BUC(\mathbb{R}^n)$. Let u_σ be the bounded viscosity solution of equation (2.1) asociated to the operator $\mathcal{I}_\sigma$ defined above.*

1. Then, u_σ converges uniformly over compact sets of $[0,T) \times \mathbb{R}^n$ as $\sigma \to 1$ to the bounded viscosity solution u of equation (2.16).

2. We further assume that f and u_0 satisfy

$$\lim_{|x|\to\infty} \sup_{t\in(0,T)} |f(t,x)| = 0 \quad \textit{and} \quad \lim_{|x|\to\infty} |u_0(x)| = 0. \tag{2.19}$$

Then, u_σ converges uniformly in $[0,T) \times \mathbb{R}^n$ as $\sigma \to 0$ to the bounded viscosity solution u of equation (2.18).

The assumption (2.19) over the data f and u_0, is sufficient to conclude that the solution of (2.1) decays to zero as $|x| \to 0$ uniformly in $t \in [0,T)$ and $\sigma \geq 0$, as it is proved in Lemma 2.3. This decay is important in the proof of the theorem since in this case the kernel flatten and a compatibility condition is required. See hypothesis (K3).

We find a convergence rate for the uniform limits obtained in Theorem 2.3 assuming regularity of the solution of the limit equation.

Theorem 2.4. *Let u_σ, u be as in Theorem 2.3.*

1. Assume $u \in C^1([0,T), C^{2+\gamma}(\mathbb{R}^n))$ with $\gamma > 0$. Then

$$||u - u_\sigma||_{L^\infty((0,T)\times\mathbb{R}^n)} \leq C\frac{1-\sigma}{\gamma}.$$

2. Assume that $u \in C^1([0,t), C^\gamma(\mathbb{R}^n))$ and

$$|u(t,x)| \leq C(1+|x|)^{-\beta}, \quad \forall\, x \in \mathbb{R}^n, \tag{2.20}$$

with $\gamma, \beta > 0$. Then

$$||u - u_\sigma||_{L^\infty((0,T)\times\mathbb{R}^n)} \leq C\Big(\frac{\sigma}{\gamma} + \sigma\log(\sigma)\Big).$$

The constants C depend on u and other parameters in the problem, but not on σ and γ.

We mention that regularity results for second order fully nonlinear equations are known when the operator is convex, but not in general. See for instance [44]. On the other hand, assuming appropriate regularity and decay estimates on the data u_0 and f we can prove that the solution of the limiting equation (2.18) satisfies the assumptions of Theorem 2.4. See Lemma 2.4.

Remark 2.1. In the first three approximation schemes, with kernels in $\mathcal{K}^1, \mathcal{K}^2$ and $\mathcal{K}^3$, we study the equations in the class of bounded functions, while in the fourth case, with kernels in $\mathcal{K}^4$, we assume the data, and consequently solutions, decay at infinity. With some extra work, we could prove Theorems 2.1 and case 1 of 2.3 in a class of decaying functions.

Remark 2.2. We would like to mention that, without a substantial change of the arguments, the results presented here can be obtained for the associated elliptic equations as

$$\lambda u(x) - \mathcal{I}[u](x) = f(x), \quad x \in \mathbb{R}^n$$

for $\lambda > 0$ and non-local operators $\mathcal{I}$.

This article is organized as follows. In Section §2 we recall the definition of viscosity solution and some other preliminaries and then we present a general convergence result, that allows to prove in a unified way Theorems 2.1 and 2.3. This general result is proved using a stability theorem discussed in Section §3 and a Comparison Principle discussed in Section §4. In Section §5 we finally prove the general theorem and in Section §6 we see how to apply it to prove the other convergence theorems. We devote Section §7 to prove Theorems 2.2 and 2.4 on rates of convergence. Finally, in Section §8 we provide alternative approximation families.

2.2 Preliminaries and the general theorem.

In this section we present some preliminaries and then we state our general theorem. This general theorem allows to prove the convergence results given in the introduction for the various approximation schemes in a unified way.

We start discussing the regularity assumptions on the function u in order to properly define the linear operators $L_K[u]$ in x, for L_K given in (2.5). Regarding the singularity of the kernel at zero, a sufficient condition for u is that it is of class $C^{1,1}(x)$ at the point x in the sense of Caffarelli and Silvestre, see [47], [48] and [108],

Definition 1. We say that a function u is of class $C^{1,1}(x)$, for $x \in \mathbb{R}^n$, if there exists $C, \delta > 0$ and $v \in \mathbb{R}^n$ such that

$$|u(x+y) - u(x) - v \cdot y| \leq C|y|^2$$

when $|y| \leq \delta$.

We notice that from this definition we have that u is derivable at x with $Du(x) = v$. We also point out that, whenever u satisfies this regularity assumption at the point x, we can write $L_K[u](x)$ in an equivalent form as

$$L_K[u](x) = 2 \int_{\mathbb{R}^n} (u(x+z) - u(x) - 1_B Du(x) \cdot z) K(z) dz$$

We will use any of these expressions whenever it is necessary.

Even though, some regularity of u is needed to evaluate the linear operator, and consequently the Bellman and Isaacs type operators defined upon the linear ones, in order to define solutions of (2.1) in the viscosity sense we just need to assume continuity or even semi-continuity.

Definition 2. An upper semi-continuous function $u : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is a *viscosity subsolution* to the problem (2.1), with $\mathcal{I}$ defined in (2.6) if

i) $u(0,x) \leq u_0(x)$ for all $x \in \mathbb{R}^n$ and

ii) for all $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$, $\varphi \in C^2((0,T) \times \mathbb{R}^n)$ such that $u - \varphi$ has a maximum at (t_0, x_0) in the set $N_\delta = (t_0 - \delta, t_0 + \delta) \times B_\delta(x_0)$, with $\delta > 0$, we have

$$\varphi_t(t_0, x_0) - \inf_\alpha \sup_\beta \left(L^{1,\delta}_{\alpha,\beta}[\varphi(t_0, \cdot)](x_0) + L^{2,\delta}_{\alpha,\beta}[p_0, u(t_0, \cdot)](x_0)\right) \le f(t_0, x_0),$$

where $p_0 = D_x\varphi(t_0, x_0)$. Here, for a given kernel K, $x, p \in \mathbb{R}^n$ and $\varphi \in C^2(\mathbb{R}^n)$ we write

$$L^{1,\delta}_K[\varphi](x) = \int_{B_\delta} (\varphi(x+y) - \varphi(x) - 1_B D\varphi(x) \cdot y) K(y) dy \quad \text{and} \tag{2.21}$$

$$L^{2,\delta}_K[p, v](x) = \int_{B^c_\delta} (v(x+y) - v(x) - 1_B p \cdot y) K(y) dy. \tag{2.22}$$

In a similar way, we define viscosity super-solution. A solution is a function which is sub and super-solution simultaneously. This definition of viscosity solution is one of the several equivalent definitions in the literature. See for instance [3] for the parabolic setting and [23], [7], [8] for the elliptic one.

Next we define the set of functions in which we study our nonlinear heat equations. This set of functions should have some flexibility in order to consider bounded functions and decaying function depending on the class of kernels in mind, according to the different approximation schemes discussed in the introduction. Consider a family of kernels $\mathcal{K}$ satisfying conditions (K1) and (K2). We say that a set of continuous functions $\mathcal{H}$ defined in $\mathbb{R}^n$ is compatible with $\mathcal{K}$ if for all $x \in \mathbb{R}^n$

$$\lim_{r \to +\infty} \sup_{K \in \mathcal{K},\ u \in \mathcal{H}} \int_{B^c_r} |u(x \pm z)| K(z) dz = 0. \tag{2.23}$$

Notice that in the first three cases considered in the introduction, the families of kernels $\mathcal{K}^1$, $\mathcal{K}^2$ and $\mathcal{K}^3$ allow to consider $\mathcal{H}$ as the set of all bounded continuous functions with a common L^∞ norm. In this case $\mathcal{H}$ and the family of kernels are compatible, since the kernels in the family of each case are bounded above by a common L^1 function away from the origin. This does not occur in the case of $\mathcal{K}^4$, where the corresponding kernels lose integrability as $\sigma \to 0^+$. However, in that case the compatibility condition is accomplished if we consider $\mathcal{H}$ consisting of all continuous functions vanishing uniformly at infinity.

Since we are interested in the uniform convergence of solutions in $\mathcal{H}$, we need to assume it is closed in the following sense

$$\begin{aligned} &\text{If } \{u_j(t,\cdot)\}_j \subset \mathcal{H}, \textit{for all } t \in [0,T) \textit{ and } u_j \textit{ converges to } u \textit{ locally} \\ &\textit{uniformly in } [0,T) \times \mathbb{R}^n, \textit{ then } u(t,\cdot) \in \mathcal{H}, \textit{for all } t \in [0,T). \end{aligned} \tag{2.24}$$

We notice that the set of uniformly bounded continuous functions and the set of uniformly vanishing continuous functions satisfy this closedness property.

Regarding the sequence of operators $(\mathcal{I}_j)$, we assume that each one is an Isaacs type operators as defined in (2.6), from a family of kernels $\mathcal{K}_j \subset \mathcal{K}$. We assume that for all j there is $w_j \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ such that

$$\forall\ K \in \mathcal{K}_j,\ K(z) \le C_r w_j(z),\ \forall |z| \ge r, \tag{2.25}$$

where C_r does not depend on j. Since each $\mathcal{I}_j$ is constructed in this way, it satisfies a suitable comparison principle for equation (2.1) as we see in Section §2.4. We notice also that each class of kernels considered in the introduction follows this structure.

A key point on our arguments is the following concept of weak convergence of operators which is a variant of the one given in [48].

Definition 3. Let $\mathcal{H}$ a fixed set of functions $u : \mathbb{R}^n \to \mathbb{R}$. We say that $\mathcal{I}_j \to \mathcal{I}$ weakly in $\mathbb{R}^n$ with respect $\mathcal{H}$ if for every $x_0 \in \mathbb{R}^n$, $\rho > 0$, every function $u \in \mathcal{H}$ and $\varphi \in C^2$, we have that for the function v defined as

$$v(x) = \begin{cases} \varphi(x), & x \in B_\rho(x_0) \\ u(x), & x \in B^c_\rho(x_0) \end{cases}$$

we have $\mathcal{I}_j[v](x) \to \mathcal{I}[v](x)$ uniformly in $B_{\rho/4}(x_0)$.

With the definitions and assumptions made above, we are in position to state our general convergence result.

Theorem 2.5. *Let u_0, f be bounded, uniformly continuous functions and $\mathcal{K}$ a family of kernels satisfying (K1) and (K2). For each $j \in \mathbb{N}$, let $\mathcal{K}_j \subset \mathcal{K}$ satisfying (2.25) relative to some $w_j \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\mathcal{I}_j$ be an operator of Isaacs type defined as in (2.6) with kernels in $\mathcal{K}_j$. Let u_j be the unique bounded viscosity solution to*

$$\begin{cases} u_t(t,x) - \mathcal{I}_j[u(t,\cdot)](x) = f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.26}$$

Assume $\{u_j(t,\cdot)\}_j \subset \mathcal{H}$, for all $t \in [0,T)$, with $\mathcal{H}$ satisfying compatibility condition (2.23), relative to $\mathcal{K}$, and the closedness condition (2.24). If $\mathcal{I}_j$ converges weakly to some operator $\mathcal{I}$ as $j \to \infty$, then u_j converges locally uniform in $[0,T) \times \mathbb{R}^n$ to the unique bounded viscosity solution of

$$\begin{cases} u_t(t,x) - \mathcal{I}[u(t,\cdot)](x) = f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n \end{cases}$$

We notice that in this result the limit operator $\mathcal{I}$ may perfectly be outside of the class $\mathcal{K}$. We devote the rest of the paper to prove this theorem and to use it to analyze the approximation schemes given in the introduction.

2.3 Stability.

We start this section with a stability result obtained by adapting to the parabolic setting the stability result for the elliptic case given by Caffarelli and Silvestre in [48]. In our general framework the precise statement is

Lemma 2.1. *Let $\mathcal{K}$ be a family of kernels satisfying conditions (K1) and (K2). Consider a sequence of operators $\mathcal{I}_j$ as in (2.6) defined with kernels in $\mathcal{K}$. Let $\mathcal{H}$ be a set of functions satisfying (2.23) relative to $\mathcal{K}$ and (2.24). Consider $u_j : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ such that $u_j(t,\cdot) \in \mathcal{H}$, for all $t \in [0,1)$, a function $u : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ and an operator $\mathcal{I}$ such that*

- *$\partial_t u_j - \mathcal{I}_j[u_j] \geq f_j$ in $(0,T) \times \mathbb{R}^n$ in the viscosity sense.*

- *$u_j \to u$ locally uniform in $(0,T)\times\mathbb{R}^n$.*
- *u is continuous in $(0,T)\times\mathbb{R}^n$.*
- *$f_j \to f$ locally uniform in $(0,T)\times\mathbb{R}^n$.*
- *$\mathcal{I}_j \to \mathcal{I}$ weakly with respect to $\mathcal{H}$.*

Then, $\partial_t u - \mathcal{I}[u] \geq f$ in $(0,T)\times\mathbb{R}^n$ in the viscosity sense.

Similar statement holds true for sub-solutions and solutions in the viscosity sense.

Proof. Let $\varphi(t,x)$ be a C^2 function touching u from below at $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$ in $N_\rho=(t_0-\rho,t_0+\rho)\times B_\rho(x_0)$ and let v be defined as

$$v(t,x)=\begin{cases}\varphi(t,x), & (t,x)\in N_\rho\\ u(t,x), & (t,x)\in N_\rho^c\end{cases}$$

Since $u_j\to u$ locally uniform in $(0,T)\times\mathbb{R}^n$, it is possible to find sequences $t_j\to t_0$, $x_j\to x_0$ and $d_j\to 0$ such that $\varphi+d_j$ touches u_j from below at (t_j,x_j). We define

$$v_j(t,x)=\begin{cases}\varphi(t,x)+d_j, & (t,x)\in N_\rho\\ u_j(t,x), & (t,x)\in N_\rho^c\end{cases}$$

and we see that $v_j(t_j,x_j)\to v(t_0,x_0)$. Let $y\in B_{\rho/4}(x_0)$ and j large so that $t_j\in(t_0-\rho/2,t_0+\rho/2)$, then we have

$$|\mathcal{I}_j[v_j(t_j,\cdot)](y)-\mathcal{I}[v(t_0,\cdot)](y)| \;\leq\; A+B+C, \tag{2.27}$$

where

$$A=|\mathcal{I}_j[v_j(t_j,\cdot)](y)-\mathcal{I}_j[v(t_j,\cdot)](y)|,\quad B=\;|\mathcal{I}_j[v(t_j,\cdot)](y)-\mathcal{I}_j[v(t_0,\cdot)](y)|$$

and

$$C=\;|\mathcal{I}_j[v(t_0,\cdot)](y)-\mathcal{I}[v(t_0,\cdot)](y)|.$$

Clearly, $C\to 0$ as $j\to\infty$ by weak convergence of the sequence of operators. To estimate A, we notice that $B_{\rho/2}(0)\subset B_\rho(x_0-y)\cap B_\rho(x_0+y)$ and that $w_j=v_j(t_j,\cdot)-v(t_j,\cdot)$ is a constant in $B_{\rho/2}(0)$, then we have

$$\delta(w_j,y,z)=0,\quad \text{for all}\quad z\in B_{\rho/2}(0)\quad\text{and}$$

$$A\;\leq\;\sup_{K\in\mathcal{K}}\Big\{\int_{B^c_{\rho/2}(0)}(|w_j(y+z)|+|w_j(y-z)|+2|w_j(y)|)K(z)dz\Big\}.$$

Using (K1), (2.23), (2.24) and the fact that $u_j\to u$ locally uniform we conclude $A\to 0$ as $j\to\infty$. To estimate B we write $\bar{w}_j=v(t_j,\cdot)-v(t_0,\cdot)$ and we see that

$$\begin{aligned}B\leq\sup_{K\in\mathcal{K}}\Big\{&\int_{B_{\rho/2}(0)}|\delta(\varphi(t_j,\cdot)-\varphi(t_0,\cdot),y,z)|K(z)dz\\&+\int_{B^c_{\rho/2}(0)}(|\bar{w}_j(y+z)|+|\bar{w}_j(y-z)|+2|\bar{w}_j(y)|)K(z)dz\Big\}.\end{aligned}$$

Using that φ is of class C^2, (K2) and the Dominated Convergence Theorem we find that the first term goes to zero. On the other hand, using the continuity of u and φ, (K1) and (2.23) we find that the second term also tends to zero with j, concluding that $B \to 0$ as $j \to \infty$.

Finally we use that u_j is super-solution to obtain

$$\begin{aligned}\partial_t\varphi(t_0,x_0) - \mathcal{I}[v(t_0,\cdot)](x_0) \quad &\leq \quad f_j(t_j,x_j) + |\partial_t\varphi(t_0,x_0) - \partial_t\varphi(t_j,x_j)| \\ &\quad\quad + |\mathcal{I}_j[v_j(t_j,\cdot](x_j) - \mathcal{I}[v(t_0,\cdot)](x_j)|.\end{aligned}$$

Using (2.27), the continuity of φ_t and convergence of f_j we conclude. □

In what follows we prove that each families of kernels considered in the four cases discussed in the introduction have the weak convergence property, in each case over an appropriate class of functions.

Lemma 2.2. *Let $C > 0$ and $\mathcal{B}_C$ the set of bounded functions whose infinity norm is bounded by C. We have*

(i) Let $\mathcal{I}_\epsilon, \mathcal{I}_0$ be as in (2.11), with kernels in $\mathcal{K}^1$ or in $\mathcal{K}^2$. Then $\mathcal{I}_\epsilon \to \mathcal{I}_0$ weakly w.r.t. $\mathcal{B}_C$ as $\epsilon \to 0$.

(ii) Let $\mathcal{I}_\sigma$ and F be as in (2.6) and (2.15), respectively. Then, $\mathcal{I}_\sigma \to F(D^2\cdot)$ weakly w.r.t. $\mathcal{B}_C$ as $\sigma \to 1$.

(iii) Let $\mathcal{I}_\sigma$ and G be as in (2.6) and (2.17), respectively. Then $\mathcal{I}_\sigma \to G$ weakly w.r.t. $\mathcal{B}_0 = \{u \in \mathcal{B}_C /\ \lim_{|x|\to\infty} |u(x)| = 0\}$ as $\sigma \to 0$.

Proof: Let $x_0 \in \mathbb{R}^n$ and v be a function with the form

$$v(x) = \begin{cases} \varphi(x), & x \in B_\rho(x_0) \\ u(x), & x \in B_\rho^c(x_0) \end{cases}$$

with $\rho > 0$, φ a C^2 function and $u \in \mathcal{H}$.

(i) We consider $\mathcal{H} = \mathcal{B}_C$ and operators $\mathcal{I}_\epsilon$ and $\mathcal{I}_0$ defined as in (2.11), with kernels of the form (2.10) and (2.8), respectively. Notice that for every $a \in \mathcal{L}_{\lambda,\Lambda}$

$$K_\epsilon(z) \leq K(z) \leq \frac{\Lambda}{|z|^{n+2\sigma}} \quad \text{for all} \quad \epsilon > 0, z \in \mathbb{R}^n.$$

For a Bellman type operator, $\epsilon > 0$, $x \in B_{\rho/4}(x_0)$ and $\eta > 0$ there exists $\alpha = \alpha(\eta, x)$ such that

$$\begin{aligned}\mathcal{I}_\epsilon[v](x) - \mathcal{I}_0[v](x) \quad &\leq \quad \eta + L_{\alpha,\epsilon}[v](x) - L_\alpha[v](x) \\ &\leq \quad \eta + \epsilon^{2(1-\sigma)}C(n,\rho,\Lambda,\sigma)||D^2\varphi||_{L^\infty(B_\rho(x_0))} \\ &\quad\quad + \epsilon^{n+2\sigma}C(n,\rho,\Lambda,\sigma)||u|z|^{-2(n+2\sigma)}||_{L^1(B^c_\rho(x_0))}.\end{aligned}$$

Since this inequality holds for any $\eta > 0$, we conclude

$$\mathcal{I}_\epsilon[v](x) - \mathcal{I}_0[v](x) \leq C(n,\sigma,p,u,\rho,\Lambda)\epsilon^{2(1-\sigma)}.$$

It is possible to obtain similar lower bound, concluding the result for Bellman operators. For Isaacs type operators, writing for each $\epsilon \in [0,1]$

$$\mathcal{I}_\epsilon[u](x) = \inf_\alpha \mathcal{I}_{\alpha,\epsilon}[u](x), \text{ with } \mathcal{I}_{\alpha,\epsilon}[u](x) = \sup_\beta L_{\alpha,\beta,\epsilon}[u](x)$$

we follow the same strategy as for the Bellman case. This completes the proof in case (i) with kernels in $\mathcal{K}^1$. Case (i) with kernels in $\mathcal{K}^2$ and Case (iii) are similar.

(iv) In this case we have $\mathcal{H} = \mathcal{B}_0$. For a Bellman type operator, we consider $\eta > 0$ arbitrary to get $\alpha = \alpha(\eta, x)$ such that

$$\mathcal{I}_\sigma[v](x) + G(v(x)) \leq \eta + L_{\alpha,\sigma}[v](x) + c_\alpha v(x) \leq \eta + A + B + C,$$

where

$$\begin{aligned} A &= \sigma\Lambda||D^2\varphi||_{L^\infty(B_\rho(x_0))} \int\limits_{B_{\rho/2}(0)} \frac{dz}{|z|^{n+2(\sigma-1)}}, \\ B &= \sigma \int\limits_{B^c_{\rho/2}(0)} (v(x+z) + v(x-z)) \frac{a_\alpha(\hat{z})}{|z|^{n+2\sigma}} dz \quad \text{and} \\ C &= \varphi(x)(c_\alpha - 2\sigma \int\limits_{B^c_{\rho/2}(0)} \frac{a_\alpha(\hat{z})}{|z|^{n+2\sigma}} dz). \end{aligned}$$

Clearly $A \to 0$ as $\sigma \to 0$. Noticing that functions in $\mathcal{H}$ decay at infinity, we also see that $B \to 0$ uniformly in $x \in B_{\rho/4}(x_0)$ as $\sigma \to 0^+$. Finally we have

$$C = \varphi(x)(c_\alpha - (\rho/2)^{-2\sigma} \int\limits_{\mathbb{S}^{n-1}} a_\alpha(\omega) d\omega) = \varphi(x) c_\alpha (1 - (\rho/2)^{-2\sigma}),$$

so that $C \to 0$ uniformly in $x \in B_{\rho/4}(x_0)$ as $\sigma \to 0^+$. Since $\eta > 0$ is arbitrary we conclude. For an Isaacs type operator we proceed as above. □

Remark 2.3. If we have $\mathcal{I}_j \to \mathcal{I}$ weakly with respect to $\mathcal{H}$ and $\mathcal{H}' \subseteq \mathcal{H}$, then, naturally, $\mathcal{I}_j \to \mathcal{I}$ weakly with respect to $\mathcal{H}'$.

2.4 A Comparison Principle.

In the proof of our general Theorem 2.5, a key role is played by the comparison principle. Besides the basic role in the existence theory for nonlinear equation as (2.1) through Perron's method, it is the main ingredient in obtaining convergence of the sequence of solutions of (2.1). In the fractional operators framework, including second order local terms in some cases, comparison principles have been addressed under different settings and for different purposes by many authors, see for instance [2], [3], [23], [47], [55], [105], among others. In this section we state in a precise way the result needed in our setting and we give a sketch of the proof for the sake of completeness.

The appropriate use of comparison principle allows to obtain a modulus of continuity of the solutions, that depends on the data u_0 and f only and not j, allowing to obtain that the sequence of solutions u_j is equicontinuous, as it shown in Section §5.

In the arguments to come, it will be crucial the use of a function ψ_h built as follows. Let $\psi(x)$ be a smooth function with $\psi, D\psi, D^2\psi$ uniformly bounded in $\mathbb{R}^n$ and such that $\psi(x) = 0$ if $|x| \leq 1$ and $\psi(x) > c$ if $|x| > 2$ with $c > 0$ a fixed constant. We define $\psi_h(x) = \psi(hx)$ and we see that

$$D\psi_h, D^2\psi_h \to 0, \text{ uniformly in } \mathbb{R}^n \text{ as } h \to 0.$$

Given a non-negative weight $w \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and a family of kernels $\mathcal{K}$ satisfying (K1), (K2) and (2.25) relative to w, then for each $K \in \mathcal{K}$, $x \in \mathbb{R}^n$ and $h < 1$ we have

$$\begin{aligned} L_K[\psi_h](x) &= \int_{B_{1/\sqrt{h}}(0)} \delta(\psi_h, x, z)K(z)dz + \int_{B^c_{1/\sqrt{h}}(0)} \delta(\psi_h, x, z)K(z)dz \\ &\leq h(A + |w|_{L^1(\mathbb{R}^n)})|D^2\psi|_\infty + |\psi|_\infty o_h(1). \end{aligned}$$

Thus we have obtained

$$L_K[\psi_h](x) \leq C(\psi, A, w)o_h(1), \tag{2.28}$$

where the constant depends on ψ, A and w, but not on $x \in \mathbb{R}^n$ or $K \in \mathcal{K}$. This type of function ψ_h is used by various authors in the proof of the comparison principle for second order fully nonlinear equations, see [66], [84] and [23].

Theorem 2.6. *Let $f \in C((0,T) \times \mathbb{R}^n)$ and $u_0 \in C(\mathbb{R}^n)$. Let $w \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\mathcal{K}$ be a family of kernels satisfying (K1), (K2) and (2.25) relative to w. Consider also a nonlocal operator $\mathcal{I}$ defined as in (2.6) with kernels in $\mathcal{K}$. Let $u, v : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ be bounded sub and super-solution of equation (2.1) associated to this operator $\mathcal{I}$, respectively. Then, $u \leq v$ in $(0,T) \times \mathbb{R}^n$.*

Proof. We proceed by contradiction, assuming that

$$M = \sup_{\substack{t \in [0,T) \\ x \in \mathbb{R}^n}} \Big(u(t,x) - v(t,x)\Big) > 0.$$

Then we double the variables and penalize considering

$$M_{\epsilon,h,\eta} = \sup_{\substack{s,t \in [0,T) \\ x,y \in \mathbb{R}^n}} \Big(u(t,x) - v(t,y) - \frac{|x-y|^2}{2\epsilon} - \frac{(s-t)^2}{2\epsilon} - \psi_h(x) - \frac{\eta}{T-t}\Big),$$

for ϵ, h and η small positive numbers and ψ_h the function defined above with $c = |u|_\infty + |v|_\infty$. We can fix h_0 and η suitably small to get

$$M_{\epsilon,h,\eta} > M/4 > 0, \tag{2.29}$$

for all ϵ and $h < h_0$. In this setting, we can assure that $M_{\epsilon,h,\eta}$ is attained at a point $(\bar{s}, \bar{t}, \bar{x}, \bar{y}) \in [0,T]^2 \times \mathbb{R}^n \times \mathbb{R}^n$. For notational simplicity we omit the dependence on ϵ, h and η. Inequality (2.29) allows to conclude that

$$|\bar{x} - \bar{y}|^2 \leq 2(|u|_\infty + |v|_\infty)\epsilon \leq C\epsilon, \tag{2.30}$$
$$|\bar{s} - \bar{t}|^2 \leq 2(|\sup u| + |\inf v|)\epsilon \leq C\epsilon \quad \text{and} \tag{2.31}$$
$$T - \bar{t} > \eta(|u|_\infty + |v|_\infty)^{-1} > 0, \tag{2.32}$$

for all $0 < h < h_0, \epsilon > 0$. The last two inequalities imply $\bar{s}, \bar{t}$ are uniformly away from 0 as $\epsilon \to 0$ when h is fixed, so that $M_{\epsilon,h,\eta}$ is in fact a maximum attained in $(0,T)^2 \times \mathbb{R}^n \times \mathbb{R}^n$ for all ϵ small enough and all $0 < h < h_0$. Next we consider

$$\varphi(s,t,x,y) = \frac{|x-y|^2}{2\epsilon} + \frac{(t-s)^2}{2\epsilon} + \psi_h(x) + \frac{\eta}{T-t}$$

and observe that the functions

$$(t,x) \mapsto u(t,x) - (v(\bar{s},\bar{y}) + \varphi(\bar{s},t,x,\bar{y})) \quad \text{and}$$
$$(s,y) \mapsto v(s,y) - (u(\bar{t},\bar{x}) - \varphi(s,\bar{t},\bar{x},y))$$

have a maximum and a minimum point at $(\bar{t},\bar{x})$ and $(\bar{s},\bar{y})$, respectively, so they can be used as test functions for our equation (2.1). Since u is a sub-solution and v is a super-solution, testing with the functions given above and subtracting we obtain

$$\begin{aligned}
&\frac{\eta}{(T-\bar{t})^2} \leq f(\bar{t},\bar{x}) - f(\bar{s},\bar{y}) \\
&+ \inf_{\alpha}\sup_{\beta} L^{1,\delta}_{K_{\alpha,\beta}}[\varphi(\bar{s},\bar{t},\cdot,\bar{y})](\bar{x}) + L^{2,\delta}_{K_{\alpha,\beta}}[D_x\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),u(\bar{t},\cdot)](\bar{x}) \\
&- \inf_{\alpha}\sup_{\beta} L^{1,\delta}_{K_{\alpha,\beta}}[-\varphi(\bar{s},\bar{t},\bar{x},\cdot)](\bar{y}) + L^{2,\delta}_{K_{\alpha,\beta}}[-D_y\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),v(\bar{s},\cdot)](\bar{y}),
\end{aligned}$$

for all $\delta > 0$. Proceeding as in the proof of Lemma 2.2, it is possible to drop the inf-sup in the last expression finding $K \in \mathcal{K}$, depending on T,η,ϵ and h such that

$$\begin{aligned}
&\frac{\eta}{(T-\bar{t})^2} \leq \frac{\eta}{2T^2} + f(\bar{t},\bar{x}) - f(\bar{s},\bar{y}) \\
&+ L^{1,\delta}_{K}[\varphi(\bar{s},\bar{t},\cdot,\bar{y})](\bar{x}) + L^{2,\delta}_{K}[D_x\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),u(\bar{t},\cdot)](\bar{x}) \\
&- L^{1,\delta}_{K}[-\varphi(\bar{s},\bar{t},\bar{x},\cdot)](\bar{y}) - L^{2,\delta}_{K}[-D_y\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),v(\bar{s},\cdot)](\bar{y}).
\end{aligned} \tag{2.33}$$

Now we proceed to estimate the right hand side of (2.33). First we look at the two terms involving $L^{1,\delta}_K$, after some calculation we find that

$$\begin{aligned}
L^{1,\delta}_{K}[\varphi(\bar{s},\bar{t},\cdot,\bar{y})](\bar{x}) \quad &- \quad L^{1,\delta}_{K}[-\varphi(\bar{s},\bar{t},\bar{x},\cdot)](\bar{y}) \\
&= \quad \frac{1}{\epsilon}\int\limits_{B_\delta(0)} |z|^2 K(z)dz + L^{1,\delta}_K[\psi_h](\bar{x}).
\end{aligned} \tag{2.34}$$

Next we analyze the two terms involving $L^{2,\delta}_K$, which can be splitted in two integrals, one over $B_1(0) \setminus B_\delta(0)$ and the other over $B_1^c(0)$. We denote these integrals as $L^{2,\delta}_{K,1}$ and $L^{2,\delta}_{K,2}$, respectively. Recalling $(\bar{s},\bar{t},\bar{x},\bar{y})$ is a global maximum point attaining $M_{\epsilon,h,\eta}$, we may write the following inequality

$$\begin{aligned}
u(\bar{t},\bar{x}+z) \quad &- \quad u(\bar{t},\bar{x}) - \Big(\frac{\bar{x}-\bar{y}}{\epsilon} + D\psi_h(\bar{x})\Big)\cdot z \;\leq\; v(\bar{s},\bar{y}+z) \\
-v(\bar{s},\bar{y}) \quad &+ \quad \psi_h(\bar{x}+z) - \psi_h(\bar{x}) - D\psi_h(\bar{x})\cdot z - \frac{\bar{x}-\bar{y}}{\epsilon}\cdot z,
\end{aligned}$$

for each $z \in \mathbb{R}^n$, then we find that

$$\begin{aligned}
L^{2,\delta}_{K,1}[D_x\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),u(\bar{t},\cdot)](\bar{x}) \quad &\leq \\
L^{2,\delta}_{K,1}[-D_y\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),v(\bar{s},\cdot)](\bar{y}) \quad &+ \quad L^{2,\delta}_{K,1}[D\psi_h(\bar{x}),\psi_h](\bar{x}).
\end{aligned} \tag{2.35}$$

Similarly, we have for each $z \in \mathbb{R}^n$

$$u(\bar{t},\bar{x}+z) - u(\bar{t},\bar{x}) \leq v(\bar{s},\bar{y}+z) - v(\bar{s},\bar{y}) + \psi_h(\bar{x}+z) - \psi_h(\bar{x})$$

and then

$$L_{K,2}^{2,\delta}[u(\bar{t},\cdot)](\bar{x}) \leq L_{K,2}^{2,\delta}[v(\bar{s},\cdot)](\bar{y}) + L_{K,2}^{2,\delta}[\psi_h](\bar{x}). \tag{2.36}$$

Then, putting together (2.34), (2.35) and (2.36) in (2.33) we obtain

$$\begin{aligned}\frac{\eta}{(T-\bar{t})^2} &\leq \frac{\eta}{2T^2} + f(\bar{t},\bar{x}) - f(\bar{s},\bar{y}) + \frac{1}{\epsilon}\int_{B_\delta(0)} |z|^2 K(z)dz \\ &+ L_K^{1,\delta}[\psi_h](\bar{x}) + L_{K,1}^{2,\delta}[D\psi_h(\bar{x}),\psi_h](\bar{x}) + L_{K,2}^{2,\delta}[\psi_h](\bar{x}).\end{aligned}$$

Using (2.28) to control the integral terms applied to ψ_h and using (K2) with $\delta \to 0$, we find

$$\frac{\eta}{(T-\bar{t})^2} \leq \frac{\eta}{2T^2} + f(\bar{t},\bar{x}) - f(\bar{s},\bar{y}) + C(A,\psi,w)o_h(1)$$

Finally, using (2.30) and (2.31) together with the continuity of f, taking $\epsilon \to 0$ and then $h \to 0$ we conclude the following inequality, which is a impossible,

$$0 < \frac{\eta}{T^2} \leq \frac{\eta}{2T^2}. \quad \Box$$

2.5 Proof of the general Theorem 2.5.

In this section we prove our general theorem applying conveniently the maximum principle and the stability property proved before. We start with the study of the existence of a solution to equation (2.26) and we find a modulus of continuity of the corresponding solution (u_j), which is independent of j. This last property is the key ingredient for the compactness of the sequence (u_j) allowing to pass to the limit. More precisely we have

Proposition 2.1. *Let u_0 and f be bounded, uniformly continuous functions, let $\mathcal{K}$ be a family of kernels satisfying (K1) and (K2) and $\mathcal{I}_j$ be a sequence of operators as in Theorem 2.5. Then, there exists a unique bounded solution to equation (2.26) for every j. Moreover, $|u_j|_\infty$ has a uniform bound C and u_j has a uniform modulus of continuity μ.*

Proof: Define

$$C_f = \sup_{\substack{t\,\in\,(0,T)\\ x\,\in\,\mathbb{R}^n}} |f(t,x)| \quad \text{and} \quad C_0 = \sup_{x\in\mathbb{R}^n} |u_0(x)|. \tag{2.37}$$

Then we may use the functions $W_\pm(t,x) = \pm(C_f t + C_0)$ as sub and super-solutions and we can apply Perron's method to conclude the existence of a solution u_j. Theorem 2.6 implies uniqueness and if we define $\bar{c} := TC_f + C_0$, then we find that $|u_j|_\infty \leq \bar{c}$, a bound which is independent on j.

We devote the rest of the proof to find the modulus of continuity μ. Here we follow the arguments of Crandall and Lions in [67] and Ishii in [81], for first order equations. First, we obtain a common modulus of continuity on the spatial variable for u_j, for t fixed. For $y \in \mathbb{R}^n$, define

$$\begin{aligned} m_{f,y} &= \sup_{\substack{t\,\in\,(0,T)\\ x\,\in\,\mathbb{R}^n}} |f(t,x+y) - f(t,x)| \quad \text{and} \\ m_{0,y} &= \sup_{x\in\mathbb{R}^n} |u_0(x+y) - u_0(x)|. \end{aligned}$$

Define also $w_j(t,x) = u_j(t,x+y)$. It is easy to see that $w_j + tm_{f,y} + m_{0,y}$ is a super-solution to (2.26), while $w_j - tm_{f,y} - m_{0,y}$ is a sub-solution of the same equation. Here we use that the operators are translation invariant. Then, using Theorem 2.6, we get

$$|u_\epsilon(t,x+y) - u_\epsilon(t,y)| \leq Tm_{f,y} + m_{0,y}, \quad \forall\, x,y \in \mathbb{R}^n,\ t \in [0,T).$$

Thus, the function $m(y) = Tm_{f,y} + m_{0,y}$ is a spatial modulus of continuity for u_j, which is independent of j and $t \in [0,T)$. Next we consider a fixed $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$. Since m is a modulus of continuity, we may assume it is a sub-linear and increasing function, then for any $\rho > 0$ we have

$$u_j(t_0,x) \leq u_j(t_0,x_0) + m(\rho) + \frac{m(\rho)}{\rho}|x - x_0|, \quad \forall\, x \in \mathbb{R}^n. \tag{2.38}$$

Consider a smooth nondecreasing function $\eta : (0,\infty) \to \mathbb{R}$ such that $\eta(r) = r$ for $r \in [0,1]$ and $\eta(r) = 2$ for $r \geq 2$, then define

$$\eta_\rho(x) = \eta(\frac{m(\rho)}{2\bar{c}\rho}\sqrt{|x|^2 + \rho^2})$$

and, for some constant $N > 0$ to be fixed, define the function

$$v_\rho(t,x) = u_j(t_0,x_0) + m(\rho) + 2\bar{c}\eta_\rho(x - x_0) + N(t - t_0).$$

Then, using the bound $\bar{c}$ for u_j and the inequality (2.38) above we have

$$u_j(t_0,x) \leq v_\rho(t_0,x) \quad \text{for all } x \in \mathbb{R}^n \text{ and } \rho > 0. \tag{2.39}$$

Noticing that v_ρ is smooth we evaluate $\mathcal{I}_j(v_\rho)$ classically. We claim that there exists a constant C_ρ independent of j and (t_0,x_0) such that

$$|\mathcal{I}_j[v_\rho](x)| \leq C_\rho, \tag{2.40}$$

for all $x \in \mathbb{R}^n$. Postponing the proof of this claim, we continue taking $N \geq C_\rho + C_f$ with C_f defined in (2.37). We have then

$$\partial_t v_\rho(t,x) - \mathcal{I}_j v_\rho(t,x) \geq f(t,x) \quad \text{in } (t_0,T) \times \mathbb{R}^n$$

and using (2.39) we see that v_ρ is a super-solution for the Cauchy problem (2.26) over $[t_0,T) \times \mathbb{R}^n$. Using Theorem 2.6 we conclude

$$u_j(t,x) \leq v_\rho(t,x) \quad \text{in } [t_0,T) \times \mathbb{R}^n$$

for all $\rho > 0$. Evaluating at $x = x_0$ we obtain

$$u_j(t,x_0) \leq u_j(t_0,x_0) + m(\rho) + 2\bar{c}\eta(\frac{m(\rho)}{2\bar{c}}) + N(t - t_0), \quad t \in [t_0,T).$$

Doing an analogous reasoning to get a lower bound and we conclude that

$$|u_j(t,x_0) - u_j(t_0,x_0)| \leq m(\rho) + 2\bar{c}\eta(\frac{m(\rho)}{2\bar{c}}) + N(t - t_0), \quad t \in [t_0,T),$$

for all $\rho > 0$. This last inequality implies that the function

$$\mu(r) = \inf\{m(\rho) + 2\bar{c}\eta(\frac{m(\rho)}{2\bar{c}}) + Nr \ : \ \rho > 0\}$$

is the desired modulus of continuity in time of u_j, which applies uniformly for all $x_0 \in \mathbb{R}^n$, completing the proof.

Now we prove our claim (2.40). We only need to estimate $\mathcal{I}_j[\eta_\rho](x)$ for each $x \in \mathbb{R}^n$, where $\mathcal{I}_j$ is an operator defined through $\mathcal{K}_j \subset \mathcal{K}$ and $\mathcal{K}$ satisfy (K1) and (K2). Let us define $c_\rho = m(\rho)/(2\bar{c}\rho)$ and for $K \in \mathcal{K}$ consider

$$A = \int_{B_{c_\rho^{-1}}} \delta(\eta_\rho, x, z)K(z)dz \quad \text{and} \quad B = \int_{B^c_{c_\rho^{-1}}} \delta(\eta_\rho, x, z)K(z)dz.$$

We see that just need to prove that A and B are bounded independent of $K \in \mathcal{K}$ and $x \in \mathbb{R}^n$. For B we just need to consider (K1) and use that $\eta \leq 2$,

$$B \leq 8 \int_{B^c_{c_\rho^{-1}}} K(z)dz \leq C_\rho.$$

On the other hand we find that

$$A \leq \max\{D^2\eta_\rho(y) \ / \ y \in B_{c_\rho^{-1}}(x)\} \int_{B_{c_\rho^{-1}}} |z|^2 K(z)dz.$$

The integral term is bounded by a constant depending on ρ but not in K by (K2). On the other hand, η, η' and η'' are bounded and η_ρ is non-constant only for $x \in \mathbb{R}^n$ such that

$$|x| < \sqrt{(2c_\rho^{-1})^2 - \rho^2}.$$

Consequently, $D^2\eta_\rho(y)$ is bounded by a constant only dependent on ρ and not on $y \in \mathbb{R}^n$. This finished the proof. $\square$

We finish the section giving the

Proof Theorem 2.5. From Proposition 2.1 we have existence and uniqueness of a bounded solutions u_j for (2.26), with a common modulus of continuity for the whole sequence. Thus the sequence converges locally uniformly in $[0, T) \times \mathbb{R}^n$ to a limit function u. Since we assumed (2.24) and $\{u_j\} \subset \mathcal{H}$, we conclude that $u \in \mathcal{H}$. The compatibility condition (2.23) between $\mathcal{H}$ and $\mathcal{K}$, together with the weak convergence of the operators $\mathcal{I}_j$ to $\mathcal{I}$ with respect to $\mathcal{H}$ allow the use Lemma 2.1 to conclude the result. $\square$

2.6 Applications of the general Theorem 2.5.

In this section we apply Theorem 2.5 to the various cases discussed in the introduction.

Proof of Theorem 2.1. In the two cases, the corresponding set of kernels $\mathcal{K}^1$ and $\mathcal{K}^2$ satisfy common properties. In particular if $\mathcal{K}$ denote any of these families, there exists a nonnegative function $w \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ such that for each $r > 0$ there exists a constant $C_r > 0$ with the property

$$\forall\ K \in \mathcal{K}, K(z) \le C_r w(z),\ \forall\ |z| \ge r.$$

Hence, condition (2.25) holds with the same w. Since Proposition 2.1 implies the uniform boundedness of the viscosity solutions of the approximating equations, we have the compatibility condition (2.23) fulfilled in each case. Clearly, the set of uniformly bounded functions satisfy condition (2.24) and then by Lemma 2.2, we apply Theorem 2.5 to conclude. $\square$

We observe that the properties of $\mathcal{K}^3$ allows to prove Part 1 of Theorem 2.3 in the same way. Thus we only need to prove Part 2 of Theorem 2.3 and we do it in what follows. We first need some preliminaries regarding the asymptotic behavior of the solutions that is a consequence of the assumptions on f and u_0. These properties are crucial in the limiting process to assure the compatibility of the class of functions $\mathcal{H}$ with the corresponding kernels in $\mathcal{K}^4$ stated in (2.23).

Our precise statement is

Lemma 2.3. *Assume f and u_0 satisfy (2.19). Let u_σ be a solution of (2.1) asociated to the operator $\mathcal{I}_\sigma$ and u be the solution of (2.18). Then*

$$\sup_{t\in(0,T)} |u_\sigma(t,x)| \le \Theta(x), \quad \textit{for all}\ \ \sigma \in [0, s_0), x \in \mathbb{R}^n, \tag{2.41}$$

where $u_0 = u$ and the function Θ satisfies $\lim_{|x|\to\infty} \Theta(x) = 0$.

Proof. By (2.19), for each $\epsilon > 0$ there exists $R_\epsilon > 1$ such that

$$\sup_{t\in(0,T)} |f(t,x)| \le \epsilon, \quad |u_0(x)| \le \epsilon,$$

for all $|x| \ge R_\epsilon$. Define for each $\epsilon, \theta > 0$ the function

$$w_{\epsilon,\theta}(t,x) = \epsilon(t+1) + Qe^{kt}(1+|x|^2)^{-\theta}$$

with $Q, k > 0$ to be fixed later. For $x \in \mathbb{R}^n$ set $a^2 = (1+|x|^2)^{-1}$, $\hat{x} = ax$. Notice that $|\hat{x}| \le 1$ and if $|x| \to \infty$, then $|\hat{x}| \to 1$. Given $K \in \mathcal{K}^4$ we have

$$|L_K[w_{\epsilon,\theta}](x)| \le \Lambda \int_{\mathbb{R}^n} |\delta(w_{\epsilon,\theta}, x, z)| \frac{dz}{|z|^{n+2\sigma}} = 2Qe^{kt}(1+|x|^2)^{-\theta-\sigma} I, \tag{2.42}$$

with

$$I = \int_{\mathbb{R}^n} |(a^2 + |\hat{x}+y|^2)^{-\theta} - 1| \frac{dy}{|y|^{n+2\sigma}} = I_1 + I_2, \tag{2.43}$$

where I_1 is the integral over $B_{1/2}(0)$ and I_2 the integral over $B_{1/2}(0)^c$. Using the mean value theorem we find that

$$|I_1| \le \max_{|y|\le 1/2} \{2\theta(a^2 + |\hat{x}+y|^2)^{-\theta-1}|\hat{x}+y|\} \int_{B_{1/2}(0)} \frac{|y|dy}{|y|^{n+2\sigma}} \le \frac{C\theta}{1-2\sigma}. \tag{2.44}$$

On the other hand,

$$|I_2| \le \int\limits_{B^c_{1/2}(0)} (a^2+|\hat{x}+y|^2)^{-\theta}\frac{dy}{|y|^{n+2\sigma}} + \frac{C}{\sigma}. \tag{2.45}$$

Here we split the domain of integration as $B^c_{1/2}(0) = U_1 \cup U_2$, with $U_1 = B^c_{1/2}(0) \cap B_{1/2}(-\hat{x})$ and $U_2 = B^c_{1/2}(0) \cap B^c_{1/2}(-\hat{x})$. We observe that

$$(a^2+|\hat{x}+y|^2)^{-\theta} \le |\hat{x}+y|^{-2\theta} \quad \text{for } y \in U_1$$

and

$$(a^2+|\hat{x}+y|^2)^{-\theta} \le C|y|^{-2\theta} \quad \text{for } y \in U_2.$$

Using these inequalities in (2.45) and assuming that $2\theta < n$ we obtain

$$\begin{aligned} |I_2| &\le \int_{U_1} |\hat{x}+y|^{-2\theta}\frac{dy}{|y|^{n+2\sigma}} + C\int_{U_2} |y|^{-2\theta}\frac{dy}{|y|^{n+2\sigma}} + \frac{C}{\sigma} \\ &\le C(1+\frac{1}{2\theta+2\sigma}+\frac{1}{\sigma}). \end{aligned} \tag{2.46}$$

Putting together (2.42), (2.43), (2.44), (2.46) and the definition of $\mathcal{I}_\sigma$ we find

$$|\mathcal{I}_\sigma[w_{\epsilon,\theta}](x)| \le CQe^{kt}(1+|x|^2)^{-(\theta+\sigma)}(1+\sigma),$$

where C is independent of $x \in \mathbb{R}^n$, θ and ϵ. Then, taking

$$Q = (|u_0|_\infty + 1)(1+R^2_\epsilon)^{\theta/2} \text{ and } \; k = 2C + (|f|_\infty+1)(1+R^2_\epsilon)^{\theta/2},$$

we see that $w_{\epsilon,\theta}$ is a super-solution of (2.1) with the operator $\mathcal{I}_\sigma$. Using the comparison principle as stated in Theorem 2.6, we conclude $u_\sigma(t,x) \le w_{\epsilon,\theta}(t,x)$ for all t,x and each $\epsilon > 0$. Similar analysis can be done for a sub-solution, so we obtain $|u_\sigma(t,x)| \le w_{\epsilon,\theta}(t,x)$ for all x,t. Defining

$$\Theta(x) = \inf_\epsilon w_{\epsilon,\theta}(T,x),$$

we get the desired inequelity (2.41) in case $\sigma > 0$.

Using $w_{\epsilon,\theta}$ again it is possible to obtain the result for $\sigma = 0$, provided a comparison principle for (2.18) is available. We obtain it next. Since the solution u of (2.18) and our sub and super-solution are regular, we just need to prove a comparison principle for u and v satisfying

$$\begin{cases} v_t(t,x) + G(v(t,x)) \ge f(t,x), & (t,x) \in (0,T)\times\mathbb{R}^n, \\ v(0,x) \ge u_0(x), & x \in \mathbb{R}^n \end{cases}$$

and

$$\begin{cases} u_t(t,x) + G(u(t,x)) \le f(t,x), & (t,x) \in (0,T)\times\mathbb{R}^n, \\ u(0,x) \le u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

in a classical way. In order to prove $u(t,x) \le v(t,x)$ for all x,t, we assume

$$\sup_{(0,T)\times\mathbb{R}^n} u(t,x) - v(t,x) > 0,$$

for contradiction. Hence, there is $x_0 \in \mathbb{R}^n$ such that, taking $\eta > 0$ sufficiently small, we have that the function

$$t \mapsto u(t, x_0) - v(t, x_0) - \eta(T - t)^{-1}$$

has a strictly positive maximum point at $t = t_0 \in [0, T]$. By the initial condition and the penalization introduced above we have $t_0 \in (0, T)$. Then

$$0 = u_t(t_0, x_0) - v_t(t_0, x_0) - \eta(T - t_0)^{-2}$$

and this implies, using the inequalities satisfied by u and v, that

$$0 \le u_t(t_0, x_0) - v(t_0, x_0) \le -G(u(t_0, x_0)) + G(v(t_0, x_0)) < 0$$

by the strict monotonicity of G, providing a contradiction. □

Now we can prove our convergence result for our fourth scheme.

Proof of Theorem 2.3, Part 2. By the definition of $\mathcal{I}_\sigma$, for each $\sigma > 0$ (2.25) holds with $w_\sigma = \sigma(1 + |x|^{n+2\sigma})^{-1}$. Next we consider the set of functions

$$\mathcal{H} = \{u : [0, T) \times \mathbb{R}^n \to \mathbb{R} / \sup_{t \in (0,T)} |u(t, x)| \le \Theta(x),\ \forall\, x \in \mathbb{R}^n\},$$

with Θ given in Lemma 2.3, and we see that $\mathcal{H}$ and the set of kernels $\mathcal{K}^4$ are compatible in the sense of (2.23) and also $\mathcal{H}$ satisfies (2.24). Hence, we can apply Theorem 2.5 using the corresponding result in Lemma 2.2, satisfied by the family of operators $\mathcal{I}_\sigma$. □

Remark 2.4. We observe that a decaying assumption as (2.41) is necessary to conclude Part 2 of Theorem 2.3. In fact, if we consider the linear case and set $u_0 \equiv a$ and $f \equiv b$, with $a, b \in \mathbb{R}$, we have that the unique solution to

$$\begin{cases} u_t(t, x) + (-\Delta)^\sigma u(t, x) = b & (t, x) \in (0, T) \times \mathbb{R}^n \\ u(0, x) = a & x \in \mathbb{R}^n \end{cases}$$

is $u_\sigma = bt + a$, for each $\sigma \in (0, \sigma_1)$. However, the limit equation is also linear and its solution is

$$u(t, x) = e^{-ct}\Big(\frac{b}{c}(e^{ct} - 1) + a\Big),$$

for a positive constant c depending only on n. Obviously, we do not have the convergence given in Theorem 2.3, not even locally uniform convergence.

2.7 Convergence rates

When the solution of the limiting problem has sufficient regularity we may use the approach of [59] to obtain convergence rates. We have to remark though, that in most of the cases we consider there is no regularity theorems available, even in the linear case. In the nonlinear case, specially for Isaacs type operators, those results are unknown even for local operators.

In this section we prove the results stated in the introduction, following the ideas in [59] in the framework of viscosity solutions, since we only assume regularity of the solution of the limiting equation. We start with

Proof Theorem 2.2. Part 1. We first see that $u_\epsilon - u$ is a viscosity super-solution for

$$\partial_t w - \mathcal{M}^-[w] = \mathcal{I}_\epsilon[u] - \mathcal{I}[u]. \tag{2.47}$$

We notice that the terms $\mathcal{I}_\epsilon[u]$ and $\mathcal{I}[u]$ are well defined by the regularity assumption on the limit function u. Let ϕ be a smooth function such that $u_\epsilon - u - \phi$ has a minimum point at $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$ and define

$$v = \begin{cases} \phi & in \quad [0,T) \times B_r(t_0, x_0) \\ u_\epsilon - u & in \quad [0,T) \times B_r(t_0,x_0)^c, \end{cases}$$

for $r > 0$. We have to prove

$$\partial_t \phi(t_0,x_0) - \mathcal{M}^-[v](t_0,x_0) \geq \mathcal{I}_\epsilon[u](t_0,x_0) - \mathcal{I}[u](t_0,x_0).$$

We observe that we can use $u + \phi$ as a test function for the equation for u_ϵ, defining $\tilde{v}$ as

$$\tilde{v} = \begin{cases} \phi + u & in \quad [0,T) \times B_r(t_0, x_0) \\ u_\epsilon & in \quad [0,T) \times B_r(t_0,x_0)^c \end{cases}$$

Using the ellipticity of $\mathcal{I}_\epsilon$ and that $\tilde{v}$ is a test function for u_ϵ, we have

$$\begin{aligned} \partial_t \phi(t_0,x_0) - \mathcal{M}^-[v](t_0,x_0) \quad &\geq \quad f(t_0,x_0) - \partial_t u(t_0,x_0) + \mathcal{I}[u](t_0,x_0) \\ &\quad\quad -\mathcal{I}[u](t_0,x_0) + \mathcal{I}_\epsilon[u](t_0,x_0), \end{aligned}$$

from where we conclude (2.47), since u is a solution to the limiting equation and it can be evaluated classically by regularity assumption. Now we estimate the right hand side of (2.47) directly. Let $(t,x) \in (0,T) \times \mathbb{R}^n$ and $\eta > 0$ arbitrary and find $\alpha = \alpha(\eta,x,t), \beta = \beta(\eta,x,t)$ such that

$$\begin{aligned} \mathcal{I}_\epsilon[u](t_0,x_0) - \mathcal{I}[u](t_0,x_0) \quad &\geq \quad -\eta + L_{\alpha,\beta,\epsilon}[u](t,x) - L_{\alpha,\beta}[u](t,x) \\ &= \quad -\eta - 2\Lambda\epsilon^{n+2\sigma}[u]_{2\sigma+\gamma} \int\limits_{\mathbb{R}^n} \frac{|z|^{-n+\gamma}}{\epsilon^{n+2\sigma} + |z|^{n+2\sigma}} dz \\ &\geq \quad -\eta - C\Lambda[u]_{2\sigma+\gamma}\epsilon^\gamma, \end{aligned}$$

where the constant C depends only on n. Here $[u]_{2\sigma+\gamma}$ denotes the Hölder constant of u and we have used that $\gamma < 1$. Since η is arbitrary, the same inequality holds for $\eta = 0$. From here we see that $(t,x) \mapsto -Ct\epsilon^\gamma$ is a sub-solution for (2.47). In a similar way it is possible to obtain a super-solution. Using Theorem 2.6, we finally obtain that

$$|u_\epsilon(t,x) - u(t,x)| \leq CT\epsilon^\gamma.$$

Part 2. Using the same argument as above, it is possible to conclude that $u_\epsilon - u$ is a viscosity super-solution for (2.47), with the appropriate operators. To estimate the right hand side, we take $\eta > 0$ and $(t,x) \in (0,T) \times \mathbb{R}^n$, so there exist $\alpha = \alpha(\eta,x,t), \beta = \beta(\eta,x,t)$ such that

$$\begin{aligned} \mathcal{I}_\epsilon[u](t,x) - \mathcal{I}[u](t,x) \quad &\geq \quad -\eta + \int\limits_{\mathbb{R}^n} \delta(u(t,\cdot),x,z)(K(z) - K_\epsilon(z))dz \\ &\geq \quad -\eta - 2\Lambda[u]_\gamma \int\limits_B |z|^{\gamma-n-\epsilon}||z|^\epsilon - 1|(1 + |\log(|z|)|)^{-\rho} dz \\ &\quad\quad +4\Lambda|u|_\infty \int\limits_{B^c} |z|^{-(n+\epsilon)}||z|^\epsilon - 1|(1+|\log(|z|)|)^{-\rho} dz \\ &= \quad -\eta - C[u]_\gamma I_1 - C|u|_\infty I_2. \end{aligned} \tag{2.48}$$

We estimate I_1 and I_2 separately. For I_1 we have, for ϵ small,

$$\begin{aligned} I_1 &= C\int_0^1 r^{\gamma-1-\epsilon}(1-r^\epsilon)(1+|\log(r)|)^{-\rho}dr \\ &\leq C\int_0^{\epsilon^{1/\gamma}} r^{\gamma-1-\epsilon}(1-r^\epsilon)dr + \int_{\epsilon^{1/\gamma}}^1 r^{\gamma-1-\epsilon}(1-r^\epsilon)dr \leq C\frac{\epsilon|\log(\epsilon)|}{\gamma^2} \end{aligned}$$

For I_2 and ϵ small we have

$$\begin{aligned} I_2 &= C\int_1^\infty r^{-1-\epsilon}(r^\epsilon-1)(1+|\log(r)|)^{-\rho}dr \\ &\leq C(\epsilon^{-\epsilon}-1)\int_1^{\epsilon^{-1}} r^{-1}(1+\log(r))^{-\rho}dr + C\int_{\epsilon^{-1}}^\infty r^{-1}(1+\log(r))^{-\rho}dr \\ &\leq C(\rho-1)^{-1}|\log(\epsilon)|^{1-\rho}. \end{aligned}$$

Then, dropping the term η in (2.48) and using the estimates for I_1 and I_2, we get for ϵ small

$$\mathcal{I}_\epsilon[u](t,x) - \mathcal{I}[u](t,x) \geq C((\rho-1)^{-1}|\log(\epsilon)|^{1-\rho} + \epsilon\gamma^{-2}|\log(\epsilon)|),$$

where C is a constant independent of t, x, ρ and ϵ. It is possible to find a similar upper bound then, using a comparison argument as in Part 1, we finish the proof. □

In a similar way we obtain the

Proof Theorem 2.4. Part 1. Proceeding as in the proof of the previous theorem we see that we only need to find that

$$|\mathcal{I}_\sigma[u] - F(D^2u)| \leq C\frac{1-\sigma}{\gamma},$$

where the constant C depends on u and other parameters, but not on σ. This inequality is obtained by direct estimate without major difficulty.

