Intercambio Restringido y Mercados Incompletos : Información y Existencia.

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Tesis para optar al grado de Doctor en Economía

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A mi familia,
a la de origen, de corazón
y a la que estamos comenzando.

C'est en forgeant qu'on devient forgeron!

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Prefacio

Este trabajo es el cierre de una seguidilla de etapas de aprendizaje que se fueron encadenando para concluir en esta tesis sobre economía matemática. Los resultados acá expuestos reflejan el trabajo, comenzado en Chile, que se vio enriquecido con mis visitas a la Universidad de Vigo y estadías en el Centro de Economía de la Sorbonne entre los años 2012 y 2014, así como también con las visitas a la Universidad de Salamanca donde logramos ir definiendo el contenido de la entrega.

Me siento orgulloso, en general, por haber tenido el privilegio de comenzar mi investigación en un área tan exigente y rica como la de equilibrio general. En particular por tres razones.

La primera, debido a la tradición de larga data comenzando con los primeros desarrollos de Walras a mediados del siglo antepasado, que sólo tuvieron respuestas al problema de existencia de equilibrio casi un siglo después. Es esta tradición que, en mi lectura, tiene por motivación más profunda la comprensión de las interacciones humanas. Esto, con el objeto de estudiar las condicionantes de una asignación de la riqueza y precios de una forma más eficiente, y por consecuencia, entender el desarrollo económico.

Es decir, como segunda y central razón : me siento orgulloso de haber podido dedicar tiempo y esfuerzo a entender un poco más el comportamiento del hombre. Ese curioso animal en sociedad que somos, que nace en la familia y que en sociedades desarrolladas—en libertad—se inserta y habita en el mundo.

Como tercer motivo, por el desafío intelectual de aprender, de resolver problemas—que aunque lejos—podrán ser algún día argumento para entender más completamente la economía en su conjunto. De esta forma, creo que se puede construir un mundo a medida de una libertad más verdadera, esa libertad que también creo está acotada por la del prójimo.

Sin duda que este desafío intelectual, me ha llevado a las fronteras de mi humanidad, de mis capacidades, y que creo con cada esfuerzo haber podido ir expandiendo en ciertas direcciones. Ese esfuerzo, nos ha permitido llegar al día de detener el trabajo sobre los manuscritos y encuadrarlos en un solo documento. Documento que, como trabajo, se debe con suerte en una muy pequeña parte a mi. En efecto, pienso que se debe inicialmente a mi familia. Son ellos quienes permitieron mi desarrollo sin bemoles a costas de los esfuerzos de todos : mis padres, mis hermanas y mi hermano. Aunque sin lugar a dudas, beneficiándonos de la clasista estructura social de nuestro país, donde nos ubicamos en el lugar de los privilegiados.

Es esta injusticia que me lleva a pensar que, en otra estructura social, no debiese haber sido yo quien dedicara su trabajo a entender esta materia sino alguien más capaz. Por esto es que tengo el deber moral de trabajar por la equidad. Es decir, trabajar por la igualdad de

oportunidades, lo que creo es el fin de un verdadero desarrollo económico. Algo, por lo que un economista debería ofrecer, más radicalmente, su vida profesional. A esta visión idealista agradezco la energía que ha empapado en mi, a través de tantas personas que el papel se haría escaso, el carisma de Chiara Lubich.

Si para la elaboración de este trabajo, mi familia ha puesto la materia prima en sus mejores condiciones, la Universidad ha sido el lugar de encuentro. En parte el lugar de formación sí, pero mayormente para mi, el lugar de encuentro entre el estudiante y quien le enseña. Agradezco a ella y a todos quienes fueron gestores de que mi supervisor Profesor Juan Pablo Torres-Martínez llegara al Departamento de Economía cuando yo iniciaba mis primeros cursos avanzados de micro. El atractivo por el bien pensar que generaron en mi sus clases y su trato fueron "EL" descubrimiento de una técnica a adquirir para formular seriamente los problemas que rondaban mi cabeza. Una técnica que, en tanto racionalista, cambió brutalmente mi forma de pensar y mi doctrina de aprendizaje. Espero algún día llegar a ver frutos de haber tratado de ejercitarla como se debe.