Part 2. We just need to estimate $\mathcal{I}_\sigma[u] - G(u)$ and proceed as before. Let $\eta > 0$, then there exist α, β depending on η, t, x such that

$$\begin{aligned} \mathcal{I}_\sigma[u] - G(u) &\leq \eta + \sigma\int_{\mathbb{R}^n} \delta(u(t,\cdot),x,z)\frac{a_{\alpha,\beta}(\hat{z})}{|z|^{n+2\sigma}}dz - c_{\alpha,\beta}u(t,x) \\ &\leq \eta + 2[u]_\gamma\frac{\Lambda\sigma}{\gamma-2\sigma} + 2\Lambda\sigma\int_{B^c}|u(t,x+z)||z|^{-(n+2\sigma)}dz, \end{aligned} \tag{2.49}$$

where we have used the definition of $c_{\alpha,\beta}$. In order to estimate the integral, we consider $\epsilon \in (0,1)$ and use the decay on u to find a constant C so that

$$|u(t,x-z)| \leq \epsilon/2, \quad \text{for } z \text{ such that } |x-z| \geq (C\epsilon)^{-1/\beta}.$$

In what follows we write $\epsilon_\beta = (C\epsilon)^{-1/\beta}$ for notational convenience. Next we consider I_1 the integral over $B^c \cap B_{\epsilon_\beta}(x)$ and I_2 the integral over $B^c \cap B^c_{\epsilon_\beta}(x)$. We study each integral separately, starting with

$$I_2 \leq \frac{\epsilon}{2} \int\limits_{B^c} |z|^{-(n+2\sigma)} dz = \frac{C\epsilon}{\sigma}. \tag{2.50}$$

For I_1, we see that

$$I_1 \leq C|u|_\infty \int\limits_{B_{\epsilon_\beta}(x)} (1+|z|)^{-(n+2\sigma)} dz,$$

If $|x| \leq 2\epsilon_\beta$ then we have

$$\begin{aligned} \int\limits_{B_{\epsilon_\beta}(x)} (1+|z|)^{-(n+2\sigma)} dz &\leq \int\limits_{B_{4\epsilon_\beta}(0)} (1+|z|)^{-(n+2\sigma)} dz \\ &\leq C(1+\frac{1-(4\epsilon_\beta)^{-2\sigma}}{\sigma}). \end{aligned} \tag{2.51}$$

If $|x| \geq 2\epsilon_\beta$ we have

$$\begin{aligned} \int\limits_{B_{\epsilon_\beta}(x)} (1+|z|)^{-(n+2\sigma)} dz &\leq \int\limits_{B_{\epsilon_\beta}(0)} (1+2\epsilon_\beta-|z|)^{-(n+2\sigma)} dz \\ &\leq C\int_0^{\epsilon_\beta} r^{n-1}(1+2\epsilon_\beta-r)^{-(n+2\sigma)} dr \\ &\leq C\int_0^{\epsilon_\beta} (1+2\epsilon_\beta-r)^{-(1+2\sigma)} dr \\ &= \frac{C}{2\sigma(1+\epsilon_\beta)^{2\sigma}}\Big\{1-\Big(\frac{1+\epsilon_\beta}{1+2\epsilon_\beta}\Big)^{2\sigma}\Big\}. \end{aligned} \tag{2.52}$$

It is not difficult to see that, when $\epsilon \in (0,1)$ and σ is small enough, then the last term is bounded by C/σ, for an appropriate constant C independent of ϵ and σ. Summarizing, from (2.51) and (2.52), we have that for all $x \in \mathbb{R}^n$,

$$I_1 \leq C(1+\frac{1-(4\epsilon_\beta)^{-2\sigma}}{\sigma}). \tag{2.53}$$

Using (2.50) and (2.53) on (2.49) and then dropping the term η, we have

$$\mathcal{I}_\sigma[u] - G(u) \leq C\frac{\sigma}{\gamma} + C\Big(\epsilon + 1 - (4\epsilon_\beta)^{-2\sigma}\Big), \tag{2.54}$$

for each $\epsilon \in (0,1)$, where C is an appropriate constant independent of γ, $\epsilon \in (0,1)$ and σ small enough. We are assuming that $\gamma \in (0,1)$.

Since $\epsilon \in (0,1)$ is arbitrary, we may minimize the right hand side of (2.54) $g(\epsilon) = \epsilon + 1 - (4\epsilon_\beta)^{-2\sigma}$. We easily see that the minimum is achieved at

$$\epsilon^* = C\Big(\frac{\sigma}{\beta}\Big)^{\frac{\beta}{\beta-2\sigma}},$$

for some C. Evaluating $g(\epsilon^*)$ for σ small we find from (2.54) that

$$\mathcal{I}_\sigma[u] - G(u) \leq C\Big(\frac{\sigma}{\gamma} + \sigma\log(\sigma)\Big). \tag{2.55}$$

Obtaining a similar inequality from above, we obtain the required inequality

$$|\mathcal{I}_\sigma[u] - G(u)| \leq C\Big(\frac{\sigma}{\gamma} + \sigma\log(\sigma)\Big). \qquad \square$$

The estimate (2.20) needed as hypothesis of Theorem 2.4, Part 2, can be obtained making some assumptions on the data u_0 and f. Next we provide the proof for completeness.

Lemma 2.4. *Assume there exists $\beta > 0$ such that*

$$|u_0(x)| + |f(t,x)| \leq c(1+|x|)^{-\beta} \tag{2.56}$$

for all $x \in \mathbb{R}^n$ and $t \in (0,T)$. Then

$$|u(t,x)| \leq C(1+|x|)^{-\beta}$$

for all $x \in \mathbb{R}^n$ and $t \in (0,T)$, with certain constant C.

Proof. We consider the function

$$w_\beta(t,x) = Qe^{kt}(1+|x|)^{-\beta}$$

with β, Q and k appropriate constants. It is easy to see that

$$\partial_t w_\beta - G[w_\beta] \geq Qke^{kt}(1+|x|)^{-\beta} - 2Q\Lambda e^{kt}(1+|x|)^{-\beta}$$

Hence, taking Q and k in terms of the constant c in (2.56), we see that w_β is a super-solution. In the same way it is possible to build a sub-solution and the result follows by comparison theorem. $\square$

2.8 Other approximation families

For each approximation scheme consider in the introduction we could consider many different families of converging operators. We end this paper discussing some of them.

In case of the first scheme we may consider the following three alternatives for the family of kernels $\mathcal{K}^1$:

1. Still considering the basic kernel given by (2.8), we could provide other type of approximation as the following two examples show. A simple truncation at 0 for a singular kernel of the form

$$K_\epsilon(z) = \min\{\frac{1}{\epsilon^{n+2\sigma}}, \frac{1}{|z|^{n+2\sigma}}\}a(\hat{z})$$

and with kernels with compact support of the form

$$K_\epsilon(z) = \left[\frac{1}{(|z|+\epsilon)^{n+2\sigma}} - \frac{1}{(\epsilon^{-1}+\epsilon)^{n+2\sigma}}\right]_+ a(\hat{z}). \tag{2.57}$$

2. Instead of considering the basic family of kernels given by (2.8), we should expect similar results for family of kernels associated to other type of processes, like tempered $2\sigma-$stable processes, hyperbolic Lévy processes or Meixner processes. See [102].

For the second approximation scheme we could also consider family of kernels different from $\mathcal{K}^2$ as we show next:

1. We let J be a non-negative continuous function with support in $B_1(0)$ and integral equal to 1. Then for $\sigma > 0$ fixed and $\epsilon \in (0,1]$ we define the kernel

$$K_\epsilon(z) = \frac{J(z)}{\min\{1, |z/\epsilon|^{n+2\sigma}\}}$$

and $K_0 = J$. We observe that for each $\epsilon > 0$, the kernel K_ϵ is non-integrable at zero and defines a fractional operator of order σ. Proceeding as before, we can prove the same result as Theorem 2.1 with operators $\mathcal{I}_\epsilon$ and $\mathcal{I}$. Nonlocal equations for such zero order operators model population dynamics and have been extensively studied during the last years. See for instance [61], [64], [65], [57] and [58], and references therein.

2. A combination of a singular kernel of order $n + 2\sigma_0$ with a kernel J as above gives

$$K_\epsilon(z) = \frac{\epsilon^2}{|z|^{n+2\epsilon}} + (\sigma_0 - \epsilon)J(z)$$

where $\epsilon \in (0, \sigma_0)$. The idea is to let $\epsilon \to 0$, starting from $\epsilon = \sigma_0$. The operator K_ϵ defines a fractional operator, for every $\epsilon \in (0,1]$ that reaches an integrable kernel J with compact support.

3. We may also consider kernels of the form

$$K_\epsilon(z) = \frac{e^{-|z|}a(\hat{z})}{|z|^{n+2(\sigma_0+\epsilon)}},$$

where $\epsilon \in (0, \sigma_1)$, with $\sigma_0 < 0$ and $\sigma_0 + \sigma_1 > 0$. The idea is to move $\epsilon \to 0$, starting at σ_1. When $\epsilon = \sigma_1$ the kernel defines a fractional operator of order $\sigma_0 + \sigma_1$, while for $\epsilon \in (\sigma_0, 0)$ these kernels are integrable and define operators of order zero.

Finally, for the third approximation scheme, we may approximate second order operators through different types of operators. For example we may consider a family of zero order operators with bounded kernel, defined by scaling as the kernels in $\mathcal{K}^1$. More precisely we consider

$$K_\sigma(z) = (1-\sigma)^{-(n+2\sigma)+1} \frac{a(\hat{z})}{1 + |z/(1-\sigma)|^{n+2\sigma}}$$

with $a \in \mathcal{L}_{\lambda,\Lambda}$. It is also possible to consider a a family of kernels with compact support, see (2.57) above. □

Chapter 3

Compactness Results for a Class of Zero-th Order Dirichlet Problems in Bounded Domains

This chapter is based in the joint work with Patricio Felmer [73].

3.1 Introduction.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain with C^2 boundary, $f \in C(\bar{\Omega})$ and $\epsilon \in (0, 1)$. In this paper we are concerned on study of the Dirichlet problem

$$-\mathcal{I}_\epsilon[u] = f \quad \text{in } \Omega, \tag{3.1}$$

$$u = 0 \quad \text{in } \Omega^c, \tag{3.2}$$

where $\mathcal{I}_\epsilon$ is a nonlocal operator approaching the fractional Laplacian as ϵ approaches 0. We focus our attention on $\mathcal{I}_\epsilon$ with the form

$$\mathcal{I}_\epsilon[u](x) := \int_{\mathbb{R}^N} [u(x+z) - u(x)] K_\epsilon(z) dz, \tag{3.3}$$

where, for $\sigma \in (0, 1)$ fixed, K_ϵ is defined as

$$K_\epsilon(z) := \frac{1}{\epsilon^{N+2\sigma} + |z|^{N+2\sigma}} = \epsilon^{-(N+2\sigma)} K_1(z/\epsilon).$$

Notice that for each $\epsilon \in (0, 1)$, K_ϵ is integrable in $\mathbb{R}^N$ with L^1 norm equal to $C\epsilon^{-2\sigma}$, where $C > 0$ is a constant depending only on N and σ. We point out that operators with kernel in L^1, like $\mathcal{I}_\epsilon$, are known in the literature as *zero order nonlocal operators*.

Operator $\mathcal{I}_\epsilon$ is a particular case of a broad class of nonlocal elliptic operators. In fact, given a positive measure μ satisfying the *Lévy condition*

$$\int_{\mathbb{R}^N} \min\{1, |z|^2\} \mu(dz) < \infty,$$

and, for each $x \in \mathbb{R}^N$ and $u : \mathbb{R}^N \to \mathbb{R}$ bounded and sufficiently smooth at x, the operator $\mathcal{I}_\mu[u](x)$ defined as

$$\mathcal{I}_\mu[u](x) = \int_{\mathbb{R}^N} [u(x+z) - u(x) - \mathbf{1}_{B_1(0)}(z)\langle Du(x), z\rangle] \mu(dz), \tag{3.4}$$

has been a subject of study in a huge variety of contexts such as potential theory ([92]), probability ([40, 102]) and analysis ([103, 104, 23, 47, 48]). An interesting point of view of our problem comes from probability, since (3.4) represents the infinitesimal generator of a *jump Lévy process*, see Sato [102]. In our setting, the finiteness of the measure is associated with the so-called *Compound Poisson Process*. Dirichlet problems with the form of (3.1)-(3.2) arise in the context of exit time problems with trajectories driven by the jump Lévy process defined by $K_\epsilon(z)dz$, and the solution u_ϵ represents the expected value of the associated cost functional, see [97].

We may start our discussion with a natural notion of solution to our problem (3.1)-(3.2): we say that a bounded function $u : \mathbb{R}^N \to \mathbb{R}$, continuous in Ω, is a solution of (3.1)-(3.2) if it satisfies (3.1) pointwise in Ω and $u = 0$ on Ω^c. As we see in Section §2, this problem has a unique solution, more interestingly, through an example we will see that such a solution may not be continuous in $\mathbb{R}^N$, since a discontinuity may appear on the boundary of Ω. See Remark 3.2.

This situation is in great contrast with the limit case $\epsilon = 0$, where the kernel becomes $K(z)dz = |z|^{-(N+2\sigma)}dz$ and the associated nonlocal operator is the *fractional Laplacian of order* 2σ, denoted by $-(-\Delta)^\sigma$, see [69]. In this case, the corresponding Dirichlet problem becomes

$$\begin{cases} C_{N,\sigma}(-\Delta)^\sigma v &= f \qquad \text{in } \Omega, \\ \qquad\qquad v &= 0 \qquad \text{in } \Omega^c, \end{cases} \tag{3.5}$$

where $C_{N,\sigma} > 0$ is a normalizing constant. In the context of the viscosity theory for nonlocal equations (see [23, 103, 104]), Barles, Chasseigne and Imbert [21] addressed a large variety of nonlocal elliptic problems including (3.5). In that paper, the authors proved the existence and uniqueness of a viscosity solution $v \in C(\bar{\Omega})$ of (3.5) satisfying $v = 0$ on $\partial\Omega$ that is, consequently, continuous when we regard it as a function on $\mathbb{R}^N$. This result is accomplished by the use of a nonlocal version of the notion of viscosity solution with *generalized boundary conditions*, see [66, 16, 31] for an introduction of this notion in the context of second-order equations.

Additionally, fractional problems like (3.5) enjoy a *regularizing effect* as in the classical second-order case. Roughly speaking, for a right-hand side which is merely bounded, the solution v of (3.5) is locally Hölder continuous in Ω, see [107]. In fact, we should mention here that interior Hölder regularity for more general fractional problems (for which (3.5) is a particular case) has been addressed by many authors, see for instance [21, 18, 35, 47, 48, 49, 108] and the classical book of Landkof [92], for a non-exhaustive list of references. The interior Hölder regularity is accomplished by well established elliptic techniques as the Harnack's inequality ([47, 36]) and the Ishii-Lions method ([21, 84]). In both cases, the nonintegrability of the kernel plays a key role. Hölder regularity for problems like (3.5) can be extended up to the boundary, as it is proved by Ros-Oton and Serra in [101], where a boundary Harnack's inequality is the key ingredient (see also [41]). Naturally, as a byproduct of these regularity results, compactness properties are available for certain families of solutions of fractional equations. For instance, the family $\{v_\eta\}$ of functions solving

$$\begin{cases} C_{N,\sigma}(-\Delta)^\sigma v_\eta &= f_\eta \qquad \text{in } \Omega \\ \qquad\qquad u_\eta &= 0, \qquad \text{in } \Omega^c, \end{cases}$$

satisfies compactness properties when $\{f_\eta\}$ is uniformly bounded in $L^\infty(\bar{\Omega})$.

For zero order problems, regularizing effects as arising in fractional problems are no longer available (see [68]). In fact, the finiteness of the kernel of zero order operators turns into degenerate ellipticity for which Ishii-Lions method cannot be applied. Thus, "regularity results" for zero order problems like (3.1)-(3.2) are circumscribed to the heritage of the modulus of continuity of the right-hand side f to the solution u_ϵ as it can be seen in [53]. However, the modulus of continuity found in [53] depends strongly on the size of the L^1 norm of K_ϵ, which explodes as $\epsilon \to 0$. A similar lack of stability as $\epsilon \to 0$ can be observed in the Harnack-type inequality results for nonlocal problems found by Coville in [61]. Hence, none of the mentioned tools are adequate for getting compactness for the family of solutions $\{u_\epsilon\}$ of problem (3.1)-(3.2), which is a paradoxical situation since, in the limit case, the solutions actually get higher regularity and stronger compactness control on its behavior.

In view of the discussion given above, a natural mathematical question is if there exists a uniform modulus of continuity in Ω, for the family of solutions $\{u_\epsilon\}$ to (3.1)-(3.2), and consequently compactness properties for it. In this direction, the main result of this paper is the following

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ a bounded domain with C^2 boundary and $f \in C(\bar{\Omega})$. For $\epsilon \in (0,1)$, let u_ϵ be a solution to problem (3.1)-(3.2). Then, there is a modulus of continuity m depending only on f, such that*

$$|u_\epsilon(x) - u_\epsilon(y)| \leq m(|x-y|), \quad \text{for } x, y \in \Omega.$$

The proof of this theorem is obtained combining the translation invariance of $\mathcal{I}_\epsilon$ and comparison principle, constructing suitable barriers to manage the discontinuities that u_ϵ may have on $\partial\Omega$ and to understand how they evolve as ϵ approaches zero, see Proposition 3.5.

As a consequence of Theorem 3.1 we have the following corollary, that actually was our original motivation to study the problem.

Corollary 3.1. Let u_ϵ be the solution to equation (3.1), with f and Ω as in Theorem 3.2, and let u be the solution of the equation (3.5), then $u_\epsilon \to u$ in $L^\infty(\bar{\Omega})$ as $\epsilon \to 0$.

We mention here that the application of the *half-relaxed limits method* introduced by Barles and Perthame in [28] (see also [29, 37, 23]) allows to obtain in a very direct way locally uniform convergence in Ω in the above corollary. At this point we emphasize on the main contribution of this paper, which is the analysis of the boundary behavior of the family $\{u_\epsilon\}$ of solutions to (3.1)-(3.2) coming from Theorem 3.1 and the subsequent *global* uniform convergence to the solution of (3.5).

There are many possible extensions of Theorem 3.1, for example, it can be readily extended to problems with the form

$$\begin{cases} -\mathcal{I}_\epsilon[u] & = f_\epsilon \quad \text{in } \Omega \\ \quad\quad u & = 0 \quad \text{in } \Omega^c, \end{cases}$$

with $\{f_\epsilon\} \subset C(\bar{\Omega})$ having a common modulus of continuity independent of $\epsilon \in (0,1)$. It can also be extended to fully nonlinear operators and to parabolic equations, as we discuss in Section §3.6. We could also consider different families of approximating zero order operators, but we do not pursue this direction. There are many other interesting lines of research that arises from this work. From the discussion given before Theorem 3.1, questions arises with respect to Harnack type inequalities and its relation with regularity and compactness properties of solutions, when $\epsilon \to 0$. Regarding operators $\mathcal{I}_\mu$, where μ might be singular with respect to the Lebesgue measure, an interesting question that arises is if the main results of this article can be extended to this case.

The paper is organized as follows: In Section §3.2 we establish the notion of pointwise solution and the comparison principle. Important estimates for the discontinuity of the solution at the boundary are given in Section §3.3, and the boundary equicontinuity result is presented in Section §3.4. The interior modulus of continuity is easily derived from the boundary equicontinuity, and therefore the proof of Theorem 3.1 is given in Section §3.5. Further related results are discussed in Section §3.6.

3.1.1 Notation

For $x \in \mathbb{R}^N$ and $r > 0$, we denote $B_r(x)$ the ball centered at x with radius r and simply B_r if $x = 0$. For a set $U \subset \mathbb{R}^N$, we denote by $d_U(x)$ the signed distance to the boundary, this is $d_U(x) = \mathrm{dist}(x, \partial U)$, with $d_U(x) \leq 0$ if $x \in U^c$. Since many arguments in this paper concerns the set Ω, we write $d_\Omega = d$. We also define

$$\Omega_r = \{x \in \Omega \ : \ d(x) < r\}$$

Concerning the regularity of the boundary of Ω, we assume it is at least C^2, so the distance function d is a C^2 function in a neighborhood of $\partial\Omega$. More precisely, there exists $\delta_0 > 0$ such that $x \mapsto d(x)$ is of class C^2 for $-\delta_0 < d(x) < \delta_0$.

In our estimates we will denote by c_i with $i = 1, 2, ...$ positive constants appearing in our proofs, depending only on N, σ and Ω. When necessary we will make explicit the dependence on the parameters. The index will be reinitiated in each proof.

3.2 Notion of Solution and Comparison Principle.

In the introduction we defined a notion of solution to problem (3.1)-(3.2), which is very natural for zero order operators and allows us to understand the main features of the mathematical problem that we have at hand. However, this notion is not suitable for a neat statement of the comparison principle and it is not adequate to understand the limit as $\epsilon \to 0$. For this reason, from now on, we adopt another notion of solution which is more adequate, that is the notion of viscosity solution with generalized boundary condition defined by Barles, Chasseigne and Imbert in [21].

We remark that results provided in this section are adequate for problems slightly more general than our problem (3.1)-(3.2). We will consider $J \in L^1(\mathbb{R}^N)$ a nonnegative function, and we define the nonlocal operator associated to J as

$$\mathcal{I}_J[u](x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)]J(z)dz, \tag{3.6}$$

for $u \in L^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, and a Dirichlet problem of the form

$$-\mathcal{I}_J[u] = f \qquad \text{in } \Omega, \tag{3.7}$$

$$u = 0 \qquad \text{on } \Omega^c, \tag{3.8}$$

with $f \in C(\bar{\Omega})$. Since we are interested in a Dirichlet problem for which the exterior data plays a role, we assume J and Ω satisfy the condition

$$\inf_{x \in \bar{\Omega}} \int_{\Omega^c - x} J(z)dz \geq \nu_0 > 0. \tag{3.9}$$

Notice that problem (3.1)-(3.2) is a particular case of (3.7)-(3.8).

In this situation, a bounded function $u : \mathbb{R}^N \to \mathbb{R}$, continuous in $\bar{\Omega}$ is a viscosity solution with generalized boundary condition to problem (3.7)-(3.8) if and only if it satisfies

$$\begin{aligned} -\mathcal{I}_J[u] &= f \qquad \text{on } \bar{\Omega}, && (3.10)\\ u &= 0 \qquad \text{in } \bar{\Omega}^c. && (3.11)\end{aligned}$$

The sufficient condition is direct from the definition and the necessary condition follows from the lemma:

Lemma 3.1. *Let $f \in C(\bar{\Omega})$ and let $u : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying*

$$-\mathcal{I}_J[u](x) \le f(x) \quad \textit{for all } x \in \Omega, \tag{3.12}$$

where the above inequality is understood pointwise. Let $x_0 \in \partial\Omega$ and assume there exists a sequence $\{x_k\} \subset \Omega$ such that

$$x_k \to x_0, \quad u(x_k) \to u(x_0) \tag{3.13}$$

and

$$\limsup_{k\to+\infty} u(x_k + z) \le u(x_0 + z), \quad \textit{a.e.} \tag{3.14}$$

Then, u satisfies (3.12) at x_0.

Here and in what follows the considered measure is the Lebesgue measure.

Proof. Consider $\{x_k\} \subset \Omega$ as in (3.13). Then, we can write

$$\int_{\mathbb{R}^N} u(x_k + z)J(z)dz - u(x_k)\int_{\mathbb{R}^N} J(z)dz \ge -f(x_k).$$

Hence, taking limsup in both sides of the last inequality, by (3.13) and the continuity of f, we arrive to

$$\int_{\mathbb{R}^N} \limsup_{k\to\infty} u(x_k + z)J(z)dz - u(x_0)\int_{\mathbb{R}^N} J(z)dz \ge -f(x_0),$$

where the exchange of the integral and the limit is justified by Fatou's Lemma. Then, using (3.14), we conclude the result. □

We continue with our analysis with an existence result for (3.7)-(3.8).

Proposition 3.1. *Let $f \in C(\bar{\Omega})$. Then, there exists a unique bounded function $u : \mathbb{R}^N \to \mathbb{R}$, continuous in $\bar{\Omega}$, which is a viscosity solution with generalized boundary condition to problem (3.7)-(3.8).*

Proof. According with our discussion above, we need to find a solution to (3.10)-(3.11). Consider the map $T_a : C(\bar{\Omega}) \to C(\bar{\Omega})$ defined as

$$T_a(u)(x) = u(x) - a\Big(||J||_{L^1(\mathbb{R}^N)}u(x) - \int_{\Omega - x} u(x+z)J(z)dz - f(x)\Big).$$

We observe that $u \in C(\bar{\Omega})$ is a fixed points of T_a if and only if u is a solution to problem (3.10)-(3.11). Therefore, the aim is to prove that for certain $a > 0$ small enough, the map T_a is a contraction in $C(\bar{\Omega})$. By (3.9), there exists $\varrho_0 >$ such that

$$||J||_{L^1(\mathbb{R}^N)} - ||J||_{L^1(\Omega - x)} \ge \varrho_0, \quad \text{for each } x \in \bar{\Omega}.$$

Let $0 < a < \min\{\varrho_0^{-1}, ||J||^{-1}_{L^1(\mathbb{R}^N)}\}$ and consider $u_1, u_2 \in C(\bar{\Omega})$. Then, for all $x \in \bar{\Omega}$ we have

$$T_a(u_1)(x) - T_a(u_2)(x) \leq \Big(1 - a||J||_{L^1(\mathbb{R}^N)} + a \int\limits_{\Omega - x} J(z)dz\Big)||u_1 - u_2||_\infty$$
$$\leq \Big(1 - a\varrho_0\Big)||u_1 - u_2||_\infty,$$

concluding that

$$||T_a(u_1) - T_a(u_2)||_\infty \leq (1 - a\varrho_0)||u_1 - u_2||_\infty,$$

that is, T_a is a contraction in $C(\bar{\Omega})$. From here existence and uniqueness follow. □

Remark 3.1. We observe that $u : \mathbb{R}^N \to \mathbb{R}$, a viscosity solution with generalized boundary condition to problem (3.7)-(3.8), may be redefined on the boundary $\partial\Omega$ as $u = 0$, to obtain a solution to (3.7)-(3.8) in the sense defined in the introduction.

Remark 3.2. Let u be a solution of (3.7)-(3.8) in the sense defined in the introduction, with $f \geq \varrho_0 > 0$. Our purpose is to show that u has a discontinuity on the boundary of Ω. Let us assume, for contradiction, that $u : \mathbb{R}^N \to \mathbb{R}$ is a continuous function.

Then $u \geq 0$ in Ω, otherwise there exists $x_0 \in \Omega$ such that $u(x_0) = \min_{\bar{\Omega}}\{u\} < 0$ and evaluating the equation at x_0 we arrive to

$$u(x_0) \int_{\Omega^c - x_0} J(z)dz \geq -\mathcal{I}_J[u](x_0) = f(x_0),$$

which is a contradiction to (3.9). Then, from the equation, we have for each $x \in \Omega$ the inequality

$$-\mathcal{I}_J[u](x) = f(x) > \varrho_0.$$

Since u and f are continuous and $u = 0$ on $\partial\Omega$ and using that $u \geq 0$ in Ω, we obtain that, for each $x \in \partial\Omega$

$$0 \geq -\int_{\Omega - x} (u(x+z) - u(x))J(z)dz = -\mathcal{I}_J[u](x) = f(x) > \varrho_0,$$

which is a contradiction. Thus, $u > 0$ on $\partial\Omega$ which implies that u is discontinuous on $\partial\Omega$.

In what follows we prove that a solution to (3.7)-(3.8), in the sense defined in the introduction, can be extended continuously to $\bar{\Omega}$.

Proposition 3.2. *Let $f \in C(\bar{\Omega})$. Let $v : \mathbb{R}^N \to \mathbb{R}$ in $L^\infty(\mathbb{R}^N) \cap C(\Omega)$ be a solution to (3.7)-(3.8), and $u : \mathbb{R}^N \to \mathbb{R}$ in $C(\bar{\Omega})$ be the viscosity solution to (3.7)-(3.8) given by Proposition 3.1. Then, $u = v$ in Ω.*

Proof: By contradiction, assume the existence of a point in Ω where u is different from v. Defining $w = u - v$, we will assume that

$$M := \sup_{\Omega}\{w\} > 0, \tag{3.15}$$

since the case $\inf_\Omega\{w\} < 0$ follows the same lines. Moreover, we assume that the supremum defining M is not attained, since this is the most difficult scenario. Let $\eta > 0$ and let $x_\eta \in \Omega \setminus \Omega_\eta$ such that

$$w(x_\eta) = \max_{\Omega \setminus \Omega_\eta}\{w\},$$

where Ω_η was defined at the end of the introduction. We clearly have $w(x_\eta) \to M$ as $\eta \to 0$ and since we assume M is not attained, then $x_\eta \to \partial\Omega$ as $\eta \to 0$. Now, using the equations for u and v at $x_\eta \in \Omega$, we can write

$$-\int_{\Omega - x_\eta} [w(x_\eta + z) - w(x_\eta)] J(z) dz + w(x_\eta) \int_{\Omega^c - x_\eta} J(z) dz \leq 0,$$

and by (3.9) and the fact that $w(x_\eta) \to M$ as $\eta \to 0$, we have

$$-\int_{\Omega - x_\eta} [w(x_\eta + z) - w(x_\eta)] J(z) dz + \nu_0 M - o_\eta(1) \leq 0, \tag{3.16}$$

where $o_\eta(1) \to 0$ as $\eta \to 0$. But writing

$$\begin{aligned} \int_{\Omega - x_\eta} [w(x_\eta + z) - w(x_\eta)] J(z) dz = & \int_{\Omega \setminus \Omega_\eta - x_\eta} [w(x_\eta + z) - w(x_\eta)] J(z) dz \\ & + \int_{\Omega_\eta - x_\eta} [w(x_\eta + z) - w(x_\eta)] J(z) dz, \end{aligned}$$

by the boundedness of w and the integrability of J, the second integral term in the right-hand side of the last equality is $o_\eta(1)$, meanwhile, using the definition of x_η we have the first integral is nonpositive. Thus, we conclude

$$\int_{\Omega - x_\eta} [w(x_\eta + z) - w(x_\eta)] J(z) dz \leq o_\eta(1),$$

and replacing this into (3.16), we arrive to

$$\nu_0 M - o_\eta(1) \leq 0.$$

By making $\eta \to 0$, we see that this contradicts (3.15), since $\nu_0 > 0$. $\square$

As a consequence of the last proposition, we have the following

Corollary 3.2. Let $f \in C(\bar{\Omega})$. Then, there exists a unique solution $v \in L^\infty(\mathbb{R}^N) \cap C(\Omega)$ to problem (3.7)-(3.8) in the sense defined in the introduction. Moreover, v is uniformly continuous in Ω and its unique continuous extension to $\bar{\Omega}$ coincides with the unique viscosity solution to (3.7)-(3.8).

The main tool in this paper is the comparison principle, and here the so-called *strong* comparison principle is the appropriate version to deal with discontinuities at the boundary.

Proposition 3.3. (Comparison Principle) *Assume $f \in L^\infty(\bar{\Omega})$. Let $u, v \in \mathbb{R}^N \to \mathbb{R}$ be bounded, upper and lower semicontinuous functions on $\bar{\Omega}$, respectively. Assume u and v satisfy*

$$-\mathcal{I}_J[u] \leq f \quad \text{and} \quad -\mathcal{I}_J[v] \geq f, \quad \text{on } \bar{\Omega}. \tag{3.17}$$

If $u \leq v$ in $\bar{\Omega}^c$, then $u \leq v$ in $\bar{\Omega}$.

Proof: Assume by contradiction that there exists $x_0 \in \bar{\Omega}$ such that

$$(u - v)(x_0) = \max_{x \in \bar{\Omega}} \{u - v\} > 0.$$

Evaluating inequalities in (3.17) at x_0 and substracting them, denoting $w = u - v$, we arrive to

$$-\int_{\Omega - x_0} [w(x_0+z) - w(x_0)]J(z)dz - \int_{\Omega^c - x_0} [w(x_0+z) - w(x_0)]J(z)dz \leq 0,$$

and therefore, using that x_0 is a maximum point for w in Ω and that $w \leq 0$ in Ω^c, we can write

$$w(x_0)\int_{\Omega^c - x_0} J(z)dz \leq 0,$$

and using (3.9) we arrive to a contradiction with the fact that $w(x_0) > 0$. □

As a first consequence of this comparison principle, we obtain an a priori $L^\infty(\bar{\Omega})$ estimate for the solutions u_ϵ of (3.1)-(3.2), independent of ϵ.

Proposition 3.4. *Let $\epsilon \in (0,1)$, $f \in C(\bar{\Omega})$ and u_ϵ be the viscosity solution of* (3.1)-(3.2). *Then, there exists a constant $C > 0$ such that*

$$||u_\epsilon||_{L^\infty(\bar{\Omega})} \leq C||f||_\infty$$

and this constant depends only on Ω, N and σ, but not on ϵ, for $\epsilon \in (0,1)$.

Proof. Consider the bounded function $\chi(x) = \mathbf{1}_{\bar{\Omega}}(x)$. We clearly have that $\chi \in C(\bar{\Omega})$ and $\chi = 0$ in $\bar{\Omega}^c$. Denote $R = diam(\Omega) > 0$ and use the definition of the operator $\mathcal{I}_\epsilon$ to see that for each $x \in \bar{\Omega}$ we have

$$-\mathcal{I}_\epsilon[\chi](x) = \int_{\Omega^c - x} K_\epsilon(z)dz \geq \int_{B^c_{R+1}} \frac{dz}{2|z|^{N+2\sigma}} = \frac{\mathrm{Vol}(B_1)(R+1)^{-2\sigma}}{2\sigma}.$$

Hence, denoting $C = (2\sigma)^{-1}\mathrm{Vol}(B_1)(R+1)^{-2\sigma}$ and $\tilde{\chi} = C^{-1}||f||_\infty \chi$, we may use the comparison principle to conclude $u_\epsilon \leq C||f||_\infty$ in $\bar{\Omega}$. A lower bound can be found in a similar way, concluding the result. □

3.3 Estimates of the Boundary Discontinuity.

The aim of this section is to estimate the discontinuity jump on $\partial\Omega$ of the solution u_ϵ of (3.1)-(3.2). For this purpose, a flattening procedure on the boundary is required.

Recall that $\delta_0 > 0$ is such that the distance function to $\partial\Omega$ is smooth in Ω_{δ_0}, and for $x \in \Omega_{\delta_0}$ we denote $\hat{x}$ the unique point on $\partial\Omega$ such that $d_\Omega(x) = |x - \hat{x}|$. We can fix δ_0 small in order to have the existence of three constants $R_0, r_0, r'_0 > 0$ depending only on the regularity of the boundary, satisfying the following properties:

(i) For each $x \in \Omega_{\delta_0}$, there exists $\mathcal{N}_x \subset \partial(\Omega - x)$, a $\partial(\Omega - x)$-neighborhood of $\hat{x} - x$, which is the graph of a C^2 function $\varphi_x : B_{R_0} \subset \mathbb{R}^{N-1} \to \mathbb{R}^N$, that is,

$$(\xi', \varphi_x(\xi')) \in \mathcal{N}_x, \quad \text{for all } \xi' \in B_{R_0}.$$

(ii) If we define the function Φ_x as

$$\Phi_x(\xi', s) = (\xi', \varphi(\xi')) + (d(x) + s)\nu_{\xi'}, \quad (\xi', s) \in B_{R_0} \times (-R_0, R_0), \tag{3.18}$$

where $\nu_{\xi'}$ is the unit inward normal to $\partial(\Omega - x)$ at $(\xi', \varphi_x(\xi'))$ and denoting $\mathcal{R}_x = \Phi_x(B_{R_0} \times (-R_0, R_0))$, then $\Phi_x : B_{R_0} \times (-R_0, R_0) \to \mathcal{R}_x$ is a C^1-diffeomorphism. Notice that $\Phi_x(0', 0)$ is the origin and therefore $\mathcal{R}_x$ is an $\mathbb{R}^N$-neighborhood of the origin.

(iii) The constant $r_0 > 0$ is such that $B_{r_0} \subset \mathcal{R}_x$ for all $x \in \Omega_{\delta_0}$.

(iv) The constant $r_0' > 0$ is such that $\Phi_x(B_{r_0'} \times (-r_0', r_0')) \subset B_{r_0}$.

We may assume $0 < r_0' \le r_0 \le \delta_0$. In addition, by the smoothness of the boundary there exists a constant $C_\Omega > 1$ such that

$$C_\Omega^{-1} K_\epsilon(\xi) \le \tilde{K}_\epsilon(\xi) \le C_\Omega K_\epsilon(\xi), \quad \xi \in \mathbb{R}^N, \tag{3.19}$$

where $\tilde{K}_\epsilon(\xi) = |\mathrm{Det}(D\Phi_x(\xi))| K_\epsilon(\Phi_x(\xi))|$.

The following Lemma is the key technical result of this paper

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $\epsilon \in (0,1)$. For $\beta \in (0,1)$, consider the function*

$$\psi(x) = \psi_\beta(x) := (\epsilon + d(x))^\beta \mathbf{1}_{\bar{\Omega}}(x), \tag{3.20}$$

where $d = d_\Omega$ is the distance function to $\partial\Omega$. Then, there exists $\bar{\delta} \in (0, \delta_0)$, $\beta_0 \in (0, \min\{1, 2\sigma\})$ and a constant $c^* > 0$, depending only on Ω, N and σ, such that, for all $\beta \le \beta_0$ we have

$$-\mathcal{I}_\epsilon[\psi](x) \ge c^*(\epsilon + d(x))^{\beta - 2\sigma}, \quad \text{for all } x \in \bar{\Omega}_{\bar{\delta}},\ \epsilon \in (0, \bar{\delta}). \tag{3.21}$$

Proof: We start considering $\bar{\delta} < r_0'$ and $x \in \Omega_{\bar{\delta}}$. We split the integral

$$\mathcal{I}_\epsilon[\psi](x) = I_0(x) + I_1(x) + I_2(x) + I_3(x),$$

where

$$\begin{aligned}
I_0(x) &:= \int_{B_{r_0}^c} [\psi(x+z) - \psi(x)] K_\epsilon(z) dz, \\
I_1(x) &:= \int_{B_{d(x)/2}} [\psi(x+z) - \psi(x)] K_\epsilon(z) dz, \\
I_2(x) &:= -(\epsilon + d(x))^{\beta - 2\sigma} \int_{(\Omega^c - x) \cap B_{r_0}} K_\epsilon(z) dz \quad \text{and} \\
I_3(x) &:= \int_{(\Omega - x) \cap B_{r_0} \setminus B_{d(x)/2}} [(\epsilon + d(x+z))^\beta - (\epsilon + d(x))^\beta] K_\epsilon(z) dz.
\end{aligned}$$

In what follows we estimate each $I_i(x), i = 0,1,2,3$. Since ψ is bounded in $\mathbb{R}^N$ independent of ϵ, β when $\epsilon, \beta \in (0,1)$, we have

$$I_0(x) \le c_1 r_0^{-2\sigma}, \tag{3.22}$$

where $c_1 > 0$ depends only on Ω and N. For $I_1(x)$, by the symmetry of K_ϵ we have

$$I_1(x) = \frac{1}{2} \int_{B_{d(x)/2}} [\psi(x+z) + \psi(x-z) - 2\psi(x)] K_\epsilon(z) dz.$$

Then we consider the function $\theta(z) = \psi(x+z) + \psi(x-z) - 2\psi(x)$, which is smooth in $\bar{B}_{d(x)/2}$ and therefore, we can write by Taylor expansion

$$\begin{aligned}\theta(z) =& \frac{\beta}{2}\Big[(\epsilon + d(x+\tilde{z}))^{\beta-1}\langle D^2 d(x+\tilde{z})z, z\rangle \\ &+ (\epsilon + d(x-\bar{z}))^{\beta-1}\langle D^2 d(x-\bar{z})z, z\rangle \\ &+ (\beta-1)(\epsilon + d(x+\tilde{z}))^{\beta-2}|\langle Dd(x+\tilde{z}), z\rangle|^2 \\ &+ (\beta-1)(\epsilon + d(x-\bar{z}))^{\beta-2}|\langle Dd(x-\bar{z}), z\rangle|^2\Big],\end{aligned}$$

where $\tilde{z}, \bar{z} \in B_{d(x)/2}$. With this, since we assume $\beta < 1$, by the smoothness of the distance function d inherited by the smoothness of $\partial\Omega$ we have

$$\theta(z) \leq c_2(\epsilon + d(x))^{\beta-1}|z|^2, \quad \text{for all } z \in B_{d(x)/2},$$

where $c_2 = C_\Omega \beta > 0$ depends on the domain, but not on ϵ or $d(x)$. From this, we get

$$I_1(x) \leq c_2(\epsilon + d(x))^{\beta-1} \int_{B_{d(x)/2}} |z|^2 K_\epsilon(z) dz,$$

and since $K_\epsilon(z) \leq K_0(z)$, we conclude that, for a constant $c_3 > 0$, we have

$$I_1(x) \leq c_3 \beta (\epsilon + d(x))^{\beta - 2\sigma + 1}. \tag{3.23}$$

Now we address the estimates of $I_2(x)$ and $I_3(x)$. For $I_2(x)$, recalling the change of variables Φ_x, we have

$$\Phi_x(B_{r_0'} \times (-r_0', -d(x))) \subset (\Omega^c - x) \cap B_{r_0}.$$

With this, using the change of variables Φ_x and applying (3.19), we have

$$I_2(x) \leq -C_\Omega^{-1}(\epsilon + d(x))^\beta \int_{B_{r_0'} \times (-r_0', -d(x))} K_\epsilon(\xi', s) d\xi' ds.$$

But there exists a constant $c_4 > 0$, depending only on N and σ, such that

$$\epsilon^{N+2\sigma} + |s|^{N+2\sigma} \leq c_4(\epsilon^{1+2\sigma} + |s|^{1+2\sigma})^{(N+2\sigma)/(1+2\sigma)},$$

and with this, defining $\rho(\epsilon, s) = (\epsilon^{1+2\sigma} + |s|^{1+2\sigma})^{1/(1+2\sigma)}$, we can write

$$\begin{aligned} I_2(x) \leq & - c_5(\epsilon + d(x))^\beta \int_{-r_0'}^{-d(x)} \frac{ds}{\rho(\epsilon, s)^{N+2\sigma}} \int_{B_{r_0'}} \frac{d\xi'}{1 + |\xi'/\rho(\epsilon, s)|^{N+2\sigma}} \\ = & - c_5(\epsilon + d(x))^\beta \int_{-r_0'}^{-d(x)} \frac{ds}{\rho(\epsilon, s)^{1+2\sigma}} \int_{B_{r_0'}/\rho(\epsilon,s)} \frac{dy}{1 + |y|^{N+2\sigma}} \\ \leq & - c_6(\epsilon + d(x))^\beta \int_{-r_0'}^{-d(x)} \frac{ds}{\epsilon^{1+2\sigma} + |s|^{1+2\sigma}}. \end{aligned}$$

Finally, making the change $t = -s/(\epsilon + d(x))$, we conclude

$$I_2(x) \le -c_6(\epsilon + d(x))^{\beta - 2\sigma} \int_{1-\tau}^{r_0'/(\epsilon + d(x))} \frac{dt}{\tau^{1+2\sigma} + |t|^{1+2\sigma}}, \tag{3.24}$$

where $\tau = \epsilon/(\epsilon + d(x)) \in (0,1)$. At this point, taking $\bar{\delta}$ small in order to have $\epsilon + d(x) < 2/r_0'$, we find that the interval

$$(1-\tau, r_0'/(\epsilon + d(x)))$$

has at least lenght 1. Hence, we conclude the existence of $c_7 > 0$, depending only on Ω, N and σ, such that

$$I_2(x) \le -c_7(\epsilon + d(x))^{\beta - 2\sigma}. \tag{3.25}$$

It remains to estimate $I_3(x)$. Defining $D_+(x) = \{z : d(x+z) \ge d(x)\}$, we clearly have

$$I_3(x) \le \int_{(\Omega - x)\cap D_+(x) \cap B_{r_0} \setminus B_{d(x)/2}} [(\epsilon + d(x+z))^\beta - (\epsilon + d(x))^\beta] K_\epsilon(z) dz,$$

and since

$$(\Omega - x) \cap B_{r_0} \cap D_+ \subset \Phi_x(B_{r_0} \times (0, r_0)),$$

we have

$$\begin{aligned} I_3(x) &\le \int_{\Phi_x(B_{r_0} \times (0,r_0)) \setminus B_{d(x)/2}} [(\epsilon + d(x+z))^\beta - (\epsilon + d(x))^\beta] K_\epsilon(z) dz \\ &= C_\Omega(\epsilon + d(x))^\beta \int_{\Phi_x(B_{r_0} \times (0,r_0)) \setminus B_{d(x)/2}} \Big[\Big(\frac{\epsilon + d(x+z)}{\epsilon + d(x)}\Big)^\beta - 1\Big] K_\epsilon(z) dz, \end{aligned}$$

Thus, making a change of variables we have

$$I_3(x) \le C_\Omega(\epsilon + d(x))^\beta \int_{B_{r_0} \times (0,r_0) \setminus \Phi_x^{-1}(B_{d(x)/2})} \Big[\Big(\frac{\epsilon + d(x + \Phi_x(\xi))}{\epsilon + d(x)}\Big)^\beta - 1\Big] \tilde{K}_\epsilon(\xi) d\xi.$$

Since Φ_x is a diffeomorphism, there exists a constant $c_8 > 0$ such that

$$d(x + \Phi_x(\xi', s)) \le d(x + \Phi_x(0, s)) + c_8|\xi'| = d(x) + s + c_8|\xi'|$$

and a constant $\lambda \in (0,1)$ small, depending only on the smoothness of $\partial\Omega$, such that $B_{\lambda d(x)} \subset \Phi_x^{-1}(B_{d(x)/2})$. Using this and (3.19) we arrive to

$$\begin{aligned} I_3(x) &\le C_\Omega(\epsilon + d(x))^\beta \int_{B_{r_0} \times (0,r_0) \setminus B_{\lambda d(x)}} [(1 + c_8|\xi/(\epsilon + d(x))|)^\beta - 1] K_\epsilon(\xi) d\xi \\ &= C_\Omega(\epsilon + d(x))^{\beta - 2\sigma} \int_{(\epsilon + d(x))^{-1} B_{r_0} \setminus B_{\lambda d(x)}} [(1 + c_8|y|)^\beta - 1] K_\tau(y) dy \\ &\le C_\Omega(\epsilon + d(x))^{\beta - 2\sigma} \int_{\lambda(1-\tau)}^{+\infty} \frac{[(1 + c_8 t)^\beta - 1] t^{N-1} dt}{\tau^{1+2\sigma} + t^{N+2\sigma}}. \end{aligned}$$

At this point, we remark that for each $M > 2$, we have

$$\int_M^{+\infty} \frac{[(1+c_8 t)^\beta - 1]t^{N-1}dt}{\tau^{1+2\sigma} + t^{N+2\sigma}} \leq c_9 M^{\beta-2\sigma},$$

where $c_9 > 0$ depends only on N, σ and Ω. On the other hand, for each $M > 2$ there exists $\beta = \beta(M) > 0$ small such that

$$\int_{\lambda(1-\tau)}^{M} \frac{[(1+c_8 t)^\beta - 1]t^{N-1}dt}{\tau^{1+2\sigma} + t^{N+2\sigma}} \leq C_\Omega^{-1} c_7/2,$$

where $c_7 > 0$ is the constant arising in (3.25). From the last two estimates, we conclude that for each $M > 2$, there exists β small such that

$$I_3(x) \leq c_7(\epsilon + d(x))^{\beta-2\sigma}/2 + c_{10} M^{\beta-2\sigma}, \tag{3.26}$$

where $c_{10} > 0$ depends only on N, σ and Ω. Putting together (3.22), (3.23), (3.25) and (3.26), and fixing $M = \max\{2, r_0\}$, we have

$$\mathcal{I}_\epsilon[\psi](x) \leq (\epsilon + d(x))^{\beta-2\sigma}(-c_7/2 + c_2\beta(\epsilon + d(x))) + c_{11} r_0^{-2\sigma},$$

where $c_{11} > 0$ depends only on N, σ and Ω. Hence, fixing $\beta > 0$ smaller if it is necessary, we can write

$$\mathcal{I}_\epsilon[\psi](x) \leq -c_7(\epsilon + d(x))^{\beta-2\sigma}/4 + c_{11} r_0^{-2\sigma}.$$

Finally, taking $\epsilon + d(x)$ small in terms of c_7, c_{11}, r_0, β and σ (and therefore, depending only on N, σ and Ω), we conclude (3.21), where $c^* = c_7/8$. $\square$

The last lemma allows us to provide the following control of the discontinuity at the boundary.

Proposition 3.5. *Let $\epsilon \in (0,1)$ and u_ϵ the solution of (3.1)-(3.2). Let $\bar{\delta} > 0$ and $\beta_0 \in (0, \min\{1, 2\sigma\})$ as in Lemma 3.2. Then, for each $d_0 \in (0, \bar{\delta})$, there exists $C_0 > 0$ satisfying*

$$|u_\epsilon(x)| \leq C_0(\epsilon + d(x))^{\beta_0} \quad \textit{for all } x \in \bar{\Omega}_{d_0}.$$

The constant C_0 depends on β_0, d_0, σ and Ω.

Proof: Let β_0 as in Lemma 3.2, $\psi = \psi_{\beta_0}$ as in (3.20) and consider the function

$$\zeta(x) = \min\{\psi(x), (\epsilon + d_0)^{\beta_0}\}, \quad x \in \mathbb{R}^N.$$

Observing that $\zeta = \psi$ in $\Omega_{d_0} \cup \Omega^c$ and $\psi \geq \zeta$ in $\mathbb{R}^N$, we easily conclude that

$$\mathcal{I}_\epsilon[\zeta](x) \leq \mathcal{I}_\epsilon[\tilde{\zeta}](x), \quad \text{for all } x \in \bar{\Omega}_{d_0},$$

and using Lemma 3.2 we get

$$-\mathcal{I}_\epsilon[\zeta](x) \geq c^*(\epsilon + d(x))^{\beta-2\sigma} \quad \text{for all } x \in \bar{\Omega}_{d_0}.$$

Let $C > 0$ be the constant in Proposition 3.4 and define the function $\tilde{z}_+ = (C d_0^{-\beta} + 2^\sigma c^{*-1})||f||_\infty \zeta$. By construction of $\tilde{z}_+$, we have

$$-\mathcal{I}_\epsilon[\tilde{z}_+] \geq ||f||_\infty \quad \text{in } \bar{\Omega}_{d_0}; \qquad \text{and} \qquad \tilde{z}_+ \geq u_\epsilon \quad \text{in } \bar{\Omega}_{d_0}^c,$$

and therefore, applying the comparison principle, we conclude $u_\epsilon \leq \tilde{z}_+$ in $\bar{\Omega}_{d_0}$. Similarly, we can conclude the function $\tilde{z}_- = -\tilde{z}_+$ satisfies $\tilde{z}_- \leq u_\epsilon$ in $\bar{\Omega}_{d_0}$, from which we get the result. $\square$

3.4 Boundary Equicontinuity.

In this section we establish the boundary equicontinuity of the family of solutions $\{u_\epsilon\}_{\epsilon\in(0,1)}$ of problem (3.1)-(3.2). The main result of this section is the following

Theorem 3.2. *Let $\epsilon \in (0,1)$ and u_ϵ be the solution to (3.1)-(3.2). There exists a modulus of continuity m_0 depending only on N, σ, f and Ω, such that*

$$|u_\epsilon(x) - u_\epsilon(y)| \leq m_0(|x-y|) \quad \textit{for all } x, y \in \bar{\Omega}_{\bar{\delta}},$$

with $\bar{\delta} > 0$ given in Lemma 3.2.

The idea of the proof is based on the fact $w(x) = u(x+y) - u(x)$, where y is fixed, satisfies an equation (near the boundary) for which the comparison principle holds. Using this, we get the result constructing a barrier to this problem, independent of ϵ and associated to m in Theorem 3.2.

In what follows we discuss the precise elements on the proof. We consider $y \in \mathbb{R}^N$ with $0 < |y| < \bar{\delta}/2$, with $\bar{\delta}$ as in Lemma 3.2. Define the sets

$$\mathcal{O} = \mathcal{O}(y) := \Omega \setminus \bar{\Omega}_{|y|}, \quad \mathcal{U} = \mathcal{U}(y) := \{x \in \mathbb{R}^N : -|y| \leq d_\Omega(x) < |y|\}.$$

and the function

$$w(x) = w_{y,\epsilon}(x) := u_\epsilon(x+y) - u_\epsilon(x), \ x \in \mathbb{R}^N. \tag{3.27}$$

Notice that $w \equiv 0$ in $\mathbb{R}^N \setminus (\bar{\mathcal{O}} \cup \mathcal{U})$ and, by Proposition 3.5, there exists $C_0, \beta_0 > 0$ such that $|w(x)| \leq C_0(\epsilon + |y|)^{\beta_0}$ for all $x \in \mathcal{U}$. Since we have that w satisfies

$$-\mathcal{I}_\epsilon[w](x) = f(x+y) - f(x) \quad \text{for all} \ \ x \in \bar{\mathcal{O}},$$

denoting by m_f the modulus of continuity of f, we conclude that $w \in C(\bar{\mathcal{O}})$ satisfies the inequality

$$-\mathcal{I}_\epsilon[w](x) \leq m_f(|y|) \quad \text{in } \bar{\mathcal{O}}, \tag{3.28}$$

and the exterior inequality

$$w(x) \leq C_0(\epsilon + |y|)^{\beta_0} \mathbf{1}_{\mathcal{U}}(x) \quad \text{in } \bar{\mathcal{O}}^c. \tag{3.29}$$

Let ζ and η the functions defined as

$$\begin{aligned} \zeta(x) =& \min\{(\epsilon + \bar{\delta} - |y|)^\epsilon, (\epsilon + d_\Omega(x) - |y|)^\epsilon\} \mathbf{1}_{\bar{\mathcal{O}}}(x) \quad \text{and} \\ \eta(x) =& C_0(\epsilon + |y|)^{\beta_0} \mathbf{1}_{\mathcal{U}}(x), \end{aligned}$$

and consider the function

$$W(x) = \eta(x) + A m(|y|) \zeta(x), \tag{3.30}$$

where $A > 0$ and m is a modulus of continuity satisfying $m(|y|) \geq m_f(|y|)$. We have the following

Proposition 3.6. *There exists $A > 0$ large, depending on Ω, N and σ, such that*

$$-\mathcal{I}_\epsilon[W](x) \geq m_f(|y|), \quad \textit{for all } x \in \bar{\mathcal{O}},$$

for all $\epsilon \in (0, \bar{\delta})$, with $\bar{\delta}$ given in Lemma 3.2.

Proof: Without loss of generality we may assume the existence of a number $0 < \alpha < \min\{1, \beta_0\}$ and a constant c_1 such that

$$m(t) \geq c_1 t^\alpha, \quad \text{for all } t \geq 0. \tag{3.31}$$

By linearity of $\mathcal{I}_\epsilon$, we have

$$\mathcal{I}_\epsilon[W](x) = \mathcal{I}_\epsilon[\eta](x) + Am(|y|)\mathcal{I}_\epsilon[\zeta](x).$$

Thus, we may estimate each term in the right-hand side separately.

1.- Estimate for $\mathcal{I}_\epsilon[\zeta](x)$*:* We first notice that for $x \in \Omega$ with $|y| \leq d_\Omega(x) \leq \bar{\delta}$ we can write

$$\zeta(x) = (\epsilon + d_\Omega(x) - |y|)^\epsilon \mathbf{1}_{\bar{\mathcal{O}}} = (\epsilon + d_{\mathcal{O}}(x))^\epsilon \mathbf{1}_{\bar{\mathcal{O}}}(x).$$

Then, applying Lemma 3.2, for all ϵ small we have

$$-\mathcal{I}_\epsilon[\zeta](x) \geq c^*(\epsilon + d(x) - |y|)^{\epsilon - 2\sigma}, \quad \text{for all } x \in \bar{\Omega}_{\bar{\delta}} \cap \bar{\mathcal{O}},$$

for some $c^* > 0$ not depending on $d(x), |y|$ or ϵ. In fact, for all $\epsilon \in (0,1)$ the term $(\epsilon + d(x) - |y|)^{-\epsilon}$ is bounded below by a strictly positive constant, independent of ϵ, driving us to

$$-\mathcal{I}_\epsilon[\zeta](x) \geq c^*(\epsilon + d(x) - |y|)^{-2\sigma}, \quad \text{for all } x \in \bar{\Omega}_{\bar{\delta}} \cap \bar{\mathcal{O}}. \tag{3.32}$$

On the other hand, when $x \in \Omega \setminus \bar{\Omega}_{\bar{\delta}}$, for all $\epsilon \in (0,1)$ we have

$$\mathcal{I}_\epsilon[\zeta](x) \leq -(\epsilon + \bar{\delta} - |y|)^\epsilon \int\limits_{(\Omega \setminus \Omega_{|y|})^c - x} K_\epsilon(z)dz \leq -\epsilon^\epsilon \int\limits_{\Omega^c - x} K_1(z)dz,$$

and therefore, there exists $c_2 > 0$, not depending on $\epsilon, d(x)$ or $|y|$, such that

$$\mathcal{I}_\epsilon[\zeta](x) \leq -c_2, \quad \text{for all } x \in \Omega \setminus \bar{\Omega}_{\bar{\delta}}.$$

Since $|y| \leq \bar{\delta}/2$, making c^* smaller if necessary, the last inequality and (3.32) drives us to

$$-\mathcal{I}_\epsilon[\zeta](x) \geq c^*(\epsilon + d(x) - |y|)^{-2\sigma}, \quad \text{for all } x \in \bar{\mathcal{O}}, \tag{3.33}$$

2.- Estimate for $\mathcal{I}_\epsilon[\eta](x)$*:* By its very definition, for $x \in \bar{\mathcal{O}}$ we have

$$\mathcal{I}_\epsilon[\eta](x) = C_0(\epsilon + |y|)^{\beta_0} \int\limits_{\mathcal{U} - x} K_\epsilon(z)dz. \tag{3.34}$$

We start considering the case $x \in \Omega \setminus \Omega_{\bar{\delta}}$, where we have $\mathrm{dist}(x, \mathcal{U}) \geq \bar{\delta}/2$ and then, there exists a constant $c_3 > 0$ depending only on $\bar{\delta}$ (which in turn depends only on the smoothness of the domain), such that $K_\epsilon(z)\mathbf{1}_{\mathcal{U}-x} \leq c_3$. Using this, we have

$$\mathcal{I}_\epsilon[\eta](x) \leq c_3(\epsilon + |y|)^{\beta_0} \int\limits_{\mathcal{U} - x} dz.$$

By the boundedness of Ω, there exists $c_4 > 0$ depending only on N such that $\mathrm{Vol}(\mathcal{U} - x) \leq c_4|y|$. Using this and (3.31), we conclude that

$$\mathcal{I}_\epsilon[\eta](x) \leq c_5 m(|y|), \tag{3.35}$$

where $c_5 > 0$ depends only on N, σ and Ω.

Now we deal with the case $x \in \mathcal{O} \cap \Omega_{\bar{\delta}}$ (notice that in this case we are assuming $d_\Omega(x) > |y|$). Using (3.34) and recalling the change of variables Φ_x introduced in (3.18), we can write

$$\mathcal{I}_\epsilon[\eta](x) \leq C_0(\epsilon + |y|)^{\beta_0}\Big(\int\limits_{(\mathcal{U}-x)\setminus B_{r_0}} K_\epsilon(z)dz + \int\limits_{(\mathcal{U}-x)\cap\mathcal{R}_x} K_\epsilon(z)dz\Big),$$

where $\mathcal{R}_x$ was defined at the beginning of Section §3. Using a similar analysis as the one leading to (3.35), there exists a universal constant $c_6 > 0$ such that

$$\mathcal{I}_\epsilon[\eta](x) \leq c_6(\epsilon + |y|)^{\beta_0}\Big(|y| + \int\limits_{(\mathcal{U}-x)\cap\mathcal{R}_x} K_\epsilon(z)dz\Big).$$

Now, we have that $(\mathcal{U} - x) \cap \mathcal{R}_x = \Phi_x(B_{R_0} \times (-d(x) - |y|, |y| - d(x)))$, and therefore, applying the change of variables Φ_x and the estimate (3.19), we arrive to

$$\mathcal{I}_\epsilon[\eta](x) \leq c_7(\epsilon + |y|)^{\beta_0}\Big(|y| + \int\limits_{B_{R_0}\times(-d(x)-|y|,|y|-d(x))} \frac{d\xi' ds}{\epsilon^{N+2\sigma} + |(\xi', s)|^{N+2\sigma}}\Big),$$

and from this, using a similar argument as the one leading to (3.24) to treat the last integral term, and applying (3.31), we conclude that

$$\mathcal{I}_\epsilon[\eta](x) \leq c_8(\epsilon + |y|)^{\beta_0}\Big(m(|y|) + \int\limits_{d(x)-|y|}^{d(x)+|y|} \frac{ds}{\epsilon^{1+2\sigma} + |s|^{1+2\sigma}}\Big). \tag{3.36}$$

Now, the core of this estimate is the computation of the last integral. Denoting

$$I(x) := (\epsilon + |y|)^{\beta_0} \int\limits_{d(x)-|y|}^{d(x)+|y|} \frac{ds}{\epsilon^{1+2\sigma} + |s|^{1+2\sigma}},$$

we claim the existence of a constant $c_9 > 0$ not depending on $\epsilon, d(x)$ or $|y|$ such that

$$I(x) \leq c_9 m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}. \tag{3.37}$$

We get this estimate considering various cases. When $|y| \leq \epsilon$ and $d(x) - |y| \leq 2\epsilon$ we write

$$I(x) = (\epsilon + |y|)^{\beta_0}\epsilon^{-2\sigma} \int\limits_{(d(x)-|y|)/\epsilon}^{(d(x)+|y|)/\epsilon} K_1(z)dz \leq 2^{\beta_0+1}\epsilon^{\beta_0-2\sigma-1}|y|,$$

and using that $m(|y|) \geq |y|^\alpha$ for some $\alpha \in (0, \beta_0)$, we have

$$\begin{aligned} I(x) &\leq 2^{\beta_0+1} m(|y|)\epsilon^{\beta_0 - 1}\epsilon^{-2\sigma}|y|^{1-\alpha} \\ &\leq 2^{\beta_0+1}3^{2\sigma} m(|y|)\epsilon^{\beta_0-\alpha}(\epsilon + d(x) - |y|)^{-2\sigma}, \end{aligned}$$

and from this, we conclude

$$I(x) \leq c_{10}\epsilon^{\beta_0-\alpha} m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}, \tag{3.38}$$

for some constant $c_{10} > 0$.

When $|y| \leq \epsilon$ and $d(x) - |y| > 2\epsilon$, we have

$$I(x) \leq 2^{\beta_0}\epsilon^{\beta_0} \int_{d(x)-|y|}^{d(x)+|y|} |z|^{-(1+2\sigma)} dz \leq 2^{\beta_0}\epsilon^{\beta_0}(d(x) - |y|)^{-(1+2\sigma)}|y|,$$

and using that $m(|y|) \geq |y|^\alpha$, we arrive to

$$I(x) \leq 2^{\beta_0-1} m(|y|)\epsilon^{\beta_0-\alpha}(d(x) - |y|)^{-2\sigma} \leq 2^{\beta_0-1+2\sigma} m(|y|)\epsilon^{\beta_0-\alpha}(\epsilon + d(x) - |y|)^{-2\sigma},$$

concluding the same estimate (3.38).

In the case $|y| > \epsilon$ and $d(x) - |y| \leq 2\epsilon$, performing the change $\xi = z/\epsilon$ in the integral defining $I(x)$, we have

$$I(x) \leq (\epsilon + |y|)^{\beta_0}\epsilon^{-2\sigma}||K_1||_{L^1} \leq ||K_1||_{L^1} 3^{2\sigma}2^{\beta_0}|y|^{\beta_0-\alpha} m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma},$$

and therefore we conclude

$$I(x) \leq C|y|^{\beta_0-\alpha} m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}. \tag{3.39}$$

Finally, in the case $|y| > \epsilon$ and $d(x) - |y| > 2\epsilon$ we have

$$I(x) \leq (\epsilon + |y|)^{\beta_0}\epsilon^{-2\sigma} \int_{(d(x)-|y|)/\epsilon}^{(d(x)+|y|)/\epsilon} K_1(z) dz \leq 2^{\beta_0-1}\sigma^{-1}|y|^{\beta_0}(d(x) - |y|)^{-2\sigma},$$

from which we arrive to (3.39). From (3.38) and (3.39) we arrive to (3.37). Hence, there exists $c_{11} > 0$ depending only on N, Ω and σ such that

$$-\mathcal{I}_\epsilon[\eta](x) \geq -c_{11} m(|y|)\Big((\epsilon + |y|)^{\beta_0} + (\epsilon + d(x) - |y|)^{-2\sigma}\Big),$$

for $x \in \mathcal{O} \cap \Omega_{\bar{\delta}}$. Taking this inequality and (3.35), since $|y| \leq \bar{\delta}/2$ there exists a constant $c_{12} > 0$ such that

$$-\mathcal{I}_\epsilon[\eta](x) \geq -c_{12} m(|y|)\Big((\epsilon + |y|)^{\beta_0} + (\epsilon + d(x) - |y|)^{-2\sigma}\Big), \tag{3.40}$$

for all $x \in \bar{\mathcal{O}}$, where the estimate for $x \in \partial\mathcal{O}$ is valid by Lemma 3.1.

3.- Conclusion: For each $x \in \bar{\mathcal{O}}$, by (3.33) and (3.40) we have

$$-\mathcal{I}_\epsilon[W](x) \geq \Big[(Ac^* - c_{12})(\epsilon + d(x) - |y|)^{-2\sigma} - c_{12}(\epsilon + |y|)^{\beta_0}\Big] m(|y|),$$

and therefore, by taking A large in terms of N, σ, c_{12}, c^* and $\mathrm{diam}(\Omega)$, we conclude by the choice of m that

$$-\mathcal{I}_\epsilon[W](x) \geq m(|y|) \geq m_f(|y|), \quad \text{for all } x \in \bar{\mathcal{O}},$$

and the proof follows. $\square$

This proposition allows us to give the

Proof of Theorem 3.2: Since w defined in (3.27) satisfies problem (3.28)-(3.29) and recalling W defined in (3.30), by Proposition 3.6 and the form of W in $\bar{\mathcal{O}}^c$, we can use the comparison principle to conclude that $w \leq W$ in $\bar{\mathcal{O}}$. This means that

$$u_\epsilon(x+y) - u_\epsilon(x) = w(x) \leq W(x) \leq c_1 A m(y), \quad x \in \bar{\Omega}_y,$$

for some constant $c_1 > 0$. Since a similar lower bound can be stated, by the arbitrariness of y we conclude the result with $m_0 = c_1 A m$. $\square$

3.5 Proof of Theorem 3.1.

Consider $\bar{\delta}$ as in Lemma 3.2, let $y \in \mathbb{R}^N$ such that $|y| \leq \bar{\delta}/8$ and consider the sets

$$\Sigma_1 = \overline{(\Omega - y) \cup \Omega}, \qquad \Sigma_2 = \Omega \cap (\Omega - y),$$
$$\Sigma_3 = \Sigma_1 \setminus \Sigma_2 \quad \text{and} \qquad \Sigma_4 = (\Omega \setminus \bar{\Omega}_{\bar{\delta}/2}) \cup ((\Omega \setminus \bar{\Omega}_{\bar{\delta}/2}) - y).$$

Notice that $\Sigma_4 \subset \Sigma_2 \subset \Sigma_1$. In addition, notice that if $z \in \Sigma_3$, then $z + y$ and z cannot be simultaneously in Ω. We also have

$$|\mathrm{dist}(z, \partial\Omega)|, |\mathrm{dist}(z+y, \partial\Omega)| \leq |y|$$

for each $z \in \Sigma_3$. Finally, observe that if $x \in \Sigma_2 \setminus \Sigma_4$, then $x, x+y \in \Omega_{\bar{\delta}}$. Thus, considering w as in (3.27), by Proposition 3.5 we can assure the existence of $C_0, \beta_0 > 0$ such that

$$w \leq C_0(\epsilon + |y|)^{\beta_0} \quad \text{in } \Sigma_3,$$

and by Theorem 3.2 we have

$$w \leq m_0(|y|) \quad \text{in } \Sigma_2 \setminus \bar{\Sigma}_4.$$

Now, consider the funtion

$$Z(x) = A m_0(|y|) \mathbf{1}_{\Sigma_2}(x) + C_0(\epsilon + |y|)^{\beta_0} \mathbf{1}_{\Sigma_3}(x),$$

where $A > 0$ is a constant to be fixed later. Notice that for each $x \in \bar{\Sigma}_3$, we have

$$\mathcal{I}_\epsilon[Z](x) = C_0(\epsilon + |y|)^{\beta_0} \int_{\Sigma_3 - x} K_\epsilon(z)dz - A m_0(|y|) \int_{\Sigma_2^c - z} K_\epsilon(z)dz. \tag{3.41}$$

At this point, we remark that there exists a constant $c_1 > 0$, independent of ϵ, y or x, such that

$$\int_{\Sigma_2^c - z} K_\epsilon(z)dz \geq c_1.$$

On the other hand, since $\mathrm{dist}(x, \Sigma_3) \geq \bar{\delta}/2$ we have $K_\epsilon(z)\mathbf{1}_{\Sigma_3 - x} \leq c_2$ for some constant $c_2 > 0$, and by the boundedness of Ω, $\mathrm{Vol}(\Sigma_3 - x) \leq c_3|y|$ for some $c_3 > 0$. Using these facts on (3.41) and applying (3.31), we arrive to

$$\mathcal{I}_\epsilon[Z](x) \leq (c_4(\epsilon + |y|)^{\beta_0} - c_1 A) m_0(|y|).$$

Thus, taking A large in terms of c_1, c_4, we conclude that $-\mathcal{I}_\epsilon[Z] \geq m_f(|y|)$ in $\bar{\Sigma}_4$.

By the very definition of w, we have

$$-\mathcal{I}_\epsilon[w] = f(x+y) - f(x), \quad \text{for } x \in \bar{\Sigma}_4.$$

Then we have that $-\mathcal{I}_\epsilon[Z] \geq -\mathcal{I}_\epsilon[w]$ in $\bar{\Sigma}_4$ and by definition of W and the bounds of w in $\bar{\Sigma}_4^c$ stated above, we conclude that $w \leq W$ in $\bar{\Sigma}_4^c$. Using the comparison principle, we conclude $w \leq W$ in $\bar{\Sigma}_4$. A similar argument states the inequality $-W \leq w$ and the result follows. □

3.6 Further Results.

3.6.1 Fully Nonlinear Equations.

The result obtained in Theorem 3.2 can be readily extended to a certain class of fully nonlinear equations. For example, consider two sets of indices $\mathcal{A}, \mathcal{B}$ and a two parameter family of radial continuous functions $a_{\alpha\beta} : \mathbb{R}^N \to \mathbb{R}$ satisfying the *uniform ellipticity condition*

$$\lambda_1 \leq a_{\alpha\beta}(z) \leq \lambda_2, \quad \forall \alpha \in \mathcal{A}, \beta \in \mathcal{B}, z \in \mathbb{R}^N \tag{3.42}$$

for certain constants λ_1, λ_2 such that $0 < \lambda_1 < \lambda_2 < +\infty$. Let us denote

$$K_{\alpha\beta,\epsilon}(z) := \frac{a_{\alpha\beta}(z)}{\epsilon^{n+2\sigma} + |z|^{n+2\sigma}}$$

and with this, for a suitable function u and $x \in \mathbb{R}^N$, define the linear operators

$$L_{\alpha\beta,\epsilon}[u](x) := \int_{\mathbb{R}^N} \delta(u, x, z) K_{\alpha\beta,\epsilon}(z) dz$$

and the corresponding *Isaacs Operator*

$$I_\epsilon[u](x) = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} L_{\alpha\beta,\epsilon}[u](x).$$

Under these definitions, we may consider the corresponding nonlinear equation

$$\begin{cases} -I_\epsilon[u] = & f \quad \text{in } \Omega \\ u = & 0 \quad \text{in } \Omega^c. \end{cases} \tag{3.43}$$

Existence and uniqueness of a pointwise solution u_ϵ to (3.43), which is continuous in $\bar{\Omega}$ can be obtained in a very similar way as in the linear case, and Proposition 3.1 can be adapted to this nonlinear setting. This allows us to use the comparison principle stated in Proposition 3.3 as well.

The lack of linearity can be handled with the positive homogeneity of these operators and the so called *extremal operators*

$$\mathcal{M}_\epsilon^+[u](x) = \sup_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} L_{\alpha\beta,\epsilon}[u](x), \quad \mathcal{M}_\epsilon^-[u](x) = \inf_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} L_{\alpha\beta,\epsilon}[u](x),$$

since, for two functions u_1, u_2 and $x \in \mathbb{R}^N$, these operators satisfy the fundamental inequality

$$\mathcal{M}_\epsilon^-[u_1 - u_2](x) \leq \mathcal{I}_\epsilon[u_1](x) - \mathcal{I}_\epsilon[u_2](x) \leq \mathcal{M}_\epsilon^+[u_1 - u_2](x).$$

A priori estimates for the solution as it is stated in Proposition 3.5 can be found using the same barriers given in the proof of that proposition, as the following useful estimates hold: For each $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $D \subset \mathbb{R}^N$

$$\int_D hK_{\alpha\beta,\epsilon}(z)dz \leq \lambda_1 \int_D hK_\epsilon(z)dz,$$

for all $h : D \to \mathbb{R}$ bounded nonnegative function, and

$$\int_D hK_{\alpha\beta,\epsilon}(z)dz \leq \lambda_2 \int_D hK_\epsilon(z)dz$$

for all $h : D \to \mathbb{R}$ bounded nonpositive function.

Using these inequalities and (3.42), we can use the same barriers appearing in the proof of Theorem 3.1 (Theorem 3.2 included) and get similar result. Moreover, the same modulus of continuity for the linear case can be obtained in this nonlinear framework, up to a factor depending on λ_1 and λ_2.

3.6.2 Parabolic Equations.

Let $T > 0$, $f : \bar{\Omega} \times [0, T] \to \mathbb{R}$ be a continuous function. A result similar to Theorem 3.2 can be readily obtained for the parabolic nonlocal equation

$$\left\{ \begin{array}{rcll} u_t - \mathcal{I}_\epsilon[u] = & f & \text{in } \Omega \times [0, T), \\ u(x,t) = & 0 & \text{in } \Omega^c \times [0, T), \\ u(x, 0) = & 0 & \text{in } \bar{\Omega}. \end{array} \right. \tag{3.44}$$

Similar problem is adressed by the authors in [72] for the Cauchy problem in all $\mathbb{R}^N$. Inspired by techniques used by Ishii in [81], a modulus of continuity in time can be derived once a modulus of continuity in space is found. So, the key fact is the modulus in x and this can be obtained in the parabolic setting noting that Theorem 3.2 readily applies considering equations with the form

$$\lambda u - \mathcal{I}_\epsilon[u] = f \quad \text{in } \Omega$$

for $\lambda > 0$ and that each time $Z(x)$ is a suitable barrier for this problem, then the function $(x, t) \mapsto e^t Z(x)$ plays the role of a barrier for the evolution problem (3.44).

3.6.3 Convergence Issues.

The proof of Corollary 3.1 is standard in the viscosity sense, once the uniform convergence is stated. However, following the ideas of Cortázar, Elgueta and Rossi in [59], and also in [72], under stronger assumptions over the regularity of u in Corollary 3.1, we can find a rate of convergence.

Theorem 3.3. *Let f, u_ϵ and u as in Corollary 3.1 and assume $u \in C^{2\sigma+\gamma}(\bar{\Omega})$ for some $\gamma > 0$. Then,*

$$||u_\epsilon - u||_{L^\infty(\bar{\Omega})} \leq C\epsilon^{\gamma_0}$$

for some $0 < \gamma_0 \leq \min\{2\sigma, \gamma\}$ and with C depending only on n and σ.

Proof. For simplicity, we will see the case $2\sigma < 1$ and $2\sigma + \gamma < 1$. Defining $w = u_\epsilon - u$, for $x \in \Omega$ we have

$$\begin{aligned} -\mathcal{I}_\epsilon[w](x) &= \mathcal{I}_\epsilon[u](x) + (-\Delta)^\sigma[u](x) \\ &= -\epsilon^{n+2\sigma} \int_{\Omega - x} \frac{u(x+z) - u(x)}{|z|^{n+2\sigma}(\epsilon^{n+2\sigma} + |z|^{n+2\sigma})} dz \\ &\quad -\epsilon^{n+2\sigma} u(x) \int_{(\Omega - x)^c} \frac{dz}{|z|^{n+2\sigma}(\epsilon^{n+2\sigma} + |z|^{n+2\sigma})} \\ &= I_1 + I_2. \end{aligned}$$

By the regularity of u we have

$$|I_1| \leq C\|u\|_{C^{2\sigma+\gamma}(\bar{\Omega})}\epsilon^\gamma,$$

where C does not depend on ϵ. On the other hand, for I_2 we split the analysis. First, if $\epsilon \leq d(x)$, then

$$|I_2| \leq C\|u\|_{C^{2\sigma+\gamma}(\bar{\Omega})}\epsilon^{n+2\sigma} d(x)^{-(n+2\sigma)+\gamma},$$

where we have used that there is no loss of boundary condition for u, hence $u = 0$ on $\partial\Omega$ and then $|u(x)| \leq [u]_{C^{2\sigma+\gamma}(\bar{\Omega})} d(x)^{2\sigma+\gamma}$. Hence, we conclude

$$|I_2| \leq C\epsilon^\gamma.$$

Second, when $d(x) < \epsilon$ we have

$$|I_2| \leq Cd(x)^{2\sigma+\gamma}\epsilon^{-2\sigma}(d(x)/\epsilon)^{-2\sigma} \leq C\epsilon^\gamma.$$

Since we know that $|w| \leq C\epsilon^{\beta_0}$ on $\partial\Omega$, by Proposition 3.5, we can get the result proceeding exactly as in the proof of Proposition 3.4. □

3.6.4 An example of a scheme without boundary equicontinuity.

In this subsection we consider the reverse scheme, that is approximating zero order equations by fractional ones and we prove the absence of uniform modulus of continuity in $\bar{\Omega}$. For this, we recall some facts of Section §3.2. Let $f \in C(\bar{\Omega})$ with $f \geq \varrho_0 > 0$, $J : \mathbb{R}^N \to \mathbb{R}_+$ integrable and $\mathcal{I}_J$ as in (3.6). Consider the associated problem (3.7)-(3.2), that is

$$\begin{cases} -\mathcal{I}_J[u] &= f \quad \text{in } \Omega \\ \quad u &= 0. \quad \text{in } \Omega^c \end{cases} \tag{3.45}$$

As we saw in Remark 3.2, the unique solution $u \in C(\bar{\Omega})$ for this problem is such that $u > 0$ in $\partial\Omega$.

Consider J, f as above, with J such that $J \geq m$ in B_r, for some $r, m > 0$. For $\epsilon \in (0,1)$ and $\alpha > 1$ consider the family of kernels

$$J_\epsilon(z) = \min\{1, |z/\epsilon|^\alpha\}^{-1} J(z),$$

which are not integrable at the origin. If we define

$$\mathcal{J}_\epsilon[u](x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)] J_\epsilon(z) dz$$

and consider the problems

$$\begin{cases} -\mathcal{J}_\epsilon[u] & = f \qquad \text{in } \Omega \\ \qquad u & = 0, \qquad \text{in } \Omega^c \end{cases} \tag{3.46}$$

it is known that the unique viscosity solution u_ϵ of (3.46) agrees the prescribed value of the equation on the boundary, and then $u_\epsilon = 0$ on $\partial\Omega$ for all $\epsilon \in (0,1)$, see for example [21]. We have $\{u_\epsilon\}$ is uniformly bounded in $L^\infty(\bar{\Omega})$ and therefore, the application of half-relaxed limits together with viscosity stability results in [23], imply $u_\epsilon \to u$ locally uniform in Ω as $\epsilon \to 0$, where u is the unique solution to (3.45). Since u is strictly positive in $\partial\Omega$, the convergence of u_ϵ to u cannot be uniform in $\bar{\Omega}$, and therefore the family $\{u_\epsilon\}$ is not equicontinuous in this case.

This example resembles the behavior of the viscosity solutions u_ϵ of the equation

$$-\epsilon u'' + u' = 1 \quad \text{in } (0,1), \quad \text{with } u(0) = u(1) = 0,$$

which approximate the solution of the equation

$$u' = 1 \quad \text{in } (0,1), \quad \text{with } u(0) = u(1) = 0.$$

In this case, the family (u_ϵ) is not equicontinuous too, see [14].

Part II

Nonlocal Elliptic Problems with Dominant Gradient Terms

Chapter 4

Existence and Uniqueness for Nonlocal Equations with Dominant Gradient Terms in Bellman Form

This chapter is based in the article [113].

Contents

4.1 Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary, $\varphi : \Omega^c \to \mathbb{R}$ a bounded continuous function, $H : \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ a continuous function, $\lambda \in \mathbb{R}$ and $\alpha \in (0,1)$. The main interest of this paper is the study of the Dirichlet problem

$$\lambda u - \mathcal{I}[u] + H(x, Du) = 0 \quad \text{in } \Omega, \tag{4.1}$$

$$u = \varphi \quad \text{on } \Omega^c. \tag{4.2}$$

The term $\mathcal{I}$ represents an integro-differential operator defined as follows: Consider $\Lambda > 0$ and $K \in L^\infty(\mathbb{R}^n)$ a nonnegative function such that $||K||_{L^\infty(\mathbb{R}^n)} \leq \Lambda$. For $x \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ bounded and in $C^{\alpha+\epsilon}(B_r(x))$ for some $r, \epsilon > 0$, we define

$$\mathcal{I}[\phi](x) = \int_{\mathbb{R}^n} [\phi(x+z) - \phi(x)] K^\alpha(z) dz,$$

where $K^\alpha(z) := K(z)|z|^{-(n+\alpha)}$ for $z \neq 0$. A particular case is when K is equal to the well-known constant $C_{n,\alpha} = \pi^{-(\alpha+n/2)}\Gamma((n+\alpha)/2)/\Gamma(-\alpha/2)$ since, in that case, $-\mathcal{I} = (-\Delta)^{\alpha/2}$, the fractional Laplacian of order α (see [69]).

The Hamiltonian H has a *Bellman form* and is defined in the following way: Let $\mathcal{B}$ be a compact metric space, $b : \bar{\Omega} \times \mathcal{B} \to \mathbb{R}^n$, $f : \bar{\Omega} \times \mathcal{B} \to \mathbb{R}$ continuous and bounded. For $x \in \bar{\Omega}$ and $p \in \mathbb{R}^n$, we denote

$$H(x,p) = \sup_{\beta \in \mathcal{B}} \{-b_\beta(x) \cdot p - f_\beta(x)\} \tag{4.3}$$

where we adopt the notation $b_\beta(x) = b(x,\beta)$ for $(x,\beta) \in \bar{\Omega} \times \mathcal{B}$ and in the same way for f. It is known that Hamiltonians as (4.3) arise in the study of Hamilton-Jacobi equations associated to optimal exit time problems (see [11], [14], [74]).

We note that λ can be negative, but for the introductory purposes of this section, the reader may think it is nonnegative. The exact assumption over λ will be given precisely in the next section, condition (**M**).

An important motivation for this problem comes from the work of Barles, Chasseigne and Imbert [21], where the authors study the Dirichlet problem for a large variety of integro-differential equations. As a particular case, for problems like (4.1)-(4.2) but with $\alpha \geq 1$, it is proven the existence of a unique viscosity solution $u \in C(\bar{\Omega})$ satisfying $u = \varphi$ on Ω^c, and therefore the solution can be regarded as a continuous function of $\mathbb{R}^n$. However, when $\alpha < 1$, the weaker diffusive effect of $\mathcal{I}$ compared with the transport effect of the first-order term may create loss of the boundary condition (that is, $u(x) \neq \varphi(x)$ for some $x \in \partial\Omega$) under certain geometric disposition of the drift at the boundary (say, when $b_\beta(x)$ is pointing strictly inside Ω for some $\beta \in \mathcal{B}$).

This feature implies that any analysis of problem (4.1)-(4.2) has to consider the eventuality of losses of the boundary condition. We deal with this difficulty incorporating condition (4.2) into the equation through $\mathcal{I}$ and therefore the problem can be reformulated as well as: finding $u \in C(\bar{\Omega})$ satisfying, for all $x \in \Omega$, the following equality in the viscosity sense

$$\begin{aligned} \lambda u(x) - \int\limits_{x+z\in\bar{\Omega}} [u(x+z) - u(x)]K^\alpha(z)dz \\ - \int\limits_{x+z\notin\bar{\Omega}} [\varphi(x+z) - u(x)]K^\alpha(z)dz + H(x, Du(x)) = 0, \end{aligned} \tag{4.4}$$

together with an appropriate notion of boundary condition to be precised later on. Hence, the solution may be different to φ at some points of $\partial\Omega$ and therefore the solution (continuous on $\bar{\Omega}$), extended naturally as φ in $\mathbb{R}^n \setminus \bar{\Omega}$, may be discontinuous as a function in $\mathbb{R}^n$.

On the other hand, since losses of the boundary condition are closely related with the behavior of the drift terms at the boundary, it is important to understand in which way it affects the elliptic nature of the equation.

An illustrative model which is similar to ours is the exhaustively studied *degenerate second-order elliptic* case (see [89] and the monographs [100], [99] for a complete survey of the also called *equations with nonnegative characteristic form*). For a continuous function a with values in the set of nonnegative matrices, f and φ continuous real valued functions, $\lambda \geq 0$ a constant and b a continuous vector field, a linear model equation

$$\left\{ \begin{array}{rcll} -\mathrm{Tr}(a(x)D^2u) - b(x)\cdot Du + \lambda u &=& f(x) & \text{in } \Omega \\ u &=& \varphi & \text{on } \partial\Omega, \end{array} \right. \tag{4.5}$$

is said to be degenerate elliptic if the matrix $a(x)$ has null eigenvalues for some $x \in \bar{\Omega}$, being of special interest the case when the degeneracy points are on the boundary. It is known (see [75]) that if a solution u of (4.5) is such that $u(x_0) > \varphi(x_0)$ for some $x_0 \in \partial\Omega$, then necessarily

$$\begin{cases} a(x_0)Dd(x_0)\cdot Dd(x_0) = 0, \ \textit{and} \\ -\mathrm{Tr}(a(x_0)D^2d(x_0)) - b(x_0)\cdot Dd(x_0) \leq 0, \end{cases} \tag{4.6}$$

where $d : \mathbb{R}^n \to \mathbb{R}$ is the signed distance function to $\partial\Omega$ which is nonnegative on $\bar{\Omega}$. Since $Dd(x_0)$ agrees with the inward unit normal of $\partial\Omega$ at x_0, first condition establishes the degeneracy of the second-order operator, depicting the absence of diffusion in the normal direction at x_0. Second condition shows that certain geometric disposition of the drift term at the boundary is necessary to loss the boundary condition. Even under this difficulty, comparison principle for bounded sub and supersolutions can be obtained for degenerate elliptic second-order problems. For instance, in the context of viscosity solutions with *generalized boundary conditions*, Barles and Burdeau in [16] prove a comparison result for a variety of quasilinear problems associated to Bellman-type Hamiltonians as (4.3). Related results can be also found in [26], [31], [94].

Condition (4.6) can be understood using the approach of stochastic exit time problems. Assume σ is a continuous function such that $a = 1/2\sigma\sigma^t$, denote $(W_t)_t$ the standard Brownian motion, let $x \in \bar{\Omega}$ and consider the stochastic differential equation

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t, \quad X_0^x = x. \tag{4.7}$$

If we denote τ the exit time from Ω of $(X_t)_t$ and $\mathbb{E}_x$ the conditional expectation with respect to the event $\{X_0 = x\}$, it is known (see [75] for instance) the *value function*

$$\mathbb{U}(x) = \mathbb{E}_x\Big\{ \int_0^\tau f(X_s)e^{-\lambda s}ds + \varphi(X_\tau)e^{-\lambda\tau} \Big\} \tag{4.8}$$

is a solution to (4.5). Thus, for a point x_0 close to the boundary, the degeneracy of the diffusion driven by σ at x_0 allows b "to push" the trajectories inside the domain and then $\mathbb{U}(x_0)$ is not necessarily equal to $\varphi(x_0)$, explaining the relation among (4.6) and the loss of the boundary condition.

An analogous interpretation can be made in the fractional framework. When $-\mathcal{I} = (-\Delta)^{\alpha/2}$, it is known that (see [102]) the fractional Laplacian is the infinitesimal generator of an $\alpha-$stable Lévy process $(Z_t)_t$, and then, if $(\beta_t)_t$ is a stochastic process $(Z_t)_t-$adapted with values in $\mathcal{B}$ and $x \in \bar{\Omega}$, the SDE

$$dX_t^\beta = b(X_t^\beta, \beta_t)dt + dZ_t, \quad X_0^\beta = x, \tag{4.9}$$

allows us to give an stochastic interpretation to our integro-differential problem as in the second-order case, where (4.9) plays the role of (4.7). We recall that losses of boundary conditions are present in both approaches. However, we would like to emphasize that the reasons for these phenomenas are qualitatively different. In the second-order case, they only arise when the diffusion *is degenerate*, preventing the trajectories driven by (4.7) reach the boundary. On the contrary, operators like $(-\Delta)^{\alpha/2}$ are *uniformly elliptic* in the sense of Caffarelli and Silvestre [47]. Roughly speaking, the form of the kernel defining the fractional Laplacian allows the trajectories defined in (4.9) to make small jumps in all directions, in particular, they can jump outside Ω when they start close to the boundary. This last feature intuitively should insure the boundary condition, but the transport effect of the drift term is strong enough to produce loss of boundary conditions, mainly due to the assumption $\alpha < 1$.

The above discussion gives rise to the main purpose of this paper, which is to establish a viscosity comparison principle for bounded sub and supersolutions of a class of integro-differential problems which seem to be uniformly elliptic, but where the functions can be discontinuous at the boundary.

We finish this introduction with a useful interpretation of our problem. For $x \in \bar{\Omega}$ and u a suitable function, define the *censored nonlocal operator* as

$$\mathcal{I}_\Omega[u](x) := \int_{x+z\in\bar{\Omega}} [u(x+z) - u(x)]K^\alpha(z)dz. \tag{4.10}$$

For $x \in \Omega$, we also define

$$\bar{\lambda}(x) = \lambda + \int_{x+z\notin\bar{\Omega}} K^\alpha(z)dz, \quad \text{and} \quad \bar{\varphi}(x) = \int_{x+z\notin\bar{\Omega}} \varphi(x+z)K^\alpha(z)dz,$$

With this, we remark that problem (4.1)-(4.2) can be seen as an elliptic problem of the type

$$\bar{\lambda}(x)u(x) - \mathcal{I}_\Omega[u](x) + H(x, Du(x)) = \bar{\varphi}(x), \quad x \in \Omega, \tag{4.11}$$

with generalized boundary condition $u = \varphi$ on $\partial\Omega$.

Note that the censored operator $\mathcal{I}_\Omega$ "localizes" inside Ω the equation, easing the analysis of the problem because, as we already explained in (4.4), an equation like (4.11) concerns unknown functions defined in $\bar{\Omega}$, incorporating the exterior condition into the equation through the function $\bar{\varphi}$. However, the function $\bar{\varphi}$ is potentially unbounded at the boundary and this creates a difficulty because we have to deal with unbounded data.

The idea is to use a censored equation like (4.11), taking advantage of the operator $\mathcal{I}_\Omega$ but taking care about the unbounded term $\bar{\varphi}$.

Organization of the Paper. In section 4.2 we introduce the principal assumptions and provide the main results. In section 4.3 we state the basic notation, give the precise notion of solution of (4.1)-(4.2) and introduce the censored problem. In section 4.4 we provide several technical results, including the key relation among our original problem and the censored one. These results allow us to undestand the behavior of the sub and supersolutions at the boundary in section 4.5, and an important improvement on its semicontinuity in section 4.6. This last property will be the key fact to prove Theorems 4.1 and 4.2 in section 4.7. Finally, some extensions are discussed in section 4.8.

4.2 Assumptions and Main Results.

Concerning the assumptions, we present them and then we give a brief explanation of each one.

Uniform ellipticity: Recalling the definition of $\mathcal{I}$ in (4.3), we assume

$$\exists\ c_1, c_2 > 0 \quad \text{such that} \quad c_1 \leq K(z) \quad \text{for any } |z| \leq c_2. \tag{\textbf{E}}$$

Nondegeneracy: We assume

$$\mu_0 \ := \ \lambda + \inf_{x\in\Omega} \int_{x+z\notin\Omega} K^\alpha(z)dz \ > \ 0. \tag{M}$$

Uniform Lipschitz continuity: Recalling H in (4.3), we assume

$$(\exists\ L>0)\ (\forall\ \beta\in\mathcal{B})\ (\forall\ x,y\in\bar{\Omega})\ :\ |b_\beta(x)-b_\beta(y)|\leq L|x-y|. \tag{L}$$

Condition (**L**) is classical for Hamilton-Jacobi equations. Condition (**E**) implies the operator $\mathcal{I}$ is *uniformly elliptic* in the sense of nonlocal operators (see [47]). Hence, $\mathcal{I}$ represents the infinitesimal generator of a jump Lévy process whose a.a. paths jump infinitely many times in each interval of time. Condition (**M**) relates the "discount rate" λ and the jumps outside Ω performed by the process associated to $\mathcal{I}$. Note that λ can be nonpositive and in that case these processes can jump outside Ω from any point with positive probability.

Next we introduce the remaining assumption over H. Because of the weak diffusion setting of our problem, we take care about the behavior of the drift terms at the boundary dividing it into representative sets. First, we consider

$$\Gamma_{in} = \{x\in\partial\Omega\ :\ \forall\beta\in\mathcal{B},\ b_\beta(x)\cdot Dd(x)>0\},$$

which can be understood as the sets of points where all the drift terms push inside Ω the trajectories defined in (4.9). For this, it is reasonable to expect a loss of the boundary condition there (see [21] for an explicit example of this fact). However, as we will see later on, it is possible to get an important improvement on the semicontinuity of sub and supersolutions on Γ_{in}, a key fact to get the comparison up to the boundary.

We also consider the set

$$\Gamma_{out} = \{x\in\partial\Omega\ :\ \forall\beta\in\mathcal{B},\ b_\beta(x)\cdot Dd(x)\leq 0\},$$

where it is reasonable to think that there is no loss of the boundary condition.

We also have in mind an intermediate situation, considering the set

$$\Gamma := \partial\Omega\setminus(\Gamma_{in}\cup\Gamma_{out}).$$

In this set, loss of the boundary condition cannot be discarded. However, by the very definition of Γ, for each $x\in\Gamma$ there exists $\bar{\beta},\underline{\beta}\in\mathcal{B}$ such that

$$b_{\underline{\beta}}(x)\cdot Dd(x)\leq 0<b_{\bar{\beta}}(x)\cdot Dd(x). \tag{4.12}$$

Dropping the supremum in the definition of H by taking $\bar{\beta}$ or $\underline{\beta}$, a subsolution of (4.1)-(4.2) is also a subsolution for the linear equation associated to these controls and therefore it enjoys the good properties of both Γ_{out} and Γ_{in} at points on Γ.

Concerning these subsets of the boundary, we assume

$$\Gamma_{in},\Gamma_{out} \text{ and } \Gamma \text{ are connected components of } \partial\Omega. \tag{H}$$

By the smoothness of $\partial\Omega$, each of these subsets is uniformly away the others, avoiding two completely different drift's behavior for arbitrarily close points.

Since the uncertainty of the value of the sub and supersolutions on the boundary is one of the main difficulties in the study of problem (4.1)-(4.2), we introduce the following definition: For u upper semicontinuous in $\bar{\Omega}$, v lower semicontinuous in $\bar{\Omega}$ (which will be thought as sub and supersolution, respectively) denote

$$\begin{aligned} \tilde{u}(x) &= \begin{cases} \limsup_{y\in\Omega,\ y\to x} u(y) & \text{if } x \in \Gamma \cup \Gamma_{in} \\ u(x) & \text{if } x \in \Omega \cup \Gamma_{out}. \end{cases} \\ \tilde{v}(x) &= \begin{cases} \liminf_{y\in\Omega,\ y\to x} v(y) & \text{if } x \in \Gamma_{in} \\ v(x) & \text{if } x \in \Omega \cup \Gamma \cup \Gamma_{out}. \end{cases} \end{aligned} \tag{4.13}$$

Clearly $\tilde{u} \le u$, $v \le \tilde{v}$, $\tilde{u}$ is upper semicontinuous in $\bar{\Omega}$ and $\tilde{v}$ is lower semicontinuous in $\bar{\Omega}$.

From now, we write usc for upper semicontinuous and lsc for lower semicontinuous functions. The main result of the article is the following

Theorem 4.1. *Assume* (**E**), (**M**), (**H**), (**L**) *hold. Let u be a bounded usc viscosity subsolution of* (4.1)-(4.2) *and v a bounded lsc viscosity supersolution of* (4.1)-(4.2). *Then*

$$u \le v \quad \text{in } \Omega.$$

Moreover, if we define $\tilde{u}, \tilde{v}$ as in (4.13), *then $\tilde{u} \le \tilde{v}$ in $\bar{\Omega}$.*

By the presence of loss of the boundary condition, this comparison principle is established by the use of the notion of generalized boundary conditions for viscosity sub and supersolution given by H. Ishii in [83] (see also [66], §7). Once the comparison holds, the configuration of problem (4.1)-(4.2) makes posible the use of Perron's method for integro-differential equations (see [3], [23], [104] and [66], [82] for an introduction on the method) to get as a corollary the following

Theorem 4.2. *Assume* (**E**), (**M**), (**H**), (**L**) *hold. Then, there exists a unique viscosity solution to problem* (4.1)-(4.2) *in $C(\bar{\Omega})$.*

4.3 Notation and Notion of Solution.

4.3.1 Basic Notation.

For $\delta > 0$ and $x \in \mathbb{R}^n$ we write $B_\delta(x)$ as the open ball of radius δ centered at x and B_δ if $x = 0$. For an arbitrary set A, we denote $d_A(x) = dist(x, \partial A)$ the signed distance function to ∂A which is nonnegative for $x \in A$ and nonpositive for $x \notin A$. For Ω we simply write $d(x) = d_{\partial\Omega}(x)$ and define the set Ω_δ as the open set of all $x \in \Omega$ such that $d(x) < \delta$. By the smoothness assumption over the domain, there exists a fixed number $\delta_0 > 0$, depending only on Ω, such that d is smooth in the set of points x such that $|d(x)| < \delta_0$. We write for $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ the sets

$$\Omega - x = \{z : x + z \in \Omega\} \quad \text{and} \quad \lambda\Omega = \{\lambda z : z \in \Omega\}.$$

Finally, for a function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $x, z \in \mathbb{R}^n$, we denote

$$\rho(\phi, x, z) = \phi(x + z) - \phi(x).$$

4.3.2 Notion of Solution.

As we mentioned in the introduction, we will understand the solutions as functions defined in $\bar{\Omega}$, fixing their value as φ outside Ω. However, we need to define properly the value on the boundary in order to fit in the definition of viscosity solution in the literature.

For an usc (resp. lsc) function $u : \bar{\Omega} \to \mathbb{R}$, we define its upper (resp. lower) $\varphi-$extension as

$$u^{\varphi}(x) \text{ (resp. } u_{\varphi}(x)) = \begin{cases} u(x) & \text{if } x \in \Omega \\ \varphi(x) & \text{if } x \in \bar{\Omega}^c \\ \max \text{ (resp. } \min)\{u(x), \varphi(x)\} & \text{if } x \in \partial\Omega, \end{cases} \tag{4.14}$$

Note that if u is usc, then u^{φ} is the usc envelope in $\mathbb{R}^n$ of the function $u\mathbf{1}_{\bar{\Omega}} + \varphi\mathbf{1}_{\bar{\Omega}^c}$. Analogously, if u is lsc, u_{φ} is the lsc envelope in $\mathbb{R}^n$ of the same function.

In what follows we consider $x \in \bar{\Omega}$, $\delta > 0$ and for any $\phi \in C^1(\bar{B}_{\rho}(x))$ and for any bounded semicontinuous function w, we define the operators

$$\mathcal{I}_{\delta}[\phi](x) = \int\limits_{B_{\delta}} \rho(\phi, x, z) K^{\alpha}(z) dz$$

$$\mathcal{I}^{\delta}[w](x) = \int\limits_{B_{\delta}^c \cap (\Omega - x)} \rho(w, x, z) K^{\alpha}(z) dz + \int\limits_{B_{\delta}^c \cap (\Omega^c - x)} [\varphi(x+z) - w(x)] K^{\alpha}(z) dz.$$

and, for each $\beta \in \mathcal{B}$

$$E_{\beta,\delta}(w, \phi, x) = \lambda w(x) - \mathcal{I}_{\delta}[\phi](x) - \mathcal{I}^{\delta}[w](x) - b_{\beta}(x) \cdot D\phi(x) - f_{\beta}(x), \tag{4.15}$$

where "E" stands for "evaluation". The nonlinear evaluation reads as

$$E_{\delta}(w, \phi, x) = \sup_{\beta \in \mathcal{B}} E_{\beta,\delta}(w, \phi, x). \tag{4.16}$$

The precise notion of solution used here is given by the following

Definition 4. (Solution to Problem (4.1)-(4.2))

An usc function $u : \bar{\Omega} \to \mathbb{R}$ is a viscosity subsolution to (4.1)-(4.2) if for each $x_0 \in \bar{\Omega}$ and each smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that x_0 is a maximum point of $u^{\varphi} - \phi$ in $B_{\delta}(x_0)$, then

$$\begin{aligned} E_{\delta}(u^{\varphi}, \phi, x_0) \leq 0 \quad & \text{if } x_0 \in \Omega, \\ \min\{E_{\delta}(u^{\varphi}, \phi, x_0), u^{\varphi}(x_0) - \varphi(x_0)\} \leq 0 \quad & \text{if } x_0 \in \partial\Omega. \end{aligned}$$

A lsc function $v : \bar{\Omega} \to \mathbb{R}$ is a viscosity supersolution to (4.1)- (4.2) if for each $x_0 \in \bar{\Omega}$ and each smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that x_0 is a minimum point of $v_{\varphi} - \phi$ in $B_{\delta}(x_0)$, then

$$\begin{aligned} E_{\delta}(v_{\varphi}, \phi, x_0) \geq 0 \quad & \text{if } x_0 \in \Omega, \\ \max\{E_{\delta}(v_{\varphi}, \phi, x_0), v_{\varphi}(x_0) - \varphi(x_0)\} \geq 0 \quad & \text{if } x_0 \in \partial\Omega. \end{aligned}$$

Finally, a solution to (4.1)-(4.2) is a function in $C(\bar{\Omega})$ which is simultaneously sub and supersolution in the above sense.

The above definition is basically the one presented in [23], [21], [103] and [104]. Written in that way we highlight the goal of this paper, which is to state the existence and uniqueness of a solution of (4.1)-(4.2) in $C(\bar{\Omega})$. However, this definition is not the most confortable to deal with the discontinuities at the boundary of u_φ and u^φ. For this reason in some situations we will look for the equation without taking into account the exterior condition through the operator (4.10), which "censores" the jumps outside Ω. Hence, we need to introduce some notation in order to write properly the viscosity inequalities concerning this operator.

If ϕ is C^1 and bounded, and if w is a bounded and semicontinuous function, the viscosity evaluation for this operator are defined by

$$\begin{aligned} \mathcal{I}_{\Omega,\delta}[\phi](x) &= \int_{B_\delta \cap (\Omega - x)} \rho(\phi, x, z) K^\alpha(z) dz, \\ \mathcal{I}_\Omega^\delta[w](x) &= \int_{B_\delta^c \cap (\Omega - x)} \rho(w, x, z) K^\alpha(z) dz. \end{aligned} \tag{4.17}$$

The idea is to use this censored operator in auxiliary problems and therefore distinct to equation (4.1). To do so, we provide a notion of viscosity solution in a rather general setting. We consider a set $\mathcal{O} \subseteq \bar{\Omega}$, a relatively open subset of $\bar{\Omega}$ and $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ a continuous function.

Definition 5. (Solution for the Censored Equation)
An usc function $u : \bar{\Omega} \to \mathbb{R}$ is a viscosity subsolution to

$$-\mathcal{I}_\Omega[w] + F(x, u, Du) = 0 \quad \text{in } \mathcal{O} \tag{4.18}$$

if, for each $x_0 \in \mathcal{O}$ and any smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that x_0 is a maximum point of $u - \phi$ in $\mathcal{O} \cap B_\delta(x_0)$, then

$$-\mathcal{I}_\Omega^\delta[u](x_0) - \mathcal{I}_{\Omega,\delta}[\phi](x_0) + F(x_0, u(x_0), D\phi(x_0)) \leq 0.$$

A lsc function $v : \bar{\Omega} \to \mathbb{R}$ is a viscosity subsolution to (4.18) if for each $x_0 \in \mathcal{O}$ and each smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that x_0 is a minimum point of $v - \phi$ in $\mathcal{O} \cap B_\delta(x_0)$, then

$$-\mathcal{I}_\Omega^\delta[v](x_0) - \mathcal{I}_{\Omega,\delta}[\phi](x_0) + F(x_0, v(x_0), D\phi(x_0)) \geq 0.$$

A viscosity solution to (4.18) is a function $u : \bar{\Omega} \to \mathbb{R}$, continuous in $\bar{\Omega}$ which is simultaneously sub and supersolution.

In what follows, when we say "solution to (4.1)-(4.2)" we mean that it is in the sense of Definition 4. Otherwise, we mean it is a solution just to an equation and then we are refering to Definition 5. The same applies for sub and supersolutions.

The following is technical notation to be used in this paper. We denote for $x, p \in \mathbb{R}^n$

$$\mathcal{H}_i(x, p) = \inf_{\beta \in \mathcal{B}} \{-b_\beta(x) \cdot p\}; \quad \mathcal{H}_s(x, p) = \sup_{\beta \in \mathcal{B}} \{-b_\beta(x) \cdot p\}. \tag{4.19}$$

In order to depict the viscosity evaluation for equations associated to the above Hamiltonians and the censored operators $\mathcal{I}_\Omega$ defined in (4.17), we write for $x \in \bar{\Omega}$, $\beta \in \mathcal{B}$, ϕ smooth and w bounded

$$\begin{aligned} &\mathcal{E}_{\delta,\beta}(w, \phi, x) = -\mathcal{I}_{\Omega,\delta}[\phi](x) - \mathcal{I}_\Omega^\delta[w](x) - b_\beta(x) \cdot D\phi(x), \\ &\mathcal{E}_\delta^i(w, \phi, x) = \inf_{\beta \in \mathcal{B}} \{\mathcal{E}_{\delta,\beta}(x, w, \phi)\} \quad \text{and} \quad \mathcal{E}_\delta^s(w, \phi, x) = \sup_{\beta \in \mathcal{B}} \{\mathcal{E}_{\delta,\beta}(x, w, \phi)\}. \end{aligned} \tag{4.20}$$

4.4 Preliminary Technical Results.

We start with the following result, which asserts that subsolutions of the problem (4.1)-(4.2) in the sense of Definition 4 are also subsolutions for an associated censored problem. This fact will be a useful tool to see problem (4.1)-(4.2) in a subtle different perspective, "localizing" it in Ω.

Lemma 4.1. *Let u be a bounded viscosity subsolution of problem (4.1)-(4.2). Then, there exists a constant $\theta_0 > 0$ such that u is a viscosity subsolution to*

$$\lambda u - \mathcal{I}_\Omega[u] + H(x, Du) = \theta_0(||\varphi||_\infty + ||u||_\infty)d^{-\alpha} \quad \text{in } \Omega. \tag{4.21}$$

Analogously, if v is a bounded viscosity supersolution of (4.1)-(4.2), then it is a viscosity supersolution to (4.21) replacing θ_0 by $-\theta_0$. The constant θ_0 depends only on n, α and Λ.

Proof: Let $x \in \Omega$ and ϕ a smooth function such that x is a maximum point of $u - \phi$ in $\Omega \cap B_\delta(x)$. For $\delta' \le \min\{\delta, d(x)\}$ we use that u is a subsolution to (4.1) to write

$$\lambda u(x) - \mathcal{I}_{\delta'}[\phi](x) - \mathcal{I}^{\delta'}[u^\varphi](x) + H(x, D\phi(x)) \le 0.$$

Note that $\rho(u, x, z) \le \rho(\phi, x, z)$ in $B_\delta \cap (\Omega - x) \setminus B_{\delta'}$. Using this inequality we can write

$$\lambda u(x) - \mathcal{I}_{\Omega,\delta}[\phi](x) - \mathcal{I}_\Omega^\delta[u](x) + H(x, D\phi(x)) \le \int_{(\Omega - x)^c} [\varphi(x+z) - u(x)]K^\alpha(z)dz.$$

Since the set $\Omega - x$ is at least at distance $d(x)$ from the origin and using the boundedness of u and φ we conclude from the above inequality that

$$\lambda u(x) - \mathcal{I}_{\Omega,\delta}[\phi](x) - \mathcal{I}_\Omega^\delta[u](x) + H(x, D\phi(x)) \le (||\varphi||_\infty + ||u||_\infty) \int_{B^c_{d(x)}} K^\alpha(z)dz,$$

where we have used notations (4.17). Using (**E**) we have

$$\int_{B^c_{d(x)}} K^\alpha(z)dz \le \Lambda|\partial B_1|d(x)^{-\alpha} \int_1^{+\infty} r^{-(1+\alpha)}dr = \Lambda|\partial B_1|\alpha^{-1}d(x)^{-\alpha},$$

where $|\partial B_1|$ is the Lebesgue measure of the unit sphere in $\mathbb{R}^n$. The result follows taking $\theta_0 = \alpha^{-1}\Lambda|\partial B_1|$. $\square$

The following technical results introduce particular functions to handle the difficulties arising in the analysis on Γ and Γ_{in} (cf. Propositions 4.2 and 4.3). As we will see, the assumption $\alpha < 1$ plays a crucial role to get them. The next lemma is intended to deal with the unpleasant term in the right hand side of (4.21).

Lemma 4.2. *Let $x_0 \in \partial\Omega$ and b a Lipschitz continuous vector field. Assume there exists $c_0 > 0$ such that b satisfies*

$$b(x_0) \cdot Dd(x_0) \ge c_0. \tag{4.22}$$

Then, for all $0 < \sigma$, there exists $0 < \bar{r} < \delta_0$[1] and $0 < \tilde{c}_0 < c_0$ such that the function d^σ is a classical subsolution to the equation

$$-\mathcal{I}_\Omega[w] - b \cdot Dw = -\tilde{c}_0 \; d^{\sigma-1} \quad \text{in } B_{\bar{r}}(x_0) \cap \Omega.$$

[1] Recall that $\delta_0 > 0$ is such that d is smooth in $\{x \in \mathbb{R}^n : |d(x)| < \delta_0\}$.

Proof: First, we take care about the nonlocal term. We define

$$\Theta_x = (\Omega - x) \cap \{z : d(x+z) < d(x)\}. \tag{4.23}$$

and note that $\rho(d^\sigma, x, z) \geq 0$ for $z \in (\Omega - x) \setminus \Theta_x$ by the monotony of the function $t \mapsto t^\sigma$. Using this and condition (**E**), we get

$$-\mathcal{I}_\Omega[d^\sigma](x) \leq \Lambda \int_{\Theta_x} -\rho(d^\sigma, x, z)|z|^{-(n+\alpha)} dz.$$

The idea is to estimate the last integral splitting the domain Θ_x. Simply dropping the nonpositive term $-d^\sigma(x+z)$, there exists $C > 0$ independent of x such that

$$\begin{aligned} -\int_{\Theta_x \setminus B_{d(x)/2}} \rho(d^\sigma, x, z)|z|^{-(n+\alpha)} dz &\leq d(x)^\sigma \int_{B^c_{d(x)/2}} |z|^{-(n+\alpha)} dz \\ &= 2^\alpha \alpha^{-1} |\partial B_1| d(x)^{\sigma - \alpha} \end{aligned}$$

meanwhile, using the differentiability of the distance function we have

$$|\rho(d^\sigma, x, z)| \leq 2^{1-\sigma} \sigma d^{\sigma-1}(x)|z|,$$

for all $z \in B_{d(x)/2}$. Thus, we conclude

$$\begin{aligned} -\int_{B_{d(x)/2}} \rho(d^\sigma, x, z)|z|^{-(n+\alpha)} dz &\leq 2^{1-\sigma} |\partial B_1| \sigma d(x)^{\sigma-1} \int_0^{d(x)/2} r^{-\alpha} dr \\ &= 2^{\alpha-\sigma}(1-\alpha)^{-1} \sigma |\partial B_1| d(x)^{\sigma-\alpha}. \end{aligned}$$

These last two estimates imply

$$-\mathcal{I}_\Omega[d^\sigma](x) \leq C d(x)^{\sigma-\alpha}, \tag{4.24}$$

for a constant depending only on the data. On the other hand, by (4.22) and the continuity of $x \mapsto b(x) \cdot Dd(x)$, we can assume $b(x) \cdot Dd(x) \geq 3c_0/4$, for all $x \in B_{\bar{r}}(x_0) \cap \Omega$. Evaluating classically and using estimate (4.24), we obtain

$$-\mathcal{I}_\Omega[d^\sigma](x) - b(x) \cdot Dd^\sigma(x) \leq d(x)^{\sigma-1}(-3\sigma c_0/4 + C d^{1-\alpha}(x)).$$

Since $\alpha < 1$, choosing $\bar{r}$ small depending on the data we get the result. □

For each $r > 0$ small enough, define the set

$$\Sigma_r = \{x \in \Omega : 0 < dist(x, \Gamma_{in}) < r\}. \tag{4.25}$$

Using the compactness of $\mathcal{B}$ it is posible to get the following nonlinear version of the last lemma, valid in Γ_{in}.

Corollary 4.1. There exists $\tilde{c}_0, \bar{r} > 0$ such that for all $\sigma > 0$, d^σ is a classical subsolution to the equation

$$-\mathcal{I}_\Omega[w] + \mathcal{H}_s(x, Dw) = -\tilde{c}_0 \ d^{\sigma-1} \quad \text{in } \Sigma_{\bar{r}},$$

where $\mathcal{H}_s$ is defined in (4.19).

Proof: We remark that, by the compactness of $\mathcal{B}$, the continuity of b and the smoothness of $\partial\Omega$, there exists a constant $c_0 > 0$ such that $b_\beta(x) \cdot Dd(x) \geq c_0$ for all $\beta \in \mathcal{B}$ and $x \in \Gamma_{in}$. Hence, we can obtain the result of the previous proposition with the same constants $\bar{r}$ and $\tilde{c}_0$ for each point of Γ_{in} and each control β. Taking supremum over $\mathcal{B}$ we conclude the result. $\square$

For the next results we introduce the function $\zeta : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\zeta(x) = \begin{cases} \log(d(x)) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c, \end{cases} \tag{4.26}$$

which is bounded above, usc in $\mathbb{R}^n$, continuous in Ω and smooth in Ω_{δ_0}. Concerning this function, we need the following estimate of its evaluation on the nonlocal operator.

Lemma 4.3. *There exists constants $0 < \delta^* < \delta_0$ and $C > 0$ such that*

$$-\mathcal{I}_\Omega[\zeta] \leq C d^{-\alpha} \quad \text{in } \Omega_{\delta^*}.$$

The constants C, δ^* depend only on the data and the smoothness of $\partial\Omega$.

The proof of this estimate is very similar to the derivation of (4.24), but we have to deal with the unboundedness of ζ so we postpone its proof to the Appendix. Since $\zeta(x) \to -\infty$ as $x \to \partial\Omega$, the idea is to use ζ as a penalization for the testings of subsolutions in order to push the maximum test points inside the domain. We can use properly this penalization argument proving ζ "behaves" like a subsolution for the problem when the drift term points strictly inside Ω.

Lemma 4.4. *Let $x_0 \in \partial\Omega$ and b a Lipschitz continuous vector field. Assume there exists $c_0 > 0$ such that b satisfies (4.22). Then, there exists $0 < \bar{r} < \delta_0$ and $0 < \tilde{c}_0 < c_0$, such that ζ is a classical subsolution to the equation*

$$-\mathcal{I}_\Omega[w] - b \cdot Dw = -\tilde{c}_0 \, d^{-1} \text{ in } B_{\bar{r}}(x_0) \cap \Omega. \tag{4.27}$$

The proof of this result follows the same lines of Lemma 4.2 using Lemma 4.3. Again by the compactness of $\mathcal{B}$, we have the following

Corollary 4.2. There exists $c_0, \bar{r} > 0$ such that ζ is a classical subsolution to the equation

$$-\mathcal{I}_\Omega[w] + \mathcal{H}_s(x, Dw) = -c_0 \, d^{-1} \quad \text{in } \Sigma_{\bar{r}},$$

where $\mathcal{H}_s$ is defined in (4.19) and Σ_r is defined in (4.25).

Remark 4.1. We can obtain the same results of Lemmas 4.2, 4.3, 4.4 and its corollaries replacing $\mathcal{I}_\Omega$ by $\mathcal{I}_{\Omega'}$ for any $\Omega' \subseteq \Omega$. Moreover, the constant arising in each property does not change.

4.5 Behavior of Sub and Supersolutions on the Boundary.

4.5.1 Classical Boundary Condition.

Here we establish sufficient conditions to get the boundary condition in the classical sense. We start with the following result whose proof follows closely the arguments of [21].

Proposition 4.1. *Assume* (**E**), (**L**) *hold. Let* $x_0 \in \partial\Omega$ *and* u, v *be bounded sub and supersolutions of problem* (4.1)-(4.2)*, respectively. For* $x \in \partial\Omega$*, define*

$$\mathcal{B}_{out}(x) = \{\beta \in \mathcal{B} \ : \ b_\beta(x) \cdot Dd(x) \le 0\}.$$

(i) If there exists $r > 0$ *such that* $\mathcal{B}_{out}(x) \ne \emptyset$ *for all* $x \in B_r(x_0) \cap \partial\Omega$*, then*

$$u(x_0) \le \varphi(x_0).$$

(ii) If there exists $r > 0$ *such that* $\mathcal{B}_{out}(x) = \mathcal{B}$ *for all* $x \in B_r(x_0) \cap \partial\Omega$*, then*

$$\varphi(x_0) \le v(x_0).$$

Note that, under the additional hypothesis (**H**), we have that each point $x \in \Gamma_{out}$ satisfies conditions (i) and (ii) in the above proposition, meanwhile each point $x \in \Gamma$ satisfies (i). Thus, the inmediate consequence of Proposition 4.1 is the following

Corollary 4.3. Assume (**E**), (**H**) and (**L**) hold. Let u, v be bounded viscosity sub and supersolution to (4.1)-(4.2), respectively, and $\tilde{u}, \tilde{v}$ as in (4.13). Then

$$\begin{aligned} \tilde{u} \le u \le \varphi \le v \le \tilde{v}, & \quad \text{in } \Gamma_{out}, \\ \tilde{u} \le u \le \varphi, \qquad & \quad \text{in } \Gamma. \end{aligned}$$

Proof of Proposition 4.1: For (i), we assume by contradiction that $u(x_0) - \varphi(x_0) = \nu$ for some $\nu > 0$, implying that $u^\varphi(x_0) = u(x_0)$. We consider C^1 functions $\chi, \psi : \mathbb{R} \to \mathbb{R}$ such that χ is even, bounded, $\chi(0) = 0$, $\chi(t) > 0$ for $t \ne 0$, $\liminf_{|t|\to\infty} \chi(t) > 0$ and such that $\chi(t) = |t|^\sigma$ with $\sigma > 1$ in a neighborhood of 0. For ψ we assume it is bounded, strictly increasing, $||\psi||_\infty \le \frac{1}{4}\nu$ and $\psi(t) = t$ in a neighborhood of 0.