Agradezco especialmente también a quienes aceptaron integrar la banca de mi tesis : Pedro Jara, Alejandro Jofré y Jorge Rivera. Aún más especialmente a Emma Moreno y Carlos Hervés que me recibieron, con la hospitalidad que sólo ellos saben donar, en mis estadías estos años en Salamanca y Vigo respectivamente.

Gracias, entonces, a mi familia, mis amigos, a Juan Pablo, a todo el personal de la Facultad de Economía y Negocios así como del Departamento de Economía, a Gema y Pamela por el apoyo logístico, a CONICYT por el financiamiento de este proyecto, a quien me acompañará sin cavilaciones por estos caminos que derivan del presente : Anto, y a esa razón que siempre ha estado.

Resumen

Abordamos el problema de la existencia de equilibrio cuando la participación financiera está restringida. En una primera aproximación, estudiamos la existencia de equilibrio cuando restricciones exógenas vienen dadas por las dotaciones de información de los individuos. Todo esto, en un entorno donde los precios revelan información induciendo una actualización endógena de la información individual. Luego, permitiendo que las restricciones en la participación dependan de variables endógenas de la economía, analizamos dos métodos para la prueba de existencia : mediante una propiedad de super-replicación de la estructura financiera y a través de un tipo de impaciencia en las preferencias de los agentes económicos.

Palabras clave

Mercados incompletos, Participación restringida, Información diferenciada

Trading Constraints Incomplete markets: Information and Existence.

Abstract

We study the equilibrium existence problem when there are restriction in the financial participation. In a first approach, we analyze equilibrium existence when exogenous restriction are given by information endowments. This is done, in an environment where prices reveal information inducing an endogenous update of private information. Then, including restrictions that depend on endogenous variables of the economy, we analyze two methods for the proof of equilibrium existence: invoking a super replication property on the financial structure and through a kind of impatience in preferences of economic agents.

Keywords

Incomplete markets, Restricted participation, Differential information

Contents

Introduction

In modern economics, economic agents are able to transfer wealth across time using financial markets in order to accommodate inter temporal consumption according preferences. Theoretically, the inclusion of financial structures in economies induces optimal resources allocation and risk-bearing, Arrow [1964]. In complete markets, where the dimension of uncertainty is equal to the one of the space of transferences generated by the financial structure, equilibria are Pareto-optimal. Nevertheless, when incomplete markets structures are considered this is not the case. In what follows we will define asset structures and their aggregation to financial structures that will be considered in the economic configurations that we analyze in the following chapters. We will pay special attention to a source of market incompleteness, the one given by the restricted participation that agents may have in financial markets.

0.1 General Financial Structures

The basic configuration of this chapter is a two period economy, with uncertainty over the realization of the states of nature over a finite set of states denoted $\mathcal{S} = \{0\} \cup S$, where zero represent the unique state in the first period and S is a finite set of states of nature that can be attained at the second period. In addition there are a finite set of durable commodities denoted by $\mathcal{L} = \{1, ..., L\}$ jointly with prices of them in each period and state of nature. Commodity prices are denoted by the vector $p = (p_0, (p_s)_{s\in\mathcal{S}}) \in \mathbb{R}^{\mathcal{L}\times\mathcal{S}} := \mathcal{P}$. Agents are represented in the set $\mathcal{I} = \{1, ..., I\}$. They are endowed with consumption sets $\mathbb{E}^i_c \subset \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+$ for each $i \in \mathcal{I}$.

0.1.1 Exogenous case

Definition 0.1. An exogenous asset structure, say asset j, is composed by returns and personalized portfolio sets for agents. Precisely, the pair (R_j, Z_j) denotes the mapping of returns $R_j : \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+ \to \mathbb{R}^S_+$ and the portfolio sets $Z_j \subset \mathbb{R}^{\mathcal{I}}$.

That is, for given commodity prices $p \in \mathcal{P}$ and consumption bundles $x := (x^1, ..., x^I) \in \prod_{i\in\mathcal{I}} \mathbb{E}^i_c$, a vector $z \in Z_j$ contains portfolios for each agent $i \in \mathcal{I}$ denoted $z^i \in Z^i_j \subseteq \mathbb{R}$. Thus, at each state of the nature in the second period the agent i will receive or pay a quantity $R_{s,j}(p) \in \mathbb{R}_+$ depending on the sign of z^i.

Furthermore, since the setting defines a general asset structure: real, numeraire and nominal securitites are also encompassed.

DEFINITION 0.2. An asset structure (R_j, Z_j) is numeraire if, for a given bundle $\zeta \in \mathbb{R}^{\mathcal{L}}$, prices $p_s \in \mathbb{R}^{\mathcal{L}}_+$ and a vector $N := (N_s)_{s \in S} \in \mathbb{R}^S_+$, the returns are specified by $R_{s,j}(p_s) = (p_s \cdot \zeta)N_s$ for each $s \in S$.

DEFINITION 0.3. An asset structure (R_j, Z_j) is nominal if, for a given vector $N := (N_s)_{s \in \mathcal{S}} \in \mathbb{R}^S_+$, the returns for each $s \in S$ are specified by $R_{s,j}(p_s) = N_s$ for all $p_s \in \mathbb{R}^{\mathcal{L}}_+$.

This configuration allow us to define particular cases as bid/ask structures, buying floors, short selling, and borrowing constraint as well as no participation at all. For instance, note that if for agent $i \in \mathcal{I}$ we have that $Z^i_j = \mathbb{R}$, then agent i face no restrictions to invest or borrow units of the asset j. Otherwise, as $Z^i_j \subsetneq \mathbb{R}$, the agent will be somehow financially restricted.

If we make the addition of countable many asset structures we will create an asset market. Let us say that there are J different asset structures that are indexed in the finite set $\mathcal{J} = \{1, ..., J\}$. Note that among those assets structures we have already fixed the same quantity of states of nature for the second period.

DEFINITION 0.4. An exogenous asset market (R, Z) is the collection of asset structures indexed by the finite set $\mathcal{J}$. Where the asset returns are given by $R : \mathbb{R}^{\mathcal{L} \times \mathcal{S}}_+ \to \mathbb{R}^{SJ}$. The feasible portfolios are denoted by $Z \subset \mathbb{R}^{JI}$. Where for given $p \in \mathcal{P}$, we have that $R(p) = [R_1(p), ..., R_J(p)]$ and $Z = [Z_1, ..., Z_J]$.

In this framework, the market incompleteness can exist because of (i) scarcity of contracts $(J < S)$ or (ii) Incomplete Financial Participation. For instance, given a complete asset structure $J = S$, if every agent is totally restricted to participate in a particular asset, then we have market incompleteness (see Polemarchakis and Siconolfi [1997]).

We say that the Asset Market is Non-Redundant if the asset returns for given prices $p \in \mathcal{P}$, $R(p)$ seen as a matrix, has full column rank, i.e., $rank\, R(p) = J$.

0.1.2 Endogenous case

When we are interested in the case where financial participation depends on endogenous variables of the economy as prices, we are not able to define asset structures independently of the whole asset market. For instance, a particular asset structure will depend on prices of commodity prices as well as prices of another asset structures. Therefore, portfolio sets will depend on the space of prices of consumption and asset prices, say $\mathbb{P} := \mathbb{R}^{\mathcal{L} \times \mathcal{S}} \times \mathbb{R}^{\mathcal{J}}$.

DEFINITION 0.5. An asset structure, say asset j, is composed by returns and personalized portfolio sets for agents. Precisely, given prices $(p, q) \in \mathbb{P}$, the pair (R_j, Z_j) denote the mapping of returns $R_j : \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \to \mathbb{R}_+^S$ and a correspondence $Z_j : \mathbb{P} \twoheadrightarrow \mathbb{R}^{\mathcal{I}}$ of personalized portfolio sets depending on commodity and the asset prices.

That is, for given prices $(p,q) \in \mathbb{P}$ and consumption bundles $x := (x^1, ..., x^I) \in \prod_{i\in\mathcal{I}} \mathbb{E}_c^i$ a vector $z \in Z_j(p,q)$ contains portfolios for each agent $i \in \mathcal{I}$ denoted $z^i \in Z_j^i(p,q) \subseteq \mathbb{R}$. Thus, at each state of the nature in the second period the agent i will receive or pay a quantity $R_{s,j}(p) \in \mathbb{R}_+$ depending on the sign of z^i.