Consider a parameter $\eta > 0$ to be sent to zero and $\epsilon = \epsilon(\eta)$ such that $\epsilon(\eta) \to 0$ as $\eta \to 0$ in a rate to be precised later on. We introduce the following penalization function

$$\Psi(y) := \epsilon^{-1}\chi(|y - x_0|) + \psi(d(y)/\eta).$$

By definition of u^φ, the function $x \mapsto u^\varphi(x) - \Psi(x)$ is upper semicontinuous in $\mathbb{R}^n$. Using that $\liminf_{|t|\to\infty} \chi(t) > 0$, taking ϵ suitably small we conclude $u^\varphi - \Psi$ attains its global maximum at a point $\bar{x} \in \mathbb{R}^n$. Using that

$$u^\varphi(\bar{x}) - \Psi(\bar{x}) \ge u^\varphi(x_0) - \Psi(x_0) = u^\varphi(x_0), \tag{4.28}$$

we conclude the term $\Psi(\bar{x})$ is bounded for all η and then $\bar{x} \to x_0$ as $\eta \to 0$. Now, since $\nu + \varphi(x_0) = u^\varphi(x_0)$, we conclude $\nu + \varphi(x_0) \le u^\varphi(\bar{x}) - 1/4\nu$, meaning that $\bar{x} \in \bar{\Omega}$ for each η small. Moreover, using again (4.28) and the upper semicontinuity of u, we have

$$d(\bar{x}) = o_1(\eta)\eta, \ |\bar{x} - x_0| = o_\eta(1), \text{ and } u^\varphi(\bar{x}) \to u(x_0), \tag{4.29}$$

as $\eta \to 0$. Since we can use the penalization as a test function for u at $\bar{x}$, for each $\beta \in \mathcal{B}$ and $\delta > 0$ we have

$$\lambda u^\varphi(\bar{x}) \le \mathcal{I}_\delta[\Psi](\bar{x}) + \mathcal{I}^\delta[u](\bar{x}) + b_\beta(\bar{x}) \cdot D\Psi(\bar{x}) + f_\beta(\bar{x}). \tag{4.30}$$

We need to estimate properly the nonlocal evaluations of the testing. For this, consider $r > 0$ small, independent of η. Considering $d(\bar{x}) < \delta \leq \mu < r$ we define the sets

$$\begin{aligned} \mathcal{A}_\delta^{ext} &= \{z \in B_r \ : \ d(\bar{x}+z) \leq d(\bar{x}) - \delta\}. \\ \mathcal{A}_{\delta,\mu} &= \{z \in B_r \ : \ d(\bar{x}) - \delta < d(\bar{x}+z) < d(\bar{x}) + \mu\}. \\ \mathcal{A}_\mu^{int} &= \{z \in B_r \ : \ \mu + d(\bar{x}) \leq d(\bar{x}+z)\}. \end{aligned}$$

We remark that $B_\delta \subset \mathcal{A}_{\delta,\mu}$ and that the constant $C > 0$ arising in each of the following estimates does not depend on r, μ, δ, η or ϵ. By (**E**), we clearly have

$$\int_{B_r^c} \rho(u^\varphi, \bar{x}, z) K^\alpha(z) dz \leq C||u^\varphi||_\infty r^{-\alpha},$$

$$\int_{\mathcal{A}_\mu^{int}} \rho(u^\varphi, \bar{x}, z) K^\alpha(z) dz \leq C||u||_\infty \mu^{-\alpha}.$$

By the continuity of φ and the last fact in (4.29), for all r and η small we have $\varphi(\bar{x}+z) \leq \varphi(x_0) + \nu/4$ for all $z \in \mathcal{A}_\delta^{ext}$ and $u(x_0) - \nu/4 \leq u^\varphi(\bar{x})$. Then, since $u(x_0) = \varphi(x_0) + \nu$ and using (**E**) we conclude

$$\int_{\mathcal{A}_\delta^{ext}} \rho(u^\varphi, \bar{x}, z) K^\alpha(z) dz \leq -\nu/2 \int_{\mathcal{A}_\delta^{ext}} K^\alpha(z) dz \leq -C\nu\delta^{-\alpha},$$

where C depends on the data and r, but not on η. Finally, using that $\bar{x}$ is a global maximum point of $u - \Psi$, we have $\rho(u^\varphi, \bar{x}, z) \leq \rho(\Psi, \bar{x}, z)$ and then we can write

$$\mathcal{I}_\delta[\Psi](\bar{x}) + \int_{\mathcal{A}_{\delta,\mu} \setminus B_\delta} \rho(u^\varphi, \bar{x}, z) K^\alpha(z) dz \leq \int_{\mathcal{A}_{\delta,\mu}} \rho(\Psi, \bar{x}, z) K^\alpha(z) dz,$$

but as it can be seen in the Appendix, by definition of Ψ we can get the inequality

$$\int_{\mathcal{A}_{\delta,\mu}} \rho(\Psi, \bar{x}, z) K^\alpha(z) dz \leq C(\eta^{-1} + \epsilon^{-1})\mu^{1-\alpha}. \tag{4.31}$$

Hence, joining the above estimates, we conclude that

$$\mathcal{I}_\delta[\Psi](\bar{x}) + \mathcal{I}^\delta[u^\varphi](\bar{x}) \leq C\Big((\eta^{-1} + \epsilon^{-1})\mu^{1-\alpha} + \mu^{-\alpha} - \delta^\alpha\Big).$$

Setting $\mu = \eta$, $\epsilon \geq \eta$ and $d(\bar{x}) < \delta \leq o_\eta(1)\eta$, we conclude from the last estimates that the nonlocal terms in (4.30) can be computed as

$$\mathcal{I}_\delta[\Psi](\bar{x}) + \mathcal{I}^\delta[u^\varphi](\bar{x}) \leq -(o_\eta(1)\eta)^{-\alpha},$$

so, replacing this in (4.30) and specifying the choice $\epsilon = \eta^\alpha$, we conclude

$$\lambda u(\bar{x}) - f_\beta(\bar{x}) \leq \eta^{-1} b_\beta(\bar{x}) \cdot Dd(\bar{x}) + C\eta^{-\alpha}|\bar{x} - x_0|^{\sigma-1} - (o_\eta(1)\eta)^{-\alpha}, \tag{4.32}$$

for all $\beta \in \mathcal{B}$. Now, denoting $\hat{\bar{x}} \in \partial\Omega$ such that $d(\bar{x}) = |\bar{x} - \hat{\bar{x}}|$, for all η small there exists $\beta_\eta \in \mathcal{B}_{out}(\hat{\bar{x}})$, by the assumption in (i). Thus, using (**L**) and the smoothness of the domain, there exists $C > 0$ independent of η such that

$$b_{\beta_\eta}(\bar{x}) \cdot Dd(\bar{x}) \leq b_{\beta_\eta}(\bar{x}) \cdot Dd(\bar{x}) - b_{\beta_\eta}(\hat{\bar{x}}) \cdot Dd(\hat{\bar{x}}) \leq C|\bar{x} - \hat{\bar{x}}| = Cd(\bar{x}),$$

and by (4.29) we conclude

$$b_{\beta_\eta}(\bar{x}) \cdot Dd(\bar{x}) \leq o_\eta(1)\eta.$$

Taking $\beta = \beta_\eta$ in (4.32), using the last inequality and (4.29) we conclude

$$\lambda u(\bar{x}) - ||f||_\infty \leq o_\eta(1) + o_\eta(1)\eta^{-\alpha} - (o_\eta(1)\eta)^{-\alpha}. \tag{4.33}$$

Letting $\eta \to 0$ we arrive to a contradiction because of the last term in the above inequality.

For (ii), we proceed in the same way using a similar contradiction argument, reversing the signs of the penalizations in order to get a minimum test point for v. The role of β_η above can be played by any $\beta \in \mathcal{B}$ by the assumption in (ii) and then the estimates are independent of β using (**L**). Hence, we arrive to an expression very similar to (4.33), with the reverse inequality, v in place of u and a plus sign in the last term in the right hand side. The contradiction is again obtained by letting $\eta \to 0$. □

Remark 4.2. The lack of information concerning v on Γ has to do with the form of H: we simply cannot drop the supremum taking $\underline{\beta}$ given in (4.12) and ensure v is still a supersolution. In fact, both cases $v \geq \varphi$ or $v < \varphi$ can happen depending on f and φ.

4.5.2 Viscosity Inequality Up to the Boundary.

Now we consider auxiliary functions associated to sub and supersolutions of problem (4.1)-(4.2) which satisfy the viscosity inequality up to the boundary for auxiliary censored problems.

Proposition 4.2. *Let $x_0 \in \partial\Omega$ and $\beta_0 \in \mathcal{B}$ such that $b = b_{\beta_0}$ satisfies condition (4.22) for some $c_0 > 0$. Let u be a bounded viscosity subsolution to (4.1)-(4.2) and $\tilde{u}$ as in (4.13). Then, for all $a > 0$ small enough, there exists $A > 0$ such that the function $U : \bar{\Omega} \to \mathbb{R}$ defined as*

$$U(x) = \tilde{u}(x) + Ad^{1-\alpha}(x) \tag{4.34}$$

is a viscosity subsolution to the equation

$$-\mathcal{I}_\Omega[w] - b \cdot Dw = 0 \quad \text{in } B_a(x_0) \cap \bar{\Omega}.$$

The constant $A > 0$ depends only on the data, $a, c_0, ||u||_\infty$ and θ_0 given in Lemma 4.1.

Proof: Since (4.22) holds, we can first choose $a < \bar{r}$ of Lemma 4.2, which depends only on c_0, but not on x_0. In what follows we split the proof depending if the test point $\bar{x}$ is in Ω or $\partial\Omega$.

Let $\bar{x} \in B_a(x_0) \cap \Omega$, $\delta > 0$ and ϕ a smooth function such that $\bar{x}$ is strict maximum point of $U - \phi$ in $B_\delta(x_0) \cap \bar{\Omega}$. Hence, we can see $\bar{x}$ as a test point for u with test function $\phi - Ad^{1-\alpha}$

by the smoothness of d near the boundary. Applying Lemma 4.1 and the definition of U we get

$$\begin{aligned} & -\mathcal{I}_{\Omega,\delta}[\phi](\bar{x}) - \mathcal{I}^{\delta}_{\Omega}[U](\bar{x}) - b(\bar{x})\cdot D\phi(\bar{x}) \\ \leq\ & \lambda||u||_\infty + f_{\beta_0}(\bar{x}) + \theta_0(||u||_\infty + ||\varphi||_\infty)d(\bar{x})^{-\alpha} \\ & + A(-\mathcal{I}_{\Omega}[d^{1-\alpha}](\bar{x}) - b(\bar{x})\cdot Dd^{1-\alpha}(\bar{x})), \end{aligned}$$

where we have used $\tilde{u} = u$ in Ω. Using Lemma 4.2 in the last inequality we conclude that there exists a constant $\tilde{c}_0 > 0$ such that

$$\mathcal{E}_{\beta_0,\delta}(U,\phi,\bar{x}) \leq d^{-\alpha}(\bar{x})\Big(Cd^{\alpha}(\bar{x}) + \theta_0(||u||_\infty + ||\varphi||_\infty) - A\tilde{c}_0\Big),$$

where we have used notation (4.20). Choosing $A > 0$ large enough and a small depending on the data but not on $\bar{x}$ we arrive to

$$\mathcal{E}_{\beta_0,\delta}(U,\phi,\bar{x}) \leq -\tilde{c}_0/2\ d(\bar{x})^{-\alpha},$$

concluding the proof in this case.

When $\bar{x} \in B_a(x_0)\cap\partial\Omega$, by definition of $\tilde{u}$ we consider a sequence $(x_k)_k$ of points in Ω such that $x_k \to \bar{x}$ and $u(x_k) \to \tilde{u}(\bar{x})$ and define $\epsilon_k = d(x_k)$. Let ϕ be a test function for U in $B_\delta(\bar{x})\cap\bar{\Omega}$ at $\bar{x}$ and consider the penalization

$$x \mapsto \Phi(x) := U(x) - (\phi(x) - \epsilon_k\zeta(x)). \tag{4.35}$$

From this, it is easy to see that for all k large there exists $\bar{x}_k \in \Omega$, maximum point of Φ in $B_\delta(\bar{x})\cap\Omega$ and using the inequality $\Phi(\bar{x}_k) \geq \Phi(x_k)$ we get

$$\bar{x}_k \to \bar{x},\ u(\bar{x}_k) \to \tilde{u}(\bar{x}), \quad \text{as } k \to \infty. \tag{4.36}$$

Now we want to use the penalization as a test function, but since ζ is unbounded close to the boundary we have to restrict the testing set, considering the last penalization as a testing in $B_{\delta'}(\bar{x}_k)$ with $\delta' < d(\bar{x}_k)$. Hence, since $\bar{x}_k \in \Omega$, arguing as above we conclude that

$$\mathcal{E}_{\beta_0,\delta'}(U,\phi,\bar{x}_k) \leq -\tilde{c}_0/2d(\bar{x}_k)^{-\alpha} + \epsilon_k\Big(-\mathcal{I}_{B_{\delta'}(\bar{x}_k)}[\zeta](\bar{x}_k) - b(\bar{x}_k)\cdot D\zeta(\bar{x}_k)\Big). \tag{4.37}$$

However, since we have $\bar{x}_k$ is a maximum point of the testing in $B_\delta(\bar{x}_k)\cap\Omega$, the inequality

$$\rho(U,\bar{x}_k,z) \leq \rho(\phi,\bar{x}_k,z) - \epsilon_k\rho(\zeta,\bar{x}_k,z) \quad \text{for all } z \in (\Omega - \bar{x}_k)\cap B_\delta$$

lead us to

$$\begin{aligned} & -\mathcal{I}^{\delta}_{\Omega}[U](\bar{x}_k) - \mathcal{I}_{\Omega,\delta}[\phi](\bar{x}_k) + \epsilon_k \int\limits_{B_\delta\setminus B_{\delta'}\cap(\Omega-\bar{x}_k)} \rho(\zeta,\bar{x}_k,z)K^{\alpha}(z)dz \\ \leq & -\mathcal{I}^{\delta'}_{\Omega}[U](\bar{x}_k) - \mathcal{I}_{\Omega,\delta'}[\phi](\bar{x}_k). \end{aligned}$$

Using this inequality in (4.37) we obtain the inequality

$$\mathcal{E}_{\beta_0,\delta}(U,\phi,\bar{x}_k) \leq -\tilde{c}_0/2d(\bar{x}_k)^{-\alpha} + \epsilon_k\Big(-\mathcal{I}_{\Omega\cap B_\delta(\bar{x}_k)}[\zeta](\bar{x}_k) - b(\bar{x}_k)\cdot D\zeta(\bar{x}_k)\Big),$$

concluding, by Lemma 4.4 (see also Remark 4.1) that

$$\mathcal{E}_{\beta_0,\delta}(U,\phi,\bar{x}_k) \leq -\tilde{c}_0/2d(\bar{x}_k)^{-\alpha} - \tilde{c}_0\epsilon_k d(\bar{x}_k)^{-1} < 0.$$

Letting $k \to \infty$, using the smoothness of ϕ, the continuity of d, and (4.36) together with the upper semicontinuity of U in $\bar{\Omega}$ we get the result. □

Remark 4.3. Note that the role of β_0 in the last proposition can be played by any control when $x_0 \in \Gamma_{in}$ and by $\bar{\beta}$ when $x_0 \in \Gamma$, with $\bar{\beta}$ as in (4.12).

For the next result, we recall that $\Sigma_r = \{x \in \Omega : dist(x, \Gamma_{in}) < r\}$.

Proposition 4.3. *Let u, v be respectively bounded viscosity sub and supersolution to problem (4.1)-(4.2) and $\tilde{u}, \tilde{v}$ as in (4.13). Recall U given in (4.34) and consider the function $V : \bar{\Omega} \to \mathbb{R}$ defined as*

$$V(x) = \tilde{v}(x) - Ad^{1-\alpha}(x). \tag{4.38}$$

Then, for all $a > 0$ small, there exist $A > 0$ such that the function U (resp. V) is a viscosity subsolution (resp. supersolution) to the equation

$$-\mathcal{I}_\Omega[w] + \mathcal{H}_s(x, Dw) = 0 \quad \text{in } \bar{\Sigma}_a.$$

The constant $A > 0$ depends only on the data, $a, ||u||_\infty$, θ_0 given in Lemma 4.1 and $\min\limits_{(x,\beta)\in\Gamma_{in}\times\mathcal{B}} \{b_\beta(x) \cdot Dd(x)\} > 0$.

Proof: As in the proof of Corollary 4.1, we have the existence of a constant $c_0 > 0$ such that $b_\beta(x) \cdot Dd(x) \geq c_0$ for all $\beta \in \mathcal{B}$ and all $x \in \Gamma_{in}$.

Let $\bar{x} \in \Sigma_a$. If ϕ is a smooth function such that $\bar{x}$ is a maximum point for $U - \phi$ in $B_\delta(\bar{x}) \cap \bar{\Omega}$, proceeding as in Proposition 4.2 and using notation (4.20) we conclude that

$$\begin{aligned} \mathcal{E}^s_\delta(U, \phi, \bar{x}) \leq \lambda ||u||_\infty + \theta_0(||u||_\varphi + ||\varphi||_\infty) d(\bar{x})^{-\alpha} + ||f||_\infty \\ + A\Big(- \mathcal{I}_\Omega[d^{1-\alpha}](\bar{x}) + \mathcal{H}_s(\bar{x}, Dd^{1-\alpha}(\bar{x}))\Big), \end{aligned}$$

and by the application of Corollary 4.1, taking A large and a small this inequality leads us to

$$\mathcal{E}^s_\delta(U, \phi, \bar{x}) \leq -\tilde{c}_0/2 \ d^{-\alpha}(\bar{x}), \tag{4.39}$$

concluding the proof of the proposition for this case. We highlight the constants a and A depend only on c_0 and not on the particular point $\bar{x}$ considered, concluding the result for Σ_a. To get the result in $\bar{\Sigma}_a$ we proceed in the same way as in Proposition 4.2, considering $\bar{x} \in \partial\Omega$ and penalizing the testing by the introduction of the function $\epsilon_k \zeta$ in order to push the testing point inside Ω (see (4.35) and its subsequent arguments). Hence, using (4.39) we arrive to

$$\mathcal{E}^s_\delta(U, \phi, \bar{x}_k) \leq -\tilde{c}_0/2 \ d^{-\alpha}(\bar{x}_k) + \epsilon_k \Big(- \mathcal{I}_{\Omega \cap B_\delta(\bar{x}_k)}[\zeta](\bar{x}_k) + \mathcal{H}_s(\bar{x}_k, D\zeta(\bar{x}_k))\Big),$$

and applying Corollary 4.2 in the last expression we conclude

$$\mathcal{E}^s_\delta(U, \phi, \bar{x}_k) \leq -\tilde{c}_0/2 \ d^{-\alpha}(\bar{x}_k) - \tilde{c}_0 \epsilon_k d^{-1}(\bar{x}_k). \tag{4.40}$$

Thus, the right-hand side is nonpositive for all k large and we conclude as in the proof of Proposition 4.2 letting $k \to \infty$ by (4.36). The result for V can be obtained in the same way. $\square$

4.6 The Cone Condition.

In this section we provide a proof for the well-known "cone condition", introduced in the pioneer paper of H.M. Soner [110] and which is known to play a key role in the uniqueness/comparison proofs: it asserts the fact of approximating the value of a subsolution to problem (4.1)-(4.2) at a point $x_0 \in \partial\Omega$ through a sequence of points lying in a cone contained in Ω whose vertex is x_0. We remark this condition is also adressed in the probabilistic approach refered as *nontangential upper semicontinuity* (see [88]), but here we follow closely the lines of the corresponding property proved in [31].

Proposition 4.4. *Let u be a bounded viscosity subsolution of (4.1)-(4.2) and $\tilde{u}$ defined in (4.13). Then, for each $x_0 \in \Gamma \cup \Gamma_{in}$ there exists a constant $C > 0$ and a sequence $(x_k)_k$ of points of Ω such that, as $k \to \infty$*

$$\begin{cases} x_k \to x_0, \\ \tilde{u}(x_k) \to \tilde{u}(x_0), \\ d(x_k) \geq C|x_k - x_0|. \end{cases} \tag{4.41}$$

Remark 4.4. In what follows, we will say that a function defined on $\bar{\Omega}$ satisfies *the cone condition at* $x_0 \in \partial\Omega$ if it satisfies condition (4.41).

Proof of Proposition 4.4: Consider $\bar{\beta}$ in (4.12) relative to x_0 and denote $b = b_{\bar{\beta}}$. Since we have $b(x_0) \cdot Dd(x_0) > 0$, we take $r > 0$ small enough such that $b(x) \cdot Dd(x) > 0$ for all $x \in \bar{\Omega} \cap \bar{B}_r(x_0)$. After rotation and translation, we can assume $x_0 = 0$ and $Dd(x_0) = e_n$ with $e_n = (0, ..., 0, 1)$, implying in particular that $b_n(0) > 0$. Finally, denote $H_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and $\mathcal{O} = \bar{H}_+ \cap \bar{\Omega} \cap \bar{B}_r$.

Recalling U defined as in (4.34), by Proposition 4.2 we have this function satisfies the equation

$$-\mathcal{I}_\Omega[w] - b \cdot Dw \leq 0 \quad \text{on } \mathcal{O}.$$

By a simple scaling argument, we conclude the function $y \mapsto U_\gamma(y) := U(\gamma y)$ defined in $\gamma^{-1}\mathcal{O}$ satisfies the equation

$$-\gamma^{1-\alpha}\mathcal{I}_{\gamma^{-1}\Omega}[w](y) - b_\gamma(y) \cdot Dw(y) \leq 0 \quad \text{on } \gamma^{-1}\mathcal{O}, \tag{4.42}$$

where $b_\gamma(y) = b(\gamma y)$ for each $y \in \gamma^{-1}\mathcal{O}$. Thus, $\bar{w} : \bar{H}_+ \to \mathbb{R}$ defined as

$$\bar{w}(x) = \limsup_{\gamma \to 0, z \to x} U(\gamma z)$$

is a viscosity subsolution for the problem

$$-b_n(0)\frac{\partial w}{\partial y_n} - b'(0) \cdot D_{y'}w = 0 \quad \text{in } \bar{H}_+. \tag{4.43}$$

In fact, by classical arguments in half-relaxed limits, for all $x \in \bar{H}_+$ and smooth function ϕ such that $\bar{w} - \phi$ has a maximum point at x, there exist sequences $\gamma_k \to 0$, $z_k \to x$ as $k \to \infty$ such that $U_{\gamma_k} - \phi$ has a maximum point at z_k, and therefore we can use inequality (4.42) associated to γ_k, evaluated on z_k. Note that U_γ is bounded in $\gamma^{-1}\Omega$, uniformly in γ. This and the integrability of K^α allows us to conclude that for each $\delta > 0$, the integral term $\mathcal{I}^\delta_{\gamma_k^{-1}\Omega}[U_{\gamma_k^{-1}}](z_k)$ is uniformly bounded in k. By the smoothness of ϕ we have the same conclusion for $\mathcal{I}_{\gamma_k^{-1}\Omega,\delta}[\phi](z_k)$. Then, the integral terms in (4.42) vanish as $k \to \infty$, and using the smoothness of ϕ and the continuity of b we arrive to (4.43). It is worth remark that (4.43) holds up to the boundary and that $b_n(0) > 0$.

The maximal solution for (4.43) with terminal data $\bar{w}(y', 1)$ (when we cast y_n as the "time" variable) is given by the function

$$W(y', y_n) = \bar{w}(y' - b_n(0)^{-1} b'(0)(y_n - 1), 1).$$

Since W is maximal, we have $\bar{w}(y) \leq W(y)$ when $0 \leq y_n \leq 1$. Now, by definition it is clear that $\bar{w}$ is upper semicontinuous and then $\bar{w}(0) = U(0)$, meanwhile by the upper semicontinuity of u at the boundary and the continuity of the distance function we have $\bar{w}(y) \leq U(0)$ for all $y \in H_+$. Then, recalling $U(0) = \tilde{u}(0)$, we conclude that

$$\tilde{u}(0) = \bar{w}(0) \leq W(0) = \bar{w}(b_n(0)^{-1} b'(0), 1) \leq \tilde{u}(0),$$

this is $\tilde{u}(0) = \bar{w}(x_b)$, with $x_b = (b_n(0)^{-1} b'(0), 1)$. By the very definition of $\bar{w}$, we have the existence of sequences $\gamma_k \to 0$, $z_k \to x_b$ such that $x_k := \gamma_k z_k$ satisfies $x_k \to 0$ and $\tilde{u}(x_k) \to \tilde{u}(0)$.

Note that by definition of the sequence $(x_k)_k$ we have $x_k = \gamma_k x_b + o(\gamma_k)$. Using this, we perform a Taylor expansion on $d(x_k)$, obtaining the existence of a point $\bar{x}_k \in H_+$ with $\bar{x}_k \to 0$ as $k \to \infty$ such that

$$d(x_k) = Dd(\bar{x}_k) \cdot (\gamma_k x_b + o(\gamma_k)).$$

Hence, since $Dd(0) = e_n$ we conclude $d(x_k) = \gamma_k + o(\gamma_k)$. Thus, using the estimates for x_k and $d(x_k)$ we get that $d(x_k) \geq (4|x_b|)^{-1}|x_k|$, for all k large enough. Recalling that $x_0 = 0$, we conclude that $(x_k)_k$ is the sequence satisfying (4.41). $\square$

Now we state the the analogous result for supersolutions, valid on Γ_{in}.

Proposition 4.5. *Let v be a bounded viscosity supersolution of (4.1)-(4.2) and let $\tilde{v}$ as in (4.13). Then, $\tilde{v}$ satisfies the cone condition on Γ_{in}.*

Proof: Following the arguments in the previous proposition, we consider this time the function

$$\underline{w}(x) = \liminf_{\gamma \to 0, z \to x} V(\gamma z),$$

where V is defined in (4.38). Using Proposition 4.3 relative to V, we prove $\underline{w}$ is a viscosity supersolution to the transport equation

$$\sup_{\beta \in \mathcal{B}} \{-b_\beta(0) \cdot Dw\} = 0 \quad \text{in } \bar{H}_+. \tag{4.44}$$

Since $x_0 \in \Gamma_{in}$, $(b_\beta)_n(0) > 0$ for all $\beta \in \mathcal{B}$ and then $\tilde{b}_\beta(0) = b_\beta(0)/(b_\beta)_n(0)$ is well defined. Now, if we denote W a minimal solution to (4.44), then it is easy to see that W is a solution to

$$-\frac{\partial w}{\partial y_n} + \sup_{\beta \in \mathcal{B}} \{\tilde{b}'_\beta(0) \cdot D_{y'} w\} = 0 \quad \text{in } \bar{H}_+.$$

By optimal control arguments (see [11], [14]), it is known the minimal solution of this equation with terminal data $W(y', 1) = \underline{w}(y', 1)$ is

$$W(y', y_n) = \inf_{\beta(\cdot)} \{\underline{w}(Y_{y'}((1 - y_n), \beta), 1)\}.$$

where $t \mapsto Y_{y'}(t, \beta(\cdot))$ is the solution of the equation $\dot{Y}(t) = \tilde{b}'(0, \beta(t))$ with initial condition $Y(0) = y'$. Note that in particular, we have

$$|Y(t)| \leq |y'| + t||b_\beta(0)||_{L^\infty(\mathcal{B})}, \tag{4.45}$$

for all $y' \in \mathbb{R}^{n-1}$ and all $t \in [0, 1]$. At this point, arguing in the same way as in the proof of Proposition 4.4, we conclude

$$\tilde{v}(0) = \inf_{\beta(\cdot)}\{\underline{w}(Y_0(1, \beta), 1)\}.$$

By this expression, we can take a minimizing sequence of controls $\beta_k(\cdot)$ and denoting $z_k = (Y_0(1, \beta_k), 1)$ we have

$$\underline{w}(z_k) \to \tilde{v}(0) \quad \text{as } k \to \infty.$$

Recalling that $V = \tilde{v} - Ad^{1-\alpha}$, by the very definition of $\underline{w}$ and the continuity of the distance function, for each k there exists sequences $\gamma_k^j \to 0$, $z_k^j \to z_k$ such that $\tilde{v}(\gamma_k^j z_k^j) \to \underline{w}(z_k)$ as $j \to \infty$. Note that by (4.45) we have the sequence $(z_k^j)_{k,j}$ is bounded and therefeore, using a diagonal argument we conclude the existence of sequences $\bar{\gamma}_k \to 0$, $\bar{z}_k \to \bar{z}$ with $\bar{z} = (\bar{z}', 1)$ and $||\bar{z}'|| \leq ||b_\beta(0)||_{L^\infty(\mathcal{B})}$ such that $\tilde{v}(\bar{\gamma}_k \bar{z}_k) \to \tilde{v}(0)$. Arguing in a similar way as in the end of the previous proposition, we conclude the sequence $(x_k)_k$ defined by $x_k = \bar{\gamma}_k \bar{z}_k$ is the desired sequence satisfying (4.41) relative to $\tilde{v}$. $\square$

Remark 4.5. No cone condition on sets as Γ_{in} is needed in the case of local PDE's (see [31] for degenerate elliptic second-order equations and [14], Lemma 5.3 for first-order equations). In fact, for such PDE's, the viscosity inequality for sub and supersolutions holds up to Γ_{in} and therefore, points on Γ_{in} can be regarded as "interior" points for uniqueness proofs. This is no longer available in our nonlocal framework, mainly due to assumption (**E**).

4.7 Proof of the Main Results.

In this section we give the proof of Theorems 4.1 and 4.2. Concerning Theorem 4.1, we mention that the way we present its proof here is slightly different from the very classical one, although the main contradiction argument is completely equivalent. We start with the following lemma, which is going to allow the contradiction argument on the comparison principle to be independent of the point where it is carried out.

Lemma 4.5. *Let u, v be respectively bounded viscosity sub and supersolutions to problem (4.1)-(4.2), and $\tilde{u}, \tilde{v}$ as in (4.13). Then, the function $\omega = \tilde{u} - \tilde{v}$ is a viscosity subsolution for the problem*

$$\begin{aligned} \lambda w - \mathcal{I}[w] + \mathcal{H}_i(x, Dw) = 0 \quad & in\ \Omega, \\ w = 0 \quad & on\ \Omega^c, \end{aligned} \tag{4.46}$$

in the sense of Definition 4.

Proof: Note that by Corollary 4.3, the boundary condition holds in the classical sense on Γ_{out} and therefore the result holds there. For shortness, we provide the proof of the viscosity inequality for a point $x_0 \in \Gamma \cup \Gamma_{out}$, since the case $x_0 \in \Omega$ follows the same ideas with far simpler computations.

Let $x_0 \in \Gamma \cup \Gamma_{in}$ and assume $\omega(x_0) > 0$. Note that in this case, we have three possibilities

(i) $u(x_0) > v(x_0) \geq \varphi(x_0)$.

(ii) $u(x_0) > \varphi(x_0) > v(x_0)$.

(iii) $\varphi(x_0) \geq u(x_0) > v(x_0)$.

Note in addition that cases (i) and (ii) cannot happen on Γ_{out} by Corollary 4.3. Hence, we give a complete proof for (iii) and after that we give some directions to conclude the remaining cases.

Denote ω^0 the extension of ω as the zero constant function in $\mathbb{R}^n \setminus \bar{\Omega}$ as in Definition 4.14, and note that since $\omega(x_0) > 0$, we have $\omega^0(x_0) = \omega(x_0)$. Consider ϕ a smooth function such that x_0 is a maximum point for $\omega^0 - \phi$ in $B_\delta(x_0)$. As before, we may assume x_0 is a strict maximum point.

By the cone condition for subsolutions given in Proposition 4.4, there exists a sequence $(x_k)_k$ satisfying (4.41) relative to u and x_0. Thus, denoting $\epsilon_k = |x_k - x_0|$ and $\nu_k = (x_k - x_0)/\epsilon_k$, up to a subsequence we can assume $\nu_k \to \nu_0$, with $Dd(x_0) \cdot \nu_0 > 0$. Following the arguments of Theorem 7.9 in [66] (see also [110]), we consider $\Phi : \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$(x, y) \mapsto \Phi(x, y) := u(x) - v_\varphi(y) - \phi_k(x, y),$$

with $\phi_k(x, y) = \phi(x) + |\epsilon_k^{-1}(x - y) - \nu_0|^2$.

Note that Φ is uppersemicontinuous in $\bar{\Omega} \times \mathbb{R}^n$ and by the boundedness of v_φ, for k large enough, there exists $(\bar{x}, \bar{y}) \in \bar{\Omega} \times \mathbb{R}^n$, maximum point for Φ. Using $\Phi(\bar{x}, \bar{y}) \geq \Phi(x_k, x_0)$ and noting that $v_\varphi(x_0) = v(x_0)$, we have

$$|\epsilon_k^{-1}(\bar{x} - \bar{y}) - \nu_0|^2 \leq u(\bar{x}) - v_\varphi(\bar{y}) - (u(x_k) - v(x_0)) + |\nu_k - \nu_0|^2. \tag{4.47}$$

By the boundedness of u and v_φ, $|\epsilon_k^{-1}(\bar{x} - \bar{y}) - \nu_0|^2$ remains bounded too and therefore $|\bar{x} - \bar{y}| \to 0$ as $k \to \infty$. Using the semicontinuity of u and v_φ, and that $u(x_k) \to u(x_0)$ into (4.47), letting $k \to \infty$ we get up to subsequences

$$\bar{x}, \bar{y} \to x_0,\ u(\bar{x}) \to u(x_0),\ v_\varphi(\bar{y}) \to v(x_0),\ |\epsilon_k^{-1}(\bar{x} - \bar{y}) - \nu_0| \to 0, \tag{4.48}$$

With this we have $\bar{x}, \bar{y}$ are valid test points for u and v, respectively, using the penalizations introduced as test functions. In fact, we have $\bar{y} \in \bar{\Omega}$, otherwise we get $v_\varphi(\bar{y}) \to \varphi(x_0) > v(x_0)$ which is a contradiction with the third statetment in (4.48). Since $\bar{y}$ is a global minimum point of $y \mapsto v_\varphi(y) - (u(\bar{x}) - \phi(\bar{x}, y))$, we can use $\bar{y}$ as a test point for v_φ even if it is on the boundary because if this is the case, $v_\varphi(\bar{y}) < \varphi(\bar{y})$ by the continuity of φ. On the other hand, again by (4.47) we have

$$\bar{x} = \bar{y} + \epsilon_k(\nu_0 + o_k(1)). \tag{4.49}$$

By a Taylor expansion of $d(\bar{x})$ we conclude the existence of $\tilde{y} \to x_0$ as $k \to \infty$ such that $d(\bar{x}) \geq d(\bar{y}) + \epsilon_k(Dd(\tilde{y}) \cdot \nu_0 + o_k(1))$. Since $Dd(x_0) \cdot \nu_0 > 0$ and by the continuity of Dd we conclude $d(\bar{x}) > 0$ for all k large enough, meaning $\bar{x} \in \Omega$ and then we use it as a test point for u. Then, since u and v are respective viscosity sub and supersolutions to problem (4.1)-(4.2), for all $0 < \delta' < d(\bar{x})$ we can write $E_{\delta'}(u^\varphi, \phi_k(\cdot, \bar{y}), \bar{x}) \leq 0 \leq E_{\delta'}(v_\varphi, -\phi_k(\bar{x}, \cdot), \bar{y})$ and then, by definition of H and using notation (4.15), for each $h > 0$ there exists $\beta_h \in \mathcal{B}$ such that

$$E_{\delta', \beta_h}(u^\varphi, \phi_\epsilon(\cdot, \bar{y}), \bar{x}) \leq 0 \quad \text{and} \quad E_{\delta', \beta_h}(v_\varphi, -\phi_\epsilon(\bar{x}, \cdot), \bar{y}) \geq -h.$$

Substracting these last inequalities we arrive to

$$\begin{aligned}&\lambda(u(\bar{x})-v(\bar{y}))-b_{\beta_h}(\bar{x})\cdot D\phi(\bar{x})-h\\ \le\;& f_{\beta_h}(\bar{x})-f_{\beta_h}(\bar{y})+2\epsilon_k^{-1}(b_{\beta_h}(\bar{x})-b_{\beta_h}(\bar{y}))\cdot(\epsilon_k^{-1}(\bar{x}-\bar{y})-\nu_0)\\ &+\mathcal{I}_{\delta'}[\phi_k(\cdot,\bar{y})](\bar{x})+\mathcal{I}^{\delta'}[u^\varphi](\bar{x})+\mathcal{I}_{\delta'}[\phi_k(\bar{x},\cdot)](\bar{y})-\mathcal{I}^{\delta'}[v_\varphi](\bar{y}),\end{aligned}\tag{4.50}$$

Now, by the continuity of f, condition (**L**) and the properties in (4.48), we conclude

$$f_{\beta_h}(\bar{x})-f_{\beta_h}(\bar{y})+2\epsilon_k^{-1}(b_{\beta_h}(\bar{x})-b_{\beta_h}(\bar{y}))\cdot(\epsilon_k(\bar{x}-\bar{y})-\nu_0)=o_k(1).$$

Using this estimate into (4.50) we get

$$\begin{aligned}&\lambda(u(\bar{x})-v(\bar{y}))-b_{\beta_h}(\bar{x})\cdot D\phi(\bar{x})-h-o_k(1)\\ \le\;&\mathcal{I}_{\delta'}[\phi_k(\cdot,\bar{y})](\bar{x})+\mathcal{I}_{\delta'}[\phi_k(\bar{x},\cdot)](\bar{y})+\mathcal{I}^{\delta'}[u^\varphi](\bar{x})-\mathcal{I}^{\delta'}[v_\varphi](\bar{y}).\end{aligned}\tag{4.51}$$

At this point, we take care about the nonlocal terms. Starting with the terms $\mathcal{I}_{\delta'}$ in the right-hand side of (4.51), by definition of ϕ_k we can write

$$\mathcal{I}_{\delta'}[\phi_k(\cdot,\bar{y})](\bar{x})+\mathcal{I}_{\delta'}[\phi_k(\bar{x},\cdot)](\bar{y})=\mathcal{I}_{\delta'}[\phi](\bar{x})+\epsilon_k^{-2}O({\delta'}^{1-\alpha}).\tag{4.52}$$

For the terms $\mathcal{I}^{\delta'}$ in the right-hand side of (4.51) we introduce the following notation

$$\begin{aligned}D_{int}&=(\Omega-\bar{x})\cap(\Omega-\bar{y}),\quad &D_{ext}&=(\Omega-\bar{x})^c\cap(\Omega-\bar{y})^c,\\ D_{int}^{\bar{x}}&=(\Omega-\bar{x})\cap(\Omega-\bar{y})^c,\quad &D_{int}^{\bar{y}}&=(\Omega-\bar{x})^c\cap(\Omega-\bar{y}).\end{aligned}$$

We highlight that each set depends both on $\bar{x}$ and $\bar{y}$, but we omit the dependence on $\bar{x}$ and/or $\bar{y}$ in some cases for a sake of simplicity. With this, we denote

$$\begin{aligned}I_{int}^a&=\int\limits_{D_{int}\setminus B_a}[u(\bar{x}+z)-v(\bar{y}+z)-(u(\bar{x})-v(\bar{y}))]K^\alpha(z)dz,\\ I_{ext}^a&=\int\limits_{D_{ext}\setminus B_a}[\varphi(\bar{x}+z)-\varphi(\bar{y}+z)-(u(\bar{x})-v(\bar{y}))]K^\alpha(z)dz,\\ I_{int,\bar{y}}^a&=\int\limits_{D_{int}^{\bar{y}}\setminus B_a}[\varphi(\bar{x}+z)-v(\bar{y}+z)-(u(\bar{x})-v(\bar{y}))]K^\alpha(z)dz,\\ I_{int,\bar{x}}^a&=\int\limits_{D_{int}^{\bar{x}}\setminus B_a}[u(\bar{x}+z)-\varphi(\bar{y}+z)-(u(\bar{x})-v(\bar{y}))]K^\alpha(z)dz,\end{aligned}\tag{4.53}$$

and note that

$$\mathcal{I}^{\delta'}[u^\varphi](\bar{x})-\mathcal{I}^{\delta'}[v_\varphi](\bar{y})=I_{int}^{\delta'}+I_{int,\bar{x}}^{\delta'}+I_{int,\bar{y}}^{\delta'}+I_{ext}^{\delta'}.\tag{4.54}$$

In what follows, our interest is to estimate each term in the right-hand side of the above inequality. For $I_{int}^{\delta'}$, since for each $z\in D_{int}$ we have $\Phi(\bar{x}+z,\bar{y}+z)\le\Phi(\bar{x},\bar{y})$, we can write

$$I_{int}^{\delta'}\le I_{int}^{\delta}+\int\limits_{D_{int}\cap B_\delta\setminus B_{\delta'}}\rho(\phi,\bar{x},z)K^\alpha(z)dz.\tag{4.55}$$

For $I^{\delta'}_{ext}$, recalling c_2 in condition (**E**) we note that for all $0 < r_1 < c_2$ small enough, by the continuity of φ we have for all k large enough, but independent of r_1 the inequality

$$\varphi(\bar{x} + z) - \varphi(\bar{y} + z) \leq \omega(x_0)/2, \quad \text{for all } z \in B_{r_1} \cap D_{ext}.$$

Recalling that $u(\bar{x}) - v(\bar{y}) \to \omega(x_0)$ as $k \to \infty$, we conclude that

$$I^{\delta'}_{ext} \leq \int_{(D_{ext} \setminus B_{\delta'}) \setminus B_{r_1}} [\varphi(\bar{x} + z) - \varphi(\bar{y} + z) - (u(\bar{x}) - v(\bar{y}))] K^{\alpha}(z) dz. \tag{4.56}$$

For $I^{\delta'}_{int,\bar{x}}$, for all $0 < r_1 < c_2$ suitably small, by the continuity of φ, the fact that $u(x_0) \leq \varphi(x_0)$ and the upper semicontinuity of u, for all k large enough but independent of r_1 we have

$$u(\bar{x} + z) - \varphi(\bar{y} + z) \leq \omega(x_0)/2 \quad \text{for all } z \in B_{r_1} \cap D^{\bar{x}}_{int}.$$

Hence, we can conclude this time that

$$I^{\delta'}_{int,\bar{x}} \leq \int_{(D^{\bar{x}}_{int} \setminus B_{\delta'}) \setminus B_{r_1}} [u(\bar{x} + z) - \varphi(\bar{y} + z) - (u(\bar{x}) - v(\bar{y}))] K^{\alpha}(z) dz.$$

But we note that the set $D^{\bar{x}}_{int}$ vanishes as $k \to \infty$, concluding from the above inequality that

$$I^{\delta'}_{int,\bar{x}} \leq r_1^{-(n+\alpha)} o_k(1). \tag{4.57}$$

At this point, by (4.49) we claim the set $D^{\bar{y}}_{int}$ is away from the origin uniformly in k, postponing the proof of this claim until the end of this Case. Using this and the fact that $D^{\bar{y}}_{int}$ vanishes as $k \to \infty$, we conclude that

$$I^{\delta}_{int,\bar{y}} = o_k(1). \tag{4.58}$$

Using (4.55)-(4.58) into (4.54), we get that

$$\begin{aligned}
&\mathcal{I}^{\delta'}[u^{\varphi}](\bar{x}) - \mathcal{I}^{\delta'}[v_{\varphi}](\bar{y}) \\
&\leq r_1^{-(n+\alpha)} o_k(1) + I^{\delta}_{int} \\
&\quad + \int_{(D_{ext} \setminus B_{\delta'}) \setminus B_{r_1}} [\varphi(\bar{x} + z) - \varphi(\bar{y} + z) - (u(\bar{x}) - v(\bar{y}))] K^{\alpha}(z) dz \\
&\quad + \int_{D_{int} \cap B_{\delta} \setminus B_{\delta'}} \rho(\phi, \bar{x}, z) K^{\alpha}(z) dz,
\end{aligned}$$

and using this estimate together with (4.52) into (4.51), for $r_1 > 0$ arbitrarily small but fixed, we let $\delta' \to 0$ and $k \to \infty$, concluding by the use of (4.48) that

$$\begin{aligned}
&\lambda \omega(x_0) - b_{\beta_h}(x_0) \cdot D\phi(x_0) \\
&\leq h + \mathcal{I}_{\Omega}[\omega](x_0) + \int_{(\Omega - x_0) \cap B_{\delta}} \rho(\phi, x_0, z) K^{\alpha}(z) dz \\
&\quad + \int_{(\Omega - x_0)^c \setminus B_{r_1}} (-\omega(x_0)) K^{\alpha}(z) dz
\end{aligned} \tag{4.59}$$

However, using that x_0 is a maximum point for $\omega^0 - \phi$ in $B_\delta(x_0)$, we clearly have that

$$-\omega(x_0) \leq \phi(x_0 + z) - \phi(x_0) \quad \text{for all } z \in (\Omega - x_0)^c \cap B_\delta(x_0).$$

Using this inequality into (4.59), we conclude by the smoothness of ϕ that

$$\lambda\omega(x_0) - \mathcal{I}_\delta[\phi](x_0) - \mathcal{I}^\delta[\omega^0](x_0) + \mathcal{H}_i(x_0, D\phi(x_0)) \leq h + Cr_1^{1-\alpha}.$$

Since h and r_1 are arbitrary numbers, we conclude the desired viscosity inequality for ω.

Now we address the claim leading to (4.58). Assume that there exists a sequence $z_k \in D_{int}^{\bar{y}}$ such that $z_k \to 0$. By definition, there exists $a_k \in \Omega$ and $b_k \in \Omega^c$ such that $z_k = a_k - \bar{y} = b_k - \bar{x}$ and by the first statement in (4.48) we have $a_k, b_k \to x_0$. Now, applying (4.49) we conclude $b_k = a_k + \epsilon_k^2(\nu_0 + o_k(1))$. Taking k large we conclude $b_k \in \Omega$, which is a contradiction.

Finally, recalling that cases (i) and (ii) can happen only on Γ_{in}, we note that by Proposition 4.5 cone condition holds for subsolutions on Γ_{in}, and then the proof of (ii) follows exactly the same lines above. For (i), by Proposition 4.5 we have conecondition for supersolutions, and then we can argue as above exchanging the roles of u and v. The proof is complete. □

With the above result, we are reasy to prove the main result of this article

Proof of Theorem 4.1 Let ω be as in Lemma 4.5 and ω^0 its extension as the constant zero function outside $\bar{\Omega}$ in the sense of Definition 4.14. By contradiction, we assume that there exists $x_0 \in \bar{\Omega}$ such that

$$\omega^0(x_0) = \sup_{\mathbb{R}^n} \omega^0 > 0.$$

Of course $x_0 \in \bar{\Omega} \setminus \Gamma_{out}$ by definition of ω^0 and Corollary 4.3. By Lemma 4.5 we can use the constant function equally zero as test function for ω at x_0, concluding for each $\delta > 0$ that

$$\lambda\omega(x_0) - \int\limits_{(\Omega - x_0)\setminus B_\delta} \rho(\omega, x_0, z) K^\alpha(z)dz + \omega(x_0) \int\limits_{(\Omega - x_0)^c \setminus B_\delta} K^\alpha(z)dz \leq 0.$$

Since x_0 is a global maximum for ω on $\bar{\Omega}$, for each $\delta > 0$ the first integral term in the right-hand side of the above inequality is nonpositive, and then we can drop this term. Using that $\omega(x_0) > 0$ and taking δ small, by (**M**) we get that

$$\mu_0/2 \ \omega(x_0) \leq 0,$$

which is a contradiction. □

Remark 4.6. Let ω as in Lemma 4.5 and $\tilde{\omega}$ as in (4.13). Using a suitable interior approximation, it is possible to prove that $\tilde{\omega}$ satisfies the viscosity inequality for subsolutions of problem (4.46) up to Γ_{in}, no matter if $\tilde{\omega}$ satisfies the boundary condition or not. Though interesting and very surprising taking into account the Remark 4.5, this result is unuseful for the purposes of Theorem 4.1, because in general we do not have $\tilde{\omega} = \omega$ on Γ_{in}, that is, it may happen that $\tilde{\omega}(x) < \tilde{u}(x) - \tilde{v}(x)$ for some $x \in \Gamma_{in}$. This technical difficulty is the main reason for the use of cone condition on Γ_{in} and therefore the inclusion of Lemma 4.5 in the proof of the comparison principle.

The following property is a direct consequence of Theorem 4.1.

Corollary 4.4. Assume (**E**), (**M**), (**H**), (**L**) hold. Consider $\varphi_1, \varphi_2 : \Omega^c \to \mathbb{R}$ continuous and bounded functions. Let u be a bounded viscosity subsolution to problem (4.1)-(4.2) with exterior data $\varphi = \varphi_1$ and v a bounded viscosity supersolution to problem (4.1)-(4.2) with exterior data $\varphi = \varphi_2$. If $\varphi_1 \leq \varphi_2$, then $u \leq v$ in Ω. Moreover, using Definition 4.13, we have $\tilde{u} \leq \tilde{v}$ in $\bar{\Omega}$.

Proof of Theorem 4.2: By the smoothness of the boundary and assumption (**L**), we can consider a Lipschitz extension to all $\mathbb{R}^n$ of the function b and continuous extension to $\mathbb{R}^n$ for f. With this, the corresponding extended Hamiltonian (which we still denote by H) can be understood as a function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Consider for each $\epsilon > 0$ continuous functions $\psi_+^\epsilon, \psi_-^\epsilon : \mathbb{R}^n \to \mathbb{R}$ with $\psi_+^\epsilon \geq \psi_-^\epsilon$ in $\mathbb{R}^n$ and $\psi_\pm^\epsilon = \varphi$ in Ω^c. With this, for $w, l \in \mathbb{R}$ and $x, p \in \mathbb{R}^n$ we define the following Hamiltonian

$$H_\epsilon(x, w, p, l) = \min\{(w - \psi_-^\epsilon(x)), \max\{(w - \psi_+^\epsilon(x)), \lambda w - l + H(x,p)\}\},$$

and its asociated problem

$$H_\epsilon(x, w(x), Dw(x), \mathcal{I}[w](x)) = 0, \ x \in \mathbb{R}^n. \tag{4.60}$$

It is easy to see that this problem admits comparison principle for bounded sub and supersolutions defined in all $\mathbb{R}^n$, see [23]. Recalling μ_0 as in (**M**), let $R = ||\varphi||_\infty + \mu_0^{-1}||f||_\infty$ and consider the function

$$g(x) = 2R\mathbf{1}_{\bar{\Omega}}(x) + R\mathbf{1}_{\bar{\Omega}^c}(x).$$

Note that $g \geq \varphi$ on Ω^c and that for each $x \in \Omega$ we have

$$\lambda g(x) - \mathcal{I}_\Omega[g](x) - \int_{\Omega - x} [\varphi(x+z) - g(x)] K^\alpha(z) dz + H(x, 0) \geq 0,$$

concluding that g is a supersolution to problem (4.60). In the same way can be proved that $-g$ is a subsolution for this problem. Thus, Perron's Method applies, concluding the existence of a solution w_ϵ for this problem. Such solution satisfies $|w_\epsilon| \leq R$, which is an estimate independent of ϵ. Now, considering $\psi_\pm^\epsilon$ such that $\psi_\pm^\epsilon \to \pm\infty$ in Ω as $\epsilon \to 0$, by the uniform boundeness of the function w_ϵ, for all $x \in \mathbb{R}^n$ the quantities

$$\bar{w}(x) := \limsup_{\epsilon \to 0, y \to x} w_\epsilon(y), \quad \underline{w}(x) := \liminf_{\epsilon \to 0, y \to x} w_\epsilon(y)$$

are well defined. The half-relaxed limits method implies $\bar{w}, \underline{w}$ are bounded sub and supersolution to (4.1) in the sense of Definition 4, respectively. By the very definition of these functions, $\underline{w} \leq \bar{w}$ in $\mathbb{R}^n$. On the other hand, considering $u = \tilde{\bar{w}}, v = \tilde{\underline{w}}$ as in (4.13) we apply Theorem 4.1 to get $u \leq v$ in $\bar{\Omega}$, concluding $u = v$ in $\mathbb{R}^n \setminus (\Gamma_{in} \cup \Gamma)$. However, for each $x_0 \in \Gamma \cup \Gamma_{in}$ we have

$$v(x_0) \leq \liminf_{x \in \Omega, x \to x_0} v(x) \leq \limsup_{x \in \Omega, x \to x_0} v(x) = \limsup_{x \in \Omega, x \to x_0} u(x) = u(x_0)$$

It is easy to see u and v are respective viscosity sub and supersolution to (4.1)-(4.2) using Corollary 4.3 and the fact that u and v differ from $\bar{w}$ and $\underline{w}$ in a set of zero Lebesgue measure. This concludes the function u (equals v) is a $C(\bar{\Omega})$ viscosity solution, with $u = \varphi$ in $\bar{\Omega}^c \cup \Gamma_{out}$. The uniqueness follows directly by comparison. $\square$

4.8 Remarks and Comments.

Comparison principle for a class of fully nonlinear equations can be obtained by natural adaptation of the arguments presented in this paper. For example, consider $0 < a < 1/2$ and $(\alpha_\beta)_{\beta\in\mathcal{B}}$ such that $\alpha_\beta \in (a, 1-a)$ for each $\beta \in \mathcal{B}$. With this, we consider the nonlocal operator

$$\mathcal{I}_\beta[\phi](x) = \int_{\mathbb{R}^n} \rho(\phi, x, z) K_\beta(z)|z|^{-(n+\alpha_\beta)} dz, \tag{4.61}$$

with $K_\beta : \mathbb{R}^n \to \mathbb{R}_+$ measurable function for all $\beta \in \mathcal{B}$ satisfying condition (**E**) for constants Λ, c_1 independent of β. We also consider a continuous function $\lambda : \bar{\Omega} \times \mathcal{B} \to \mathbb{R}_+$ and introduce the following generalization of assumption (**M**): For all $\beta \in \mathcal{B}$

$$\inf_{x\in\bar{\Omega}} \left\{ \lambda_\beta(x) + \int_{\Omega - x} K_\beta(z) dz \right\} > 0. \tag{M'}$$

Let $\mathcal{C}$ be the set of real valued functions ϕ defined in all $\mathbb{R}^n$ for which (4.61) is well defined for all $\beta \in \mathcal{B}$ and define $F : \mathbb{R}^n \times \mathbb{R} \times \mathcal{C} \times \mathbb{R}^n \to \mathbb{R}$ as

$$F(x, z, \phi, p) = \sup_{\beta\in\mathcal{B}} \{\lambda_\beta(x) z - \mathcal{I}_\beta[\phi](x) - b_\beta(x) \cdot p - f_\beta(x)\}.$$

The fully nonlinear version of problem (4.1)-(4.2) reads as

$$\left\{ \begin{array}{rll} F(x, u(x), u, Du(x)) & = 0 & \text{in } \Omega. \\ u & = \varphi & \text{in } \Omega^c. \end{array} \right.$$

Note that the definition of $\Gamma_{in}, \Gamma_{out}$ and Γ depends only on the configuration of the drift term, so we shall use them exactly in the same form. By the assumption over the numbers α_β, the order of the nonlocal operator is less than the order of the drift so the results concerning section 4.4 can be readily adapted. On the other hand, the fractional nature of the operator is still present in this case by the nonintegrability of the kernels, so results in section 4.5 and in paticular we can conclude cone condition for this type of equations. The conclusions in the contradiction arguments used to prove Theorem 4.1 can be obtained in the same way in this setting by (**M'**).

The situation is different for nonlocal operators which are not uniformly elliptic. We address here two examples: For the first one, consider $J \in L^1(\mathbb{R}^n)$ a positive function and denote

$$\mathcal{J}[u](x) = \int_{\mathbb{R}^n} \rho(u, x, z) J(z) dz.$$

The second one is the case of operators with censored jumps as

$$\mathcal{I}_+[u](x) = \int_{H_+} \rho(u, x, z)|z|^{-(n+\alpha)} dz,$$

with $\alpha \in (0, 1)$ and $H_+ = \{(x', x_n) : x_n > 0\}$. It is known that in Dirichlet problems where $\mathcal{J}$ plays the diffusive role there exists loss of the boundary condition even in absence of drift term (see [53]). A similar feature arises in equations related to operator $\mathcal{I}_+$, where despite the nonintegrability of the kernel, its null diffusive feature in some particular directions may create loss of the boundary condition at points where the censored direction is normal to the boundary. Hence, in both cases we cannot prescribe the value of the solution of the equation associated to these operators since definitions of Γ_{out} and Γ do not provide any information. However, we can obtain comparison results for these type of operators making stronger assumptions over Γ_{out} and Γ. Define

$$\Gamma'_{out} = \{x \in \partial\Omega \ : \ \forall\ \beta \in \mathcal{B},\ b_\beta(x) \cdot Dd(x) < 0\},$$

consider the set Γ_{in} as in the introduction and $\Gamma = \partial\Omega \setminus (\Gamma'_{out} \cup \Gamma_{in})$. Assuming the conditions

(H') Γ'_{out}, Γ_{in} and Γ are connected components of $\partial\Omega$, and

(HΓ) For each $x \in \Gamma$, there exists $\bar{\beta}, \underline{\beta} \in \mathcal{B}$ such that

$$b_{\underline{\beta}}(x) \cdot Dd(x) < 0 < b_{\bar{\beta}}(x) \cdot Dd(x),$$

then we can obtain strong comparison principle for Dirichlet problems associated to degenerate elliptic nonlocal operators. This is due to the stronger assumption over the controls on Γ_{out} and Γ, since at one hand this implies Corollary 4.3 holds, and on the other, by the fact we are still in the framework of low diffusive influence of the operator compared with the drift, technical results of section 4.5 and cone condition can be proved as they were presented here without substantial changes.

4.9 Appendix.

4.9.1 Proof of Lemma 4.3.

Let us first consider the illustrative case of a flat boundary, namely $\Omega = H_+ = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\}$. In this case, $d(x) = x_n$ and then for $x \in H_+$ we have

$$\begin{aligned}
-\mathcal{I}_\Omega[\zeta](x) &= -\int_{-x_n<z_n} \rho(\zeta, x, z) K^\alpha(z)dz \\
&\leq -\int_{-x_n<z_n<0} \rho(\log, x_n, z_n) K^\alpha(z)dz \\
&\leq -\Lambda \int_{-x_n<z_n<0} \log(1 + z_n/x_n)|z|^{-(n+\alpha)}dz \\
&= -\Lambda \int_{-x_n<z_n<0} \log(1 + z_n/x_n)|z_n|^{-(n+\alpha)} \int_{\mathbb{R}^{n-1}} (1 + |z'/|z_n||^2)^{-(n+\alpha)/2} dz' dz_n.
\end{aligned}$$

Then, performing the change $y = z'/|z_n|$ and using the integrability of the function $(1 + |y|^2)^{-(n+\alpha)/2}$ in $\mathbb{R}^{n-1}$, we conclude

$$-\mathcal{I}_\Omega[\zeta](x) \leq -\Lambda C_{n,\alpha} \int_{-x_n<z_n<0} \log(1 + z_n/x_n)|z_n|^{-(1+\alpha)} dz_n. \tag{4.62}$$

Thus, applying the change of variables $t = z_n/x_n$ in the integral of the right-hand side we conclude

$$-\mathcal{I}_\Omega[\zeta](x) \leq -\Lambda C_{n,\alpha} x_n^{-\alpha} \int_{-1}^{0} \log(1 + t)|t|^{-(1+\alpha)}dt.$$

Note that in $[-1/2, 0]$ we have $|\log(1+t)||t|^{-(1+\alpha)} \leq 3/4|t|^{-\alpha}$ and since $\alpha < 1$ this term is integrable. Meanwhile in $[-1, -1/2]$ the term $|t|^{-(1+\alpha)}$ is uniformly bounded and by the integrability of the log function at zero we conclude the lemma for the flat boundary case.