This configuration allows us to include additional cases as price-dependent borrowing constraints, income-based financial access, exclusive credit lines and security-exchanges. An endogenous asset market can be defined according to Definition 0.4.

0.2 Restriction Classes

In what follows we will analyze the classes of restrictions that may arise at individual level, so let us fix an individual $i \in \mathcal{I}$.

0.2.1 Functional restrictions

We may define the portfolio set of i by means of functions $g^i(p,q) : \mathbb{P} \to \mathbb{R}^{\mathcal{J}}$ such that for all $(p,q) \in \mathbb{P}$:

$$Z_j^i(p,q) = \{z_j^i \in \mathbb{R} : z^i + g_j^i(p,q) \geq 0\}.$$

In other words, agent i can take $g_j^i(p,q)$ as her smallest position in asset j (analogously it is possible to define upper bounds).

0.2.2 Polyhedral

Polyhedral restrictions appear if Z^i defines a polyhedral convex subset of $\mathbb{R}^J$, i.e., subsets defined by finitely many inequalities.

EXAMPLE 0.6 (Polyhedral). If Z^i is equal to: $\{0\}$ we have no participation in financial markets, $\mathbb{R}^J$ no restrictions.

Polyhedral constraints are compatible with settings including buying floors, and short selling or linear constraints. Furthermore, for $Z_j^i = \mathbb{R}_+$ (resp. $Z_j^i = \mathbb{R}_-$) we will say that asset j is an *ask* security (resp. *bid* security).

EXAMPLE 0.7 (Non Polyhedral convex). Non-linear constraints or Borrowing constraints

0.2.3 Convex sets

The most general kind of restrictions are those given by portfolio sets Z^i is a closed convex set containing zero.

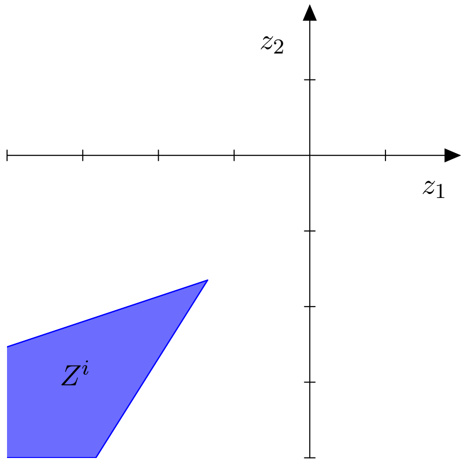

Figure 1: Example of Polyhedral restrictions where $Z^i \subset \mathbb{R}^2$

0.3 Literature

Restricted Participation was studied first in papers by Radner [1972], Siconolfi [1989] and Cass [2006]. The latter as a journal-published version of the 1984' working paper. More recent interest is developed in Balasko, Cass, and Siconolfi [1990] for the nominal assets case and Polemarchakis and Siconolfi [1997] for real assets.

The papers by Cass [2006] and Siconolfi [1989] explore the most general framework when there is a nominal asset market, i.e., restrictions given by closed and convex sets containing the zero vector. Further developments in nominal asset as Balasko, Cass, and Siconolfi [1990] explore real indeterminacy and arguments to avoid intermediation when restrictions are real homogeneous equality constraints. The same sort of constraints but in real asset markets are studied by Polemarchakis and Siconolfi [1997]. Meanwhile, a general form of constraints as the ones given by quasi concave inequalities in nominal asset market is the subject of Cass, Siconolfi, and Villanacci [2001] taking a differentiable approach requiring, obviously, differentiability condition on the functional restrictions.

Even if in Cass [2006] the kind of portfolio constraint is general enough, it is required that one agent with unconstrained portfolio set behaves at equilibrium as if she were in complete markets. What will be called the *Cass trick*. Martins-da Rocha and Triki [2005] try to disentangle this approach requiring portfolio set be subsets of a finite dimensional vector space in a multi-period configuration.