Now we deal with the general case. Take $x \in \Omega$ close to the boundary and consider Θ_x as in (4.23). By the definition of ζ in (4.26) we can use same arguments as at the begining of the proof of Lemma 4.2, to conclude

$$\mathcal{I}_\Omega[\zeta](x) \leq \Lambda \int_{\Theta_x} [\log(d(x)) - \log(d(x+z))]|z|^{-(n+\alpha)}.$$

Denote by $\hat{x}$ the unique point of $\partial\Omega$ such that $d(x) = |x - \hat{x}|$. After a rotation we can assume $x - \hat{x} = d(x)e_n$, where $e_n = (0, ..., 0, 1)$. By the smoothness of the boundary, there exists $R > 0$, an open set $\mathcal{O} \subset \mathbb{R}^{n-1}$ containing the origin and a smooth function $F : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\{(x', x_n) \in \mathcal{O} \times \mathbb{R} : x_n = F(x')\} = \partial\Omega \cap B_R(\hat{x}),$$

that is, near $\hat{x}$ the boundary is the graph of a smooth function. We can assume without loss of generality that $F(0) = \hat{x}$. With this, we define the change of variables

$$\begin{array}{rccl} \Phi : & \mathcal{O} \times [0, d(x)] & \to & \mathbb{R}^n \\ & (y, s) & \mapsto & F(y) - x + (d(x) - s)Dd(F(y)). \end{array}$$

Regarding this function, by the smoothness of $\partial\Omega$, Φ is a diffeomorphism of $\mathcal{U}$ into $\Phi(\mathcal{U})$, where $\mathcal{U} \subset \mathbb{R}^n$ is an open set such that $\bar{\mathcal{O}} \times [0, d(x)] \subset \mathcal{U}$. This allows us to consider Φ as a change of variables that flattens $\partial\Omega - x$. Note that Φ is C^1 and $D\Phi(s, y)$ is uniformly bounded in $\bar{\mathcal{O}} \times [0, d(x)]$, independent of x. Since $x = \hat{x} + d(x)Dd(\hat{x})$, then $\Phi(0) = 0$ and by well-known results, there exists $\kappa > 0$ independent of x such that

$$\kappa^{-1}|(y, s)| \leq |\Phi(y, s)| \leq \kappa|(y, s)|, \tag{4.63}$$

for all $(y, s) \in \bar{\mathcal{O}} \times [0, d(x)]$. Denoting $\Theta_x^0 = \Phi(\mathcal{O} \times [0, d(x)])$, we have $\Theta_x^0 \subset \Theta_x$ and for each $z \in \Theta_x^0$ there exists a unique $(y, s) \in \mathcal{O} \times [0, d(x)]$ such that $z = \Phi(y, s)$ and then $d(x+z) = d(x) - s$. Applying Φ as a change of variables we can write

$$\begin{aligned} &\int_{\Theta_x^0} -\rho(\zeta, x, z)|z|^{-(n+\alpha)}dz \\ &= -\int_{\mathcal{O}\times[0,d(x)]} [\log(d(x) - s) - \log(d(x))]|\Phi(y, s)|^{-(n+\alpha)}|\det(D\Phi(y, s))|dyds \\ &\leq -C\kappa^{n+\alpha} \int_0^{d(x)} \int_{\mathcal{O}} [\log(d(x) - s) - \log(d(x))]|(y, s)|^{-(n+\alpha)}dyds, \end{aligned}$$

where, in the last inequality, we have used the boundedness of $|\det(D\Phi)|$ and (4.63). Thus, we can integrate over y in a similar way as in the flat case (see (4.62)) and making $t = -s$ we conclude

$$\int_{\Theta_x^0} -\rho(\zeta, x, z)|z|^{-(n+\alpha)}dz \leq -C\kappa^{n+\alpha} \int_{-d(x)}^0 \log(1 + t/d(x))|t|^{-(1+\alpha)}dt,$$

arriving to the same integral obtained in the flat case. We conclude that the integral over Θ_x^0 is bounded by $Cd(x)^{-\alpha}$, where C depends on n, α and the smoothness of $\partial\Omega$.

Since the remaining portion $\Theta_x \setminus \Theta_x^0$ is at distance at least $R/2$ from the origin, the term $|z|^{-(n+\alpha)}$ is bounded on $\Theta_x \setminus \Theta_x^0$ and then we have

$$\int_{\Theta_x \setminus \Theta_x^0} -\rho(\zeta, x, z)|z|^{-(n+\alpha)}dz \leq C_R \int_{\Theta_x \setminus \Theta_x^0} -\rho(\zeta, x, z)dz,$$

where C_R depends on R but not in x. Using similar flattening arguments and the integrability of the log function near zero we conclude the last integral is just bounded independently of x. This concludes the proof. $\square$

4.9.2 Proof of Estimate (4.31).

We recall that

$$\mathcal{A}_{\delta,\mu} = \{z \in B_r \ : \ d(x) - \delta < d(x+z) < d(x) + \mu\},$$

and that the function Ψ is defined as

$$\Psi(x) = \eta^{-1}d(x) + \epsilon^{-1}|x - x_0|^2$$

in a neighborhood of x_0. Since $\bar{x} \to x_0$, taking r, δ and μ suitably small in the definition of $\mathcal{A}_{\delta,\mu}$, the second integral in (4.31) can be estimated as

$$\int_{\mathcal{A}_{\delta,\mu}} \rho(\Psi, \bar{x}, z)K^\alpha(z)dz \leq C(\eta^{-1} + \epsilon^{-1}) \int_{\mathcal{A}_{\delta,\mu}} |z|^{-(n+\alpha-1)}dz \tag{4.64}$$

with C independent of $\bar{x}, \eta$ and ϵ. From this point we can use a flattening argument very similar to the previous one, building a change of variables which allows us to estimate the whole integral by the one dimensional integral in the normal direction to the boundary. With this, we can conclude that

$$\int_{\mathcal{A}_{\delta,\mu}} |z|^{-(n+\alpha-1)}dz \leq C \int_{-\delta}^{\mu} |s|^{-\alpha}ds,$$

and since we assume $\delta \leq \mu$ then we can bound above the integral by $C\mu^{1-\alpha}$. This concludes (4.31). $\square$

Chapter 5

Regularity Results and Large Time Behavior for Integro-Differential Equations with Coercive Hamiltonians

This chapter is based in a joint work with Guy Barles, Shigeaki Koike and Olivier Ley which can be found in the preprint [24].

Contents

5.1 Introduction.

In [50], Capuzzo-Dolcetta, Leoni and Porretta prove a surprising regularity result for *sub-solutions* of superquadratic second-order elliptic equations which can be described in the following way. We consider the model equation

$$\lambda v - Tr(A(x)D^2 v(x)) + b(x)|Dv(x)|^m = f(x) \quad \text{in } \Omega, \tag{5.1}$$

where Ω is an open subset of $\mathbb{R}^N$, A, b, f are continuous functions in Ω, A taking values in the set of nonnegative matrices and b, f are real valued, with $b(x) \geq b_0 > 0$ in Ω, $m > 2$ and $\lambda \geq 0$. The function $v : \Omega \to \mathbb{R}$ is a real-valued solution and $Dv, D^2 v$ denote its gradient and Hessian matrix. In [50], the authors prove that, if $u : \Omega \to \mathbb{R}$ is a bounded viscosity subsolution of (5.1) then u is locally Hölder continuous with exponent $\alpha := (m-2)(m-1)^{-1}$ and the local Hölder seminorm depends only on the datum (L^∞ bounds on A, f and b_0) but not on any L^∞ bound nor oscillation of u. Actually this result provides, in many interesting situations, an estimate on the L^∞ norm of u.

The starting point of the present work was to investigate how such a result could be extended to the case of nonlocal elliptic equations like

$$\lambda u(x) - I_x(u,x) + H(x, Du(x)) = 0 \quad \text{in } \Omega, \tag{P}$$

where $\lambda \geq 0$ and $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a continuous nonlinearity having the same properties as $b(x)|p|^m - f(x)$ above. The term I_x is a nonlocal operator playing the role of the diffusion, defined as follows: for $x, y \in \mathbb{R}^N$ and $\phi : \mathbb{R}^N \to \mathbb{R}$ a bounded continuous function which is C^2 in a neighborhood of y, we write

$$I_x(\phi, y) = \int_{\mathbb{R}^N} [\phi(y+z) - \phi(y) - \mathbf{1}_B \langle D\phi(y), z\rangle]\nu_x(dz), \tag{5.2}$$

where B denotes the unit ball and $\{\nu_x\}_{x\in\mathbb{R}^N}$ is a family of *Lévy measures*, see (M1)-(M2) below for precise assumptions. An important example of such nonlocal operator is the case when $\nu_x = \nu$ for all $x \in \mathbb{R}^N$, with

$$\nu(dz) = C_{N,\sigma}|z|^{-(N+\sigma)}dz,$$

where $\sigma \in (0,2)$ and $C_{N,\sigma}$ is a normalizing constant. In that case, for all $x \in \mathbb{R}^N$, $-I_x = (-\Delta)^{\sigma/2}$ is the fractional Laplacian of order σ (see [69]). By the form of I_x in (5.2), we point out that subsolutions of (P) must be defined on $\mathbb{R}^N$ or at least in a large enough domain (depending on ν_x) in order that the nonlocal operator is well-defined.

In [50] and even more in the simplified version given in [15], the authors take advantage of the superquadratic gradient term to construct *locally* a strict supersolution to (5.1) using power-like functions. The power profile of such supersolutions gives the (local) Hölder regularity for bounded *subsolutions* of the equation. This proof is based on the leading effect of the gradient term more than on the ellipticity, resembling the behavior of first-order coercive equations (see [14]). The Hölder exponent $(m-2)(m-1)^{-1}$ just comes from a simple balance of powers in (5.1) and this Hölder regularity can be extended up to the boundary of the domain if it is regular enough (see also [15]).

All these arguments seem extendable to the nonlocal framework, and in particular, if we think the nonlocal term as an operator of order $\sigma \in (0,2)$. But here is a key difference which is going to play a double role : first, depending on the support of the measure $\nu_x(dz)$, the operator may use values of u outside Ω. This arises, typically, when equation (P) is complemented by an exterior Dirichlet condition (see [21]). Of course, and this is very natural in the case of exterior Dirichlet condition, these outside values cannot be controlled by the equation. Hence, in that case, it is clearly impossible to have results which are independent of the L^∞ norm or oscillation of u.

On the contrary, this analysis shows that, in principle, this could be possible in the case when the support of the measure is such that the integral of $I_x(u,y)$ only takes into account points such that $y + z \in \Omega$, typically when

$$I_x(\phi, y) = C_{N,\sigma} \int_{y+z\in\Omega} [\phi(y+z) - \phi(y) - \mathbf{1}_B \langle D\phi(y), z\rangle]|z|^{-(N+\sigma)}dz\ . \tag{5.3}$$

These type of operators are related to "censored processes" in the probabilistic literature : in this context, it means that the jumps processes cannot jump from Ω to Ω^c. We refer to e.g. [42, 77, 79, 80, 90, 86] for more details on such processes. In [42, 80], the censored fractional Laplacian appears in connection with Dirichlet forms; they also appear in the analysis literature as regional Laplacians ([85]) and very naturally in the study of Neumann boundary conditions ([20]). We therefore call *censored operators (with respect to Ω)* the operators which satisfy

$$x + \mathrm{supp}\{\nu_x\} \subset \Omega, \quad \text{for all } x \in \Omega.$$

Actually, we remark that we can always reduce to the case of a censored operator by incorporating the integral over the complement of Ω into the right-hand side f (see Lemma 5.1 below and/or [113]). This "censoring" procedure modifies the right-hand side into a function which blows up at the boundary of Ω with a rate which is controlled in terms of the singularity of the measure (the σ in the fractional Laplacian case) and the oscillation of u. Thus, as it can be seen in [50], the presence of these unbounded ingredients in the equation restricts the expected values of the Hölder exponent if we wish a result which holds up to the boundary. Moreover, the same effect arises even for nonlocal operators which are originally censored, since the proof of the Hölder regularity consists in localizing, typically in some ball included in Ω and, at this step also, the values of u outside the ball creates essentially the same difficulty as the one described above : if we want to write the nonlocal equation as a censored equation in the ball, then this mechanically changes the "natural" Hölder exponent because of the right-hand side which blows up at the boundary of the ball.

All these difficulties explain all the different formulations we give for some results but also the nature of the Hölder exponent we obtain. To be more specific, we consider the basic model equation

$$\lambda u(x) + a(x)(-\Delta)^{\sigma/2}u(x) + b(x)|Du(x)|^m = f(x) \quad \text{in } \Omega, \tag{5.4}$$

where λ, b, f are as in (5.1) and a is a continuous real-valued function with $a \geq 0$ in Ω. The role of the superquadraticity in (5.1) is played by a *superfractional growth condition on the gradient*, which is encoded by m in (5.4) through the assumption

$$m > \sigma, \tag{5.5}$$

and the strict positivity requirement on b. The difficulties we mention above on the nonlocality have a price and this price is a "less natural" Hölder exponent $(m-\sigma)/m$ for subsolutions to (5.4). Nevertheless, we can get interior Hölder regularity results with "more natural" exponents $(m-\sigma)/(m-1)$ if $\sigma > 1$, Lipschitz continuity if $\sigma < 1$, and any exponent in $(0,1)$ for $\sigma = 1$, since localization arguments are unnecessary in this situation. Finally, we point out that in the case of censored operators (here if $(-\Delta)^{\sigma/2}$ is replaced by the operator given by (5.3)), we recover a complete control on the oscillation of u on $\overline{\Omega}$ as a consequence of the form of the estimates (see Corollary 5.1 below).

It is worth pointing out that our results share (with some limitations we described above) the same interesting consequence as the ones of [50], namely a control on the oscillation of (sub)solutions to (5.1) inside Ω (i.e. at least locally) which is stable as $\lambda \to 0^+$. This feature has important applications on the study of large time behavior for associated parabolic problems and homogenization because of the importance of the ergodic problem.

We are able to provide *global* oscillation bounds satisfying this stability property for some class of problems (P) as, for example, equations associated to censored operators and obviously for equations set in the whole space $\mathbb{R}^N$. This contrasts with the results obtained by

Cardaliaguet and Rainer [51] (see also [49]), where the authors obtain very interesting regularity results for (parabolic) superquadratic integro-differential equations using a probabilistic approach, but where their Hölder estimates depend on the L^∞ norm of the solution.

In the second part of this paper, we present an application of our regularity results to the study of the large time behavior for Cauchy problems

$$\partial_t u(x,t) - I_x(u(\cdot,t),x) + H(x, Du(x,t)) = 0 \quad \text{in } Q, \tag{CP}$$

where $Q = \mathbb{R}^N \times (0,+\infty)$. The asymptotic behavior of the nonlocal evolution problem is also motivated by its second-order parallel, as the model equation

$$\partial_t u(x,t) - \mathrm{Tr}(A(x)D^2u(x,t)) + b(x)|Du(x,t)|^m = f(x) \quad \text{in } Q. \tag{5.6}$$

In the superquadratic case $m > 2$, this evolution equation is also influenced by the stronger effect of the first-order term. This can be seen in the paper of Barles and Souganidis [32], where the authors study general equations including (5.1) and (5.6), obtain Lipschitz bounds for the solutions and prove that, in the periodic setting, the solution approaches to the solution of the so-called *ergodic problem* as $t \to +\infty$. This ergodic problem is solved by passing to the limit as $\lambda \to 0^+$ in equation (5.1), which is possible by the compactness given by the Lipschitz bounds which are independent of λ. A second key ingredient in the analysis of the ergodic problem and the large time behavior of (5.6) is the Strong Maximum Principle ([12]).

Similar methods and results to [32] are obtained in [112] in the context of Cauchy-Dirichlet second-order evolution problems in bounded domains. In the nonlocal context, analogous ergodic large time behavior for evolution problems are available. For instance, in [19] the authors follow the arguments of [32], using the Lipschitz regularity results given in [18], which allows to "linearize" the equation in order to apply the Strong Maximum Principle of [55].

In this paper we also follow the lines of [32] to prove the ergodic asymptotic behavior. However, contrarily to [32] or [19], we do not use the Strong Maximum Principle in the same way : we do not perform any "linearization" of the equation (which would have required Lipschitz bounds) and therefore we are able to provide results which just use the Hölder regularity of the solutions. This proof requires slightly stronger assumptions on the nonlocal operator since we have to be able to use the Strong Maximum Principle à la Coville [62, 63] and to do so, we need the support of the measure defining the nonlocal operator to satisfy an "iterative covering property". Though a restriction, this property allows us to study the large time behavior for equations associated to very degenerate x-dependent nonlocal operators and x-dependent Hamiltonians with a higher degree of coercivity.

Of course, comparison principles are of main importance in this method and for this reason we should focus on a particular class of x-dependent nonlocal operators in *Lévy-Ito* form (see (5.50)). We refer to [23] for comparison results associated to these operators.

The paper is organized as follows: Section 5.2 is entirely devoted to the regularity results for the stationary problem. In section 5.3 we provide the comparison principle and well-possedness of the evolution problem. Finally, the large time behavior for this problem is presented in section 5.4, where the mentioned version of the strong maximum principle is established.

Basic Notation. For $x \in \mathbb{R}^N$ and $r > 0$, we denote $B_r(x)$ as the open ball centered at x with radius r. We just write B_r for $B_r(0)$ and B for $B_1(0)$.

Let $\Omega \subset \mathbb{R}^N$. We denote as d_Ω the signed distance function to $\partial\Omega$ which is nonnegative in $\bar{\Omega}$. For $\delta > 0$, we also denote $\Omega^\delta = \{x \in \Omega : d_\Omega(x) > \delta\}$. For any $u : \Omega \to \mathbb{R}$, the oscillation of u over Ω is defined by

$$\mathrm{osc}_\Omega u = \sup_\Omega u - \inf_\Omega u.$$

For $x, \xi, p \in \mathbb{R}^N$, $A \subset \mathbb{R}^N$ and ϕ a bounded function, we define

$$I_\xi[A](\phi, x, p) = \int_{\mathbb{R}^N \cap A} [\phi(x+z) - \phi(x) - \mathbf{1}_B \langle p, z\rangle] \nu_\xi(dz). \tag{5.7}$$

We write in a simpler way $I_\xi[A](\phi, x) = I_\xi[A](\phi, x, D\phi(x))$ when $\phi \in L^\infty(\mathbb{R}^N) \cap C^2(B_\delta)$ for some $\delta > 0$, $I_\xi(\phi, x, p) = I_\xi[\mathbb{R}^N](\phi, x, p)$ when $A = \mathbb{R}^N$ and $I = I_\xi$ if $\nu_\xi = \nu$ does not depend on ξ. Note that with these notations, $I_x(\phi, x) = I_x[\mathbb{R}^N](\phi, x, D\phi(x))$ for ϕ bounded and smooth at x (see (5.2)).

This paper is based on the viscosity theory to get the results. We refer to [66, 14, 91] for the definition and main results of the classical theory, and to [23, 21, 3, 103, 104] for the nonlocal setting. Following the definition introduced in the mentioned references, we always assume a viscosity subsolution is upper semicontinuous and a viscosity supersolution is lower semicontinuous in the set where the equation takes place.

5.2 Regularity.

5.2.1 Assumptions and Main Regularity Results.

Let $\sigma \in (0, 2)$ fixed. Recalling I_x defined in (5.2), we assume the following conditions over the family $\{\nu_x\}_x$

(M1) *For all $R > 0$ and $\alpha \in [0, 2]$, there exists a constant $C_R > 0$ such that, for all $\delta > 0$ we have*

$$\sup_{x \in \bar{B}_R} \int_{B_\delta^c} \min\{1, |z|^\alpha\} \nu_x(dz) \leq C_R h_{\alpha,\sigma}(\delta),$$

where $h_{\alpha,\sigma}(\delta)$ is defined for $\delta > 0$ as

$$h_{\alpha,\sigma}(\delta) = \begin{cases} \delta^{\alpha-\sigma} & \text{if } \alpha < \sigma \\ |\ln(\delta)| + 1 & \text{if } \alpha = \sigma \\ 1 & \text{if } \alpha > \sigma, \end{cases} \tag{5.8}$$

and where we use the convention $|z|^\alpha = 1, z \in \mathbb{R}^N$ when $\alpha = 0$.

(M2) *For all $R > 0$ and $\alpha \in (\sigma, 2]$ there exists a constant $C_R > 0$ such that, for all $\delta \in (0, 1)$ we have*

$$\sup_{x \in \bar{B}_R} \int_{B_\delta} |z|^\alpha \nu_x(dz) \leq C_R \delta^{\alpha - \sigma}.$$

Assumptions (M1) and (M2) say the nonlocal operator I_x is at most of order σ, locally in $x \in \mathbb{R}^N$. Concerning this last fact, we remark that in the case ν_x is symmetric and $\sigma \in (0, 1)$, I_x defined in (5.2) can be written as

$$I_x(\phi, y) = \int_{\mathbb{R}^N} [\phi(y+z) - \phi(y)] \nu_x(dz), \tag{5.9}$$

for all $y \in \mathbb{R}^N$ and ϕ bounded and C^1 in a neighborhood of y. Since our interest is to keep I_x as a nonlocal operator of order σ, we adopt this formula as a definition for I_x in the case $\sigma \in (0,1)$, even if ν_x is not symmetric.

In order to expand the application of our results, we consider an open set $\Omega \subseteq \mathbb{R}^N$ not necessarily bounded, and H satisfying the growth condition

$$H(x,p) \geq b_0|p|^m - A(d_\Omega(x)^{-\theta} + 1), \quad \text{for } x \in \Omega,\ p \in \mathbb{R}^N, \tag{5.10}$$

where $b_0, A > 0$ and $0 \leq \theta < m$.

We first concentrate in regularity results in the **superlinear** case

$$m > \max\{1, \sigma\},$$

which encodes the coercivity of the Hamiltonian, see Theorems 5.1 and 5.2 below. Note that, in Section 5.2.5, we state also a result in the sublinear case and, in Section 5.2.6, we extend our results in the superlinear case to Lévy-Ito operators.

Over the exponent θ, we assume $0 \leq \theta < m$ in order to state the blow-up behavior at the boundary of the right-hand side. Thus, our arguments rely over the (more general) equation

$$-I_x(u,x) + b_0\,|Du(x)|^m = A(d_\Omega(x)^{-\theta} + 1), \quad x \in \Omega. \tag{P'}$$

In principle, due to the nonlocal nature of I_x, the function u satisfying the above equation should be defined not only in Ω but on the set

$$\Omega_\nu = \Omega \cup \bigcup_{x \in \bar{\Omega}} \{x + \mathrm{supp}\{\nu_x\}\}, \tag{5.11}$$

which, loosely speaking, represents the reachable set from Ω through ν.

The following result states the regularity up to the boundary for *subsolutions* of problem (P').

Theorem 5.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, $A, b_0 > 0$ and $\sigma \in (0,2)$. Let $\{\nu_x\}_{x \in \mathbb{R}^N}$ be a family of measures satisfying (M1)-(M2) relative to σ, and I_x defined as in (5.2) if $\sigma \geq 1$ and as (5.9) if $\sigma < 1$, associated to $\{\nu_x\}_{x\in\mathbb{R}^N}$. Let $m > \max\{1,\sigma\}$, $\theta \in [0,m)$, and define*

$$\gamma_0 = \min\{(m-\sigma)/m, (m-\theta)/m\}. \tag{5.12}$$

Then, any bounded viscosity subsolution $u : \mathbb{R}^N \to \mathbb{R}$ to the problem (P') is locally Hölder continuous in Ω with Hölder exponent γ_0 as in (5.12), and Hölder seminorm depending on Ω, the data and $\mathrm{osc}_{\Omega_\nu}(u)$, with Ω_ν defined as in (5.11).

Moreover, if Ω has a $C^{1,1}$ boundary, then u can be extended to $\bar{\Omega}$ as a Hölder continuous function of exponent γ_0.

A second result states interior Hölder regularity for subsolutions of (P') with a Hölder exponent which is more natural to the balance between the order of the nonlocal operator and the Hamiltonian.

Theorem 5.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, $A, b_0 > 0$ and $\sigma \in (0,2)$. Let $\{\nu_x\}_{x \in \mathbb{R}^N}$ be a family of measures satisfying (M1)-(M2) relative to σ, and I_x defined as in (5.2) if $\sigma \geq 1$ and as (5.9) if $\sigma < 1$, associated to $\{\nu_x\}_{x\in\mathbb{R}^N}$. Let $m > \max\{1,\sigma\}$ and $\theta \in [0,m)$. Define*

$$\tilde{\gamma}_0 = \tilde{\gamma}_0(\sigma, m) = \begin{cases} (m-\sigma)/(m-1) & \text{if } \sigma > 1 \\ \in (0,1) & \text{if } \sigma = 1 \\ 1 & \text{if } \sigma < 1, \end{cases} \tag{5.13}$$

and consider

$$\gamma_0 = \min\{\tilde{\gamma}_0, (m-\theta)/m\}. \tag{5.14}$$

Then, any bounded viscosity subsolution $u : \mathbb{R}^N \to \mathbb{R}$ to the equation (P') is locally Hölder continuous in Ω with exponent γ_0 given by (5.14), and Hölder seminorm depending on the data, Ω and $\mathrm{osc}_{\Omega_\nu}(u)$, where Ω_ν is defined as in (5.11).

Note that for the same data, γ_0 defined in (5.14) is always bigger or equal than γ_0 defined in (5.12), and therefore, the interior Hölder exponent given by Theorem 5.2 is better than the one given by Theorem 5.1.

Remark 5.1. Theorems 5.1 and 5.2 can be extended to unbounded domains. In fact, if Ω is unbounded, arguing over a bounded set $\Omega' \subset \Omega$ we can apply the method used in the above theorems to conclude the corresponding local Hölder regularity results for Ω. Moreover, if $\partial\Omega$ has uniform $C^{1,1}$ bounds, and if (M1)-(M2) hold with C_R independent of R, then we have global Hölder estimates for bounded subsolutions to (P'), in the flavour of Theorem 5.1.

Since our aim is to include in our regularity results nonlocal operators of censored nature, we provide here a more accurate definition of such an operator. Recalling definition (5.11), we say that I_x is of censored nature relative to Ω if the family $\{\nu_x\}_{x\in\mathbb{R}^N}$ defining I_x satisfies the condition

$$\Omega_\nu = \Omega. \tag{5.15}$$

The idea is to set up the problem to provide an unified proof of Theorem 5.1 for censored and noncensored operators. This is possible after a "censoring" procedure we explain now. Let $\{\nu_x\}_{x\in\mathbb{R}^N}$ a family of Lévy measures and $\Omega \subseteq \mathbb{R}^N$ an open set. For each $\xi \in \mathbb{R}^N$ we define the *censored measure respect to* Ω *and* ξ as

$$\tilde{\nu}_\xi(dz) = \mathbf{1}_{\Omega-\xi}(z)\nu_\xi(dz). \tag{5.16}$$

For $\xi, x \in \mathbb{R}^N$, $\delta > 0$ and a bounded function $\phi \in C^2(\bar{B}_\delta(x))$, we define

$$\begin{aligned} \tilde{I}_\xi(\phi, x) &= \int_{\mathbb{R}^N} [\phi(x+z) - \phi(x) - \mathbf{1}_B\langle D\phi(x), z\rangle]\tilde{\nu}_\xi(dz) \\ &= \int_{\Omega-\xi} [\phi(x+z) - \phi(x) - \mathbf{1}_B\langle D\phi(x), z\rangle]\nu_\xi(dz). \end{aligned} \tag{5.17}$$

Of special interest is the *censored operator* I_Ω defined as

$$I_\Omega(\phi, x) = \tilde{I}_x(\phi, x), \quad x \in \bar{\Omega}, \tag{5.18}$$

from whose definition we note that $I_\Omega(\phi, x) = I_x[\Omega - x](\phi, x)$.

Note that if $\{\nu_x\}_{x\in\mathbb{R}^N}$ satisfies (M1) and (M2), then $\{\tilde{\nu}_x\}_{x\in\mathbb{R}^N}$ satisfies (M1) and (M2) with the same constants C_R. Thus, the next lemma allows us to reduce general nonlocal equations like (P') to the censored case.

Lemma 5.1. (Censoring the Equation) *Let $\Omega \subset \mathbb{R}^N$ open and bounded, $\sigma \in (0, 2)$ and $\{\nu_x\}_{x\in\mathbb{R}^N}$ a family of measures satisfying (M1)-(M2) related to σ. Let I_x be as in (5.2), (5.9) associated to $\{\nu_x\}_{x\in\mathbb{R}^N}$. Let $m > \sigma, \beta_0 > 0$ and for $f : \Omega \to \mathbb{R}$ locally bounded, let $u : \mathbb{R}^N \to \mathbb{R}$ be a bounded viscosity subsolution to*

$$-I_x(u, x) + \beta_0|Du(x)|^m = f(x), \quad x \in \Omega. \tag{5.19}$$

Then, there exists $C > 0$ (depending on Ω and β_0) such that the function u restricted to Ω satisfies, in the viscosity sense, the inequality

$$-I_\Omega(u,x) + \frac{\beta_0}{2}|Du(x)|^m \le f(x) + C(\mathrm{osc}_{\Omega_\nu}(u)+1)d_\Omega(x)^{-\sigma}, \quad x \in \Omega,$$

where I_Ω is defined in (5.18) and Ω_ν is defined in (5.11).

Proof: For simplicity, we present the proof for classical subsolutions. The rigorous proof follows easily by using classical viscosity techniques (for instance, see [113]). We also focus on the case $\sigma \ge 1$.

Using (5.19), for each $x \in \Omega$ we have

$$\begin{aligned}
&- I_\Omega(u,x) + \beta_0|Du(x)|^m \\
&\le f(x) + \int_{\Omega^c - x}(u(x+z)-u(x))\nu_x(dz) + |Du(x)|\int_{B\cap(\Omega^c-x)}|z|\nu_x(dz) \\
&\le f(x) + C\Big(\mathrm{osc}_{\Omega_\nu}(u)d_\Omega(x)^{-\sigma} + |Du(x)|h_{1,\sigma}(d_\Omega(x))\Big),
\end{aligned}$$

where $C > 0$ comes from the application of (M1) and depends only on Ω. Now, by Young's inequality, there exists $C(\beta_0)$ such that

$$|Du(x)|d_\Omega(x)^{1-\sigma} \le \frac{\beta_0}{2}|Du(x)|^m + C(\beta_0)h_{1,\sigma}(d_\Omega(x))^{m/(m-1)}.$$

At this point, we note that since $m > \sigma$ we have $m(1-\sigma)/(m-1) \ge -\sigma$. Then, if $\sigma > 1$, using (5.8) we can write

$$h_{1,\sigma}(d_\Omega(x))^{m/(m-1)} = d_\Omega(x)^{m(1-\sigma)/(m-1)} \le d_\Omega(x)^{-\sigma},$$

meanwhile if $\sigma = 1$, we get

$$h_{1,\sigma}(d_\Omega(x))^{m/(m-1)} = (|\log(d_\Omega(x))|+1)^{m/(m-1)} \le Cd_\Omega(x)^{-\sigma},$$

where $C > 0$ depends only on m. Thus, using these estimates we conclude the result for the case $\sigma \ge 1$.

The case $\sigma < 1$ follows the same ideas but with easier computations because of the first order finite difference of the integrand defining I_x, see (5.9). $\square$

5.2.2 Key Technical Lemmas.

We start with some notation: for $r > 0$ and $x_0 \in \mathbb{R}^N$, define

$$d_0(x) = |x - x_0| \quad \text{and} \quad d_r(x) = r - d_0(x), \tag{5.20}$$

that is, for $x \in B_r(x_0)$, $d_0(x)$ represents the distance of x to the center of the ball, meanwhile $d_r(x) = d_{B_r(x_0)}(x)$ is the distance of x to the boundary of the ball. We define w as

$$w = w_1 + w_2, \tag{5.21}$$

where, for $C_1, \gamma > 0$ and $C_2 \geq 0$ we consider

$$\begin{aligned} w_1(x) &= \begin{cases} C_1 d_0(x)^\gamma & x \in \bar{B}_r(x_0) \\ C_1 r^\gamma & x \in \bar{B}_r^c(x_0) \end{cases} \\ w_2(x) &= \begin{cases} C_1(r^\gamma - d_r(x)^\gamma) & x \in \bar{B}_r(x_0) \\ C_1 r^\gamma + C_2 & x \in \bar{B}_r^c(x_0). \end{cases} \end{aligned} \tag{5.22}$$

We note that w_1 and w_2 (when $C_2 = 0$) are Hölder continuous in $\mathbb{R}^N$ with exponent γ. If $C_2 > 0$, w_2 is γ-Hölder in $B_r(x_0)$ and it has a discontinuity on $\partial B_r(x_0)$. In any case, both w_1 and w_2 (for any $C_2 \geq 0$) are smooth in $B_r(x_0) \setminus \{0\}$.

For $x \in B_r(x_0)$ consider ϱ defined as

$$\varrho(x) = \frac{1}{4} \min\{d_0(x), d_r(x)\}. \tag{5.23}$$

Of course, w depends on the particular choice of γ, r, x_0, C_1, C_2, meanwhile ϱ depends on r and x_0, but we omit these dependences for simplicity of the notation.

We remark that if $|x - x_0| \leq r/2$ then $\varrho(x) = d_0(x)/4$, meanwhile if $|x - x_0| > r/2$ we have $\varrho(x) = d_r(x)/4$.

The goal is to prove that w is a supersolution of (P'). The following key lemma gives us a first useful estimate for the nonlocal term applied to w.

Lemma 5.2. *Let $\sigma \in (0,2)$ and a family of measures $\{\nu_x\}_{x \in \mathbb{R}^N}$ satisfying (M1), (M2) relative to σ. Let I_x as in (5.2), (5.9) associated to $\{\nu_x\}_{x \in \mathbb{R}^N}$. Let $x_0 \in \mathbb{R}^N$, $r \in (0,1)$, $\gamma \in (0,1]$, $C_1 > 0$, $C_2 \geq 0$, and consider w as in (5.21) and ϱ as in (5.23) associated to these parameters. Then, there exists a constant $C > 0$ (not depending on r, C_1 and C_2) such that*

$$\sup_{\xi \in B_1(x)} \{I_\xi(w, x)\} \leq C \begin{cases} C_1 \varrho^{\gamma - 1}(x) h_{1,\sigma}(\varrho(x)) & \text{if } C_2 = 0, \sigma \geq 1 \\ C_1 h_{\gamma,\sigma}(\varrho(x)) & \text{if } C_2 = 0, \sigma < 1 \\ (C_1 + C_2)\varrho(x)^{-\sigma} & \text{if } C_2 > 0 \end{cases}, \tag{5.24}$$

for each $x \in B_r(x_0) \setminus \{x_0\}$.

Proof: Denote $R = |x_0| + 1$. We remark that C_R in the arguments to come is a generic constant depending on R through the constants arising in (M1) and (M2). The constant C arising in the proof is a positive constant independent of x, R, r, C_1 or C_2.

Consider $x \in B_r(x_0) \setminus \{x_0\}$. For each $\xi \in B_1(x)$, by definition of w we can write

$$I_\xi(w, x) = I_\xi(w_1, x) + I_\xi(w_2, x),$$

where w_i, $i = 1, 2$ are defined in (5.22). In what follows, we are going to estimate the integrals in the right-hand side of the above expression.

1.- *Estimate for $I_\xi(w_1, x)$.* We can split this integral term as

$$I_\xi(w_1, x) = I_\xi[B_{\varrho(x)}](w_1, x) + I_\xi[B^c_{\varrho(x)}](w_1, x).$$

Note that for each $z \in B_{\varrho(x)}$ we have

$$\begin{aligned} w_1(x + z) - w_1(x) &= \langle Dw_1(x + tz), z\rangle, \\ w_1(x + z) - w_1(x) - \langle Dw_1(x), z\rangle &= \frac{1}{2}\langle D^2 w_1(x + sz)z, z\rangle, \end{aligned}$$

for some $s, t \in (0,1)$. We recall that the first equality is used in the integral defining $I_\xi[B_{\varrho(x)}](w_1, x)$ when $\sigma < 1$, and the second is used in the case $\sigma \geq 1$. Now, direct computations on the derivatives of w_1 drives us to

$$\begin{aligned} \langle D^2 w_1(x+sz)z, z\rangle &\leq C_1\gamma d_0(x)^{\gamma-2}|z|^2 \\ \langle Dw_1(x+tz), z\rangle &\leq C_1\gamma d_0(x)^{\gamma-1}|z|. \end{aligned}$$

for all $z \in B_{\varrho(x)}$, $s, t \in (0,1)$. Thus, using these inequalities on the corresponding form of $I_\xi[B_{\varrho(x)}](w_1, x)$, using that $\varrho(x) \leq d_0(x)$ and applying (M2), we arrive at

$$I_\xi[B_{\varrho(x)}](w_1, x) \leq C_R C_1 \varrho(x)^{\gamma-\sigma}. \tag{5.25}$$

Concerning the estimate of $I_\xi[B^c_{\varrho(x)}](w_1, x)$, we write

$$I_\xi[B^c_{\varrho(x)}](w_1, x) \leq \int\limits_{B^c_{\varrho(x)}} [w_1(x+z) - w_1(x)]\nu_\xi(dz) + |Dw_1(x)| \int\limits_{B\setminus B_{\varrho(x)}} |z|\nu_\xi(dz),$$

and we suppress the last integral term in the case $\sigma < 1$. Using the definition of w_1 we get from the above inequality that

$$\begin{aligned} I_\xi[B^c_{\varrho(x)}](w_1, x) \leq & \int_{B\setminus B_{\varrho(x)}} [w_1(x+z) - w_1(x)]\nu_\xi(dz) + C_1 r^\gamma \int_{B^c} \nu_\xi(dz) \\ & + C_1\gamma d_0(x)^{\gamma-1} \int_{B\setminus B_{\varrho(x)}} |z|\nu_\xi(dz), \end{aligned}$$

where, as before, the last integral does not exist if $\sigma < 1$. Since w_1 is $\gamma-$Hölder continuous we have $w_1(x+z) - w_1(x) \leq C_1|z|^\gamma$. Using this together with (M1) (see (5.8)) we can write

$$I_\xi[B^c_{\varrho(x)}](w_1, x) \leq C_R C_1 \Big(h_{\gamma,\sigma}(\varrho(x)) + r^\gamma + d_0(x)^{\gamma-1} h_{1,\sigma}(\varrho(x)) \Big),$$

where the last term inside the parentheses is suppressed if $\sigma < 1$. Noting that $\varrho(x) \leq d_0(x) < r < 1$, we conclude that

$$I_\xi[B^c_{\varrho(x)}](w_1, x) \leq C_R C_1 \left\{ \begin{array}{ll} h_{\gamma,\sigma}(\varrho(x)) + \varrho(x)^{\gamma-1} h_{1,\sigma}(\varrho(x)), & \text{if } \sigma \geq 1 \\ h_{\gamma,\sigma}(\varrho(x)), & \text{if } \sigma < 1. \end{array} \right.$$

At this point, we note that if $\sigma \geq 1$ and $\gamma \in (0,1]$, we always have $h_{\gamma,\sigma}(\varrho) \leq \varrho^{\gamma-1} h_{1,\sigma}(\varrho)$, for all $\varrho \in (0,1)$. Taking this into account we get

$$I_\xi[B^c_{\varrho(x)}](w_1, x) \leq C_R C_1 \left\{ \begin{array}{ll} \varrho(x)^{\gamma-1} h_{1,\sigma}(\varrho(x)), & \text{if } \sigma \geq 1 \\ h_{\gamma,\sigma}(\varrho(x)), & \text{if } \sigma < 1. \end{array} \right.$$

and joining this last inequality and (5.25) we conclude that

$$I_\xi(w_1, x) \leq C_R C_1 \left\{ \begin{array}{ll} \varrho(x)^{\gamma-1} h_{1,\sigma}(\varrho(x)), & \text{if } \sigma \geq 1 \\ h_{\gamma,\sigma}(\varrho(x)), & \text{if } \sigma < 1. \end{array} \right. \tag{5.26}$$

2.- *Estimate for* $I_\xi(w_2, x)$. Analogously as the previous estimate, we write

$$I_\xi(w_2, x) = I_\xi[B_{\varrho(x)}](w_2, x) + I_\xi[B^c_{\varrho(x)}](w_2, x).$$

We start with $I_\xi[B_{\varrho(x)}](w_2,x)$. By recalling (5.22), direct computations drive us to

$$\begin{aligned} Dw_2(x) &= C_1\gamma d_r^{\gamma-1}(x)Dd_0(x), \\ D^2w_2(x) &= C_1\gamma d_r(x)^{\gamma-2}d_0(x)^{-1} \\ &\qquad \times \Big(d_r(x)I_N + [(1-\gamma)d_0(x) - d_r(x)]Dd_0(x)\otimes Dd_0(x)\Big), \end{aligned}$$

and therefore, using the above computations as a Taylor expansion of the finite difference in the integral defining $I_\xi[B_{\varrho(x)}](w_2,x)$, we claim that

$$I_\xi[B_{\varrho(x)}](w_2,x) \le C_R C_1 \varrho(x)^{\gamma-\sigma}. \tag{5.27}$$

In fact, when $\sigma < 1$, using (5.9) and the above expression for Dw_2, we have

$$I_\xi[B_{\varrho(x)}](w_2,x) = C_1\gamma \int_0^1 \int_{B_{\varrho(x)}} d_r^{\gamma-1}(x+sz)\langle Dd_0(x+sz), z\rangle \nu_\xi(dz)ds,$$

but for all $s\in(0,1)$ and $z\in B_{\varrho(x)}$, we have $d_r(x+sz)\ge\varrho(x)$. Thus, we have

$$I_\xi[B_{\varrho(x)}](w_2,x) \le CC_1\varrho^{\gamma-1}(x)\int_{B_{\varrho(x)}} |z|\nu_\xi(dz),$$

and applying (M1) we conclude (5.27).

Now we deal with the case $\sigma\ge1$. Since in this case

$$I_\xi[B_{\varrho(x)}](w_2,x) = \frac{1}{2}\int_0^1\int_{B_{\varrho(x)}} \langle D^2w_2(x+sz)z,z\rangle\nu(dz)ds,$$

using the explicit form of D^2w_2 we get

$$\begin{aligned} &I_\xi[B_{\varrho(x)}](w_2,x) \\ &\le CC_1 r\int_0^1\int_{B_{\varrho(x)}} d_r(x+sz)^{\gamma-2}d_0(x+sz)^{-1}|z|^2\nu_\xi(dz)ds, \end{aligned} \tag{5.28}$$

and we estimate this last integral by cases. If $d_0(x)\ge r/2$ we have $\varrho(x)=d_r(x)/4$. Then, for $z\in B_{\varrho(x)}$ and $s\in(0,1)$ we have $3\varrho(x)\le d_r(x+sz)$ and $r/4\le d_0(x+sz)$. Using these estimates into (5.28), we conclude

$$I_\xi[B_{\varrho(x)}](w_2,x) \le CC_1\varrho(x)^{\gamma-2}\int_{B_{\varrho(x)}}|z|^2\nu_\xi(dz) \le C_RC_1\varrho(x)^{\gamma-\sigma},$$

where we have used (M2). On the other hand, if $d_0(x)<r/2$ we have $\varrho(x)=d_0(x)/4$. Then, for $z\in B_{\varrho(x)}$ and $s\in(0,1)$ we have $r/4\le d_r(x+sz)$ and $3\varrho(x)\le d_0(x+sz)$. Using these estimates into (5.28), we get

$$I_\xi[B_{\varrho(x)}](w_2,x) \le CC_1r^{\gamma-1}\varrho(x)^{-1}\int_{B_{\varrho(x)}}|z|^2\nu_\xi(dz) \le C_RC_1\varrho(x)^{\gamma-\sigma},$$

where we have used that $\varrho(x)\le r$ and (M2). This concludes (5.27).

Concerning the estimate of $I_\xi[B^c_{\varrho(x)}](w_2, x)$, we should be careful with the fact that C_2 may be strictly positive.

At one hand, if $C_2 = 0$, then as in the computations relative to w_1, we have

$$w_2(x+z) - w_2(x) \leq C_1|z|^\gamma \quad \text{for all } z \in B^c_{\varrho(x)},$$

and therefore, we can write

$$\begin{aligned} & I_\xi[B^c_{\varrho(x)}](w_2, x) \\ & \leq \int_{B^c_{\varrho(x)}} [w_2(x+z) - w_2(x)]\nu_\xi(dz) + |Dw_2(x)| \int_{B\setminus B_{\varrho(x)}} |z|\nu_\xi(dz) \\ & \leq C_1 \int_{B^c_{\varrho(x)}} |z|^\gamma \nu_\xi(dz) + C_1 d_r(x)^{\gamma-1} \int_{B\setminus B_{\varrho(x)}} |z|\nu_\xi(dz), \end{aligned}$$

where the last integral is suppressed if $\sigma < 1$. Thus, applying (M1) and using that $r < 1$, we obtain from the above inequality that

$$I_\xi[B^c_{\varrho(x)}](w_2, x) \leq C_R C_1 \Big(h_{\gamma,\sigma}(\varrho(x)) + d_r(x)^{\gamma-1} h_{1,\sigma}(\varrho(x)) \Big),$$

where the last term does not exist if $\sigma < 1$. Finally, since $\varrho(x) \leq d_r(x)$ we conclude

$$I_\xi[B^c_{\varrho(x)}](w_2, x) \leq C_R\, C_1 \begin{cases} \varrho(x)^{\gamma-1} h_{1,\sigma}(\varrho(x)), & \text{if } \sigma \geq 1 \\ h_{\gamma,\sigma}(\varrho(x)), & \text{if } \sigma < 1. \end{cases} \tag{5.29}$$

On the other hand, if $C_2 > 0$, then we have the inequality

$$w_2(x+z) - w_2(x) \leq C_1 + C_2 \quad \text{for all } z \in B^c_{\varrho(x)}.$$

Using this, now we can write

$$\begin{aligned} & I_\xi[B^c_{\varrho(x)}](w_2, x) \\ & \leq \int_{B^c_{\varrho(x)}} [w_2(x+z) - w_2(x)]\nu_\xi(dz) + |Dw_2(x)| \int_{B\setminus B_{\varrho(x)}} |z|\nu_\xi(dz) \\ & \leq (C_1 + C_2) \int_{B^c_{\varrho(x)}} \nu_\xi(dz) + C_1 d_r(x)^{\gamma-1} \int_{B\setminus B_{\varrho(x)}} |z|\nu_\xi(dz), \end{aligned}$$

where the last integral is suppressed if $\sigma < 1$. Applying (M1) and using that $\varrho(x) \leq d_r(x)$ we conclude in this case that

$$I_\xi[B^c_{\varrho(x)}](w_2, x) \leq C_R(C_1 + C_2)\varrho(x)^{-\sigma} + C_R C_1 \varrho^{\gamma-1}(x) h_{1,\sigma}(\varrho(x)),$$

where the last term does not exist if $\sigma < 1$. Thus, since $\gamma > 0$ we get

$$I_\xi[B^c_{\varrho(x)}](w_2, x) \leq C_R(C_1 + C_2)\varrho(x)^{-\sigma}. \tag{5.30}$$

In summary, when $C_2 = 0$, joining (5.29) and (5.27) we have

$$I_\xi(w_2, x) \leq C_R\, C_1 \begin{cases} \varrho(x)^{\gamma-1} h_{1,\sigma}(\varrho(x)), & \text{if } \sigma \geq 1 \\ h_{\gamma,\sigma}(\varrho(x)), & \text{if } \sigma < 1, \end{cases} \tag{5.31}$$

meanwhile, when $C_2 > 0$, using (5.30) and (5.27) we conclude that

$$I_\xi(w_2, x) \le C_R(C_1 + C_2)\varrho(x)^{-\sigma}. \tag{5.32}$$

3.- *Conclusion.* The estimate (5.24) comes from (5.26) and (5.31) when $C_2 = 0$, and from (5.26) and (5.32) when $C_2 > 0$. The proof is complete. □

Using the last lemma we are able to prove w is a strict supersolution for a problem ad-hoc to (P'). This is established in the following two lemmas, whose main difference is whether C_2 is strictly positive or not.

Lemma 5.3. (Strict Supersolution, Case $\mathbf{C_2} > 0$) *Let $x_0 \in \mathbb{R}^N$, $\sigma \in (0, 2)$ and $\{\nu_x\}_{x \in \mathbb{R}^N}$ a family of measures satisfying (M1), (M2) relative to σ. Consider I_x as in (5.2), (5.9) associated to $\{\nu_x\}_{x \in \mathbb{R}^N}$. Let $m > \max\{1, \sigma\}$, $\theta \in [0, m)$ and γ_0 given in (5.12).*

Then, for each $A, b_0, C_2 > 0$, there exists $C_1 > 0$ large enough such that, for all $r \in (0, 1)$ and $\gamma \in (0, \gamma_0]$, the function w defined in (5.21) (relative to x_0, γ, C_1, C_2 and r) satisfies the inequality

$$-\sup_{\xi \in B_1(x)} \{I_\xi(w, x)\} + b_0|Dw(x)|^m \ge A\varrho(x)^{-\theta} \quad \text{for } x \in B_r(x_0) \setminus \{x_0\}, \tag{5.33}$$

where ϱ defined in (5.23) is associated to x_0 and r.

Proof: Let $x \in B_r(x_0) \setminus \{x_0\}$. Direct computations over w_1, w_2 defined in (5.22) give us the expression

$$Dw(x) = C_1\gamma(d_0(x)^{\gamma-1} + d_r(x)^{\gamma-1})\frac{x - x_0}{|x - x_0|},$$

concluding that

$$|Dw(x)| = C_1\gamma(d_0(x)^{\gamma-1} + d_r(x)^{\gamma-1}) \ge CC_1\varrho(x)^{\gamma-1}.$$

Using this together with the estimates given by Lemma 5.2 for the nonlocal term in the case $C_2 > 0$, we obtain the existence of an universal constant $\bar{C} > 0$ such that for all C_1, C_2 and b_0, and for all $x \in B_r(x_0) \setminus \{x_0\}$ we have

$$\begin{aligned} &-\sup_{\xi \in B_1(x)} \{I_\xi(w, x)\} + b_0|Dw(x)|^m \\ &\ge \bar{C}\Big(b_0C_1^m\varrho(x)^{m(\gamma-1)} - (C_1 + C_2)C_R\varrho(x)^{-\sigma}\Big). \end{aligned} \tag{5.34}$$

But since $\gamma_0 = \min\{m - \sigma, m - \theta\}/m$ and $\gamma \le \gamma_0$ we have $m(\gamma - 1) \le \min\{-\sigma, -\theta\}$. Then, we conclude from (5.34) that

$$-\sup_{\xi \in B_1(x)} \{I_\xi(w, x)\} + b_0|Dw(x)|^m \ge \bar{C}\varrho(x)^{m(\gamma-1)}\Big(b_0C_1^m - (C_1 + C_2)C_R\Big).$$

Hence, we arrive at (5.33) by taking

$$C_1 = (4A(\bar{C}b_0)^{-1})^{1/m} + (4C_2C_Rb_0^{-1})^{1/m} + (2C_Rb_0^{-1})^{1/(m-1)},$$

that is, we should take C_1 satisfying

$$C_1 \ge C(A^{1/m} + C_2^{1/m} + 1), \tag{5.35}$$

where $C > 0$ is a constant not depending on C_2 or A. □

Next lemma deals with the case $C_2 = 0$.

Lemma 5.4. (Strict Supersolution, Case $C_2 = 0$) *Let $x_0 \in \mathbb{R}^N$, $\sigma \in (0,2)$ and $\{\nu_x\}_{x\in\mathbb{R}^N}$ a family of measures satisfying (M1), (M2) relative to σ. Consider I_x as in (5.2), (5.9) associated to $\{\nu_x\}_{x\in\mathbb{R}^N}$. Let $m > \max\{1,\sigma\}$, $\theta \in [0,m)$ and γ_0 defined in (5.14). Assume $C_2 = 0$.*

Then, for each $A, b_0 > 0$, there exists $C_1 > 0$ large enough such that, for all $r \in (0,1)$ and $\gamma \in (0,\gamma_0]$, the function w defined in (5.21) (relative to x_0, γ, C_1 and r) satisfies the inequality (5.33).

The proof of this lemma follows exactly as Lemma 5.3 using the estimate given by Lemma 5.2 in the case $C_2 = 0$ and the definition of γ_0 given in (5.14).

Remark 5.2. As we mentioned in the introduction, the power profile of w gives us the Hölder regularity for subsolutions to (P'). The different uses of Lemmas 5.3 and 5.4 can be described as follows: as it can be seen in the proof of Theorem 5.1 below, the application of Lemma 5.3 under a correct choice of $C_2 > 0$ allows us to localize the arguments to obtain an interior Hölder regularity with a Hölder seminorm (cast by C_1) which is independent of the distance to the boundary, a key fact to conclude the regularity up to the boundary. However, the discontinuity of w due to $C_2 > 0$ implies a "worse" bound for $I_x(w)$ (see Lemma 5.2), restricting the values of the Hölder exponent if we look for regularity up to the boundary, no matter the nonlocal operator has censored nature or not.

On the other hand, Lemma 5.4 is used in the proof of Theorem 5.2, where no localization is needed. Thus, the "better" bounds for $I_x(w)$ given by Lemma 5.2 allows to obtain interior Hölder regularity with "more natural" exponents.

5.2.3 Proofs of the Main Theorems.

We start with the regularity result up to the boundary.

Proof of Theorem 5.1: Applying Lemma 5.1, we see that u satisfies the censored equation

$$-I_\Omega(u,x) + \frac{b_0}{2}|Du|^m \leq A(d_\Omega(x)^{-\theta} + 1) + C(\mathrm{osc}_{\Omega_\nu}(u) + 1)d_\Omega(x)^{-\sigma}, \quad x \in \Omega,$$

where $C > 0$ is the constant given in Lemma 5.1. If we define $\eta = \max\{\sigma, \theta\}$, in particular we see that u satisfies the viscosity inequality

$$-I_\Omega(u,x) + \frac{b_0}{2}|Du|^m \leq \tilde{A} d_\Omega(x)^{-\eta}, \quad x \in \Omega, \tag{5.36}$$

where

$$\tilde{A} = A(1 + \mathrm{diam}(\Omega)^\eta) + C(\mathrm{osc}_{\Omega_\nu}(u) + 1). \tag{5.37}$$

From this point, we will argue over equation (5.36).

Let $x_0 \in \Omega$ and denote $R = |x_0| + 1$. Consider γ_0 as in (5.12), and for $C_1, C_2 > 0$ to be fixed later and $r = \min\{1, d_\Omega(x_0)\}/4$, consider w as in (5.21) (with $\gamma = \gamma_0$) associated to these parameters.

Denote

$$M := \sup\{u(x) - u(x_0) - w(x) : x \in \bar{\Omega}\}.$$

The aim is to prove that for suitable $C_1 > 0$ we get $M \leq 0$, which implies easily the Hölder continuity of u. We argue by contradiction, assuming that $M > 0$. Choosing

$$C_2 \geq \mathrm{osc}_{\Omega_\nu}(u), \tag{5.38}$$

by definition of w, for each $x \in \bar{\Omega} \setminus \bar{B}_r(x_0)$ we have

$$u(x) - u(x_0) - w(x) \leq \mathrm{osc}_{\Omega_\nu}(u) - (2C_1 r^\gamma + \mathrm{osc}_{\Omega_\nu}(u)) < 0.$$

Hence, by the upper semicontinuity of $u - w$, it follows that the supremum defining M is attained in $\bar{B}_r(x_0)$. Moreover, since $w(x_0) = 0$, the point attaining the maximum in M is in $\bar{B}_r(x_0) \setminus \{x_0\}$.

Let $A_0 > 0$ be fixed later. By Lemma 5.3, we can consider C_1 large enough in order to have

$$- \sup_{\xi \in B_1(x)} \{\tilde{I}_\xi(w, x)\} + \frac{b_0}{2}|Dw(x)|^m \geq A_0 \varrho(x)^{-\eta}, \quad x \in B_r(x_0) \setminus \{x_0\}, \tag{5.39}$$

in fact, by (5.35) it is sufficient to take

$$C_1 \geq C(A_0^{1/m} + C_2^{1/m} + 1) \tag{5.40}$$

for some universal constant $C > 0$. Doubling variables and penalizing, we consider

$$M_\epsilon := \sup\{\Phi(x, y) : (x, y) \in \bar{\Omega} \times \bar{\Omega}\},$$

where $\Phi(x, y) = u(x) - u(x_0) - w(y) - \epsilon^{-2}|x - y|^2$.

By classical arguments in the viscosity theory, we have $M_\epsilon \geq M > 0$ for all $\epsilon > 0$ and the supremum in M_ϵ is attained at $(\bar{x}, \bar{y}) \in \bar{\Omega} \times \bar{\Omega}$ with $\bar{y} \in \bar{B}_r(x_0) \setminus \{x_0\}$, which in addition satisfies the following properties

$$\epsilon^{-2}|\bar{x} - \bar{y}|^2 \to 0; \quad \bar{x}, \bar{y} \to x^*; \quad u(\bar{x}) \to u(x^*), \quad \text{as } \epsilon \to 0, \tag{5.41}$$

where $x^* \in \bar{B}_r(x_0) \setminus \{x_0\}$ attains the supremum defining M. In particular, $\bar{y} \neq x_0$ for all $\epsilon > 0$. Moreover, note that the function

$$-\Phi(\bar{x}, \cdot) : \, y \mapsto w(y) - (u(\bar{x}) - u(x_0) - \epsilon^{-2}|\bar{x} - y|^2)$$

has a global minimum point at $\bar{y} \in \bar{B}_r(x_0) \setminus \{x_0\}$ for all $\epsilon > 0$. We claim that this fact implies $\bar{y} \notin \partial B_r(x_0)$ for each $\epsilon > 0$. Otherwise, denoting $\xi = (x_0 - \bar{y})/|x_0 - \bar{y}|$ we have $\bar{y} + s\xi \in B_r(x_0)$ for each $0 < s < r$. Therefore $-\Phi(\bar{x}, \bar{y}) \leq -\Phi(\bar{x}, \bar{y} + s\xi)$, which implies by definition of w in (5.21)

$$0 \leq s^{-1}(w(\bar{y}) - w(\bar{y} + s\xi)) \leq \epsilon^{-2}(-2\langle \bar{x} - \bar{y}, \xi \rangle + s)$$

and

$$0 \leq C_1(s^{-1}(r^\gamma - (r - s)^\gamma) + s^{\gamma - 1}) \leq \epsilon^{-2}(-2\langle \bar{x} - \bar{y}, \xi \rangle + s).$$

Making $s \to 0$ we arrive at a contradiction, concluding the claim. Hence, for all $\epsilon > 0$, there exists $r_\epsilon \in (0, r)$ such that $r_\epsilon < |\bar{y} - x_0| < r - r_\epsilon$.

On the other hand, using that $(\bar{x}, \bar{y})$ is a maximum point for Φ, denoting $h = x - y$ and $\bar{h} = \bar{x} - \bar{y}$ we have

$$u(h + y) - w(y) - \epsilon^{-2}|h|^2 \leq u(\bar{h} + \bar{y}) - w(\bar{y}) - \epsilon^{-2}|\bar{h}|^2,$$

for each $y \in \bar{\Omega}$ and h such that $y + h \in \bar{\Omega}$. Hence, we conclude

$$\bar{u}(y) - w(y) \leq \bar{u}(\bar{y}) - w(\bar{y}) \quad \text{for all } y \in \Omega - \bar{h},$$

where $\bar{u}(y) := u(\bar{h} + y)$ for each $y \in \Omega - \bar{h}$. In particular, $\bar{y}$ is a maximum point for $\bar{u} - w$ in $\Omega - \bar{h}$. Now, a simple translation argument over equation (5.36) allows us to prove that $\bar{u}$ satisfies the equation

$$-\tilde{I}_{x+\bar{h}}(\bar{u}, x) + \frac{b_0}{2}|D\bar{u}(x)|^m \leq \tilde{A} d_\Omega^{-\eta}(x + \bar{h}), \quad x \in \Omega - \bar{h},$$

in the viscosity sense. Since $|\bar{h}| \to 0$ as $\epsilon \to 0$, for all ϵ small enough we have $\bar{y} \in B_r(x_0) \subset \Omega - \bar{h}$. Recalling w is smooth at $\bar{y}$ we can use it as a test function for $\bar{u}$ at $\bar{y}$, concluding the inequality

$$-\tilde{I}_{\bar{y}+\bar{h}}(w, \bar{y}) + \frac{b_0}{2}|Dw(\bar{y})|^m \leq \tilde{A} d_\Omega^{-\eta}(\bar{y} + \bar{h}),$$

but since $\bar{y} + \bar{h} \in B_1(\bar{y})$ for ϵ small enough, using (5.39) we get

$$A_0 \varrho^{-\eta}(\bar{y}) \leq \tilde{A} d_\Omega^{-\eta}(\bar{y} + \bar{h}).$$

Note that for each $x \in B_r(x_0)$ we have $\varrho(x) \leq d_\Omega(x)$ and since $\eta \geq 0$, we get from the above inequality that

$$A_0 d_\Omega^{-\eta}(\bar{y}) \leq \tilde{A} d_\Omega^{-\eta}(\bar{y} + \bar{h}).$$

At this point, recalling $\bar{h} \to 0$ and $\bar{y} \to x^* \in \bar{B}_r(x_0)$ as $\epsilon \to 0$, taking limits in the above inequality we arrive at a contradiction previously fixing

$$A_0 \geq \tilde{A} + 1. \tag{5.42}$$

Thus, for each $x_0 \in \Omega$ and $r \leq d_\Omega(x_0)/4$, we have

$$|u(x) - u(y)| \leq C_1 |x - y|^{\gamma_0} \quad \text{for all } x, y \in B_r(x_0),$$

from which we conclude the local Hölder continuity. In the case the boundary is $C^{1,1}$, from the above inequality we note that for each $B_r(x_0) \subset \Omega$, the Hölder exponent and seminorm of u in $B_r(x_0)$ does not depend on r, and applying the method used by Barles in [15] (see also [50]) we can extend the Hölder regularity up to the boundary.

Finally, we recall that by (5.42), (5.37), (5.40) and the choice of C_2 in (5.38), the constant C_1 leading to the contradiction has the form

$$C_1 \geq C(A^{1/m} + \mathrm{osc}_{\Omega_\nu}(u)^{1/m} + 1), \tag{5.43}$$

for some constant $C > 0$ depending on the data. □

A very important consequence of the previous result is the following control of the oscillation.

Corollary 5.1. **(Oscillation Bound)** Let $\Omega \subset \mathbb{R}^N$ be open and bounded with a $C^{1,1}$ boundary, and assume the hypotheses of Theorem 5.1 hold. Assume further the nonlocal operator has a censored nature, that is, the family of measures $\{\nu_x\}_{x \in \mathbb{R}^N}$ satisfies the censored condition (5.15). Then, there exists $K > 0$ such that, for each bounded viscosity subsolution u of (P'), we have

$$\mathrm{osc}_\Omega(u) \leq K.$$

Proof: The choice of C_1 given by (5.43) in Theorem 5.1 leads us to

$$|u(x) - u(y)| \leq C(A^{1/m} + \mathrm{osc}_{\Omega_\nu}(u)^{1/m} + 1)|x - y|^{\gamma_0}, \quad \text{for all } x, y \in \bar{\Omega},$$

where γ_0 is given by (5.12). Now, by (5.15) we have $\mathrm{osc}_{\Omega_\nu}(u) = \mathrm{osc}_\Omega(u)$ and by compactness of $\bar{\Omega}$, there exists $\underline{x}, \bar{x} \in \bar{\Omega}$ such that $\mathrm{osc}_\Omega(u) = u(\bar{x}) - u(\underline{x})$. Then, we can write

$$\mathrm{osc}_\Omega(u) \leq C(A^{1/m} + \mathrm{osc}_\Omega(u)^{1/m} + 1),$$

from where we obtain the result since $m > 1$. □

Note that for noncensored problems, we can provide global oscillation bounds as in the last corollary if we a priori know that $\mathrm{osc}_{\Omega_\nu}(u) = \mathrm{osc}_\Omega(u)$.

Proof of Theorem 5.2: Let $x_0 \in \Omega$, denote $R = |x_0| + 1$ and fix $r = \min\{1, d_\Omega(x_0)\}/4$. Consider γ_0 as in (5.14) and for $C_1 > 0$ to be fixed later, define w as in (5.21) (with $\gamma = \gamma_0$) associated to these parameters.

Since the proof follows the same lines of Theorem 5.1, we will be sketchy in the current proof bringing light on its contrasts. The first difference is that this time we do not censorize the equation (since it would restrict the Hölder exponent, see Lemma 5.1).

Denote

$$M := \sup\{u(x) - u(x_0) - w(x) : x \in \mathbb{R}^N\}. \tag{5.44}$$

The aim is to prove that for suitable $C_1 > 0$ we get $M \leq 0$. We argue by contradiction, assuming that $M > 0$. Note that choosing

$$C_1 r^{\gamma_0} \geq \mathrm{osc}_{\Omega_\nu}(u), \tag{5.45}$$

and by the upper semicontinuity of $u - w$ we have the supremum defining M is attained in $\bar{B}_r(x_0)$.

Let $A_0 > 0$ be fixed later. Enlarging C_1 if it is necessary, by Lemma 5.4 we can write

$$-\sup_{\xi \in B_1(x)} \{I_\xi(w, x)\} + b_0|Dw(x)|^m \geq A_0 \varrho(x)^{-\theta}, \quad x \in B_r(x_0) \setminus \{x_0\}. \tag{5.46}$$

Doubling variables and penalizing, we consider

$$M_\epsilon := \sup\{\Phi(x, y) : (x, y) \in \mathbb{R}^N \times \mathbb{R}^N\},$$

where $\Phi(x, y) = u(x) - u(x_0) - w(y) - \epsilon^{-2}|x - y|^2$. By classical arguments in the viscosity theory, we have $M_\epsilon \geq M > 0$ for all $\epsilon > 0$ and the supremum in M_ϵ is attained at $(\bar{x}, \bar{y})$ with $\bar{x}, \bar{y} \in \mathbb{R}^N$ with $\bar{y} \in \bar{B}_r(x_0) \setminus \{x_0\}$, which in addition satisfies (5.41) where $x^* \in B_r(x_0) \setminus \{x_0\}$ attains the supremum in (5.44).

If $\gamma_0 < 1$, then we can prove that $\bar{y} \notin \partial B_r(x_0)$ in the same way as in Theorem 5.1 using that w satisfies a state constraint problem on $\partial B_r(x_0)$. If $\gamma_0 = 1$ (which is the case of $\theta = 0$ and $\sigma < 1$), then we consider w with $\gamma < \gamma_0$ and continue with the proof, taking into account that the Hölder seminorm does not change as $\gamma \to \gamma_0$.

From this point, we follow the remaining lines of Theorem 5.1, taking A_0 large in terms of A arising in (P') . □

5.2.4 Examples.

In this section we provide some examples of nonlocal terms and Hamiltonians for which our results hold.

We start with the assumptions over the nonlocal term. As we mentioned before, assumptions (M1) and (M2) are intended as a restriction on the order of the operator, which is less or equal than σ. In the case of x-independent operators, that is the case when there exists a measure ν such that the family $\{\nu_x\}_{x\in\mathbb{R}^N}$ defining I_x satisfies $\nu_x = \nu$ for each $x \in \mathbb{R}^N$, the operator may range from zero order operators (when ν is finite, see [53]) to the factional Laplacian of order s for $s \leq \sigma$, passing through operators which are not uniformly elliptic in the sense of Caffarelli and Silvestre [47], as it is the case of measures with the form

$$\nu(dz) = \mathbf{1}_{\mathbb{H}_+}(z)|z|^{-(N+s)}dz,$$

where $0 < s \leq \sigma$ and $\mathbb{H}_+ = \{(z', z_N) \in \mathbb{R}^N : z_N > 0\}$. Another interesting example of such non-uniformly elliptic operators is given by operators with "orthogonal diffusion", for example in the case ν has the form

$$\nu(dz) = |z_2|^{-(N+s_2)}dz_2 \otimes \delta_0(z_1)dz_1 + |z_1|^{-(N+s_1)}dz_1 \otimes \delta_0(z_2)dz_2 \tag{5.47}$$

where $z = (z_1, z_2)$ with $z_i \in \mathbb{R}^{d_i}, i = 1, 2$ and $N = d_1 + d_2$, and $0 < s_1, s_2 \leq \sigma$. Here δ_0 denotes the Dirac measure supported at 0 and $\otimes$ denotes the measure product. In this case, such a measure gives rise to an operator which is the sum of fractional Laplacians in each direction $z_i, i = 1, 2$.

Concerning x-dependent nonlocal operators, the classical example comes from measures ν_x with the form

$$\nu_x(dz) = K(x, z)\nu(dz),$$

where ν is an x-independent Lévy measure and $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative function such that $K(\cdot, z) \in L^\infty_{loc}(\mathbb{R}^N)$ for all $x \in \mathbb{R}^N$, and $K(x, \cdot) \in L^\infty(\mathbb{R}^N)$ for all $x \in \mathbb{R}^N$. As a particular case we have the weighted fractional Laplacian

$$-I_x(u, x) = K(x)(-\Delta)^\sigma u(x),$$

where K is bounded and nonnegative.

We highlight that in view of Lemma 5.1, the regularity results apply to censored operators defined in (5.18), where we recall that the measures defining them has the form (5.16).

Concerning H, we note that the structure of the Hamiltonian is encoded by the inequality (5.10). Thus, given σ and $m > \max\{1, \sigma\}$, our results apply to H with the form

$$H(x, p) = b(x)|p|^m + a_1(x)|p|^l + \langle a_2(x), p\rangle - f(x), \tag{5.48}$$

where $x, p \in \mathbb{R}^N$, $b \geq b_0 > 0$, $0 < l < m$ and a_1, a_2, f bounded. In the case $m \leq 1$ we can consider

$$H(x, p) = b(x)|p|^m + a_1(x)|p|^l - f(x), \tag{5.49}$$

with b, a_1, l and f as above. Of course, we can replace the main power $|p|^m$ by $\phi(x, p)|p|^m$, where the function $\phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies $\phi \geq \phi_0$ for some constant $\phi_0 > 0$.

5.2.5 Regularity Results for the Sublinear Case.

In this subsection we provide a regularity results in the case $\sigma < m \leq 1$.

Theorem 5.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and $\sigma \in (0,1)$. Let I_x as in (5.9) associated to a family of measures $\{\nu_x\}_{x \in \mathbb{R}^N}$ satisfying (M1), (M2) relative to σ. Let $m \in (\sigma, 1]$, $\theta \in [0, m)$ and γ_0 as in (5.12).*

Then, for each $b_0, A > 0$ and $\gamma < \gamma_0$, any bounded viscosity subsolution $u : \mathbb{R}^N \to \mathbb{R}$ to the equation (P') is locally Hölder continuous in Ω with Hölder exponent γ. If Ω has $C^{1,1}$ boundary, then u is γ-Hölder continuous in Ω and can be extended as a Hölder continuous function on $\bar{\Omega}$.

The Hölder seminorm depends on the data and $\mathrm{osc}_{\Omega_\nu}(u)$, where Ω_ν is defined in (5.11).

Proof: As in Theorem 5.1, we start with the analogous of Lemma 5.3. Let $r > 0$, consider $x_0 \in \mathbb{R}^N$, define d_0, d_r as in (5.20) and ϱ as in (5.23). Let w defined in (5.21) associated to these parameters and $\gamma < \gamma_0$. Let $A, b_0 > 0$. Performing the same computations as in Lemma 5.3 we arrive at inequality (5.34), that is

$$
\begin{aligned}
&- \sup_{\xi \in B_1(x)} \{I_\xi(w, x)\} + b_0 |Dw(x)|^m \\
&\geq \bar{C}\Big(b_0 C_1^m \varrho(x)^{m(\gamma - 1)} - (C_1 + C_2) C_R \varrho(x)^{-\sigma}\Big),
\end{aligned}
$$

for all $x \in B_r(x_0) \setminus \{x_0\}$. Since this time $m(\gamma - 1) < -\sigma$ and $\varrho(x) \leq r$ for each $x \in B_r(x_0)$, we can take $r = r(C_1, C_2, b_0)$ small such that

$$
- \sup_{\xi \in B_1(x)} \{I_\xi(w, x)\} + b_0 |Dw(x)|^m \geq \frac{\bar{C} b_0 C_1^m}{2} \varrho(x)^{m(\gamma-1)}, \quad x \in B_r(x_0) \setminus \{x_0\}.
$$

By the choice of $\gamma < \gamma_0$, we see that $m(\gamma - 1) \leq -\theta$, and therefore w satisfies

$$
- \sup_{\xi \in B_1(x)} \{I_\xi(w, x)\} + b_0 |Dw(x)|^m \geq \bar{C} C_1^m \varrho(x)^{-\eta}, \quad x \in B_r(x_0) \setminus \{x_0\},
$$

with $\eta = \max\{\sigma, \theta\}$. From this point, we proceed exactly as in the proof of Theorem 5.1, where the last inequality plays the role of (5.39), concluding the result by taking C_1 large in terms of A. $\square$

Remark 5.3. Since $m \leq 1$, the parameter r depends on C_2 in the proof of Theorem 5.3 and therefore we have a Hölder seminorm which does not give a control of the oscillation in the general case.

Interior regularity results for the sublinear case in the flavour of Theorem 5.2 can be obtained in the same way as the previous theorem.

Theorem 5.4. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Let $\sigma \in (0,1)$, I_x as in (5.9) associated to a family of measures $\{\nu_x\}_{x \in \mathbb{R}^N}$ satisfying (M1), (M2) relative to σ. Let $m \in (\sigma, 1]$, $\theta \in [0, m)$ and γ_0 as in (5.14).*

Then, for each $b_0, A > 0$ and $\gamma < \gamma_0$, any bounded viscosity subsolution $u : \mathbb{R}^N \to \mathbb{R}$ to the equation (P') is locally Hölder with Hölder exponent γ. Moreover, for each $\delta > 0$, the Hölder seminorm of u in Ω^δ depends on the data and $\mathrm{osc}_{\Omega_\nu}(u)\delta^{-\gamma}$.

5.2.6 Extension to Lévy-Ito Operators.

We present an important extension of our regularity results over equations associated to nonlocal operators in *Lévy-Ito form*: for $x \in \mathbb{R}^N$ and a bounded function $\phi \in C^2(\bar{B}_\delta(x))$ for some $\delta > 0$, we consider I_x^j defined as

$$I_x^j(u,x) = \int_{\mathbb{R}^N} [u(x + j(x,z)) - u(x) - \mathbf{1}_B \langle Du(x), j(x,z)\rangle]\nu(dz), \tag{5.50}$$

where ν is a positive regular measure in $\mathbb{R}^N$. The function $j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ should be understood as a *jump function*, whose basic assumption concerns the following bound for the jumps, which is uniform in x.

(J1) There exists $C_j > 0$ such that, for all $x \in \mathbb{R}^N$

$$|j(x,z)| \leq C_j |z|.$$

We remark that given ν and j as above, it is possible to define the associated x-dependent measure ν_x^j as the push forward of the measure ν through the function $j(x,\cdot)$. That is, ν_x^j is defined as

$$\int_{\mathbb{R}^N} f(y)\nu_x^j(dy) = \int_{\mathbb{R}^N} f(j(x,z))\nu(dz), \tag{5.51}$$

for each measurable function f satisfying $|f(z)| \leq C \min\{1, |z|^2\}$ for some $C > 0$. It is important to remark that if ν satisfies (M1),(M2) and j satisfies (J1), then $\{\nu_x^j\}_{x \in \mathbb{R}^N}$ satisfies (M1), (M2) too, where the associated constants now depend on C_j.

We also notice that in the case the family of measures $\{\nu_x^j\}_x$ satisfies (M1)-(M2) with $\sigma \in (0,1)$, then we do not need to compensate the integrand and I_x^j is defined as

$$I_x^j(u,x) = \int_{\mathbb{R}^N} [u(x + j(x,z)) - u(x)]\nu(dz). \tag{5.52}$$

For sake of shortness, from this point we mainly argue over I_x^j with the form (5.50), but all the results are valid for I_x^j with the form (5.52) when $\sigma \in (0,1)$.

The following result states the regularity result up to the boundary for Lévy-Ito problems.

Theorem 5.5. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, $A, b_0 > 0$, $\sigma \in (0,2)$, a measure ν satisfying (M1)-(M2) relative to σ, and a jump function j satisfying (J1). Let I_x^j as in (5.50), (5.52) associated to ν and j. Let $m > \max\{1, \sigma\}$ and $\theta \in [0, m)$.*

Then, any bounded viscosity subsolution $u : \mathbb{R}^N \to \mathbb{R}$ to the problem

$$-I_x^j(u,x) + b_0\, |Du(x)|^m \leq A d_\Omega(x)^{-\theta}, \quad x \in \Omega \tag{5.53}$$

is locally Hölder continuous in Ω with Hölder exponent γ_0 given in (5.12), and Hölder semi-norm depending on Ω, the data and $\mathrm{osc}_{\Omega_{\nu^j}}(u)$, where Ω_{ν^j} is defined as in (5.11) relative to the familty of measures $\{\nu_x^j\}_{x \in \mathbb{R}^N}$ given by (5.51).

Moreover, if Ω has a $C^{1,1}$ boundary, then u can be extended as a Hölder continuous function to $\bar{\Omega}$ with Hölder exponent γ_0.

Proof: This proof follows the lines of Theorem 5.1 and therefore we provide only a sketch of the proof in order to show how to treat the Lévy-Ito form.

1.- Technical lemmas in the Lévy-Ito context. Under the current assumptions, considering $x_0 \in \mathbb{R}^N$, $C_1, C_2, r > 0$ and γ_0 as in (5.12), w defined in (5.21) (with $\gamma = \gamma_0$) satisfies the inequality

$$\sup_{\xi \in B_1(x)} \{I_\xi^j(w,x)\} \le C(C_1 + C_2)\varrho^{-\sigma}(x), \quad \text{for all } x \in B_r(x_0) \setminus \{x_0\},$$

where ϱ is defined in (5.23) and this time the constant C depends also on C_j arising in (J1). This is accomplished replacing ϱ by

$$\tilde{\varrho}(x) = \min\{d_0(x), d_r(x)\}/(4C_j),$$

in the proof of Lemma 5.2. Once we get this estimate, taking $C_1 > 0$ as in (5.43) (with C now depending on C_j) we conclude

$$-\sup_{\xi \in B_1(x)} \{I_\xi^j(w,x)\} + b_0|Dw(x)|^m \ge A\varrho^{-\theta}(x) \quad \text{for } x \in B_r(x_0) \setminus \{x_0\},$$

following directly the arguments given in Lemma 5.3.

2.- Censored Lévy-Ito operators. Let u be a bounded subsolution to (5.53). Arguing as in Lemma 5.1, the Lévy-Ito analogous to inequality (5.36) reads as

$$-I_\Omega^j(u,x) + \frac{b_0}{2}|Du|^m \le Ad_\Omega(x)^{-\theta} + C(\mathrm{osc}_{\Omega_\nu^j}(u) + 1)d_\Omega^{-\sigma}(x), x \in \Omega,$$

where C depends on C_j and the censored Lévy-Ito operator I_Ω^j is defined as

$$I_\Omega^j(u,x) = \int\limits_{x + j(x,z) \in \Omega} [u(x + j(x,z)) - u(x) - 1_B\langle Du(x), j(x,z)\rangle]\nu(dz).$$

3.- Conclusion. Once we localize the equation inside Ω, we follow exactly the same lines of the proof of Theorem 5.1. The corresponding inequality (5.43) this time reads as

$$C_1 \ge C(A^{1/m} + \mathrm{osc}_{\Omega_\nu^j}(u) + 1), \tag{5.54}$$

where C depends on C_j. $\square$

The immediate consequence of this theorem is the corresponding control of the oscillation. Its proof follows the same lines of the one of Corollary 5.1 by using the above theorem.

Corollary 5.2. Let $\Omega \subset \mathbb{R}^N$ open and bounded, and assume the hypotheses of Theorem 5.5 hold. Assume further the nonlocal operator has a censored nature, that is, the family of measures $\{\nu_x^j\}_{x \in \mathbb{R}^N}$ defined in (5.51) satisfies the censored condition (5.15). Then, there exists $K > 0$ such that, for each bounded viscosity solution of (5.53) we have

$$\mathrm{osc}_\Omega(u) \le K.$$

Following the directions given in Theorem 5.5, it is possible to provide an interior regularity result in the flavour of Theorem 5.2, as well as regularity results for sublinear Hamiltonians in the flavour of Theorems 5.3 and 5.4, both in the Lévy-Ito framework. Additionally, we can provide extensions for the mentioned results associated to Lévy-Ito operators when the domain is unbounded (see Remark 5.1). We omit the details.

5.3 Well-Posedness for the Cauchy Problem in Lévy-Ito Form.

The x-dependence of the nonlocal term represents a serious difficulty in the statement of the comparison principle for integro-differential equations (see [23]), and this comparison principle is a key tool in the study of the large time behavior of evolution equations. However, we are able to prove it in the interesting case of nonlocal operators in *Lévy-Ito* form defined in (5.50) and (5.52). It is why, from now on, we consider the Cauchy problem in Lévy-Ito form

$$\begin{aligned} \partial_t u(x,t) - I_x^j(u(\cdot,t),x) + H(x, Du(x,t)) &= 0 \quad (x,t) \in Q, & (5.55)\\ u(\cdot,0) &= u_0 \quad x \in \mathbb{R}^N, & (5.56) \end{aligned}$$

where we recall that $Q = \mathbb{R}^N \times (0,\infty)$.

We start with the assumptions. Over ν we require the classical assumption

(M) There exists $C_\nu > 0$ such that

$$\int_{\mathbb{R}^N} \min\{1, |z|^2\}\nu(dz) \leq C_\nu.$$

We also require the following compatibility condition among j and ν.

(J2) For each $\delta > 0$, there exists $C_\delta > 0$ such that, for each $x, y \in \mathbb{R}^N$ we have

$$\int_{B_\delta} |j(x,z) - j(y,z)|^2 \nu(dz) \leq C_\delta |x-y|^2,$$

$$\int_{B\setminus B_\delta} |j(x,z) - j(y,z)| \nu(dz) \leq C_\delta |x-y|.$$

Concerning the Hamiltonian we assume the following conditions.

(H1) There exists $m > 1$ and moduli of continuity ζ_1, ζ_2 such that, for all $x, y, p, q \in \mathbb{R}^N$ we have

$$H(y, p+q) - H(x,p) \leq \zeta_1(|x-y|)(1 + |p|^m) + \zeta_2(|q|)|p|^{m-1}.$$

(H2) Let m be as in (H1). There exists $A, b_0 > 0$ such that for all $\mu \in (0,1)$ we have

$$H(x,p) - \mu H(x, \mu^{-1}p) \leq (1-\mu)\Big(b_0(1-m)|p|^m + A\Big).$$

Note that a measure ν satisfying (M1)-(M2) satisfies (M).

Concerning (J1)-(J2), let us give an example. Consider

$$j(x,z) = g(x)z \quad \text{for all } x, z \in \mathbb{R}^N. \tag{5.57}$$

If $g : \mathbb{R}^N \to \mathbb{R}$ is bounded then (J1) holds but (J2) may fail. If, in addition, g is Lipschitz continuous and the measure $|z|\nu(dz)$ is finite away the origin, then (J2) holds.

If $m > 1$, assumption (H2) implies (5.10). Examples of Hamiltonians satisfying (H1) and (H2) are provided in subsection 5.2.4, see (5.48), (5.49).

Remark 5.4. In this section we will argue over nonlocal operators I_x^j with the form (5.50) (that is, nonlocal operators of order $\sigma \geq 1$). However, the same arguments can be used to get the results related to I_x^j with the form (5.52), replacing (M) by the condition

$$\int_{\mathbb{R}^N} \min\{1, |z|\}\nu(dz) \leq C_\nu < +\infty.$$

Our comparison principle reads as follows

Proposition 5.1. *Let ν be a Lévy measure satisfying (M), j satisfying (J1) and both satisfying (J2). Let I_x^j defined as in (5.50) associated to ν and j. Assume H satisfies (H1),(H2) and $u_0 \in C_b(\mathbb{R}^N)$.*

For each $T > 0$, denote $Q_T = \mathbb{R}^N \times (0, T]$. Let $u, v \in L^\infty(\bar{Q}_T)$ for each $T > 0$ be respective viscosity sub and supersolution to (5.55)-(5.56). Then, $u \leq v$ in $\bar{Q}$.

We would like to mention that comparison principles for problem (5.55)-(5.56) for the sublinear case (that is $m \leq 1$ in (H1)) are proven in [23] and for this reason we concentrate only in the superlinear case.

The following lemma states the initial condition for viscosity sub and supersolutions is satisfied in the classical sense.

Lemma 5.5. *Let I_x^j defined in (5.50) with ν satisfying (M), j satisfying (J1) and H satisfying (H1). Let u, v be respectively a viscosity sub and supersolution to problem (5.55)-(5.56), satisfying local boundedness in Q. Then, $u(x, 0) \leq u_0(x) \leq v(x, 0)$ for all $x \in \mathbb{R}^N$.*

We refer to [94] for a proof of the corresponding result in the second-order setting. The proof for the current case can be obtained by adjusting the arguments showed in [94] to the nonlocal framework.

We prove Proposition 5.1 in a rather indirect way by using the following lemma, which will be also used to prove a version of the Strong Maximum Principle valid for our problem in Section 5.4.1.

Lemma 5.6. *Let $\sigma \in (0, 2)$ and let I_x^j defined in (5.50) with ν satisfying (M), j satisfying (J1) and both satisfying (J2). Assume further that $j(\cdot, z) \in C(\mathbb{R}^N)$ for each $z \in \mathbb{R}^N$. Let H satisfying (H1),(H2). Let $u, v \in L^\infty(\bar{Q}_T)$ for all $T > 0$ be respectively a sub and supersolution to (CP). Then, there exists $\bar{c} > 0$ such that, for each $\mu \in (0, 1)$, the function*

$$\omega(x, t) := \mu u(x, t) - v(x, t)$$

satisfies, in the viscosity sense, the equation

$$\partial_t \omega - I_x^j(\omega(\cdot, t), x) - \bar{c}\frac{\zeta_2(|D\omega|)^m}{(1-\mu)^{m-1}} \leq CA(1-\mu) \quad \text{in } Q, \tag{5.58}$$

where $A > 0$ appears in (H2), ζ_2 appears in (H1), $\bar{c} = (m^m b_0^{m-1})^{-1}$ and $C > 0$ is an universal constant.

Proof: We start noting that if u is a viscosity subsolution to (5.55), denoting $\bar{u} = \mu u$ we have

$$\partial_t \bar{u} - I_x^j(\bar{u}, x) + \mu H(x, \mu^{-1} D\bar{u}) \leq 0 \quad \text{in } Q, \tag{5.59}$$

in the viscosity sense.

Let $(x_0, t_0) \in Q$ and ϕ a smooth function such that $\omega - \phi$ has a strict maximum point at (x_0, t_0). Let $\epsilon > 0$. Doubling variables we consider the function

$$\Phi(x, y, s, t) := \bar{u}(x, s) - v(y, t) - \tilde{\phi}(x, y, s, t),$$

where $\tilde{\phi}(x, y, s, t) = \phi(y, t) + \epsilon^{-2}|x - y|^2 + \epsilon^{-2}(s - t)^2$. By its upper semicontinuity, Φ attains its maximum over the set

$$\mathcal{K} := \bar{B}_{2C_j}(x_0) \times \bar{B}_{2C_j}(x_0) \times [0, t_0 + 1] \times [0, t_0 + 1]$$

at a point $(\bar{x}, \bar{y}, \bar{s}, \bar{t})$. Moreover, classical argument in the viscosity theory allows us to get that, as $\epsilon \to 0$

$$\begin{aligned} &\bar{x}, \bar{y} \to x_0; \quad \bar{s}, \bar{t} \to t_0; \quad \epsilon^{-2}|\bar{x} - \bar{y}|^2,\ \epsilon^{-2}(\bar{s} - \bar{t})^2 \to 0; \\ &\bar{u}(\bar{x}, \bar{s}) \to \bar{u}(x_0, t_0),\ v(\bar{y}, \bar{t}) \to v(x_0, t_0), \end{aligned} \tag{5.60}$$

concluding that for all ϵ suitably small, $\bar{s}, \bar{t} \in (0, t_0 + 1)$ and $\bar{x}, \bar{y} \in \bar{B}_{2C_j}(x_0)$. Hence, using that $(x, s) \mapsto \Phi(x, \bar{y}, s, \bar{t})$ has a local maximum point at $(\bar{x}, \bar{s})$ and $(y, t) \mapsto \Phi(\bar{x}, y, \bar{s}, t)$ has a local minimum point at $(\bar{y}, \bar{t})$, we can subtract the viscosity inequality for v at $(\bar{y}, \bar{t})$ to the viscosity inequality for $\bar{u}$ (given by (5.59)) at $(\bar{x}, \bar{s})$ to conclude, for each $\delta' > 0$, the inequality

$$\mathcal{A} - I^{\delta'} \leq 0, \tag{5.61}$$

where for $\delta' > 0$ we denote

$$\begin{aligned} I^{\delta'} = &I^j_{\bar{x}}[B^c_{\delta'}](\bar{u}(\cdot, \bar{s}), \bar{x}, \bar{p}) - I^j_{\bar{y}}[B^c_{\delta'}](v(\cdot, \bar{t}), \bar{y}, \bar{q}) \\ &+ I^j_{\bar{x}}[B_{\delta'}](\tilde{\phi}(\cdot, \bar{y}, \bar{s}, \bar{t}), \bar{x}) - I^j_{\bar{y}}[B_{\delta'}](-\tilde{\phi}(\bar{x}, \cdot, \bar{s}, \bar{t}), \bar{y}), \end{aligned}$$

with

$$\begin{aligned} \bar{p} &:= D_x\tilde{\phi}(\bar{x}, \bar{y}, \bar{s}, \bar{t}) = 2\epsilon^{-2}(\bar{x} - \bar{y}), \\ \bar{q} &:= -D_y\tilde{\phi}(\bar{x}, \bar{y}, \bar{s}, \bar{t}) = \bar{p} - D\phi(\bar{y}, \bar{t}), \end{aligned}$$

and

$$\mathcal{A} = (\partial_t\tilde{\phi} - \partial_s\tilde{\phi})(\bar{x}, \bar{y}, \bar{s}, \bar{t}) + \mu H(\bar{x}, \mu^{-1}\bar{p}) - H(\bar{y}, \bar{q}).$$

We estimate each term of the inequality (5.61) to get the result. We start with $\mathcal{A}$, noting that taking $\epsilon = \epsilon(\mu)$ small enough, we have

$$(1 - \mu)(m - 1)b_0 - \zeta_1(|\bar{x} - \bar{y}|) > 0.$$

Then, from (H1),(H2) we get

$$\begin{aligned} &\mu H(\bar{x}, \mu^{-1}\bar{p}) - H(\bar{y}, \bar{q}) \\ &\geq \mu H(\bar{x}, \mu^{-1}\bar{p}) - H(\bar{x}, \bar{p}) + H(\bar{x}, \bar{p}) - H(\bar{y}, \bar{q}) \\ &\geq (1 - \mu)(m - 1)b_0|\bar{p}|^m - A(1 - \mu) - \zeta_1(|\bar{x} - \bar{y}|)(1 + |\bar{p}|^m) - \zeta_2(|D\phi(\bar{y}, \bar{t})|)|\bar{p}|^{m-1} \\ &\geq \inf_{\theta \geq 0} \Big\{ \Big((1 - \mu)(m - 1)b_0 - \zeta_1(|\bar{x} - \bar{y}|) \Big) \theta^{m/(m-1)} - \zeta_2(|D\phi(\bar{y}, \bar{t})|)\theta \Big\} \\ &\quad - A(1 - \mu) - \zeta_1(|\bar{x} - \bar{y}|), \end{aligned}$$

that is, denoting $\tilde{c} = (m-1)^{m-1}/m^m$, we obtain

$$\begin{aligned}\mu H(\bar{x}, \mu^{-1}\bar{p}) - H(\bar{y}, \bar{q}) \geq & - \tilde{c}\frac{\zeta_2(|D\phi(\bar{y}, \bar{t})|)^m}{((1-\mu)(m-1)b_0 - \zeta_1(|\bar{x}-\bar{y}|))^{m-1}} \\ & - A(1-\mu) - \zeta_1(|\bar{x}-\bar{y}|),\end{aligned}$$

from which we conclude

$$\begin{aligned}\mathcal{A} \geq & \ \partial_t \phi(\bar{y}, \bar{t}) - \tilde{c}\frac{\zeta_2(|D\phi(\bar{y}, \bar{t})|)^m}{((1-\mu)(m-1)b_0 - \zeta_1(|\bar{x}-\bar{y}|))^{m-1}} \\ & - A(1-\mu) - \zeta_1(|\bar{x}-\bar{y}|).\end{aligned} \tag{5.62}$$

Now we address the estimate for $I^{\delta'}$ in (5.61). Using the smoothness of ϕ, (M) and (J1) we clearly have

$$\begin{aligned}& I^j_{\bar{x}}[B_{\delta'}](\tilde{\phi}(\cdot, \bar{y}, \bar{s}, \bar{t}), \bar{x}) - I^j_{\bar{y}}[B_{\delta'}](-\tilde{\phi}(\bar{x}, \cdot, \bar{s}, \bar{t}), \bar{y}) \\ & \leq I^j_{\bar{y}}[B_{\delta'}](\phi(\cdot, \bar{t}), \bar{y}) + \epsilon^{-2} o_{\delta'}(1).\end{aligned} \tag{5.63}$$

On the other hand, since $(\bar{x}, \bar{y}, \bar{s}, \bar{t})$ is a maximum point for Φ in $\mathcal{K}$, and since $\bar{x}, \bar{y} \to x_0$ as $\epsilon \to 0$, for all ϵ small enough, by (J1) we have the inequality

$$\begin{aligned}& \bar{u}(\bar{x} + j(\bar{x}, z), \bar{s}) - v(\bar{y} + j(\bar{y}, z), \bar{t}) - (\bar{u}(\bar{x}, \bar{s}) - v(\bar{y}, \bar{t})) \\ & \leq \phi(\bar{y} + j(\bar{y}, z), \bar{t}) - \phi(\bar{y}, \bar{t}) + \epsilon^{-2}(|\bar{x} - \bar{y} + j(\bar{x}, z) - j(\bar{y}, z)|^2 - |\bar{x} - \bar{y}|^2),\end{aligned}$$

for each $z \in B_1$. Hence, for each $0 < \delta' < \delta < 1$, using this inequality we conclude that

$$\begin{aligned}& I^j_{\bar{x}}[B^c_{\delta'}](\bar{u}(\cdot, \bar{s}), \bar{x}, \bar{p}) - I^j_{\bar{y}}[B^c_{\delta'}](v(\cdot, \bar{t}), \bar{y}, \bar{q}) \\ & \leq J^\delta - \int_{B \setminus B_\delta} \langle \bar{p}, j(\bar{x}, z) - j(\bar{y}, z) \rangle \nu(dz) \\ & \quad + I^j_{\bar{y}}[B_\delta \setminus B_{\delta'}](\phi(\cdot, \bar{t}), \bar{y}) + 2\epsilon^{-2} \int_{B_\delta \setminus B_{\delta'}} |j(\bar{x}, z) - j(\bar{y}, z)|^2 \nu(dz),\end{aligned}$$

where

$$\begin{aligned}J^\delta = \int_{B^c_\delta} \Big[& \bar{u}(\bar{x} + j(\bar{x}, z), \bar{s}) - v(\bar{y} + j(\bar{y}, z), \bar{t}) - (\bar{u}(\bar{x}, \bar{s}) - v(\bar{y}, \bar{t})) \\ & - \mathbf{1}_B \langle D\phi(\bar{y}, \bar{t}), j(\bar{y}, z) \rangle \Big] \nu(dz).\end{aligned} \tag{5.64}$$

Fixing $\delta > 0$ and using (J2) together with (5.60), we conclude that

$$\begin{aligned}& I^j_{\bar{x}}[B^c_{\delta'}](\bar{u}(\cdot, \bar{s}), \bar{x}, \bar{p}) - I^j_{\bar{y}}[B^c_{\delta'}](v(\cdot, \bar{t}), \bar{y}, \bar{q}) \\ & \leq J^\delta + I^j_{\bar{y}}[B_\delta \setminus B_{\delta'}](\phi(\cdot, \bar{t}), \bar{y}) + C_\delta o_\epsilon(1).\end{aligned}$$

Hence, joining the last inequality and (5.63) in the definition of $I^{\delta'}$, we conclude that for all $0 < \delta' < \delta$

$$I^{\delta'} \leq J^\delta + I^j_{\bar{y}}[B_\delta](\phi(\cdot, \bar{t}), \bar{y}) + C_\delta \ o_\epsilon(1) + \epsilon^{-2} o_{\delta'}(1),$$

with J^δ defined in (5.64). Replacing the last inqueality and (5.62) into (5.61), we conclude that

$$\begin{aligned} &\partial_t\phi(\bar{y},\bar{t}) - I^j_{\bar{y}}[B_\delta](\phi(\cdot,\bar{t}),\bar{y}) - J^\delta - \tilde{c}\frac{\zeta_2(|D\phi(\bar{y},\bar{t})|)^m}{((1-\mu)(m-1)b_0 - \zeta_1(|\bar{x}-\bar{y}|))^{m-1}} \\ \leq &(1-\mu)A + C_{j,\delta}o_\epsilon(1) + \epsilon^{-2}o_{\delta'}(1) + \zeta_1(|\bar{x}-\bar{y}|). \end{aligned} \tag{5.65}$$

But by (J2), the continuity assumption over j, the semicontinuity and boundedness of $\bar{u}, v$ in each $\bar{Q}_T$, by using (5.60) we apply Fatou's Lemma concluding that for each $\delta > 0$ fixed, we get

$$\limsup_{\epsilon\to 0} J^\delta \leq I^j_{x_0}[B^c_\delta](\omega(\cdot,t_0), x_0, D\phi(x_0,t_0)).$$

Hence, letting $\delta' \to 0$ and $\epsilon \to 0$ in (5.65), and recalling (5.60) we conclude the desired viscosity inequality leading to (5.58). □

We also require the following

Lemma 5.7. *Let I^j_x defined in (5.50) with ν satisfying (M) and j satisfying (J1). Let $\psi \in C^2_b(\mathbb{R}^d)$ satisfying $||\psi||_{C^2(\mathbb{R}^d)} \leq \Lambda$ for some $\Lambda > 0$. For $\beta > 0$, define the function*

$$\psi_\beta(x) = \psi(\beta^2 x), \quad x \in \mathbb{R}^N. \tag{5.66}$$

Then, ψ_β satisfies

$$||D\psi_\beta||_\infty \leq \Lambda\beta^2, \quad ||D^2\psi_\beta||_\infty \leq \Lambda\beta^4, \quad ||I^j_x(\psi_\beta,\cdot)||_\infty \leq \Lambda o_\beta(1),$$

where $o_\beta(1) \to 0$ as $\beta \to 0$.

Proof: The estimates concerning $D\psi_\beta, D^2\psi_\beta$ are direct. Concerning the estimate of the nonlocal term, for each $x \in \mathbb{R}^d$ we have

$$I^j(\psi_\beta, x) \leq \Lambda\beta^4 \int_B |j(x,z)|^2\nu(dz) + \Lambda\beta^2 \int_{B_{1/\beta}\setminus B} |j(x,z)|\nu(dz) + 2\Lambda \int_{B^c_{1/\beta}} \nu(dz).$$

Hence, using (M) and (J1) in the right-hand side of the last inequality, we get

$$I^j_x(\psi_\beta, x) \leq C_j^2 C_\nu \Lambda\beta^4 + C_j\Lambda\beta^2 \int_{B_{1/\beta}\setminus B} |z|\nu(dz) + 2\Lambda\ o_\beta(1).$$

Finally, using that $|z| \leq 1/\beta$ in the integral term of the last inequality and applying (M), we conclude the estimate for the nonlocal term. □

Using the last three lemmas we are in position to prove the comparison principle for (5.55)-(5.56).

Proof of Proposition 5.1: Let $T > 0$. We will argue over the finite horizon problem

$$\left\{ \begin{array}{rll} \partial_t u - I^j_x(u,x) + H(x,Du) & = 0 & \text{in } Q_T \\ u(x,0) & = u_0(x) & x \in \mathbb{R}^N, \end{array} \right.$$

from which the general result follows by the fact that T is arbitrary.

We assume by contradiction that

$$M := \sup_{Q_T}\{u - v\} > 0. \tag{5.67}$$

Denote $R = 2(||u||_{L^\infty(\bar{Q}_T)} + ||v||_{L^\infty(\bar{Q_T})})$ and consider $\psi \in C_b^2(\mathbb{R}^N)$ a nonnegative function with $\psi = 0$ in B, $R \leq \psi \leq 2R$ in B_2^c and satisfying $||D\psi||_\infty, ||D^2\psi||_\infty \leq \Lambda$ for some $\Lambda > 0$. For this function ψ and $\beta > 0$, consider ψ_β as in (5.66).

Now, for $\eta, \mu \in (0,1)$, consider the function

$$\bar{\omega}(x,t) = \mu u(x,t) - v(x,t) - \eta t, \quad (x,t) \in Q.$$

Noting that $\bar{\omega} - \psi_\beta \to u - v$ locally uniform in $\bar{Q}_T$ as $\eta, \beta \to 0$ and $\mu \to 1$, by (5.67) we see that $\bar{\omega} - \psi_\beta$ is strictly positive at some point in $\bar{Q}_T$ for all η, β close to 0 and μ close to 1. Hence, by construction of ψ_β, $\bar{\omega} - \psi_\beta$ attains its maximum in $\bar{Q}_T$ at some point (x^*, t^*), and by Lemma 5.5, taking η, β smaller and μ larger if it is necessary, we have $t^* > 0$ for all such as parameters. At this point, we fix $\eta > 0$ satisfying the above facts.

Now, by Lemma 5.6, $\bar{\omega}$ is a viscosity subsolution of

$$\partial_t \bar{\omega} - I_x^j(\bar{\omega}(\cdot,t),x) - \bar{c}\frac{\zeta_2(|D\bar{\omega}|)^m}{(1-\mu)^{m-1}} \leq CA(1-\mu) - \eta \quad \text{in } Q_T,$$

and therefore we can use ψ_β as a test function for $\bar{\omega}$ at (x^*, t^*), concluding that

$$-I_x^j(\psi_\beta, x^*) - \bar{c}\frac{\zeta_2(|D\psi_\beta(x^*)|)^m}{(1-\mu)^{m-1}} \leq CA(1-\mu) - \eta.$$

Using Lemma 5.7, we conclude from the above inequality that

$$-(1 + \bar{c}(1-\mu)^{1-m})o_\beta(1) \leq CA(1-\mu) - \eta.$$

Letting $\beta \to 0$ and then $\mu \to 1$, we get the contradiction with the fact that $\eta > 0$. $\square$

As it is classical in the viscosity solution's theory, Proposition 5.1 allows the application of Perron's method to conclude the existence. In this task, we introduce the additional asumption

(H0) *There exists a constant $H_0 > 0$ such that $||H(\cdot,0)||_\infty \leq H_0$.*

This assumption allows us to build sub and supersolutions for (5.55). The existence result is the following

Corollary 5.3. Let I_x^j defined as in (5.50), with ν satisfying (M), j satisfying (J1) and both satisfying (J2). Assume $H \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfies (H0)-(H2). Let $u_0 \in C_b(\mathbb{R}^N)$. Then, there exists a unique viscosity solution $u \in C(\bar{Q}) \cap L^\infty(\bar{Q}_T)$ for all $T > 0$ to problem (5.55)-(5.56).

A priori bounds for the solution given in Corollary 5.3 can be derived from the application of comparison principle. Using ad-hoc sub and supersolutions, if u is the solution of (5.55)-(5.56), then

$$||u(\cdot,t)||_{L^\infty(\mathbb{R}^N)} \leq H_0\ t + ||u_0||_\infty, \tag{5.68}$$

which means that for fixed time t, the function $x \mapsto u(x,t)$ is globally bounded in $\mathbb{R}^N$.

Similar results can be given for the stationary problem (P) in the Lévy-Ito setting, namely equations with the form

$$\lambda u - I_x^j(u,x) + H(x, Du) = 0 \quad \text{in } \mathbb{R}^N. \tag{5.69}$$

Proposition 5.2. *Let $\lambda > 0$, I_x^j defined in (5.50) with ν satisfying (M), j satisfying (J1) and both satisfying (J2). Assume H satisfies (H0)-(H2). Let u, v be bounded viscosity sub and supersolution to equation (5.69). Then, $u \le v$ in $\mathbb{R}^N$.*

Moreover, if in addition we assume (H0), then there exists a unique viscosity solution $u \in C_b(\mathbb{R}^N)$ to equation (5.69), which satisfies

$$||u||_\infty \le \lambda^{-1} H_0. \tag{5.70}$$

5.4 Application to Periodic Equations: Large Time Behavior.

In this section we provide the large time behavior result for the problem (5.55)-(5.56) in the case the data are $\mathbb{Z}^N-$periodic. Hence, we will argue over the problem

$$\begin{aligned} \partial_t u - I_x^j(u(\cdot,t),x) + H(x,Du) &= 0 \qquad \text{in } \mathcal{Q} := \mathbb{T}^N \times (0,+\infty), & (5.71)\\ u(\cdot,0) &= u_0 \qquad \text{in } \mathbb{T}^N, & (5.72) \end{aligned}$$

where I_x^j is a nonlocal operator in Lévy-Ito form defined in (5.50) (replacing $\mathbb{R}^N$ by $\mathbb{T}^N$). Of course, the results obtained in this section can be readily extended to the case the Lévy-Ito operator has the form (5.52), provided the measure ν is such that I_x^j has order strictly less than 1 (see Remark 5.4).

Since problem (5.71)-(5.72) is a particular case of (5.55)-(5.56), comparison principle, existence and uniqueness hold for this problem under the conditions on the data given in the statement of Proposition 5.1. In particular, for the solution u of (5.71)-(5.72) we have the a priori estimate (5.68).

5.4.1 Strong Maximum Principle.

We need some notation for the statement of the Strong Maximum Principle: let ν, j in the definition of I_x^j and for $x \in \mathbb{R}^N$ we define inductively

$$X_0(x) = \{x\}, \quad X_{n+1}(x) = \bigcup_{\xi \in X_n(x)} \{\xi + j(\xi, \mathrm{supp}\{\nu\})\}, \quad \text{for } n \in \mathbb{N},$$

and

$$\mathcal{X}(x) = \overline{\bigcup_{n\in\mathbb{N}} X_n}. \tag{5.73}$$

The Strong Maximum Principle presented here relies in the nonlocallicity of the operator under the “iterative covering property”

$$\mathcal{X}(x) = \mathbb{T}^N, \quad \text{for all } x \in \mathbb{T}^N. \tag{5.74}$$

We can provide three interesting examples where this condition clearly holds. Of course, (5.74) depends on both ν and j, but we mainly focus on the structure of ν for which this condition is valid, and therefore we assume in the following examples that $j(x,z) = z$ for all $x, z \in \mathbb{R}^N$. In this context, the most basic example is the case where there exists $r > 0$ such that

$$B_r \subset \mathrm{supp}\{\nu\}.$$

A second example where the previous property does not hold, but (5.74) remains valid, is when ν has the form (5.47), namely

$$\nu(dz) = |z_2|^{-(N+\sigma)} dz_2 \otimes \delta_0(z_1)dz_1 + |z_1|^{-(N+\sigma)} dz_1 \otimes \delta_0(z_2)dz_2,$$

where δ_0 is the Dirac measure supported at 0 and $\otimes$ is the measure product.

The third example strongly takes into account the topology of the torus. In (say) $\mathbb{T}^2$, consider $L \subset \mathbb{T}^2$ a line of irrational slope, that is, $L : z_2 = \alpha z_1$, with α irrational. Let $\tilde{\nu}$ be the 1-dimensional Haussdorff measure in $\mathbb{T}^2$ and let $l \subset L$ with $\tilde{\nu}(l) > 0$. Then, the measure $\nu = \mathbf{1}_l(z)\tilde{\nu}(dz)$ satisfies the assumption (5.74).

The strong maximum principle is stated through the following

Proposition 5.3. *Let $\sigma \in (0,2)$ and let I_x^j defined in (5.50) with ν satisfying (M), j satisfying (J1) with $j(\cdot, z) \in C(\mathbb{T}^N)$ for each $z \in \mathbb{R}^N$, and ν, j satisfying (J2) and (5.74). Consider H satisfying (H0)-(H2), with ζ_2 in (H1) such that $\zeta_2(s) = c|s|$ for some $c > 0$. Let u be a $\mathbb{Z}^N$-periodic viscosity subsolution to (5.71), and v a $\mathbb{Z}^N$-periodic viscosity supersolution to (5.71), such that there exists $(x_0, t_0) \in \mathcal{Q}$ satisfying*

$$(u - v)(x_0, t_0) = \sup_{\mathcal{Q}} \{u - v\}.$$

Then, the function $u - v$ is constant in $\mathbb{T}^n \times [0, t_0]$. Moreover, we have

$$(u - v)(x, t) = \sup_{x \in \mathbb{T}^N} \{u(x, 0) - v(x, 0)\}, \quad \textit{for all } (x, t) \in \bar{\mathcal{Q}}.$$

The following lemma is a consequence of the comparison principle, see [32].

Lemma 5.8. *Assume assumptions of Proposition 5.1 hold. Let u, v be locally bounded sub and supersolution to equation (5.71) and for $t \in [0, +\infty)$, define*

$$\kappa(t) = \sup_{x \in \mathbb{T}^N} \{u(x, t) - v(x, t)\}. \tag{5.75}$$

Then, for all $0 \le s \le t$, we have $\kappa(t) \le \kappa(s)$.

Now we are in position to prove the strong maximum principle.

Proof of Propostion 5.3: We divide the proof in several parts.

1.- Preliminaries. Under the definition of κ in (5.75), we must prove that for each $(x, t) \in \mathbb{T}^N \times [0, t_0]$

$$(u - v)(x, t) = \kappa(0).$$

However, since $\kappa(t_0)$ is a global maximum value of κ in $[0, +\infty)$, by Lemma 5.8 we have $\kappa(t) = \kappa(0)$ for all $t \in [0, t_0]$. Hence, it is sufficient to prove that for each $\tau \in (0, t_0)$ we have

$$u(x, \tau) - v(x, \tau) = \kappa(\tau), \quad \text{for all } x \in \mathbb{T}^N,$$

which implies the result up to $\tau = 0$ and $\tau = t_0$ by upper-semicontinuity.

We fix $\tau \in (0, t_0)$ and define the set

$$\mathcal{M}_\tau = \{x \in \mathbb{T}^N : (u - v)(x, \tau) = \kappa(\tau)\},$$

which is nonempty by upper-semicontinuity of $u - v$. Hence, with the above facts the proof follows by proving that $\mathcal{M}_\tau = \mathbb{T}^N$.

2.- Localization on time τ. For $\eta > 0$ we consider the function

$$(x,t) \mapsto \tilde{W}(x,t) := u(x,t) - v(x,t) - \eta(t-\tau)^2.$$

Note that for each $(x,t) \in \mathcal{Q}$, we have

$$\tilde{W}(x,t) \le \kappa(t) - \eta(t-\tau)^2 \le \kappa(\tau) = (u-v)(x_1,\tau) = \tilde{W}(x_1,\tau),$$

for some $x_1 \in \mathcal{M}_\tau$, and therefore the supremum of $\tilde{W}$ in $\mathcal{Q}$ is achieved, and each such as maximum point has the form (x,τ) for some $x \in \mathcal{M}_\tau$. Hence, we clearly have

$$\kappa(\tau) = \sup_{(x,t)\in\mathcal{Q}} \tilde{W}(x,t).$$

3.- Localization around a point in $\mathcal{M}_\tau$. From this point we fix $x_\tau \in \mathcal{M}_\tau$ and introduce a function $\psi \in C_b^2(\mathbb{R})$ with $\psi(0) = 0$, $\psi > 0$ in $\mathbb{R}\setminus\{0\}$ and $\psi(x) = 4R$ if $|x| \ge 1$, with

$$R = ||u||_{L^\infty(\mathbb{T}^N\times[0,t_0+1])}.$$

For $\epsilon > 0$, $x \in \mathbb{T}^N$ define $\psi_\epsilon(x) = \psi(|x-x_\tau|/\epsilon)$. We remark that $\psi_\epsilon \in C_b^2(\mathbb{T}^N)$, $\psi_\epsilon(x_\tau) = 0$, $\psi_\epsilon > 0$ in $\mathbb{R}^N\setminus\{x_\tau\}$ and for each $\epsilon > 0$ its first and second derivatives are bounded, depending on ϵ.

We take $0 < \mu < 1$, denote $\bar{u} = \mu u$ and $\omega_\mu = \bar{u} - v$ as in Lemma 5.6, and consider the function

$$(x,t) \mapsto W_\mu(x,t) := \omega_\mu(x,t) - \eta|t-\tau|^2 - (1-\mu)\psi_\epsilon(x).$$

By upper-semicontinuity of W_μ, there exists $(x_\mu,t_\mu) \in \mathbb{T}^N \times [0,t_0+1]$ such that

$$W_\mu(x_\mu,t_\mu) = \sup_{\mathbb{T}^N\times[0,t_0+1]} W_\mu,$$

and since $W_\mu \to \tilde{W}$ locally uniform on $\bar{\mathcal{Q}}$ as $\mu \to 1$ we have, up to subsequences, $(x_\mu,t_\mu) \to (x^*,\tau)$ as $\mu \to 1$, where $x^* = x^*(\epsilon) \in \mathcal{M}_\tau$.

In fact, since (x_μ,t_μ) is maximum for W_μ, for all $(x,t) \in \mathbb{T}^N \times [0,t_0+1]$ we have

$$\begin{aligned} W_\mu(x_\mu,t_\mu) =& (u-v)(x_\mu,t_\mu) + (\mu-1)(u+\psi_\epsilon)(x_\mu,t_\mu) - \eta(t_\mu-\tau)^2 \\ \ge& (u-v)(x,t) + (\mu-1)(u+\psi_\epsilon)(x,t) - \eta(t-\tau)^2. \end{aligned}$$

In particular, taking the point $(x,t) = (x_\tau,\tau)$ in the right-hand side we obtain

$$(u-v)(x_\mu,t_\mu) + (\mu-1)(u+\psi_\epsilon)(x_\mu,t_\mu) \ge \kappa(\tau) + (\mu-1)u(x_\tau,\tau). \tag{5.76}$$

Now, since $t_\mu \in [0,t_0+1]$ for all μ close to 1, we have

$$(u-v)(x_\mu,t_\mu) \le \kappa(t_\mu) \le \kappa(\tau),$$

and replacing this into (5.76) we get

$$u(x_\mu,t_\mu) + \psi(|x_\mu - x_\tau|/\epsilon) \le u(x_\tau,\tau),$$

that is $\psi(|x_\mu - x_\tau|/\epsilon) \leq 2R$. By the choice of ψ we conclude that $x_\mu \in B_\epsilon(x_\tau)$ for all μ close to 1. Since $x_\mu \to x^* \in \mathcal{M}_\tau$, we conclude $x^* \in \bar{B}_\epsilon(x_\tau)$.

4.- Using the viscosity inequality for ω_μ. From the above facts, we see that the function $(x,t) \mapsto \phi(x,t) := (1-\mu)\psi_\epsilon(x) + \eta(t-\tau)^2$ is a test function for ω_μ at (x_μ, t_μ). Then, by Lemma 5.6, for each $\delta, \epsilon > 0$ we have

$$\begin{aligned} 2\eta(t_\mu - \tau) - I^j_{x_\mu}[B^c_\delta](\omega_\mu(\cdot, t_\mu), x_\mu) - I^j_{x_\mu}[B_\delta]((1-\mu)\psi_\epsilon, x_\mu) \\ -\bar{c}(1-\mu)|D\psi_\epsilon(x_\mu)|^m \leq CA(1-\mu), \end{aligned}$$

but by (M) and (J1) we have

$$I^j_{x_\mu}[B_\delta](\psi_\epsilon, x_\mu) \leq C_j |D^2\psi_\epsilon|_\infty.$$

From this, it follows that

$$\begin{aligned} 2\eta(t_\mu - \tau) - I^j_{x_\mu}[B^c_\delta](\omega_\mu(\cdot, t_\mu), x_\mu) \\ -(1-\mu)\Big(C_j|D^2\psi_\epsilon|_\infty + \bar{c}|D\psi_\epsilon(x_\mu)|^m + CA\Big) \leq 0. \end{aligned} \tag{5.77}$$

Note that for all $\epsilon > 0$, by the smoothness of ψ_ϵ the term in parenthesis in (5.77) remains bounded as $\mu \to 1$, meanwhile $t_\mu \to \tau$. On the other hand, by the continuity of j and (M), by Dominated Convergence Theorem we get

$$I^j_{x_\mu}[B^c_\delta](\omega_\mu(\cdot, t_\mu), x_\mu) \to I^j_{x^*}[B^c_\delta]((u-v)(\cdot, \tau), x^*) \quad \text{as } \mu \to 1,$$

where $x^* \in \mathcal{M}_\tau$ is such that $x^* \in \bar{B}_\epsilon(x_\tau)$. Recalling that $(u-v)(x^*, \tau) = \kappa(\tau)$, letting $\mu \to 1$ in (5.77) we arrive at

$$\int_{B^c_\delta} [(u-v)(x^* + j(x^*, z), \tau) - \kappa(\tau)]\nu(dz) = 0,$$

and since $x^* \in \bar{B}_\epsilon(x_\tau)$, letting $\epsilon \to 0$ we finally conclude

$$\int_{B^c_\delta} [(u-v)(x_\tau + j(x_\tau, z), \tau) - \kappa(\tau)]\nu(dz) = 0. \tag{5.78}$$

5.- Conclusion. Since $\delta > 0$ is arbitrary, we conclude $(u-v)(x,\tau) = \kappa(\tau)$ for all $x \in X_1(x_\tau)$. Hence, we can proceed in the same way as above, concluding by induction that $(u-v)(x,\tau) = \kappa(\tau)$ for all $x \in \bigcup_{n \in \mathbb{N}} X_n(x_\tau)$. Finally, by upper-semicontinuity of $u-v$ and (5.74) we conclude the result. □

Remark 5.5. In Proposition 5.3, the assumption on the continuity of j can be dropped. For instance, it is used to pass to the limit in (5.78). In this direction, note that if $g \in C(\mathbb{T}^N)$ we can write

$$|g(x^* + j(x^*, z)) - g(x_\tau + j(x_\tau, z))| \leq \zeta(x^* + j(x^*, z) - x_\tau - j(x_\tau, z)),$$

where ζ is the modulus of continuity of g. However, it is known that a modulus of continuity may be assumed to satisfy that $\zeta(t) \leq \zeta(\rho) + \rho^{-1}t$ for each $t, \rho > 0$ (see [81]). Using this, we conclude

$$|g(x^* + j(x^*, z)) - g(x_\tau + j(x_\tau, z))| \leq \zeta(\rho) + \rho^{-1}(|x^* - x_1| - |j(x^*, z) - j(x_\tau, z)|)$$

for all $\rho > 0$. Hence, using (J1) we can make $x^* \to x_\tau$ and then letting $\rho \to 0$ to get the desired convergence without asking continuity on j.

Additionally, instead of assuming $\zeta_2(s) = c|s|$, it is enough to ask that

$$\zeta_2(s)s^{(1-m)/m} \to 0 \quad \text{as } s \to 0.$$

5.4.2 The Ergodic Problem.

Roughly speaking, solving the *ergodic problem* means pass to the limit as $\lambda \to 0$ in the stationary periodic problem

$$\lambda u - I_x^j(u,x) + H(x,Du) = 0 \quad x \in \mathbb{T}^N, \tag{5.79}$$

whose existence and uniqueness for $\lambda > 0$ holds by Proposition 5.2. Hence, the required compactness of the family of solutions $\{u_\lambda\}$ is typically obtained by regularity results which are independent of λ.

Proposition 5.4. *Let $\sigma \in (0,2)$ and I_x^j defined in (5.50) with ν satisfying (M1), (M2) associated to σ, j satisfying (J1) with $j(\cdot,z) \in C(\mathbb{T}^N)$ for each $z \in \mathbb{R}^N$, and that ν, j satisfy (J2) and (5.74). Assume H satisfies (H0)-(H2), with $m > \max\{1,\sigma\}$ in (H1). Then, there exists a unique constant $c \in \mathbb{R}$ for which the stationary ergodic problem*

$$-I_x^j(u,x) + H(x,Du) = -c, \quad \text{in } \mathbb{T}^N \tag{5.80}$$

has a solution $w \in C^{(m-\sigma)/m}(\mathbb{T}^N)$. Moreover, w is the unique continuous solution of (5.80), up to an additive constant.

Proof: Let $\lambda > 0$ and consider the periodic stationary problem (5.79). By Proposition 5.2 we have the existence and uniqueness of a solution u_λ to this problem which, by (5.70), satisfies the estimate $||u_\lambda||_\infty \le \lambda^{-1}H_0$. Thus, by Theorem 5.5 we show that $u_\lambda \in C^{(m-\sigma)/m}(\mathbb{T}^N)$ with Hölder seminorm independent of λ or $||u_\lambda||_\infty$.

Now, denote $w_\lambda = u_\lambda - u_\lambda(0)$ which satisfies the equation

$$\lambda u - \mathcal{I}_x^j(u,x) + H(x,Du) = -\lambda u_\lambda(0), \quad \text{in } \mathbb{T}^N. \tag{5.81}$$

Using Theorem 5.5 we see that the family $\{w_\lambda\}_{\lambda\in(0,1)}$ is uniformly bounded and that this family is equi-Hölder with exponent $(m-\sigma)/m$. Hence, by Arzela-Ascoli Theorem, there exists $w \in C^{(m-\sigma)/m}(\mathbb{T}^N)$ such that $w_\lambda \to w$ as $\lambda \to 0$, uniformly on $\mathbb{T}^N$. Additionally, we have the existence of a constant $c \in \mathbb{R}$ such that $\lambda u_\lambda(0) \to c$ as $\lambda \to 0$. By standard stability results for viscosity solutions (see [23], [3] and [66]), we have the pair (w,c) found above is a (viscosity) solution to (5.80).

If (w_i,c_i), $i=1,2$ are two solutions for (5.80), then we see that $v_i(x,t) = w_i(x,t) + c_i t, i = 1,2$ are two solutions to the Cauchy problem (5.71) with initial data w_i. Hence, by comparison principle we conclude that

$$v_1(x,t) - ||w_1 - w_2||_\infty \le v_2(x,t), \quad \text{for all } (x,t) \in \mathcal{Q},$$

and therefore, we obtain $(c_1 - c_2)t \le 2||w_1 - w_2||_\infty$. Dividing by t and letting $t \to +\infty$ we obtain that $c_1 \le c_2$. Exchanging the roles of w_1 and w_2, we get $c_1 = c_2 = c$ and therefore c is unique. Moreover, for each $t \in [0,+\infty)$ we have

$$\sup_{x\in\mathbb{T}^N}\{v_1(x,t) - v_2(x,t)\} = \sup_{\mathcal{Q}}\{v_1 - v_2\} = \sup_{\mathbb{T}^N}\{w_1 - w_2\} =: m,$$

and therefore, by Proposition 5.3 we conclude that for each $x \in \mathbb{T}^N$

$$w_1(x) = w_2(x) + m,$$

concluding the proof. $\square$

5.4.3 Large Time Behavior.

The main result of this section is the following

Theorem 5.6. *Assume assumptions of Proposition 5.4 hold. Let u be the unique solution to problem (5.71)-(5.72). Then, there exists a pair (w, c) solution to (5.80) such that*

$$u(x,t) - ct - w(x) \to 0, \quad \text{as } t \to +\infty,$$

uniformly in $\mathbb{T}^N$.

Proof: Here we follow closely the arguments given in [32],[112] in the local framework and [19] in the nonlocal one.

We assume first that $u_0 \in C^2(\mathbb{T}^N)$. In this case, by using comparison principle it is possible to prove that u is Lipschitz in t (see [112]), with Lipschitz constant

$$C^* = || - I_x^j(u_0, \cdot) + H(\cdot, Du_0)||_{L^\infty(\mathbb{T}^N)} < \infty.$$

Now, by recalling that (H2) implies (5.10), for each $t \in (0, +\infty)$ the function $x \mapsto u(x,t)$ is a viscosity subsolution to the problem

$$-I_x^j(u, x) + b_0|Du|^m \leq C^* + H_0,$$

with H_0 given by (H0). Using Theorem 5.5 we conclude the unique solution to u of problem (5.71)-(5.72) is in $C^{\gamma_0,1}(\mathcal{Q})$, with γ_0 defined in (5.12).

Note that u and the function $(x,t) \mapsto w(x) + ct$ are solutions to (5.71). Hence, by comparison principle we have

$$||u(\cdot,t) - w - ct||_\infty \leq ||u_0 - w||_\infty, \tag{5.82}$$

meanwhile, if we define

$$\kappa(t) = \max_{\mathbb{T}^N}\{u(\cdot,t) - w - ct\}, \tag{5.83}$$

by Lemma 5.8 we see that κ is nonincreasing. Since in addition it is bounded there exists $\bar{\kappa} \in \mathbb{R}$ such that $\kappa(t) \to \bar{\kappa}$ as $t \to +\infty$.

Now, define the function $(x,t) \mapsto v(x,t) := u(x,t) - ct$. Using (5.82) we obtain

$$||v(\cdot,t)||_\infty \leq ||w||_\infty + ||u_0 - w||_\infty, \quad \text{for each } t \geq 0,$$

and by the fact that the family $\{v(\cdot,t)\}_t$ is equi-Hölder (with exponent γ_0), by Arzela-Ascoli we can extract a subsequence $\{v(\cdot,t_k)\}_k$ with $t_k \to \infty$ as $k \to \infty$ such that

$$v(\cdot,t_k) \to \bar{v}, \quad \text{uniformly in } \mathbb{T}^N \text{ as } k \to +\infty.$$

Define $v_k(x,t) = v(x, t + t_k)$. Recalling that v_k is solution to

$$\left\{ \begin{array}{rll} \partial_t v_k - I_x^j(v_k(\cdot,t),x) + H(x, Dv_k) & = -c & \text{in } \mathcal{Q} \\ v_k(x,0) & = v(x,t_k) & x \in \mathbb{T}^N, \end{array} \right.$$

and using comparison principle we conclude $\{v_k\}_k$ satisfies the inequality

$$||v_k - v_{k'}||_{L^\infty(\mathcal{Q})} \leq ||v(\cdot, t_k) - v(\cdot, t_{k'})||_\infty, \tag{5.84}$$

for all $t \geq 0$ and $k, k' \in \mathbb{N}$. Hence, $\{v_k\}_k$ is an uniformly bounded Cauchy sequence in $C(\mathcal{Q})$ and therefore, up to a subsequence, we conclude $v_k \to \tilde{v}$ in $C(\mathcal{Q})$ as $k \to \infty$, where $\tilde{v}$ solves

$$\left\{ \begin{array}{rll} \partial_t \tilde{v} - I_x^j(\tilde{v}(\cdot,t),x) + H(x, D\tilde{v}) & = -c & \text{in } \mathcal{Q} \\ \tilde{v}(x,0) & = \bar{v} & x \in \mathbb{T}^N, \end{array} \right.$$

Using the definition of κ given in (5.83), for each $t \geq 0$ we obtain

$$\kappa(t + t_k) = \max_{\mathbb{T}^N}\{v_k(\cdot,t) - w\},$$

and since $\{v_k\}_k$ is uniformly convergent, we can pass to the limit as $k \to \infty$ concluding that

$$\bar{\kappa} = \max_{\mathbb{T}^N}\{\tilde{v}(\cdot,t) - w\} \quad \text{for each } t \in [0, +\infty),$$

and applying Proposition 5.3, for each $(x,t) \in \mathcal{Q}$ we have

$$\tilde{v}(x,t) = w(x) + \bar{\kappa},$$

and therefore we have $\bar{v} = w + \bar{\kappa}$ in $\mathbb{T}^N$. This implies that $v(x,t) \to w + \bar{\kappa}$. But by using the definition of v we have

$$||u(\cdot,t) - ct - w - \bar{\kappa}||_\infty = ||v(\cdot,t) - v - \bar{\kappa}||_\infty \to 0$$

as $t \to \infty$. Replacing w by $w + \bar{\kappa}$, we conclude the result in the case the initial data is smooth.

The general result for $u_0 \in C(\mathbb{T}^N)$ follows by an approximation argument using a sequence of smooth initial data u_0^ϵ satisfying $u_0^\epsilon \to u_0$ uniformly in $\mathbb{T}^N$ as $\epsilon \to 0$. We refer to [112] for details. □

Chapter 6

Existence, Uniqueness and Asymptotic Behavior for Nonlocal Parabolic Problems with Dominant Gradient Terms

This chapter is based in the joint work with Guy Barles which can be found in the preprint [33].

Contents

6.1 Introduction.

In this paper we are concerned with the existence, uniqueness and asymptotic behavior for the solution of the following Cauchy problem set in $Q = \Omega \times (0, +\infty)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary

$$\left\{ \begin{array}{rll} \partial_t u - \mathcal{I}(u(\cdot,t),x) + H(x,t,u,Du) = & 0, & \text{in } Q \\ u(x,t) = & \varphi(x,t), & \text{in } Q^{ext} \\ u(x,0) = & u_0(x), & \text{in } \bar{\Omega}. \end{array} \right. \tag{CP}$$

where $u : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ stands for the unknown function depending on the "space" variable $x \in \mathbb{R}^n$ and the "time" variable $t \in [0, +\infty)$, $\partial_t u$ is the derivative of u with respect to t and Du is its gradient with respect to x. We denote by $Q^{ext} = \Omega^c \times (0, +\infty)$ and the function $\varphi : \bar{Q}^{ext} \to \mathbb{R}$ is assumed to be continuous and bounded; it represents the prescribed value of u in Q^{ext} ("Dirichlet boundary condition").

For $\alpha \in (0, 2)$ fixed, $\mathcal{I}$ represents an *integro-differential operator of order less or equal than* α, defined in the following way: for $x \in \mathbb{R}^n$ and ϕ regular enough at x and bounded in $\mathbb{R}^n$, $\mathcal{I}(\phi, x)$ has the general form

$$\mathcal{I}(\phi, x) = \int_{\mathbb{R}^n} [\phi(x+z) - \phi(x) - \mathbf{1}_B \langle D\phi(x), z\rangle] K(z)|z|^{-(n+\alpha)} dz, \tag{6.1}$$

where $K : \mathbb{R}^n \to \mathbb{R}$ is a measurable, nonnegative and bounded function. Such an operator is called *elliptic*, and ranges from *zero-th order non local operators* in the case $K(z)|z|^{-(n+\alpha)}$ has finite measure (see [53]) to the fractional Laplacian of order α, which is the case when K is equal to a well-known constant $C_{n,\alpha} > 0$ (see [69]).

Our main interest is to prove the well-posedness of problem (CP) in the context of loss of the boundary condition, namely existence and uniqueness of a viscosity solution in $C(\bar{Q})$ which does not agree with φ on $\partial\Omega \times (0, +\infty)$. Such losses of boundary conditions were studied in [21] whose main result was that, if H has some natural growth depending on the ellipticity properties of $\mathcal{I}$, then there is no loss of boundary condition. Our key assumptions on α and H will imply that our framework is exactly the opposite, i.e. the H term will be (in a suitable sense) stronger than the $\mathcal{I}$ one.

We recall that, in the second-order case, there are two well-known examples of problems developing this kind of loss of boundary conditions. The first case is the case of the degenerate parabolic problems where the $\mathcal{I}$ is replaced by a second-order linear operator: the equation becomes

$$\partial_t u - \frac{1}{2}\mathrm{Tr}(a(x)D^2u(x)) + H(x, t, u, Du) = 0 \quad \text{in } \Omega \times [0, +\infty),$$

but we assume that the operator is degenerate, i.e. the symmetric matrix $a(x)$ is nonnegative for any x but can have 0 eigenvalues. Such problems, in particular in the linear case where studied by Keldysh [89] and Radkevich [100, 99] by pde methods (solutions in a weak sense) and by Freidlin [75] through a probabilistic approach. The first general results by a viscosity solutions' approach handling real losses of Dirichlet boundary conditions for second-order equations appears in [16] following some previous results for first-order equations (see [29, 30]). More specifically, in problems which arise from the study of optimal exit time problems, one is led to Hamilton-Jacobi equations where H has the Bellman form

$$H(x, t, u, p) = \lambda u + \sup_{\beta \in \mathcal{B}} \{-b(x, t, \beta) \cdot p - f_\beta(x, t, \beta)\}, \tag{6.2}$$

where $\lambda \geq 0$, $\mathcal{B}$ is a compact metric space (the control-space) and b, f are continuous and bounded functions (see [11] and [74] for the connections between control problems and such equations).

Loss of boundary conditions may arise at some point $x_0 \in \bar{\Omega}$ when $a(x_0)$ is singular, and more precisely when $a(x_0)n(x_0) = 0$ where $n(x_0)$ the unit outer normal vector to $\partial\Omega$ at x_0. This condition indicates the lack of diffusion in the normal direction at x_0. In this context, in order to decide if there is (or not) a loss of boundary condition, one has to examine the first-order term in the equation together with the geometrical properties of the boundary :

we do not give details here and refer instead to [16]. Despite of the difficulty connected to the loss of boundary conditions, existence and uniqueness for such problems can be obtained in the context of viscosity solutions with *generalized boundary condition* (see [16], [26], [94] [31] and references therein).

The second example, and in some sense which can be seen as being closer to our framework, is the case of uniformly parabolic second-order problem associated to a Hamiltonian with superquadratic growth in Du, namely equations with the form

$$\partial_t u - \Delta u + H(x, u, Du) = 0 \quad \text{in } \Omega \times [0, +\infty), \tag{6.3}$$

where

$$H(x, t, u, p) = \lambda u + |p|^m - f(x), \quad m > 2, \tag{6.4}$$

where $\lambda \geq 0$ and $f \in C(\bar{\Omega})$. In this case, losses of boundary conditions come from the relative strenght of the second-order term and the $|Du|^m$-term : in the superquadratic case, the $|Du|^m$-term may impose such losses of boundary data. In [27], [112], the existence and uniqueness of solutions is obtained (taking into account these losses of Dirichlet boundary conditions) and the asymptotic behavior of the solution of the problem as $t \to +\infty$ is also studied in [112]. In this task, the discount rate λ in problems with Hamiltonians as (6.2) or (6.4) is determinant on the asymptotic behavior. For instance, as it can be seen in [112], if $\lambda > 0$ then the asymptotic behavior of problems like (6.3) is the uniform convergence in $C(\bar{\Omega})$ as $t \to +\infty$ to the solutions of the associated stationary problem. However, if the case $\lambda = 0$ different behaviors may arise and it is well-known that the *ergodic problem* plays a key role, see [32]. We mention here that such as ergodic behavior for nonlocal operators is studied by the authors in collaboration with S. Koike and O. Ley [24], see also [19].

This (very brief and incomplete) state-of-the-art on parabolic Dirichlet problems with loss of boundary conditions allows us to be more specific on the contents of this paper : we obtain the well-posedness of problem (CP) in two cases which can be understood as the extension of the both types of second-order problems we presented above. The first one concerns *coercive* Hamiltonians as (6.4) for which the superquadratic condition has to be replaced in our context by the *superfractional condition* $m > \alpha$, making the first-order term the leading term in the equation. We remark that we have no other additional restriction to m (in particular, we can deal with $m < 1$) and then we allow the study of Hamiltonians which are concave in Du.

On the other hand, in the case of problem (CP) associated to Bellman-type Hamiltonians with the form (6.2), the diffusive role of $\mathcal{I}$ defined in (6.1) is of weaker order than the first-order term when we assume $\alpha < 1$. However, in contrast with the *degenerate* second-order case, losses of boundary conditions arise even if we impose an *uniform ellipticity condition* in the sense of Caffarelli and Silvestre [47], which is related with the nonintegrability of K^α at the origin (see assumption (UE) below). As in [113], the well-posedness of (CP) is obtained through a careful examination of the effects of the drift b at each point of $\partial\Omega \times (0, +\infty)$ and suitable assumptions.

Organization of the Paper: In Section 6.2 we provide the notion of solution for (CP). In section 6.3 we precise what we mean with (CP) in coercive and Bellman form, introduce the assumptions of each problem and present the main results. In section 6.4 we study the behavior of sub and supersolutions on the parabolic boundary. Section 6.5 is devoted to regularity issues for each problem. The proof of the main results are given in section 6.6 and the existence, uniqueness and large time behavior is addressed in section 6.7.

6.2 Basic Notation and Notion of Solution.

We start with the basic notation. For $\delta > 0$ and $x \in \mathbb{R}^n$ we write $B_\delta(x)$ as the ball of radius δ centered at x and B_δ if $x = 0$. For an arbitrary set A, we denote $d_A(x) = dist(x, \partial A)$ the signed distance function to ∂A which is nonnegative for $x \in A$ and nonpositive for $x \notin A$. For Ω we simply write $d(x) = d_{\partial\Omega}(x)$ and define the set Ω_δ as the open set of all $x \in \Omega$ such that $d(x) < \delta$. By the smoothness of the domain, there exists a fixed number $\delta_0 > 0$, depending only on Ω, such that d is smooth in the set of points x such that $|d(x)| < \delta_0$ (see [78]). For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we write

$$\Omega - x = \{z : x + z \in \Omega\} \quad \text{and} \quad \lambda\Omega = \{\lambda z : z \in \Omega\}.$$

By a modulus of continuity ω we mean a nondecreasing, sublinear, continuous function $\omega : [0, +\infty) \to \mathbb{R}$ such that $\omega(0) = 0$.

Given a set $A \subset \mathbb{R}^n$, we denote USC(A) the set of real valued, upper semicontinuous (usc for short) functions. In the analogous way, we write LSC(A) the set of real valued, lower semicontinuous (lsc for short) functions.

Before presenting the viscosity evaluation, we need to introduce some notation related with the nonlocal term $\mathcal{I}$. For $\alpha \in (0, 2)$, we denote

$$K^\alpha(z) = K(z)|z|^{-(n+\alpha)}, \quad \text{for} \quad z \neq 0.$$

As we mentioned in the introduction, we are interested in the case α represents the order of $\mathcal{I}$ and therefore, in the case $\alpha \in (0, 1)$, for each $x \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ bounded and smooth at x, we write

$$\mathcal{I}(\phi, x) = \int_{\mathbb{R}^n} [\phi(x+z) - \phi(x)] K^\alpha(z) dz. \tag{6.5}$$

We remark that in the case K is symmetric (that is, $K(z) = K(-z)$ for $z \in \mathbb{R}^n$), then (6.1) is equivalent to (6.5) when $\alpha \in (0, 1)$.

For $x, p \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$ and ϕ a bounded function, we define

$$\mathcal{I}[A](\phi, x, p) = \int_{\mathbb{R}^n \cap A} [\phi(x+z) - \phi(x) - \mathbf{1}_B \langle p, z \rangle] K^\alpha(z) dz. \tag{6.6}$$

We write in a simpler way $\mathcal{I}[A](\phi, x) = \mathcal{I}[A](\phi, x, D\phi(x))$ when $\phi \in L^\infty(\mathbb{R}^n) \cap C^2(B_\delta)$ for some $\delta > 0$, $\mathcal{I}(\phi, x, p) = \mathcal{I}[\mathbb{R}^n](\phi, x, p)$ when $A = \mathbb{R}^n$. In the case $\alpha \in (0, 1)$, the presence of the compensator (namely, the term $\mathbf{1}_B \langle D\phi(x), z \rangle$) is not necessary to give a sense to the nonlocal term and for this reason we drop it in (6.6).

If $\phi \in C^2(B_\delta(x) \times (t - \delta, t + \delta))$ and $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a bounded measurable function, we define

$$\begin{aligned} &E_\delta(w, \phi, x, t) \\ &= \partial_t \phi(x,t) - \mathcal{I}[B_\delta](\phi(\cdot, t), x) - \mathcal{I}[B_\delta^c](w(\cdot, t), x, D\phi(x,t)) \\ &\quad + H(x, t, w(x,t), D\phi(x,t)), \end{aligned} \tag{6.7}$$

where "E" stands for "evaluation".

For $T > 0$, we define the sets

$$Q_T = \Omega \times (0, T]; \quad \partial^l Q_T = \partial\Omega \times (0, T]; \quad Q_T^{ext} = \Omega^c \times (0, T].$$

We are going to consider finite time horizon problem associated with (CP)

$$\begin{cases} \partial_t u - \mathcal{I}[u(\cdot,t)] + H(x,t,u,Du) & = 0 \quad \text{in } Q_T \\ u(x,t) & = \varphi(x,t) \quad \text{in } Q_T^{ext} \\ u(x,0) & = u_0(x) \quad \text{in } \bar{\Omega}, \end{cases} \tag{CP$_\mathrm{T}$}$$

For a function $u \in \mathrm{USC}(\bar{\Omega}\times[0,T])$ (resp $u \in \mathrm{LSC}(\bar{\Omega}\times[0,T])$), we define its upper (resp. lower) φ-extension as the function defined in $\mathbb{R}^n \times [0,T]$ by

$$\begin{aligned} & u^{\varphi}(x,t) \text{ (resp. } u_{\varphi}(x,t)) \\ = & \begin{cases} u(x,t) & \text{if } (x,t) \in \Omega\times[0,T] \\ \varphi(x,t) & \text{if } (x,t) \in \bar{\Omega}^c\times[0,T] \\ \max \text{ (resp. } \min)\{u(x,t),\varphi(x,t)\} & \text{if } (x,t) \in \partial\Omega\times[0,T], \end{cases} \end{aligned} \tag{6.8}$$

We provide a definition of solution to problem (CP$_\mathrm{T}$) which can be extended naturally to (CP).

Definition 6. A function $u \in \mathrm{USC}(\bar{\Omega}\times[0,T])$ is a viscosity subsolution of (CP$_\mathrm{T}$) if for any smooth function $\phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, any maximum point $(x_0,t_0) \in \bar{\Omega}\times[0,T]$ of $u^{\varphi} - \phi$ in $B_\delta(x_0)\times(t_0-\delta,t_0+\delta)\cap\mathbb{R}^n\times[0,T]$ with $\delta > 0$, we have the inequality

$$\begin{aligned} E_\delta(u^{\varphi},\phi,x_0,t_0) \le 0 \qquad & \text{if } (x_0,t_0) \in Q_T, \\ \min\{E_\delta(u^{\varphi},\phi,x_0,t_0), u(x_0,t_0)-\varphi(x_0,t_0)\} \le 0 \qquad & \text{if } x_0 \in \partial\Omega, \\ \min\{E_\delta(u^{\varphi},\phi,x_0,t_0), u(x_0,t_0)-u_0(x_0)\} \le 0 \qquad & \text{if } t_0 = 0, \end{aligned}$$

where E_δ is defined in (6.7).

A function $v \in \mathrm{LSC}(\bar{\Omega}\times[0,T])$ is a viscosity supersolution of (CP$_\mathrm{T}$) if for any smooth function $\phi : \mathbb{R}^n\times[0,T] \to \mathbb{R}$, any minimum point $(x_0,t_0) \in \bar{\Omega}\times[0,T]$ of $v_\varphi - \phi$ in $B_\delta(x_0)\times(t_0-\delta,t_0+\delta)\cap\mathbb{R}^n\times[0,T]$ with $\delta > 0$, we have the inequality

$$\begin{aligned} E_\delta(v_{\varphi},\phi,x_0,t_0) \ge 0 \qquad & \text{if } (x_0,t_0) \in Q_T, \\ \max\{E_\delta(v_{\varphi},\phi,x_0,t_0), v(x_0,t_0)-\varphi(x_0,t_0)\} \ge 0 \qquad & \text{if } x_0 \in \partial\Omega, \\ \max\{E_\delta(u^{\varphi},\phi,x_0,t_0), u(x_0,t_0)-u_0(x_0)\} \ge 0 \qquad & \text{if } t_0 = 0. \end{aligned}$$

Finally, a viscosity solution of (CP$_\mathrm{T}$) is a function whose upper and lower semicontinuous envelopes are sub and supersolution of the problem, respectively.

The above definition is basically the same as the one presented in [3], [21], [23], [103] and [104]. Written in that way we highlight the goal of this paper, which is to state the existence and uniqueness of a solution of (CP) in $C(\bar{Q})$.

We note that Definition 6 interprets the points at $\Omega\times\{T\}$ as *interior points*, which is consistent with the classical definition of the Cauchy problem for parabolic equations (see [70],[76]). Of course, a weaker definition of viscosity solution (concerning functions defined only in $\bar{\Omega}\times[0,T)$) can be set, obtaining the same results presented in this paper. However, we avoid this extra difficulty here since its consideration has no significant contribution to the development of our problem.

6.3 Assumptions and Main Results.

As we mentioned in the introduction, in this paper we study the well-posedness for problem (CP) in two cases, depending on the features of H. Basically, we are interested in the case when H has a coercive nature in the gradient term, and the case H has a Bellman form and therefore it is not necessarily coercive.

6.3.1 Coercive Hamiltonian and Examples.

In this case we restrict the time dependence of H by the assumption

(A0) *There exists $H_0 : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous and $f : \bar{\Omega} \times [0, +\infty) \to \mathbb{R}$ uniformly continuous and bounded such that*

$$H(x,t,r,p) = H_0(x,r,p) - f(x,t),$$

for all $x \in \bar{\Omega}, t \geq 0, r \in \mathbb{R}$ and $p \in \mathbb{R}^n$.

Let $\alpha \in (0,2)$ and $\mathcal{I}$ as in (6.1), (6.5). We will consider *superfractional* coercive Hamiltonians, where the gradient growth is given by H_0 through the basic assumption

(A1) *There exists $m > \alpha$ and $C_0 > 0$ such that, for all $R > 0$ there exists $C_R > 0$ satisfying*

$$H_0(x,r,p) \geq C_0|p|^m - C_R,$$

for all $x \in \bar{\Omega}$, $p \in \mathbb{R}^n$ and $|r| \leq R$.

However, we must be careful if the coercivity is sub or superlinear. For this, we split the analysis depending on the gradient growth of H_0, that is

• *Sublinear Coercivity:* Assume (A0) holds. We say that H is *sublinearly coercive* if H_0 satisfies (A1) with $m \leq 1$, and the following continuity condition holds

(A2-a) *For all $R > 0$, there exists a modulus of continuity ω_R satisfying*

$$H_0(y,r,p) - H_0(x,r,p+q) \leq \omega_R(|x-y|(1+|p|) + |q|),$$

for all $x, y \in \bar{\Omega}$, $|r| \leq R$, $p, q \in \mathbb{R}^n$, $|q| \leq 1$.

• *Superlinear Coercivity:* Assume (A0) holds. We say that H is *superlinearly coercive* if H_0 satisfies

(A1-b) *There exists $m > \max\{1, \alpha\}$ and $a_0 > 0$ such that, for all $R > 0$, there exists a constant C_R such that*

$$H_0(x,r,p) - \mu H_0(x, \mu^{-1}r, \mu^{-1}p) \leq \Big((1-m)a_0|p|^m + C_R\Big)(1-\mu)$$

for all $\mu < 1$, $x \in \bar{\Omega}$, $p \in \mathbb{R}^n$, $|r| \leq R$.

(A2-b) *If m is given by Assumption (A1-b), for all $R > 0$, there exists a modulus of continuity ω_R satisfying*

$$H_0(y,r,p) - H_0(x,r,p+q) \leq \omega_R(|x-y|)(1+|p|^m) + |p|^{m-1}\omega_R(|q|),$$

for all $x, y \in \bar{\Omega}$, $|r| \leq R$, $p, q \in \mathbb{R}^n$, $|q| \leq 1$.

Remark 6.1. Note that Condition (A1-b) gives us the gradient coercivity of H_0 since it implies (A1) with $m > \max\{1, \alpha\}$.

In order to describe the kind of Hamiltonians we have in mind, we introduce the following examples : in the first one, we assume $m \leq 1$ and consider

$$H(x,t,r,p) = a_1(x)|p|^m + a_2(x)|p|^l + \lambda(x)r - f(x,t),$$

while in the second case, we suppose $m > 1$ and

$$H(x,t,r,p) = a_1(x)|p|^m + a_2(x)|p|^l + b(x) \cdot p + \lambda(x)r - f(x,t).$$

In both cases, $l < m$, $a_1, a_2, \lambda, f : \bar{\Omega} \to \mathbb{R}$ are continuous functions with $\lambda \geq 0$. We assume in addition that a_1, a_2 are Lipschitz continuous and $a_1 \geq C_0$ for some fixed constant $C_0 > 0$.

These Hamiltonians are coercive in Du and in the case $m > 1$ we can include transport terms with a Lipschitz continuous vector field $b : \bar{\Omega} \to \mathbb{R}^n$. The above assumptions are easily checkable in both cases.

6.3.2 Bellman Hamiltonian.

Let $\mathcal{B}$ a compact metric space, $b : \bar{\Omega} \times [0,+\infty) \times \mathcal{B} \to \mathbb{R}^n$ and $f, \lambda : \bar{\Omega} \times [0,+\infty) \times \mathcal{B} \to \mathbb{R}$ continuous and bounded functions. We say that H has a *Bellman form* if, for $t \in [0,+\infty)$, $r \in \mathbb{R}$, $x \in \bar{\Omega}, p \in \mathbb{R}^n$, $H(x,t,r,p)$ can be written as

$$H(x,t,r,p) = \sup_{\beta \in \mathcal{B}} \{\lambda_\beta(x,t)r - b_\beta(x,t) \cdot p - f_\beta(x,t)\}, \tag{$H_\mathcal{B}$}$$

and satisfies the assumptions (L) and (Σ) below. In ($H_\mathcal{B}$) we have adopted the abuse of notation $b_\beta(x,t) = b(x,t,\beta)$ and in the same way for the other functions.

For H with the form ($H_\mathcal{B}$) we impose the uniform space-time Lipschitz asumption:

There exists $L > 0$ such that, for all $\beta \in \mathcal{B}$, $(x,s),(y,t) \in \bar{\Omega} \times [0,+\infty))$, we have

$$|b_\beta(x,s) - b_\beta(y,t)| \leq L(|x-y| + |s-t|). \tag{L}$$

Then we introduce the notation

$$\begin{aligned}
\Gamma_{in} &= \{(x,t) \in \partial^l Q \ : \ \forall\ \beta \in \mathcal{B},\ b_\beta(x,t) \cdot Dd(x) > 0\},\\
\Gamma_{out} &= \{(x,t) \in \partial^l Q \ : \ \forall \beta \in \mathcal{B},\ b_\beta(x,t) \cdot Dd(x) \leq 0\},\\
\Gamma &= \partial^l Q \setminus (\Gamma_{in} \cup \Gamma_{out}),
\end{aligned}$$

and with this, we consider the following condition over the behavior of the drisf terms on $\partial^l Q$

$$\Gamma_{in}, \Gamma_{out} \text{ and } \Gamma \text{ are unions of connected components of } \partial^l Q. \tag{Σ}$$

We remark that, in the current Bellman setting, the nonlocal term $\mathcal{I}$ is assumed to be of order $\alpha < 1$. Therefore it has a weaker effect compared with the first-order terms. In particular, on the boundary, the behavior of the drift plays a determinant role. In this direction, the set Γ_{out} should be understood as the set where the classical boundary condition holds, meanwhile on Γ_{in} may arise losses of the boundary condition due to the "stronger" influence of the transport term compared with the nonlocal diffusion. Finally, on Γ, we do not have a transport effect anymore : the value of the different costs (boundary or running cost) decides of the choice of the control and of the loss or no loss of boundary condition.

We introduce assumption (Σ) in order to avoid have different behaviors of the b_β's on the same connected component, which could be a source of discontinuities for the solution (the reader may think in term of transport equation to be convinced by this claim). On Γ, it can be seen as a controllability assumption in the normal direction. Similar assumptions of the boundary are made in [16], [31] in the degenerate second-order setting and [113] for the nonlocal one.

6.3.3 Structural Assumptions and Main Results.

As it is classical for Cauchy-Dirichlet problems, the initial and boundary data satisfy the following *compatibility condition* at $t = 0$

(H0) $$u_0(x) = \varphi(x, 0), \text{ for all } x \in \partial\Omega.$$

The properness of the problem is encoded by the following two conditions

(H1) *For all $R > 0$, there exists $h_R \in C(\bar{\Omega})$ such that, for all $x \in \bar{\Omega}$, $u, v \in \mathbb{R}$, $0 \le t \le R$, and $p \in \mathbb{R}^n$, we have*
$$H(x, t, u, p) - H(x, t, v, p) \ge h_R(x)(u - v).$$

(H2) *For all $R > 0$, the function h_R in (H1) satisfies*
$$\inf_{x\in\bar{\Omega}} \{h_R(x) + \int_{\Omega^c - x} K^\alpha(z)dz\} \ge 0.$$

As it is classical in problems where loss of the boundary condition arises, Strong Comparison Principle needs the introduction of a modification of sub and supersolutions. For a function u bounded and usc in $\bar{Q}$ (which will be thought as subsolution) we denote

$$\tilde{u}(x, t) = \begin{cases} \limsup\limits_{Q\ni(y,s)\to(x,t)} u(y, s) & \text{if } (x, t) \in \partial^l Q \\ u(x, t) & \text{if } (x, t) \in \bar{Q} \setminus \partial^l Q. \end{cases} \tag{6.9}$$

Theorem 6.1. (Strong Comparison Principle - Coercive Case) *Let $\varphi \in C_b(\bar{Q}^{ext})$ and $u_0 \in C(\bar{\Omega})$. Assume (H0) holds and that H has a coercive form satisfying (H1)-(H2). If u, v are bounded viscosity sub and supersolution to problem (CP) respectively, then*
$$u \le v \quad \text{in } Q \cup \bar{\Omega} \times \{0\}.$$

Moreover, if $\tilde{u}$ is defined as in (6.9), then $\tilde{u} \le v$ in $\bar{Q}$.

The result concerning the Bellman needs also a redefinition of sub and supersolutions at the boundary. Of course, in this control framework, the different part of the boundary $\{\Gamma_{in}, \Gamma_{out}, \Gamma\}$ play different roles.

For bounded functions u and v, u usc in $\bar{Q}$, v lsc in $\bar{Q}$, we denote

$$\begin{aligned} \tilde{u}(x,t) &= \begin{cases} u(x,t) & \text{if } (x,t) \in \bar{Q} \setminus (\Gamma_{in} \cup \Gamma) \\ \limsup\limits_{Q \ni (y,s) \to (x,t)} u(y,s) & \text{if } (x,t) \in \Gamma_{in} \cup \Gamma \end{cases} \\ \tilde{v}(x,t) &= \begin{cases} v(x,t) & \text{if } (x,t) \in \bar{Q} \setminus \Gamma_{in} \\ \liminf\limits_{Q \ni (y,s) \to (x,t)} v(y,s) & \text{if } (x,t) \in \Gamma_{in}. \end{cases} \end{aligned} \tag{6.10}$$

In the Bellman case, we will require the stronger ellipticity assumption

$$\text{There exists } c_1, c_2 > 0 \text{ such that } c_2 \leq K(z) \text{ for all } |z| \leq c_1. \tag{UE}$$

Theorem 6.2. (Strong Comparison Principle - Bellman Case) *Let $\varphi \in C_b(\bar{Q}^{ext})$ and $u_0 \in C(\bar{\Omega})$. Assume $\alpha < 1$, (UE), (H0) hold and let H with Bellman form satisfying (H1)-(H2). If u, v are bounded viscosity sub and supersolution of (CP) respectively, then*

$$u \leq v \quad \text{in } Q \cup \bar{\Omega} \times \{0\}.$$

Moreover, if $\tilde{u}, \tilde{v}$ are defined as in (6.10), then $\tilde{u} \leq \tilde{v}$ in $\bar{Q}$.

The result of Theorem 6.2 can be obtained without the uniform ellipticity assumption (UE) by slightly changing the definition of $\Gamma_{in}, \Gamma_{out}$ and Γ. Indeed, in this setting, only the assumptions on the drift term determine the loss or not loss of the boundary condition of the solution on $\Gamma_{in}, \Gamma_{out}$ and Γ and they have to be strong enough to compensate the lack of the ellipticity effect of $\mathcal{I}$.

6.4 Initial and Boundary Condition.

We also remark that, considered as a part of the parabolic boundary, we ask the initial condition is satisfied in the generalized sense. However, the initial condition is satisfied in the classical sense on $\Omega \times \{0\}$. Moreover, mainly because of (H0), the condition holds classically on $\bar{\Omega} \times \{0\}$.

Lemma 6.1. *Assume that $H \in C(\bar{\Omega} \times [0, +\infty) \times \mathbb{R} \times \mathbb{R}^n)$, $\varphi \in C_b(\bar{Q}^{ext})$, $u_0 \in C(\bar{\Omega})$ satisfying (H0). If u, v are respectively a bounded, usc viscosity subsolution and a bounded, lsc viscosity supersolution to (CP), then $u(x,0) \leq u_0(x) \leq v(x,0)$ for all $x \in \bar{\Omega}$.*

The proof of this lemma follows the same lines of the analogous result for the second-order case presented in [94], with subtle modifications concerning the nonlocal operator.

Now we look for the behavior of sub and supersolutions at the lateral parabolic boundary.

Lemma 6.2. *Assume that $H \in C(\bar{\Omega} \times [0, +\infty) \times \mathbb{R} \times \mathbb{R}^n)$ and $\varphi \in C_b(\bar{Q}^{ext})$. If $(x_0, t_0) \in \partial^l Q$ and u, v are respectively a bounded, usc viscosity subsolution and a bounded, lsc viscosity supersolution to (CP), then*

(i) *We have $u(x_0, t_0) \leq \varphi(x_0, t_0)$ if one of the following conditions hold:*

(i.1) There exists $C_0, \rho > 0$ and $m > \alpha$ such that for all $R > 0$, there exists $C_R > 0$ satisfying

$$H(x, t, r, k\eta^{-1} Dd(x)(1 + o_\eta(1))) \geq C_0 (k\eta^{-1})^m - C_R$$

for all $k, \eta > 0$, $x \in B_\rho(x_0)$ and $t, |r| \leq R$.

(i.2) Condition (UE) *with* $\alpha < 1$ *holds, and there exists* $c_0, \rho > 0$ *such that, for all* $R > 0$ *there exists* C_R *satisfying*

$$H(x, t, r, k\eta^{-1}Dd(x)(1 + o_\eta(1))) \geq -c_0 k\eta^{-1}d(x) - C_R$$

for all $k, \eta > 0$, $x \in B_\rho(x_0)$ *and* $t, |r| \leq R$.

(ii) *We have* $v(x_0, t_0) \geq \varphi(x_0, t_0)$ *if condition* (UE) *with* $\alpha < 1$ *holds, and there exists* $c_0, \rho > 0$ *such that, for all* $R > 0$ *there exists* C_R *satisfying*

$$H(x, t, r, -k\eta^{-1}Dd(x)(1 + o_\eta(1))) \leq c_0 k\eta^{-1}d(x) + C_R$$

for all $k, \eta > 0$, $x \in B_\rho(x_0)$ *and* $t, |r| \leq R$.

Proof: We concentrate on **(i)** since **(ii)** is an adaptation to $(i.2)$. By contradiction, we assume $u(x_0, t_0) - \varphi(x_0, t_0) = \nu$ for some $\nu > 0$. This implies in particular that $u^\varphi(x_0, t_0) = u(x_0, t_0)$. We consider $\sigma \in (\max\{1, \alpha\}, 2)$ and $C^{1,\sigma-1}$ functions $\chi, \psi : \mathbb{R} \to \mathbb{R}$ such that χ is even, bounded, $\chi(0) = 0$, $\chi(t) > 0$ for $t \neq 0$, $\liminf_{|t|\to\infty} \chi(t) > 0$ and such that $\chi(t) = |t|^\sigma$ in a neighborhood of 0. For ψ we assume it is bounded, strictly increasing, $\psi \geq -\frac{1}{4}\nu$ and such that for some $k > 0$, $\psi(t) = kt$ for all $|t| \leq 1$. We consider a parameter η and $\epsilon = \epsilon_\eta \to 0$ as $\eta \to 0$ to be fixed later, and introduce the test function

$$\Psi(y, t) := \psi(d(y)/\eta) + \epsilon^{-1}\chi(|y - x_0|) + \epsilon^{-1}\chi(|t - t_0|).$$

By our assumption on u, φ, χ and ψ, the function $(x, t) \mapsto u^\varphi(x, t) - \Psi(x, t)$ has a maximum point $(\bar{x}, \bar{t}) \in \mathbb{R}^n \times (0, T)$ for η small enough. Of course, $(\bar{x}, \bar{t})$ depends on η but we drop the dependence on η to simplify the notations. From the maximum point property, $u^\varphi(\bar{x}, \bar{t}) - \Psi(\bar{x}, \bar{t}) \geq u^\varphi(x_0, t_0) - \Psi(x_0, t_0)$ which implies

$$u^\varphi(\bar{x}, \bar{t}) - \psi(d(\bar{x})/\eta) - \epsilon^{-1}\chi(|\bar{x} - x_0|) - \epsilon^{-1}\chi(|\bar{t} - t_0|) \geq \varphi(x_0) + \nu.$$

Using this inequality, classical arguments show that $\bar{x} \to x_0$ and $\bar{t} \to t_0$ as $\eta \to 0$. And from the same inequality we obtain $\bar{x} \in \bar{\Omega}$ for η small enough because $\psi \geq -1/4\nu$ and φ is continuous. Finally, using properly the usc of u^φ we conclude

$$d(\bar{x}) = o_1(\eta)\eta, \ |\bar{x} - x_0|, |\bar{t} - t_0| = o_\eta(1), \text{ and } u^\varphi(\bar{x}, \bar{t}) \to u(x_0, t_0), \tag{6.11}$$

as $\eta \to 0$. Hence, picking some $\delta > 0$, we can use the viscosity inequality for subsolutions, concluding that

$$\begin{aligned} \partial_t \Psi(\bar{x}, \bar{t}) \leq \mathcal{I}[B_\delta](\Psi(\cdot, \bar{t}), \bar{x}) + \mathcal{I}[B_\delta^c](u^\varphi(\cdot, \bar{t}), \bar{x}, D\Psi(\bar{x}, \bar{t})) \\ - H(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D\Psi(\bar{x}, \bar{t})), \end{aligned} \tag{6.12}$$

where in view of the first and second statement in (6.11), for η small enough we can write

$$D\Psi(\bar{x}, \bar{t}) = k\eta^{-1}Dd(\bar{x}) + \epsilon^{-1}|\bar{x} - x_0|^{\sigma-2}(\bar{x} - x_0). \tag{6.13}$$

We start with the estimates concerning the nonlocal terms in (6.12). To do this, we consider $r \leq 1$ independent of η and $d(\bar{x}) < \delta \leq \mu < r$. We define the sets

$$\begin{aligned} \mathcal{A}_\delta^{ext} &= \{z \in B_r \ : \ d(\bar{x} + z) \leq d(\bar{x}) - \delta\}. \\ \mathcal{A}_{\delta,\mu} &= \{z \in B_r \ : \ d(\bar{x}) - \delta < d(\bar{x} + z) < d(\bar{x}) + \mu\}. \\ \mathcal{A}_\mu^{int} &= \{z \in B_r \ : \ \mu + d(\bar{x}) \leq d(\bar{x} + z)\}. \end{aligned}$$

We remark that $B_\delta \subset \mathcal{A}_{\delta,\mu}$ and using that $\bar{x}$ is a global maximum point of $u - \Psi$, in particular we have $\delta(u^\varphi(\cdot,\bar{t}),\bar{x},z) \le \delta(\Psi(\cdot,\bar{t}),\bar{x},z)$ in $\mathcal{A}_{\delta,\mu} \setminus B_\delta$. Using this last fact we can write

$$\begin{aligned} &\mathcal{I}[B_\delta^c](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) + \mathcal{I}[B_\delta](\Psi(\cdot,\bar{t}),\bar{x}) \\ &\le \mathcal{I}[B_r^c](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) + \mathcal{I}[\mathcal{A}_\delta^{ext}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \\ &\quad + \mathcal{I}[\mathcal{A}_\mu^{int}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) + \mathcal{I}[\mathcal{A}_{\delta,\mu}](\Psi(\cdot,\bar{t}),\bar{x}), \end{aligned}$$

and from this we estimate each term in the right-hans side of the above inequality separately. The constant $C > 0$ arising in each of the following estimates does not depend on μ, δ, η or ϵ.

Using the expression (6.13), we have

$$\begin{aligned} \mathcal{I}[B_r^c](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \le 2||u^\varphi||_\infty \int_{B_r^c} K^\alpha(z)dz \\ + (k\eta^{-1} + \epsilon^{-1} o_\eta(1)) \int_{B\setminus B_r} K^{\alpha-1}(z), \end{aligned}$$

where the last integral does not exists if $\alpha < 1$. Thus, we get

$$\mathcal{I}[B_r^c](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \le C||u^\varphi||_\infty r^{-\alpha} + C(\eta^{-1} + \epsilon^{-1} o_\eta(1)) r^{1-\alpha},$$

and similarly, we have

$$\mathcal{I}[\mathcal{A}_\mu^{int}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \le C||u^\varphi||_\infty \mu^{-\alpha} + C(\eta^{-1} + \epsilon^{-1} o_\eta(1)) \mu^{1-\alpha}.$$

At this point, we consider $\mu = \eta$. Thus, for all η small enough and $z \in \mathcal{A}_{\delta,\mu}$ we have $\psi(d(\bar{x}+z)/\eta) = k\eta^{-1} d(\bar{x}+z)$ and applying the definition of Ψ we get

$$\begin{aligned} &\Psi(\bar{x}+z,\bar{t}) - \Psi(\bar{x},\bar{t}) \le C(\eta^{-1} + \epsilon^{-1})|z|, \\ &\Psi(\bar{x}+z,\bar{t}) - \Psi(\bar{x},\bar{t}) - \langle D\Psi(\bar{x},\bar{t}), z\rangle \le C(\eta^{-1} + \epsilon^{-1})|z|^2, \end{aligned}$$

from which we can get

$$\mathcal{I}[\mathcal{A}_{\delta,\mu}](\Psi(\cdot,\bar{t}),\bar{x}) \le C(\eta^{-1} + \epsilon^{-1}) \varrho_\alpha(\mu),$$

where

$$\varrho(\mu) = \begin{cases} \mu^{2-\alpha} & \text{if } \alpha > 1 \\ \mu \ln(\mu) & \text{if } \alpha = 1 \\ \mu^{1-\alpha} & \text{if } \alpha < 1. \end{cases} \tag{6.14}$$

Thus, recalling that we have chosen $\mu = \eta$ and taking $\epsilon \ge \eta^{\min\{\alpha,1\}}$, by the above estimates we can write

$$\begin{aligned} &\mathcal{I}[B_\delta^c](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) + \mathcal{I}[B_\delta](\Psi(\cdot,\bar{t}),\bar{x}) \\ &\le C\eta^{-\alpha} + C\eta^{-1}\varrho_\alpha(\mu) + \mathcal{I}[\mathcal{A}_\delta^{ext}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})). \end{aligned} \tag{6.15}$$

where the constant C depends only on the data and $||u^\varphi||_\infty$.

Under the above choice of ϵ and using (6.11), we have $\partial_t \Psi(\bar{x},\bar{t}) \ge \eta^{-\alpha} o_\eta(1)$. Using this estimate and (6.15) into (6.12) we can write

$$\begin{aligned} \eta^{-\alpha} o_\eta(1) \le C(r^{-\alpha} + \eta^{-\alpha}) + \mathcal{I}[\mathcal{A}_\delta^{ext}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \\ - H(\bar{x},\bar{t},u(\bar{x},\bar{t}),D\Psi(\bar{x},\bar{t})), \end{aligned} \tag{6.16}$$

where $D\Psi(\bar{x},\bar{t}) = \epsilon^{-1}o_\eta(1) + k\eta^{-1}Dd(\bar{x})$.

Since $u(x_0,t_0) = \varphi(x_0) + \nu$, by the continuity of φ and the last fact in (6.11), for all η small enough, using that $u^\varphi = \varphi$ in Q^{ext} we can write

$$\begin{aligned}\mathcal{I}[\mathcal{A}_\delta^{ext}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \leq & -C\nu\int_{\mathcal{A}_\delta^{ext}} K^\alpha(z)dz \\ & + C(\epsilon^{-1}+\eta^{-1})\int_{\mathcal{A}_\delta^{ext}} |z|K^\alpha(z)dz.\end{aligned}$$

where we supress the last integral term when $\alpha < 1$. Using the definition of K^α, and recalling the choice of ϵ above, we conclude from the above inequality that

$$\mathcal{I}[\mathcal{A}_\delta^{ext}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \leq -C\nu\int_{\mathcal{A}_\delta^{ext}} K^\alpha(z)dz + C\eta^{-1}\tilde{\varrho}_\alpha(\delta), \tag{6.17}$$

where $\tilde{\varrho}(\delta) = \delta^{1-\alpha}$ if $\alpha > 1$, $\tilde{\varrho}(\delta) = |\ln(\delta)| + 1$ when $\alpha = 1$ and $\tilde{\varrho}(\delta) = 0$ if $\delta < 1$.

At this point we split the analysis. When we consider case $(i.1)$, we just have condition K is nonnegative and bounded, and therefore we only can insure that

$$-C\nu\int_{\mathcal{A}_\delta^{ext}} K^\alpha(z)dz \leq 0.$$

Using this into (6.17) we get

$$\mathcal{I}[\mathcal{A}_\delta^{ext}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \leq \tilde{\varrho}(\delta),$$

and replacing this into (6.16), we choose $\delta = \eta$. Applying the definition of $\tilde{\varrho}$ and using the condition over H in $(i.1)$, we arrive at

$$\eta^{-\alpha}o_\eta(1) \leq C(r^{-\alpha} + \eta^{-\alpha}) - C_0k^m\eta^{-m} + \tilde{C},$$

where $\tilde{C}$ depends only on $||u||_\infty$ and the data. We fix $r > 0$ and since $k > 0$ and $m > \alpha$, by choosing η small enough, we reach the contradiction.

For the case $(i.2)$, recalling that $\alpha < 1$ and the strong ellipticity assumption (UE), we have from (6.17) that

$$\mathcal{I}[\mathcal{A}_\delta^{ext}](u^\varphi(\cdot,\bar{t}),\bar{x},D\Psi(\bar{x},\bar{t})) \leq -C\nu\delta^{-\alpha}$$

with $C > 0$ independent of η and δ. We replace this estimate into (6.16) to conclude this time that

$$\eta^{-\alpha}o_\eta(1) \leq C(r^{-\alpha} + \eta^{-\alpha}) - C\nu\delta^{-\alpha} - H(\bar{x},\bar{t},u(\bar{x},\bar{t}),k\eta^{-1}Dd(\bar{x})(1+o_\eta(1))).$$

At this point we choose $d(\bar{x}) < \delta < \eta o_\eta(1)$ and applying the condition over the Hamiltonian for this case together with (6.11), we arrive at

$$\eta^{-\alpha}o_\eta(1) \leq C(r^{-\alpha} + \eta^{-\alpha}) - C\nu\delta^{-\alpha} + c_0k\eta^{-1}d(\bar{x}) + \tilde{C},$$

where $\tilde{C}$ depends only on $||u||_\infty$ and the data. Fixing $r > 0$ and recalling that $\eta^{-1}d(\bar{x}) = o_\eta(1)$, we reach the contradiction by choosing η small enough. This concludes the proof. $\square$

As a corollary of this lemma we have the following

Proposition 6.1. *Let $\varphi \in C_b(\bar{Q}^{ext})$, $\mathcal{I}$ as in (6.1) and H with coercive form. Let u be a bounded viscosity subsolution for the problem (CP) and let $(x_0, t_0) \in \partial^l Q$. Then, $u(x_0, t_0) \leq \varphi(x_0, t_0)$. In particular, $\tilde{u}$ defined in (6.9) satisfies $\tilde{u}(x_0, t_0) \leq \varphi(x_0, t_0)$.*

By Remark 6.1, this result holds since it fits into the case $(i.1)$ in Lemma 6.2. Concerning the Bellman structure of the problem, we have

Proposition 6.2. *Let $\varphi \in C_b(\bar{Q}^{ext})$, $\alpha < 1$, $\mathcal{I}$ as in (6.5) satisfying (UE), and H with Bellman form. Let Let u, v be bounded viscosity sub and supersolution for (CP), respectively, and $\tilde{u}$, $\tilde{v}$ as in (6.10). Then*

$$\begin{aligned} \tilde{u} \leq u \leq \varphi \leq v \leq \tilde{v} \quad & \text{on } \Gamma_{out}, \\ \tilde{u} \leq u \leq \varphi \quad & \text{on } \Gamma. \end{aligned}$$

This result holds since it fits into the cases $(i.2)$ and (ii) in Lemma 6.2.

6.5 Regularity Issues for Coercive and Bellman Problems.

6.5.1 Regularity for Coercive Problem.

We consider the stationary equation associated to the coercive version of (CP)

$$\begin{cases} -\mathcal{I}[u] + H_0(x, u, Du) &= A \quad \text{in } \Omega \\ u &= \varphi \quad \text{in } \Omega^c, \end{cases} \tag{6.18}$$

where $A > 0$, $\varphi \in C_b(\Omega^c)$, $\mathcal{I}$ is a nonlocal operator of order α with the form (6.1) or (6.5) and H_0 defined in (A0) has a coercive form (sub or superlinear).

As it can be seen in [24], the superfractional assumption (A1) makes the gradient term the leading one in equation (6.18), and therefore regularity results can be obtained in an analogous way as in the case of first and second-order equations with coercive Hamiltonians in Du (see [15], [14], [50] and references therein). This regularity result is presented here through the following

Proposition 6.3. ([24]) *Let u be a bounded usc viscosity subsolution in Ω to Equation (6.18). Then, there exists a constant C such that, for all $x, y \in \Omega$*

$$|u(x) - u(y)| \leq C|x - y|^{\frac{m-\alpha}{m}}$$

where C depends on the data, A and $||u^{\varphi}||_{\infty}$. In particular, u can be extended up to $\bar{\Omega}$ as a Hölder continuous function with Hölder exponent $(m - \alpha)/m$.

Using this result we can obtain a regularity result for parabolic equations which is sufficient to get the comparison principle. To do so, we need to introduce some notations: for $E \subseteq \mathbb{R}^n$ closed and $g : E \times [0, T] \to \mathbb{R}$ a bounded usc function, we define the time sup-convolution of g with parameter $\gamma > 0$ as the function g^{γ} given by

$$g^{\gamma}(x, t) := \sup_{s \in [0,T]} \{g(x, s) - \gamma^{-1}(s - t)^2\}, \quad \text{for } x \in E, t \in [0, T]. \tag{6.19}$$

It is well-known that, for each $\gamma > 0$ and $x \in E$, $t \mapsto g^{\gamma}(x, t)$ is Lipschitz continuous in $[0, T]$, with Lipschitz constant $C_{\gamma} := 4T\gamma^{-1}$. In addition, if $g \in C(E \times [0, T])$, $g^{\gamma} \to g$ locally uniformly in $E \times [0, T]$ as $\gamma \to 0$.

Lemma 6.3. *Let $\varphi \in C_b(\bar{Q}_T^{ext})$, $\mathcal{I}$ as in (6.1) or (6.5) and H with coercive form. Let u be a bounded viscosity subsolution to problem ($\mathrm{CP_T}$). Then, there exists a constant $a_\gamma > 0$, $a_\gamma \to 0$ as $\gamma \to 0$, such that u^γ is a viscosity subsolution in $\Omega \times [a_\gamma, T]$ of the problem*

$$\begin{cases} \partial_t u^\gamma - \mathcal{I}(u^\gamma) + H(x,t,u^\gamma, Du^\gamma) & = o_\gamma(1) \qquad \text{in } \Omega \times [a_\gamma, T] \\ u^\gamma & = \varphi^\gamma \qquad \text{in } \Omega^c \times [a_\gamma, T], \end{cases}$$

where $o_\gamma(1)$ depends only on the time modulus of continuity of the function f given in (A0).

Proof: By the upper semicontinuity of u, for each $(x,t) \in \bar{Q}_T$ there exists $t_\gamma \in [0,T]$ depending on x and γ such that

$$u^\gamma(x,t) = u(x,t_\gamma) - \gamma^{-1}(t - t_\gamma)^2.$$

Since u is bounded, we also have that $|t_\gamma - t| \leq (2||u||_{L^\infty(\bar{Q}_T)}\gamma)^{1/2}$ and then we initially set a_γ as twice this last constant.

We start noting that by applying Proposition 6.1, for each $(x,t) \in \partial^l Q_T$ we can write

$$u^\gamma(x,t) \leq \varphi(x,t_\gamma) - \gamma^{-1}(t-t_\gamma)^2 \leq \varphi^\gamma(x,t),$$

and therefore, the (lateral) boundary condition holds in the classical sense.

Now we address the viscosity inequality in Q_T. Let $(\bar{x},\bar{t}) \in Q_T$ and ϕ a smooth test-funtion such that $(\bar{x},\bar{t})$ is a maximum for $u^\gamma - \phi$ in $B_{\delta_1}(\bar{x}) \times (\bar{t} - \delta_2, \bar{t} + \delta_2)$ for some $\delta_1, \delta_2 > 0$. Without loss of generality we can assume $\delta_1 < d(\bar{x})$.

Denote as $\bar{t}_\gamma$ the time attaining the supremum in the definition of $u^\gamma(\bar{x},\bar{t})$ and $\tilde{\phi}(x,s) = \phi(x, s + \bar{t} - \bar{t}_\gamma)$. Using the definition of u^γ and performing a translation argument in time, we conclude that

$$u(\bar{x},\bar{t}_\gamma) - \tilde{\phi}(\bar{x},\bar{t}_\gamma) \geq u(x,s) - \tilde{\phi}(x,s), \quad \text{for all } (x,s) \in B_{\delta_1}(\bar{x}) \times (\bar{t}_\gamma - \delta_2, \bar{t}_\gamma + \delta_2),$$

which is a testing for u at $(\bar{x},\bar{t}_\gamma)$ with test-function $\tilde{\phi}$. Applying the viscosity inequality for u, we can write

$$E_\delta(u^\varphi, \tilde{\phi}, \bar{x}, \bar{t}_\gamma) \leq 0. \tag{6.20}$$

Now, using the definition of sup-convolution we have

$$\begin{aligned} u(\bar{x}+z, \bar{t}_\gamma) - \gamma^{-1}(\bar{t}_\gamma - \bar{t})^2 &\leq u^\gamma(\bar{x}+z,\bar{t}), \quad z \in \Omega - \bar{x}, \\ \varphi(\bar{x}+z, \bar{t}_\gamma) - \gamma^{-1}(\bar{t}_\gamma - \bar{t})^2 &\leq \varphi^\gamma(\bar{x}+z,\bar{t}), \quad z \in \Omega^c - \bar{x}, \end{aligned}$$

meanwhile using that $u^\gamma(\bar{x},\bar{t}) = u(\bar{x},\bar{t}_\gamma) - \gamma^{-1}(\bar{t}_\gamma - \bar{t})^2$ we conclude

$$\mathcal{I}[B_{\delta_1}^c](u(\cdot,\bar{t}_\gamma), \bar{x}, D\tilde{\phi}(\bar{x},\bar{t}_\gamma)) \leq \mathcal{I}[B_{\delta_1}^c]((u^\gamma)^{\varphi^\gamma}(\cdot,\bar{t}), \bar{x}, D\tilde{\phi}(\bar{x},\bar{t}_\gamma)).$$

Finally, by definition of $\tilde{\phi}$ we have

$$\partial_t \tilde{\phi}(\bar{x},\bar{t}_\gamma) = \partial_t \phi(\bar{x},\bar{t}) \quad \text{and} \quad D\tilde{\phi}(\bar{x},\bar{t}_\gamma) = D\phi(\bar{x},\bar{t}).$$

Using these facts into (6.20) and using the uniform continuity of f, we arrive to the desired viscosity inequality for u^γ. □

Joining Lemmas 6.3 and 6.3 we conclude the following

Lemma 6.4. *Let $\varphi \in C_b(\bar{Q}_T^{ext})$, $\mathcal{I}$ as in (6.1) or (6.5), and H with coercive form. Then, for all u bounded viscosity subsolution to problem ($\mathrm{CP_T}$), there exists $\gamma_0 > 0$ such that, for all $\gamma \le \gamma_0$, $u^\gamma \in C^{1-\alpha/m,1}(\Omega \times [a_\gamma, T])$, where u^γ is defined in (6.19) and a_γ is the constant given in Lemma 6.3.*

Moreover, under the above assumptions, $\tilde{u}^\gamma \in C^{1-\alpha/m,1}(\bar{\Omega} \times [a_\gamma, T])$, where $\tilde{u}$ is defined in (6.9).

Proof: The regularity in t comes from the definition of the sup-convolution. For the Hölder regularity in x the idea is to prove that for each $t \in [a_\gamma, T]$, $x \mapsto u^\gamma(x,t)$ is a viscosity solution to a problem like (6.18). Let $x_0 \in \Omega$, $t_0 \in (a_\gamma, T)$ and ϕ a test-function for $u^\gamma(t,\cdot)$ at x_0. For $\epsilon > 0$ small, we incorporate the time variable in the following way

$$(x,s) \mapsto \Phi(x,s) := u^\gamma(x,s) - \phi(x) - \epsilon^{-1}(s-t_0)^2.$$

The function Φ being bounded and upper semicontinuous in $\bar{Q}_T$, has a maximum point $(\bar{x},\bar{s}) \in \bar{Q}_T$. Since $\Phi(\bar{x},\bar{s}) \ge \Phi(x_0,t_0)$, we have $(\bar{s}-t_0)^2 \le 2||u||_\infty \epsilon$, concluding that $\bar{s} \to t_0$ as $\epsilon \to 0$. Then, using the upper semicontinuity of u^γ, we get $\bar{x} \to x_0$ as $\epsilon \to 0$ too.

Using Lemma 6.3, we conclude that

$$\begin{aligned} 2\epsilon^{-1}(\bar{s}-t_0) - \mathcal{I}[B_\delta](\phi,\bar{x}) - \mathcal{I}[B_\delta^c](u^\gamma(\bar{s},\cdot),\bar{x},D\phi(\bar{x})) \\ + H(\bar{x},\bar{s},u^\gamma(\bar{x},\bar{s}),D\phi(\bar{x})) \le o_\gamma(1), \end{aligned}$$

but we remark that $2\epsilon^{-1}(\bar{s}-t_0) \ge C_\gamma$ because of the Lipschitz continuity of u^γ (recall that $2\epsilon^{-1}(\bar{s}-t_0)$ is in the time superdifferential of u^γ at $(\bar{x},\bar{s})$). Letting $\epsilon \to 0$ and controlling the integral terms by the use of Fatou's Lemma, we conclude that $x \mapsto u^\gamma(t,x)$ is a subsolution to the problem

$$-\mathcal{I}(u,x) + H_0(x,u,Du) \le ||f||_\infty + C_\gamma + o_\gamma(1) \quad \text{in } \Omega$$

for all $t \in [a_\gamma, T]$. Using Proposition 6.3, we conclude the result.

Concerning the last part of the lemma, assume $u = \tilde{u}$. Then, to prove that $u^\gamma \in C^{1-\alpha/m,1}(\bar{\Omega} \times [a_\gamma, T])$, it is sufficient to show that u^γ is continuous up to the lateral boundary. In fact, for $(x_0,t_0) \in \partial\Omega \times [a_\gamma, T-a_\gamma]$, by definition of u^γ and since $u = \tilde{u}$, we can write

$$u^\gamma(x_0,t_0) = u(x_0,s) - \gamma^{-1}(s-t_0)^2 = u(x_k,s_k) - \gamma^{-1}(s-t_0)^2 + o_k(1)$$

for some s depending on (x_0,t_0), $x_k \to x_0$, $x_k \in \Omega$ and $s_k \to s$. Then

$$\begin{aligned} u^\gamma(x_0,t_0) &= u(x_k,s_k) - \gamma^{-1}(s_k-t_0)^2 - \gamma^{-1}o_k(1) \\ &\le u^\gamma(x_k,t_0) - \gamma^{-1}o_k(1), \end{aligned}$$

concluding that

$$u^\gamma(x_0,t_0) \le \limsup_{\Omega \ni x \to x_0, t \to t_0} u^\gamma(x,t) = \lim_{\Omega \ni x \to x_0, t \to t_0} u^\gamma(x,t), \tag{6.21}$$

where the last equality comes from u^γ is $C^{1-\alpha/m,1}(\Omega \times [a_\gamma, T-a_\gamma])$.

Now, taking $\Omega \ni x_k \to x_0$, we clearly have $u^\gamma(x_k,t_0) = u(x_k,s_k) - \gamma^{-1}(s_k-t_0)^2$, for some s_k depending on t_0 and x_k. We see that (s_k) is bounded and therefore it converges to some $\bar{s} \in [0,T]$. Dedefining a_γ smaller, we have $\bar{s} \in [a_\gamma, T-a_\gamma]$. Now, using the usc of u we have

$$u^\gamma(x_k,t_0) \le u(x_0,\bar{s}) - \gamma^{-1}(\bar{s}-t_0)^2 \le u^\gamma(x_0,t_0),$$

from which we get the reverse inequality in (6.21). This concludes the proof. $\square$

6.5.2 Cone Condition for the Bellman Problem.

The comfortable Hölder continuity property for subsolutions in the coercive case is hardly available in the Bellman case. However, this property can be replaced by the weaker "cone condition" which is sufficient to apply Soner's argument and to get the desired comparison results, see [27], [31], [66].

Proposition 6.4. *Let $\varphi \in C_b(\bar{Q}^{ext})$, $\alpha < 1$, $\mathcal{I}$ as in (6.5) and H with Bellman form. Let u be a bounded viscosity subsolution to (CP) and let $\tilde{u}$ as in (6.10). Then, for each $(x_0, t_0) \in \Gamma \cup \Gamma_{in}$, there exists $C > 0$ and a sequence $(x_k, t_k) \in Q$ such that, as $k \to \infty$*

$$\begin{cases} (x_k, t_k) \to (x_0, t_0); \ \tilde{u}(x_k, t_k) \to \tilde{u}(x_0, t_0), \\ |x_k - x_0| \geq Cd(x_k), \\ |t_k - t_0| \geq Cd(x_k). \end{cases} \tag{6.22}$$

We provide the proof of the above cone condition for completeness. However, we note that the results of this section are the direct extensions to the parabolic framework of the results presented in [113] and therefore we will omit most of the proofs.

To get Proposition 6.4, we need to introduce notation and give an intermediate result. For $x \in \bar{\Omega}$, a function $\phi : \bar{\Omega} \to \mathbb{R}$ bounded, in $C^1(\bar{B}_r(x))$ for some $r > 0$, we define the *censored operator* $\mathcal{I}_\Omega(\phi, x)$ as

$$\mathcal{I}_\Omega(\phi, x) = \int_{\Omega - x} [\phi(x+z) - \phi(x)] K^\alpha(z) dz.$$

Associated to this operator, we have the following proposition

Lemma 6.5. *Let $\varphi \in C_b(\bar{Q}^{ext})$, $\alpha < 1$, $\mathcal{I}$ as in (6.5) and H with Bellman form. Let u be a bounded viscosity subsolution to (CP) and let $\tilde{u}$ as in (6.10). Let $(x_0, t_0) \in \partial^l Q$ and $\beta_0 \in \mathcal{B}$ such that*

$$b_{\beta_0}(x_0, t_0) \cdot Dd(x_0) \geq c_0 \tag{6.23}$$

for some $c_0 > 0$, and consider the function $U : \bar{Q} \to \mathbb{R}$ defined as

$$U(x,t) = \tilde{u}(x,t) + Ad^{1-\alpha}(x)$$

Then, there exists $A, a > 0$ such that U is a viscosity subsolution of the equation

$$\partial_t u - \mathcal{I}_\Omega(u(\cdot, t)) - b_{\beta_0} \cdot Du = 0 \quad \text{in } B_a(x_0) \times (t_0 - a, t_0 + a).$$

We remark that the notion of viscosity subsolution for censored equations is analogous to the one presented in Definition 6.

Using this result, we are in position to prove cone condition.

Proof of Proposition 6.4: Note that, if either $x_0 \in \Gamma$ or $x_0 \in \Gamma_{in}$, there exists a control $\beta_0 \in \mathcal{B}$ satisfying (6.23) for some $c_0 > 0$. Thus, denoting $b = b_{\beta_0}$ we can take $r > 0$ small enough such that $b(x,t) \cdot Dd(x) > c_0/2$ for all $x \in \bar{\Omega} \cap \bar{B}_r(x_0)$ and $|t - t_0| < r$. After rotation in the x variable and a translation in (x,t), we can assume $t_0 = 0$, $x_0 = 0$ and $Dd(x_0) = e_n$ with $e_n = (0, ..., 0, 1)$, implying in particular that $b_n(0,0) > 0$. Finally, denote $\mathbb{H}_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and $A = \bar{\mathbb{H}}_+ \cap \bar{\Omega} \cap \bar{B}_r$.

Recalling the function U defined in Lemma 6.5, we have this function satisfies the equation

$$\partial_t U - \mathcal{I}_\Omega(U(\cdot,t)) - b \cdot DU \leq 0 \quad \text{on } \bar{A} \times (-r, r).$$

By a simple scaling argument, we conclude the function $(y, s) \mapsto U(\gamma y, \gamma s)$ defined in $\gamma^{-1}(A \times (-r, r))$ satisfies the equation

$$\partial_t w - \gamma^{1-\alpha}\mathcal{I}_{\gamma^{-1}\Omega}(w, y) - b_\gamma(y) \cdot Dw(y) \leq 0 \quad \text{on } \gamma^{-1}(A \times (-r, r)), \tag{6.24}$$

where $b_\gamma(y, s) = b(\gamma y, \gamma s)$ for each $(y, s) \in \gamma^{-1}(A \times (-r, r))$. Thus, the function $\bar{w} : \bar{\mathbb{H}}_+ \times \mathbb{R} \to \mathbb{R}$ defined as

$$\bar{w}(x, t) = \limsup_{\gamma \to 0, (z,s) \to (x,t)} U(\gamma z, \gamma s)$$

is a viscosity subsolution for the problem

$$\partial_t w - b_n(0, 0)\frac{\partial w}{\partial y_n} - b'(0, 0) \cdot D_{y'} w = 0 \quad \text{in } \bar{\mathbb{H}}_+ \times \mathbb{R},$$

by classical arguments in half-relaxed limits applied over the equation (6.24). It is worth remark that by Lemma 6.5 this equation holds up to the boundary and that $b_n(0, 0) > 0$.

The maximal solution for the last transport equation with terminal data $\bar{w}(y', 1, \tau)$ (when we cast y_n as the new "time" variable) is given by the function

$$W(y', y_n, s) = \bar{w}(y' - b_n(0)^{-1}b'(0)(y_n - 1), 1, s + b_n(0)^{-1}(y_n - 1)).$$

Since W is maximal, we have $\bar{w}(y, s) \leq W(y, s)$ when $0 \leq y_n \leq 1$. Now, by definition it is clear that $\bar{w}$ is upper semicontinuous and then $\bar{w}(0, 0) = U(0, 0)$, meanwhile by the upper semicontinuity of u at the boundary and the continuity of the distance function we have $\bar{w}(y, s) \leq U(0, 0)$ for all $y \in \mathbb{H}_+$. Then, recalling $U(0, 0) = \tilde{u}(0, 0)$, we conclude that

$$\tilde{u}(0, 0) = \bar{w}(0, 0) \leq W(0, 0) = \bar{w}(b_n(0)^{-1}b'(0), 1, -b_n(0)^{-1}) \leq \tilde{u}(0, 0),$$

this is $\tilde{u}(0, 0) = \bar{w}(x_b, t_b)$, with $x_b = (b_n(0)^{-1}b'(0), 1)$ and $t_b = -b_n(0)^{-1}$. By the very definition of $\bar{w}$, we have the existence of sequences $\gamma_k \to 0$, $t_k \to t_b$, $z_k \to x_b$ such that $(x_k, t_k) := (\gamma_k z_k, \gamma_k t_k)$ satisfies $(x_k, t_k) \to (0, 0)$ and $\tilde{u}(x_k, t_k) \to \tilde{u}(0, 0)$.

Note that by definition of the sequence $(x_k)_k$ we have $x_k = \gamma_k x_b + o(\gamma_k)$. Using this, we perform a Taylor expansion on $d(x_k)$, obtaining the existence of a point $\bar{x}_k \in \mathbb{H}_+$ with $\bar{x}_k \to 0$ as $k \to \infty$ such that

$$d(x_k) = Dd(\bar{x}_k) \cdot (\gamma_k x_b + o(\gamma_k)).$$

Hence, since $Dd(0) = e_n$ we conclude $d(x_k) = \gamma_k + o(\gamma_k)$. Thus, using the estimates for x_k and $d(x_k)$ we get that $d(x_k) \geq (4|x_b|)^{-1}|x_k|$, for all k large enough. Recalling that $x_0 = 0$, we conclude that $(x_k)_k$ is the sequence satisfying (6.22). Finally, for the t variable we have $t_k = \gamma_k t_b + o(\gamma_k)$ and then we get $|t_k| \leq (4|t_b|)^{-1}d(x_k)$ for all k large. Recalling $x_0 = 0$ we conclude the result. □

Remark 6.2. It is important to note that, considering (6.22) and its proof, the time and space variables are playing the same role regarding the cone's condition property. This fact explains why we cannot weaken the time Lipschitz continuity of H given in assumption (H).

Following the same ideas given in Proposition 6.4, it is possible to conclude the cone condition for supersolutions in Γ_{in}.

Proposition 6.5. *Let $\varphi \in C_b(\bar{Q}^{ext})$, $\alpha < 1$, $\mathcal{I}$ as in (6.5) and H with Bellman form. Let v a bounded viscosity supersolution to (CP) and let $\tilde{v}$ as in (6.10). Then, for each $(x_0, t_0) \in \Gamma_{in}$, there exists a sequence $(x_k, t_k)_k$ of points of Q satisfying (6.22) relative to $\tilde{v}$.*

To get the last proposition, a similar result as Lemma 6.5 is needed for supersolutions. This time we cannot get rid of the nonlinearity of H because of the Bellman form, but this can be handled because all the drift terms are pointing "strictly inside" Ω. See [113] for details.

6.6 Proof of The Comparison Results.

6.6.1 Strong Comparison Principle for the Coercive Case.

We start with the following

Lemma 6.6. *Let $\varphi \in C_b(Q_T^{ext})$, $\mathcal{I}$ as in (6.1), and H with coercive form satisfying (H1)-(H2). Let u, v be bounded, respective sub and supersolution to the problem*

$$\begin{cases} \partial_t u - \mathcal{I}(u,x) + H(x,t,u,Du) &= 0 \quad \text{in } Q_T \\ u &= \varphi \quad \text{in } Q_T^{ext}, \end{cases} \tag{6.25}$$

and let $\tilde{u}$ as in (6.9).

Let $\gamma \in (0,1)$ and $\mu \in (0,1)$ if H is superlinearly coercive, $\mu = 1$ if H is sublinearly coercive. Define $\bar{u} = \mu \tilde{u}^\gamma$ where u^γ as in (6.19), and $w = \bar{u} - v$. Then, w is a viscosity subsolution for the problem

$$\begin{cases} \partial_t w + h_R w - \mathcal{I}(w,x) - \bar{\omega}_R(|Dw|) &= \bar{C}_R(1-\mu) + o_\gamma(1) \quad \text{in } Q_T \\ w &= \bar{\varphi} - \varphi \quad \text{in } Q_T^{ext}, \end{cases} \tag{6.26}$$

where a_γ is given in Lemma 6.3, $o_\gamma(1)$ depends only on the modulus of continuity of f, $R = ||\bar{u}||_\infty + ||v||_\infty$, $\bar{\omega}_R$ is a modulus of continuity depending on R and the data, h_R arises in (H1), $\bar{C}_R$ depends on R and $||f||_\infty$, and $\bar{\varphi} = \mu\varphi^\gamma$.

Proof: We omit the superscript $\sim$ for simplicity and we address the superlinear case; the sublinear case follows the same ideas with easier computations.

Note that by Lemma 6.3 and direct arguments of the viscosity theory, we have $\bar{u}$ is a viscosity subsolution to the problem

$$\partial_t \bar{u} - \mathcal{I}(\bar{u}, x) + \mu H(x,t,\mu^{-1}\bar{u}, \mu^{-1}D\bar{u}) = o_\gamma(1) \quad \text{in } \Omega \times [a_\gamma, T]$$
$$\bar{u} = \bar{\varphi}, \quad \text{in } \Omega^c \times [a_\gamma, T],$$

where $o_\gamma(1) \to 0$ as $\gamma \to 0$ uniformly on $\mu \in (0,1)$. Moreover, by Lemma 6.4, we see that $\bar{u} \in C^{1-\alpha/m,1}(\bar{\Omega} \times [a_\gamma, T])$.

The aim is prove that w is a subsolution to (6.26) in the viscosity sense with generalized boundary condition, and the most difficult scenario is when we study the subsolution's obstacle requirement at the lateral boundary.

Let $(x_0, t_0) \in \partial^l Q_T$. If $w(x_0,t_0) \leq (\bar{\varphi} - \varphi)(x_0,t_0)$, then the boundary condition for subsolutions is satisfied in the classical sense and we get the result. For this, we assume $w(x_0,t_0) > (\bar{\varphi} - \varphi)(x_0,t_0)$ and the rest of the proof is devoted to conclude the subsolution's viscosity inequality at (x_0,t_0). In this case, $w^{\bar{\varphi}-\varphi}(x_0,t_0) = w(x_0,t_0)$, and by Lemma 6.1 we see that

$$v(x_0,t_0) < \bar{u}(x_0,t_0) - \bar{\varphi}(x_0,t_0) + \varphi(x_0,t_0) \leq \varphi(x_0,t_0). \tag{6.27}$$

Let ϕ smooth such that $w^{\bar{\varphi}-\varphi} - \phi$ has a strict maximum point in $\bar{Q}_T$ at (x_0,t_0). Define $\nu_0 = (Dd(x_0), 0)$ and for all $\epsilon > 0$ we consider the function

$$\phi_\epsilon(x,y,s,t) = \phi(y,t) + |\epsilon^{-1}((x,s) - (y,t)) - \nu_0|^2.$$

Now we look for maximum points of the function $\Phi : \bar{\Omega} \times \mathbb{R}^n \times [0,T]^2 \to \mathbb{R}$ defined as

$$\Phi(x,y,s,t) := \bar{u}(x,s) - v(y,t) - \phi_\epsilon(x,y,s,t).$$

Note that by the boundedness and the upper semicontinuity of Φ, there exists a point $(\bar{x},\bar{y},\bar{s},\bar{t}) \in \bar{\Omega} \times \mathbb{R}^n \times [0,T]^2$ attaining the maximum of Φ in this set. Then, using the inequality

$$\Phi(\bar{x},\bar{y},\bar{s},\bar{t}) \geq \Phi(x_0 + \epsilon Dd(x_0), x_0, t_0, t_0),$$

together with the continuity of $\bar{u}$ given by Lemma 6.4, classical arguments in viscosity solution's theory allows us to write

$$\begin{aligned} &(\bar{x},\bar{s}), (\bar{y},\bar{t}) \to (x_0,t_0), \quad |\epsilon^{-1}(\bar{x}-\bar{y}, \bar{s}-\bar{t}) - \nu_0| \to 0, \\ &\text{and} \quad \bar{u}(\bar{x},\bar{s}) \to \bar{u}(x_0,t_0),\ v_\varphi(\bar{y},\bar{t}) \to v(x_0,t_0), \end{aligned} \tag{6.28}$$

as $\epsilon \to 0$. Moreover, if ϵ is small enough, we have $\bar{y} \in \bar{\Omega}$, since otherwise, by the continuity of φ, we would have

$$v_\varphi(\bar{y},\bar{t}) = \varphi(\bar{x},\bar{t}) \to \varphi(x_0,t_0)$$

as $\epsilon \to 0$, which is a contradiction to (6.27) in view of the last fact in (6.28). Moreover, by the continuity of φ we see that $v_\varphi(\bar{y},\bar{t}) < \varphi(\bar{y},\bar{t})$ for all ϵ small and therefore, even if $\bar{y} \in \partial\Omega$, we have a viscosity supersolution inequality associated to v_φ at $(\bar{y},\bar{t})$.

On the other hand, by the second property in (6.28) we have

$$\bar{x} = \bar{y} + \epsilon Dd(x_0) + o_\epsilon(\epsilon), \tag{6.29}$$

A simple Taylor expansion on the distance function implies that $d(\bar{x}) \geq d(\bar{y}) + \epsilon(1 - o_\epsilon(1))$ for all ϵ small enough, concluding that $\bar{x} \in \Omega$. We consider $0 < \delta' < \delta$ and we subtract the viscosity inequality for v at $(\bar{y},\bar{t})$ to the viscosity inequality for $\bar{u}$ at $(\bar{x},\bar{s})$, concluding that

$$\mathcal{A} - \mathcal{I}^{\delta'} \leq o_\gamma(1), \tag{6.30}$$

where

$$\begin{aligned} \mathcal{I}^{\delta'} = {} & \mathcal{I}[B_{\delta'}](\phi_\epsilon(\cdot,\bar{y},\bar{s},\bar{t}),\bar{x}) - \mathcal{I}[B_{\delta'}](-\phi_\epsilon(\bar{x},\cdot,\bar{s},\bar{t}),\bar{y}) \\ & + \mathcal{I}[B_{\delta'}^c](\bar{u}(\cdot,\bar{s}),\bar{x},\bar{p}) - \mathcal{I}[B_{\delta'}^c](v(\cdot,\bar{t}),\bar{y},\bar{q}), \end{aligned}$$

with

$$\begin{aligned} \bar{p} &= D_x\phi_\epsilon(\bar{x},\bar{y},\bar{s},\bar{t}) = \epsilon^{-1}(\epsilon^{-1}((\bar{x},\bar{s}) - (\bar{y},\bar{t})) - \nu_0) \\ \bar{p} &= -D_y\phi_\epsilon(\bar{x},\bar{y},\bar{s},\bar{t}) = \bar{p} - D\phi(\bar{y},\bar{t}), \end{aligned}$$

and
$$\mathcal{A} = (\partial_s \phi_\epsilon - \partial_t \phi_\epsilon)(\bar{x}, \bar{y}, \bar{s}, \bar{t}) + \mu H(\bar{x}, \bar{s}, \mu^{-1}\bar{u}(\bar{x}, \bar{s}), \mu^{-1}\bar{p}) - H(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), \bar{q}).$$

Now we estimate each term in (6.30), starting with $\mathcal{A}$. We have

$$(\partial_s \phi_\epsilon - \partial_t \phi_\epsilon)(\bar{x}, \bar{y}, \bar{s}, \bar{t}) = \partial_t \phi(\bar{y}, \bar{t}), \tag{6.31}$$

and then it remains to estimate the difference among the Hamiltonians to complete the bound for $\mathcal{A}$. Using (A0) and the first statement in (6.28), we readily have

$$\mu H(\bar{x}, \bar{s}, \mu^{-1}\bar{u}(\bar{x}, \bar{s}), \mu^{-1}\bar{p}) - H(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), \bar{q}) \geq (\mu - 1)||f||_\infty - o_\epsilon(1) + \mathcal{H}_0, \tag{6.32}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$ uniformly in the rest of the variables and $\mathcal{H}_0$ is defined as

$$\mathcal{H}_0 = \mu H_0(\bar{x}, \mu^{-1}\bar{u}(\bar{x}, \bar{s}), \mu^{-1}\bar{p}) - H_0(\bar{y}, v(\bar{y}, \bar{t}), \bar{q}).$$

Now, using (H1),(A1-b) and (A2-b) we have

$$\begin{aligned} \mathcal{H}_0 \geq\ & h_R(\bar{x})(\bar{u}(\bar{x}, \bar{s}) - v(\bar{y}, \bar{t})) + (1 - \mu)\Big((m-1)a_0|\bar{p}|^m - C_R\Big) \\ & - \omega_R(|\bar{x} - \bar{y}|)(1 + |\bar{p}|^m) - \omega_R(|D_y\phi(\bar{y}, \bar{t})|)|\bar{p}|^{m-1}, \end{aligned}$$

where $R = ||\bar{u}||_\infty + ||v||_\infty$. Thus, using the first fact in (6.28), for all ϵ small in terms on $1 - \mu$ we can write

$$\begin{aligned} \mathcal{H}_0 \geq\ & (1 - \mu)(m-1)a_0|\bar{p}|^m/2 - \omega_R(|D_y\phi(\bar{y}, \bar{t})|)|\bar{p}|^{m-1} \\ & + h_R(\bar{x})(\bar{u}(\bar{x}, \bar{s}) - v(\bar{y}, \bar{t})) - C_R(1 - \mu) - o_\epsilon(1) \\ \geq\ & \inf_{p \geq 0}\{(1 - \mu)(m-1)a_0 p^m/2 - \omega_R(|D_y\phi(\bar{y}, \bar{t})|)p^{m-1}\} \\ & + h_R(\bar{x})(\bar{u}(\bar{x}, \bar{s}) - v(\bar{y}, \bar{t})) - C_R(1 - \mu) - o_\epsilon(1). \end{aligned}$$

We notice that the infimum in the last expression is attained, from which we conclude that

$$\begin{aligned} \mathcal{H}_0 \geq\ & - c_{m,\mu}\omega_R(|D\phi(\bar{y}, \bar{t})|)^m \\ & + h_R(\bar{x})(\bar{u}(\bar{x}, \bar{s}) - v(\bar{y}, \bar{t})) - C_R(1 - \mu) - o_\epsilon(1), \end{aligned}$$

where $c_{m,\mu} = \frac{(2(m-1))^{m-1}}{m^m}((1-\mu)(m-1)a_0)^{1-m}$. Replacing this into (6.32) and recalling (6.31), we conclude the following estimate for $\mathcal{A}$

$$\begin{aligned} \mathcal{A} \geq\ & \partial_t\phi(\bar{y}, \bar{t}) + h_R(\bar{x})(\bar{u}(\bar{x}, \bar{s}) - v(\bar{y}, \bar{t})) - c_{m,\mu}\omega_R(|D\phi(\bar{y}, \bar{t})|)^m \\ & + (\mu - 1)(||f||_\infty + C_R) - o_\epsilon(1), \end{aligned} \tag{6.33}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$ if we keep μ, R fixed.

Now we addres the estimates for $\mathcal{I}^{\delta'}$. We start noting that

$$\mathcal{I}[B_{\delta'}](\phi_\epsilon(\cdot, \bar{y}, \bar{s}, \bar{t}), \bar{x}) - \mathcal{I}[B_{\delta'}](-\phi_\epsilon(\bar{x}, \cdot, \bar{s}, \bar{t}), \bar{y}) \leq \epsilon^{-2} o_{\delta'}(1), \tag{6.34}$$

where $o_{\delta'}(1)$ is independent of ϵ. To estimate the integral terms outside $B_{\delta'}$, we consider the sets

$$\begin{aligned} D_{int} &= (\Omega - \bar{x}) \cap (\Omega - \bar{y}), \quad D_{ext} = (\Omega - \bar{x})^c \cap (\Omega - \bar{y})^c, \\ D_{int}^{\bar{x}} &= (\Omega - \bar{x}) \cap (\Omega - \bar{y})^c, \quad D_{int}^{\bar{y}} = (\Omega - \bar{x})^c \cap (\Omega - \bar{y}), \end{aligned} \tag{6.35}$$

and then we can write

$$\mathcal{I}[B^c_{\delta'}](\bar{u}(\cdot,\bar{s}),\bar{x},\bar{p}) - \mathcal{I}[B^c_{\delta'}](v(\cdot,\bar{t}),\bar{y},\bar{q}) = \mathcal{I}^{\delta'}_{int} + \mathcal{I}^{\delta'}_{int,\bar{x}} + \mathcal{I}^{\delta'}_{int,\bar{y}} + \mathcal{I}^{\delta'}_{ext},$$

where

$$\begin{aligned}
\mathcal{I}^{\delta'}_{int} &= \int_{D_{int}\setminus B_{\delta'}} [\bar{u}(\bar{x}+z) - v(\bar{y}+z) - (\bar{u}(\bar{x}) - v(\bar{y})) - \mathbf{1}_B\langle D\phi(\bar{y}), z\rangle] K^\alpha(z)dz \\
\mathcal{I}^{\delta'}_{int,\bar{x}} &= \int_{D^{\bar{x}}_{int}\setminus B_{\delta'}} [\bar{u}(\bar{x}+z) - \varphi(\bar{y}+z) - (\bar{u}(\bar{x}) - v(\bar{y})) - \mathbf{1}_B\langle D\phi(\bar{y}), z\rangle] K^\alpha(z)dz \\
\mathcal{I}^{\delta'}_{int,\bar{y}} &= \int_{D^{\bar{y}}_{int}\setminus B_{\delta'}} [\bar{\varphi}(\bar{x}+z) - v(\bar{y}+z) - (\bar{u}(\bar{x}) - v(\bar{y})) - \mathbf{1}_B\langle D\phi(\bar{y}), z\rangle] K^\alpha(z)dz \\
\mathcal{I}^{\delta'}_{ext} &= \int_{D_{ext}\setminus B_{\delta'}} [\bar{\varphi}(\bar{x}+z) - \varphi(\bar{y}+z) - (\bar{u}(\bar{x}) - v(\bar{y})) - \mathbf{1}_B\langle D\phi(\bar{y}), z\rangle] K^\alpha(z)dz.
\end{aligned}$$

We estimate each integral term separately. For $\mathcal{I}^{\delta'}_{int}$, using that $(\bar{x},\bar{y},\bar{s},\bar{t})$ is a maximum point for Φ in $\bar{\Omega}\times\mathbb{R}^n\times[0,T]^2$, for all $z\in D_{int}$ we see that

$$\bar{u}(\bar{x}+z) - v(\bar{y}+z) - (\bar{u}(\bar{x}) - v(\bar{y})) \le \phi(\bar{y}+z) - \phi(\bar{y}),$$

and therefore we can write

$$\mathcal{I}^{\delta'}_{int} \le \mathcal{I}[D_{int}\cap B_\delta \setminus B_{\delta'}](\phi,\bar{y}) + \mathcal{I}^{\delta}_{int}.$$

We can use the same argument for $I^{\delta'}_{int,\bar{x}}$, concluding that

$$\mathcal{I}^{\delta'}_{int,\bar{x}} \le \mathcal{I}[D_{int,\bar{x}}\cap B_\delta \setminus B_{\delta'}](\phi,\bar{y}) + \mathcal{I}^{\delta}_{int,\bar{x}},$$

but in this case we note that keeping $\delta>0$ fixed, $\mathbf{1}_{D_{int,\bar{x}}\setminus B_\delta}(z)K^\alpha(z)$ is an integrable kernel, uniformly in ϵ. Since $|D_{int,\bar{x}}|\to 0$ as $\epsilon\to 0$, we conclude that

$$\mathcal{I}^{\delta'}_{int,\bar{x}} \le \mathcal{I}[D_{int,\bar{x}}\cap B_\delta \setminus B_{\delta'}](\phi,\bar{y}) + o_\epsilon(1).$$

For $\mathcal{I}^{\delta'}_{ext}$, we recall that $w(x_0,t_0) - (\bar{\varphi}-\varphi)(x_0,t_0) > 0$. Then, by the last fact in (6.28), the continuity of $\bar{\varphi},\varphi$ and the boundedness of $D\phi(\bar{y})$, there exists $0<r_0<\delta$ small not depending on ϵ,δ,δ' such that, for all $r<r_0$ and for all ϵ small enough, we have the inequality

$$\bar{\varphi}(\bar{x}+z) - \varphi(\bar{y}+z) - (\bar{u}(\bar{x}) - v(\bar{y})) - \langle D\phi(\bar{y}), z\rangle \le 0, \quad \text{for all } z\in D_{ext}\cap B_r,$$

and therefore, we arrive at

$$\mathcal{I}^{\delta'}_{ext} \le \mathcal{I}^{r}_{ext}.$$

We finish the estimates for the nonlocal term with $I^{\delta'}_{int,\bar{y}}$. We claim that $D^{\bar{y}}_{int}$ is away from the origin uniformly in ϵ and δ'. This fact is less obvious so we postpone its proof until the end. Thus, since $\mathbf{1}_{D_{int}\setminus B_{\delta'}}(z)K\alpha(z)$ is an integrable kernel, uniformly in δ' and ϵ, and since $|D^{\bar{y}}_{int}|\to 0$ as $\epsilon\to 0$, we conclude

$$I^{\delta}_{int,\bar{y}} = o_\epsilon(1).$$

Thus, joining the above inequalities concerning the integral terms outside $B_{\delta'}$ and (6.34), we conclude that

$$\mathcal{I}^{\delta'} \le \mathcal{I}[(D_{int}\cup D_{int,\bar{x}})\cap B_\delta \setminus B_{\delta'}](\phi,\bar{y}) + \mathcal{I}^{\delta}_{int} + \mathcal{I}^{r}_{ext} + o_\epsilon(1) + \epsilon^{-2}o_{\delta'}(1),$$

and replacing this and (6.33) into (6.30), we arrive to

$$\begin{aligned}&\partial_t\phi(\bar y,\bar t)+h_R(\bar x)(\bar u(\bar x,\bar s)-v(\bar y,\bar t))-c_{m,\mu}\omega_R(|D\phi(\bar y,\bar t)|)^m\\&\quad-\mathcal{I}[(D_{int}\cup D_{int,\bar x})\cap B_\delta\setminus B_{\delta'}](\phi,\bar y)-\mathcal{I}^\delta_{int}-\mathcal{I}^r_{ext}\\&\leq(1-\mu)(||f||_\infty+C_R)+o_\gamma(1)+o_\epsilon(1)+\epsilon^{-2}o_{\delta'}(1).\end{aligned}$$

At this point, letting $\delta'\to 0$ and then $\epsilon\to 0$, by (6.28), the smoothness of ϕ, the continuity of h_R,ω_R and using Dominated Convergence Theorem, we arrive at

$$\begin{aligned}\partial_t\phi(x_0,t_0)+h_R(x_0)w(x_0,t_0)-c_{m,\mu}\omega_R(|D\phi(x_0,t_0)|)^m&\\-\mathcal{I}[(\Omega-x_0)\cap B_\delta](\phi,x_0)&\\-\mathcal{I}[(\Omega-x_0)\setminus B_\delta](w,x_0,D\phi(x_0))&\\-\mathcal{I}[(\Omega-x_0)^c\setminus B_r](w^{\bar\varphi-\varphi},x_0,D\phi(x_0))&\leq\bar C_R(1-\mu)+o_\gamma(1),\end{aligned}$$

where $\bar C_R=||f||_\infty+C_R$. Using that (x_0,t_0) is a maximum point for $w^{\bar\varphi-\varphi}-\phi$, we can write

$$\begin{aligned}\partial_t\phi(x_0,t_0)+h_R(x_0)w(x_0,t_0)-c_{m,\mu}\omega_R(|D\phi(x_0,t_0)|)^m&\\-\mathcal{I}[(\Omega-x_0)\cap B_\delta](\phi,x_0)&\\-\mathcal{I}[B_\delta^c](w^{\bar\varphi-\varphi},x_0,D\phi(x_0))&\\-\mathcal{I}[(\Omega-x_0)^c\cap B_\delta\setminus B_r](\phi,x_0)&\leq\bar C_R(1-\mu)+o_\gamma(1),\end{aligned}$$

and from this, by the smoothness of ϕ we can let $r\to 0$, concluding that

$$\begin{aligned}\partial_t\phi(x_0,t_0)+h_R(x_0)w(x_0,t_0)-c_{m,\mu}\omega_R(|D\phi(x_0,t_0)|)^m&\\-\mathcal{I}[B_\delta](\phi,x_0)-\mathcal{I}[B_\delta^c](w^{\bar\varphi-\varphi},x_0,D\phi(x_0))&\leq\bar C_R(1-\mu)+o_\gamma(1),\end{aligned}$$

from which we conclude the result.

Now we address the claim leading to the estimate of $\mathcal{I}^{\delta'}_{int,\bar y}$. Assume that there exists a sequence $\epsilon_k\to 0$ and $z_k\in D^{\bar y}_{int}$ such that $z_k\to 0$. By definition, there exists $a_k\in\Omega$ and $b_k\in\Omega^c$ such that $z_k=a_k-\bar y=b_k-\bar x$ and by the first property in (6.28) we have $a_k,b_k\to x_0$. Now, applying (6.29) we conclude $b_k=a_k+\epsilon_k(Dd(x_0)+o_{\epsilon_k}(1))$. Taking k large we conclude $b_k\in\Omega$, which is a contradiction. □

With the above lemma, we are in position to prove the comparison principle.

Proof of Theorem 6.1: We argue over the redefined function given by (6.9), but we omit the superscript $\sim$ for simplicity. We start assuming by contradiction that

$$2M:=\sup_{\bar Q_T}\{u-v\}>0.$$

Then, taking $\eta>0$ small in terms of M, we have

$$\sup_{(x,t)\in\bar Q_T}\{u(x,t)-v(x,t)-\eta t\}=:M>0.\tag{6.36}$$

By the upper semicontinuity of $u-v$ in $\bar Q_T$, this supremum is attained at some point $(x_0,t_0)\in\bar Q_T$. By Lemma 6.1, taking η smaller if it is necessary, for each (x_0,t_0) attaining M we have $t_0>0$.

For the superlinear coercive case, we consider $\eta, \gamma, \mu > 0$, denote $\bar{u} = \mu u^\gamma$ and note that $\bar{u} - v - \eta t \to u^\gamma - v$ as $\eta \to 0^+$, $\mu \to 1^-$ uniformly in $\bar{Q}_T$. Since $u^\gamma \geq u$ in $\bar{Q}_T$, for all η close to 0 and $\mu < 1$ close to 1, we have

$$\sup_{(x,t)\in\bar{Q}_T} \{\bar{u}(x,t) - v(x,t) - \eta t\} \geq M/2. \tag{6.37}$$

This supremum is attained at some point $(\tilde{x}, \tilde{t}) \in \bar{Q}_T$. Using that $u \leq u^\gamma$, by the upper semicontinuity of u and the lower semicontinuity of v, we have

$$\begin{aligned} M &\leq \liminf_{\gamma\to 0,\mu\to 1} \{\bar{u}(x_0,t_0) - v(x_0,t_0) - \eta t_0\} \\ &\leq \liminf_{\gamma\to 0,\mu\to 1} \{\sup_{\bar{Q}_T}\{\bar{u} - v - \eta t\}\} \\ &= \liminf_{\gamma\to 0,\mu\to 1} \{\bar{u}(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) - \eta\tilde{t}\} \\ &\leq \limsup_{\gamma\to 0,\mu\to 1} \{\bar{u}(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) - \eta\tilde{t}\} \leq M, \end{aligned}$$

and therefore we have $w(\tilde{x},\tilde{t}) \to w(x_0,t_0)$ as $\eta, \gamma \to 0$ and $\mu \to 1$, for some (x_0,t_0) attaining M in (6.36). In particular, for all γ small enough, $\tilde{t} > a_\gamma$, with a_γ given in Lemma 6.3.

The idea is to use the function $(x,t) \mapsto \eta t$ as test function for $w = \bar{u} - v$ at $(\tilde{x},\tilde{t})$ and the corresponding viscosity inequality given by Lemma 6.6. We can use it at once if $\tilde{x} \in \Omega$ for all μ, γ. On the contrary, in the case $\tilde{x} \in \partial\Omega$ we note that $M/2 \leq w(\tilde{x},\tilde{t}) = \bar{u}(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t})$, and by continuity of φ, we have $\bar{\varphi} \to \varphi$ locally uniformly in $\Omega^c \times (0,T)$ as $\mu \to 1$ and $\gamma \to 0$. Thus, we can take μ close to 1 and γ close to 0 in order to have

$$w(\tilde{x},\tilde{t}) > (\bar{\varphi} - \varphi)(\tilde{x},\tilde{t}),$$

which says that we can test the equation at $(\tilde{x},\tilde{t})$ even if this point is on the lateral boundary. Note that this last inequality implies additionally that $w(\tilde{x},\tilde{t}) = w^{\bar{\varphi}-\varphi}(\tilde{x},\tilde{t})$.

Thus, for each $\delta > 0$ we can write

$$\eta + h_R(\tilde{x})w(\tilde{x},\tilde{t}) - \mathcal{I}[B_\delta^c](w^{\bar{\varphi}-\varphi}(\cdot,\tilde{t}),\tilde{x},0) \leq \bar{C}_R(1-\mu) + o_\gamma(1),$$

where $R = ||\bar{u}||_\infty + ||v||_\infty$. Using that $(\tilde{x},\tilde{t})$ attains the supremum in (6.37) we have

$$\eta + h_R(\tilde{x})w(\tilde{x},\tilde{t}) - \mathcal{I}[B_\delta^c \cap (\Omega^c - \tilde{x})](w^{\bar{\varphi}-\varphi}(\cdot,\tilde{t}),\tilde{x},0) \leq \bar{C}_R(1-\mu) + o_\gamma(1),$$

and from this we see that

$$\begin{aligned} &\eta + \Big(h_R(\tilde{x}) + \int_{(\Omega^c-\tilde{x})\setminus B_\delta} K^\alpha(z)dz\Big)w(\tilde{x},\tilde{t}) \\ &- \int_{(\Omega^c-\tilde{x})\setminus B_\delta} (\bar{\varphi}(\tilde{x}+z) - \varphi(\tilde{x}+z))K^\alpha(z)dz \leq \bar{C}_R(1-\mu) + o_\gamma(1), \end{aligned}$$

But using that $\bar{\varphi} \to \varphi$ locally uniform in $\Omega^c \times (0,T)$ as $\mu \to 1$ and $\eta \to 0$, using Dominated Convergence Theorem, the continuity of h_R and that $w(\tilde{x},\tilde{t}) \to M$, taking $\eta, \gamma \to 0$ and $\mu \to 1$ we arrive at

$$\eta + \Big(h_R(x_0) + \int_{(\Omega^c-x_0)\setminus B_\delta} K^\alpha(z)dz\Big)w(x_0,t_0) \leq 0,$$

where (x_0, t_0) is a point attaining the supremum in (6.36). Finally, by (H1) we can take $\delta > 0$ small in order to have

$$\eta/2 \leq 0,$$

which is a contradiction. □

6.6.2 Strong Comparison Principle for the Bellman Case.

The analogous to Lemma 6.6 for the Bellman case reads as follows

Lemma 6.7. *Let $\varphi \in C_b(Q_T^{ext})$, $\alpha < 1$, $\mathcal{I}$ as in (6.5), with K satisfying (UE) and H with Bellman form, satisfying (H1)-(H2). Let u, v be bounded, respective viscosity sub and supersolution to (6.25), and consider $\tilde{u}, \tilde{v}$ as in (6.10). Then, $w := \tilde{u} - \tilde{v}$ is a viscosity subsolution for the problem*

$$\begin{aligned} \partial_t w + h_R(x)w - \mathcal{I}(w, x) - \beta|Dw| = 0 \quad & \text{in } Q_T \\ w = 0 \quad & \text{in } Q_T^{ext} \end{aligned}$$

where $\beta = \sup_{\beta \in \mathcal{B}} |b_\beta(x_0, t_0)|$, $R = ||\bar{u}||_\infty + ||v||_\infty$, $\bar{\omega}$ is a modulus of continuity depending on b and h_R arises in (H1).

We require the following result which states the viscosity inequality holds on Γ_{in} for the redefined functions $\tilde{u}, \tilde{v}$.

Lemma 6.8. *Assume the conditions of Lemma 6.7 hold. Let $(x_0, t_0) \in \Gamma_{in}$ and assume $\tilde{u}(x_0, t_0) > \varphi(x_0, t_0)$. Then, for each ϕ smooth such that (x_0, t_0) is a maximum point for $\tilde{u}^\varphi - \phi$ in $B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$, then $E_\delta(\tilde{u}^\varphi, \phi, x_0, t_0) \leq 0$. The analogous result holds for $\tilde{v}$.*

Proof: Let $(x_k, t_k) \to (x_0, t_0)$ such that $\tilde{u}(x_k, t_k) \to \tilde{u}(x_0, t_0)$, with $x_k \in \Omega$. Define $\epsilon_k = d(x_k)$ and consider the function

$$(x, t) \mapsto \tilde{u}^\varphi(x, t) - \phi(x, t) + \epsilon_k \ln(d(x))\mathbf{1}_\Omega(x).$$

For k large enough, we have this function has a maximum point $(\bar{x}_k, \bar{t}_k)$ in $B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$, with $(\bar{x}_k, \bar{t}_k) \to (x_0, t_0)$, $\tilde{u}(\bar{x}_k, \bar{t}_k) \to \tilde{u}(x_0, t_0)$ and $\bar{x}_k \in \Omega$. Using this and since $u^\varphi = \tilde{u}^\varphi$ up to a set of zero Lebesgue measure, we can write the viscosity inequality for u at $(\bar{x}_k, \bar{t}_k)$

$$E_\delta(\tilde{u}^\varphi, \phi - \epsilon_k \ln(d)\mathbf{1}_\Omega, \bar{x}_k, \bar{t}_k) \leq 0.$$

But using that $(x_0, t_0) \in \Gamma_{in}$, there exists $c_0 > 0$ such that, for all k large enough we have $b_\beta(\bar{x}_k, \bar{t}_j) \cdot Dd(\bar{x}_k) \geq c_0$. Thus, we arrive at

$$\epsilon_k \Big(- \mathcal{I}[B_\delta \cap (\Omega - \bar{x}_k)](\ln(d), \bar{x}_k) + c_0 d^{-1}(\bar{x}_k) \Big) + E_\delta(\tilde{u}^\varphi, \phi, \bar{x}_k, \bar{t}_k) \leq 0.$$

Here we mention that there exists a constant $c > 0$ such that

$$\mathcal{I}[B_\delta \cap (\Omega - \bar{x}_k)](\ln(d), \bar{x}_k) \leq c d^{-\alpha}(\bar{x}_k),$$

see [113] for a proof of this result. Thus, for all k large we have

$$E_\delta(\tilde{u}^\varphi, \phi, \bar{x}_k, \bar{t}_k) \leq 0,$$

and recalling that $\tilde{u}^\varphi = \tilde{u}$ in a neighborhood of (x_0, t_0), taking $k \to \infty$ together with Dominated Convergence Theorem to control the integral terms, we get the result. $\square$

Proof of Lemma 6.7: We concentrate in the viscosity inequality on the lateral boundary. By Lemma 6.2, the interesting case is when the test point $(x_0, t_0) \in \Gamma \cup \Gamma_{in}$ is such that $w(x_0, t_0) > 0$. Note that $w^0(x_0, t_0) = w(x_0, t_0)$ in this case.

Consider ϕ a smooth function such that $w^0 - \phi$ has a strict maximum point in $\bar{Q}_T$ at (x_0, t_0).

If $(x_0, t_0) \in \Gamma$, Proposition 6.2 allows us to conclude $\tilde{u}(x_0, t_0) \leq \varphi(x_0, t_0)$ and Proposition 6.4 implies the existence of a sequence satisfying (6.22). In particular, denoting $\epsilon_k = \sqrt{|x_k - x_0|^2 + (t_k - t_0)^2}$, up to a subsequences we have $\epsilon_k^{-1}(x_k, t_k) \to \nu_0$ satisfying $\nu_0 \cdot (Dd(x_0), 0) \geq c_0$, for some $c_0 > 0$. This time, for $k \in \mathbb{N}$ we double variables and use the penalization

$$\tilde{u}(x, s) - \tilde{v}(y, t) - \phi(y, t) - |\epsilon_k^{-1}((x, s) - (y, t)) - \nu_0|^2,$$

and from this point we argue exactly as in Lemma 6.6, arriving at inequality (6.30), where $\mathcal{I}^{\delta'}$ is managed in the same way as in the coercive case, but $\mathcal{A}$ in this case has the form

$$\mathcal{A} \geq \partial_t \phi(x_0, t_0) + h_R(x_0) w(x_0, t_0) - \beta |D\phi(x_0, t_0)| - o_k(1),$$

where $o_k(1) \to 0$ as $k \to \infty$. From this, we proceed exactly as in the proof of Lemma 6.6 to conclude the result.

If $(x_0, t_0) \in \Gamma_{in}$, we consider two sub-cases: if $\tilde{v}(x_0, t_0) < \varphi(x_0, t_0)$, then we argue exactly as in the case of Γ because cone condition also holds for subsolutions on Γ_{in}. On the other hand, if $\varphi(x_0, t_0) \leq \tilde{v}(x_0, t_0)$, we can exchange the roles of u and v in the proof of the case $(x_0, t_0) \in \Gamma$ since cone condition holds for supersolution on Γ_{in} as it is stated in Proposition 6.5. We remark that by Lemma 6.8 we can use the viscosity inequality on Γ_{in} for $\tilde{u}$ and/or $\tilde{v}$ if they do not satisfy the boundary condition in the classical sense. $\square$

Proof of Theorem 6.2: We argue by contradiction as in the proof of Theorem 6.1, where this time the linearization precedure is played by Lemma 6.7. We omit the details. $\square$

6.7 Existence and Large Time Behavior.

6.7.1 Existence and Uniqueness Issues.

For both coercive and Bellman case, the application of Perron's method on a sequence of finite-time horizon problems with the form ($\mathrm{CP_T}$) with $T \to \infty$ and the strong comparison principle allows us to get the existence of a solution which is defined for all time.

For reasons that will be made clear in the next theorem , we introduce the following nondegeneracy condition:

(H2') *There exists $\mu_0 > 0$ and a continuous function $h : \bar{\Omega} \to \mathbb{R}$ satisfying*

$$\inf_{x \in \bar{\Omega}} \left\{ h(x) + \int_{x+z \notin \Omega} K^\alpha(z) dz \right\} \geq \mu_0,$$

such that, for all $R > 0$, h_R defined in (H1) satisfies $h_R \geq h$.

Theorem 6.3. (Existence and Uniqueness) *Let $\alpha \in (0,2)$, $u_0 \in C(\bar{\Omega})$, $\varphi \in C_b(\bar{Q}^{ext})$ satisfying (H0). Assume (CP) has a*

- *Coercive Form: $\mathcal{I}$ as in (6.1) (as in (6.5) if $\alpha < 1$), and H has coercive form.*
- *Bellman Form: $\alpha < 1$, $\mathcal{I}$ as in (6.5) satisfying (UE), and H has Bellman form.*

In both cases, we further assume that H satisfies (H1)-(H2). Then, there exists a unique viscosity solution $u \in C(\bar{Q}) \cap L^\infty(\bar{Q}_T)$ for all $T > 0$, to problem (CP).

Moreover, if (H2') holds, then the unique solution $u \in C(\bar{Q}) \cap L^\infty(\bar{Q}_T)$ for all $T > 0$, to problem (CP), is uniformly bounded in $\bar{Q}$.

Theorem 6.3 for the finite time horizon problem ($\mathrm{CP_T}$) follows from the application of Perron's method over an extended problem over $\mathbb{R}^n \times [0,T]$. For this auxiliary problem, the role of the global sub and supersolution present in Perron's method is played by functions with the form $(x,t) \mapsto C_1 t + C_2$, for suitable constants C_1, C_2 depending on the data and T. On the other hand, under the assumption (H2') these global sub and supersolution can be taken as constant functions depending on the data, but not on T, concluding the uniform boundedness. See [21], [113] for details.

Assumption (H2') also allows us to get the strong comparison principle and therefore the existence and uniqueness for the associated stationary problem.

Theorem 6.4. *Let $\bar{\varphi} \in C_b(\Omega^c)$, $\bar{H} \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ and consider*

$$\left\{ \begin{array}{rcll} -\mathcal{I}(u) + \bar{H}(x,u,Du) & = & 0 & \text{in } \Omega, \\ u & = & \bar{\varphi} & \text{in } \Omega^c. \end{array} \right. \tag{6.38}$$

Assume this problem has coercive or Bellman form in the sense of Theorem 6.3 in the time independent framework, with $\bar{H}$ satisfying (H1), (H2) and (H2'). Then, there exists a unique viscosity solution $u \in C(\bar{\Omega})$ for (6.38).

6.7.2 Large Time Behavior.

Once the existence and uniqueness for problem (CP) is obtained, it arises the natural question of the asymptotic behavior of the solution as $t \to +\infty$. For our models, the answer is contained in the following

Theorem 6.5. *Let $u_0 \in C(\bar{\Omega})$ and $\varphi \in C_b(\bar{Q}^{ext})$ satisfying (H0). Assume (CP) has coercive or Bellman form in the sense of Theorem 6.3, with H satisfying (H1),(H2) and (H2'). Assume there exist continuous functions $\bar{H} : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\bar{\varphi} : \Omega^c \to \mathbb{R}$ satisfying*

$$\begin{array}{l} H(\cdot,t,\cdot,\cdot) \to \bar{H} \quad \text{in } C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n), \\ \varphi(\cdot,t) \to \bar{\varphi} \quad \text{in } C(\Omega^c), \end{array} \tag{6.39}$$

as $t \to \infty$. Then, the unique viscosity solution u of (CP) converges uniformly in $\bar{\Omega}$ to u_∞, the unique viscosity solution of the problem (6.38).

Proof: The proof of this theorem can be framed in the general context of parabolic equations for which the limit problem satisfied the comparison principle. For each $(x,t) \in \bar{\Omega} \times [0,+\infty)$, define the functions

$$\bar{u}(x,t) = \limsup_{\epsilon \to 0, z \to x, z \in \Omega} u(z, t/\epsilon),$$

$$\underline{u}(x,t) = \liminf_{\epsilon \to 0, z \to x, z \in \Omega} u(z, t/\epsilon),$$

which are well defined by the uniform boundedness of u. The application of the half-relaxed limits method proves that for all $t > 0$, the functions $x \mapsto \bar{u}(x,t)$ and $x \mapsto \underline{u}(x,t)$ are respectively viscosity sub and supersolution for problem (6.38). Then, by comparison principle for Dirichlet problems we have $\bar{u} = \underline{u}$ in $\bar{Q}$ and consequently $\bar{u}(t,x) = \underline{u}(t,x) = u_\infty(x)$ for all $(x,t) \in \bar{Q}$ by the uniqueness of problem (6.38). This concludes the result. $\square$

We can provide a rate of convergence in the particular case that H is time independent and φ converges uniformly to $\bar{\varphi}$ as $t \to \infty$.

Proposition 6.6. *Let $u_0 \in C(\bar{\Omega})$, $\varphi \in C_b(\bar{Q}^{ext})$ satisfying (H0), and $H \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is time independent. Assume problem (CP) has coercive or Bellman form in the sense of Theorem 6.3, with H satisfying (H1), (H2) and (H2'). Assume there exists $\bar{\varphi} \in C_b(\Omega^c)$ such that*

$$\varphi(\cdot,t) \to \bar{\varphi}$$

uniformly in Ω^c as $t \to \infty$.

Let u be the unique solution to problem (CP), and u_∞ be the unique bounded viscosity solution to (6.38) asociated to $\bar{H} = H$ and $\bar{\varphi}$. Then,

$$||u(\cdot,t) - u_\infty||_{L^\infty(\bar{\Omega})} \leq e^{-\mu_0 t}\Big(||u_0 - u_\infty||_{L^\infty(\bar{\Omega})} + \mu_0 \int_{-\infty}^{t} g(s) e^{\mu_0 s} ds\Big),$$

where g is defined as

$$g(t) = \begin{cases} \sup\limits_{\tau \geq t} ||\varphi(\cdot,\tau) - \bar{\varphi}||_{L^\infty(\Omega^c)} & \text{for } t \geq 0 \\ \sup\limits_{\tau \geq 0} ||\varphi(\cdot,\tau) - \bar{\varphi}||_{L^\infty(\Omega^c)} & \text{for } t < 0. \end{cases}$$

Proof: Note that $g(t) \leq ||\varphi||_\infty + ||\bar{\varphi}||_\infty$ and then, the function

$$G(t) = \mu_0 \int_{-\infty}^{t} g(s) e^{\mu_0 s} ds$$

is well defined. Note also that g is decreasing in t and this implies that

$$e^{-\mu_0 t} G(t) \geq \mu_0 e^{-\mu_0 t} g(t) \int_{-\infty}^{t} e^{\mu_0 s} ds \geq g(t). \tag{6.40}$$

With this, consider the function

$$U(x,t) = u_\infty(x) + e^{-\mu_0 t} \tilde{G}(t).$$

where $\tilde{G}(t) = G(t) + ||u_0 - u_\infty||_{L^\infty(\bar{\Omega})}$. We claim U is a supersolution for the problem satisfied by u. In fact, for all $x \in \bar{\Omega}$ we clearly have

$$U(x,0) \geq u_\infty(x) + ||u_0 - u_\infty||_{L^\infty(\bar{\Omega})} \geq u_0(x).$$

Let $(x_0,t_0) \in Q$ and let ϕ be a smooth function such that (x_0,t_0) is a minimum point of $U_\varphi - \phi$ in $B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta)$. At one hand, from this testing we have

$$\partial_t \phi(x_0,t_0) = -\mu_0 e^{-\mu_0 t_0} \tilde{G}(t_0) + \mu_0 g(t_0). \tag{6.41}$$

On the other hand, we get that x_0 is a minimum point for the function

$$x \mapsto (u_\infty)_{\bar{\varphi}}(x) - \Big(- e^{-\mu_0 t_0}\tilde{G}(t_0) + \phi(x, t_0)\Big)$$

in $B_\delta(x_0)$. Hence, we use this as a testing for u_∞, which is a supersolution for the problem (6.38) at x_0. Using the viscosity inequality for u_∞, the definition of U, the equality (6.41) and the assumption (H1), we arrive to

$$\begin{aligned} &\partial_t \phi(x_0, t_0) - \mathcal{I}[B_\delta](\phi(\cdot, t_0), x_0) - \mathcal{I}[B_\delta^c](U^\varphi(\cdot, t_0), x_0, D\phi(x_0, t_0)) \\ \geq &- H(x_0, U(x_0, t_0), D\phi(x_0, t_0)) + A_0, \end{aligned} \tag{6.42}$$

where

$$\begin{aligned} A_0 = &- \mu_0 e^{-\mu_0 t_0}\tilde{G}(t_0) + \mu_0 g(t_0) + h(x_0) e^{-\mu_0 t_0}\tilde{G}(t_0) \\ &+ \int_{(\Omega - x_0)^c \setminus B_\delta} [e^{-\mu_0 t_0}\tilde{G}(t_0) - (\varphi(x_0 + z, t_0) - \bar{\varphi}(x_0 + z))] K^\alpha(z) dz. \end{aligned}$$

But clearly we have

$$A_0 \geq (e^{-\mu_0 t_0}\tilde{G}(t_0) - g(t_0)) \Big(\int_{(\Omega - x_0)^c \setminus B_\delta} K^\alpha(z) dz + h(x_0) - \mu_0 \Big),$$

and applying (H2') and (6.40), we obtain $A_0 \geq 0$. This concludes the claim when $(x_0, t_0) \in Q$. For $(x, t) \in \partial^l Q$ and $U(x, t) < \varphi(x, t)$, by definition we have

$$u_\infty(x) < \varphi(x, t) - e^{-\mu_0 t} G(t).$$

Using the inequality (6.40) and the definition of g, we conclude

$$u_\infty(x) < \bar{\varphi}(x),$$

concluding that in this case we can use the corresponding viscosity inequality for u_∞, concluding the claim.

In the same way a subsolution can be constructed, and the result follows by comparison principle. $\square$

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