0.4 Requirements on restrictions

Given that prices are chosen in a bounded set, a key step in the proof of existence derives from showing that portfolios belong to a bounded set also. Indeed, the fact that portfolios belong to a bounded set is a standard argument in the absence of redundant assets. But with

Table 1: Restrictions References

Exogenous Restrictions			
Reference	Assets	Restricted Participation	Financial Survival
Siconolfi [1989]		Convex constraints	
Balasko, Cass, and Siconolfi [1990]	Nominal		
Polemarchakis and Siconolfi [1997]	Real		
Angeloni and Cornet [2006]	Real	Convex, Compact	Zero-neighborhood at least one agent
Aouani and Cornet [2009]	Numeraire	Convex polyhedral FN1	Every agent FN2
	Nominal		
	Neither Nominal nor Numeraire	Nonredundancy F3	
Cornet and Gopalan [2010]	Nominal	Closed, Convex	
Endogenous Restrictions			
Reference	Assets	Dependences of Restricted Participation	
Cass, Siconolfi, and Villanacci [2001]	Nominal	Prices	
		Differentiability and Regularity	
Carosi et al. [2009]	Numeraire	Commodity and asset prices	
		Homogeneity, Differentiability and Regularity	
Seghir and Torres-Martínez [2011]		Commodity purchases	

Aouani and Cornet [2009] normalize commodity and asset prices in first period to a simplex, as well as commodity prices in the second period. Cornet and Gopalan [2010] requires spanning condition (closed convex cone generated by union of portfolio be a linear space). Seghir and Torres-Martínez [2011] assume an impatience condition (A2).

portfolio constraints, for instance when the source of redundancy is financial intermediation, there is no reason to assume the absence of redundant securities, see Balasko, Cass, and Siconolfi [1990]. In conclusion, if there is financial intermediation, it is not possible to simply assume that the asset market is non-redundant and the literature has accommodated non-redundancy like assumptions.

0.4.1 Non-redundancy

The existence of bounds on attainable allocations is directly related with the non- redundancy of the financial structure. In fact, non-redundancy implies the non-existence of unbounded sequences of portfolios that do not generate commitments. This is exactly what Siconolfi [1989] requires for nominal asset and exogenous constraints that are captured by convex sets:

[Siconolfi, 1989, (A5)]: For every agent, $\{z \in Z^i : z \neq 0, Rz = 0, \lambda z \in Z^i \text{ for } \lambda > 0\} \neq \emptyset$.

Note that non-redundancy is not innocuous since a linearly dependent asset can give wealth transfers (opportunities) to restricted agents [Cass, Siconolfi, and Villanacci, 2001, Remark 1]. In the same spirit, another assumption that implies non-redundancy is:

[Aouani and Cornet, 2009, (F3)]:

$$\sum_{i\in\mathcal{I}}(AZ^i \cap \{R(p) \geq 0\}) \quad \bigcap \quad -\sum_{i\in\mathcal{I}}(AZ^i \cap \{R(p) \geq 0\}) = \{0\},$$

where $AZ^i := \{\lim_n \lambda_n z_n : (\lambda_n)_n \downarrow 0 \,\text{and}\, z_n \in Z^i \,, \forall\, n\}$.

In addition to the problem of choosing allocations in bounded spaces, it is necessary to find bounds on asset prices. If the price space—including commodity and asset prices—is defined by a simplex, then the solution to search bounds on prices induces another problem. This is because as prices concentrate on particular assets that are restricted, the price normalization may produce empty interior of the budget sets. A solution to the problem is to assume financial survival conditions.

0.4.2 Financial Survival

When consumption sets $\mathbb{E}_c^i$ are different from $\mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$, equilibrium existence requires what is known as a survival assumption in consumption. More precisely, if an agent $i \in \mathcal{I}$ has endowments denoted by $w^i := (w_s^i)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$, equilibrium existence will require that w^i belong to the interior of $\mathbb{E}_c^i$ relative to $\mathbb{R}^{\mathcal{L}\times\mathcal{S}}$. This implies that consumer i should have a positive quantity of each commodity as endowments in order to ensure the non-emptiness of her budget set. As a parallel way, when there are restrictions in the financial participation of agent i, i.e., $Z^i \neq \mathbb{R}^{\mathcal{J}}$, we need to ensure the non-emptiness of the budget set in some way. A first approach to the problem will require that a neighborhood of the zero portfolio have to also belong to the portfolio set, i.e. zero vector belongs to the interior of Z^i. This is what the literature calls Financial Survival. This kind of hypothesis used in the literature are given by:

[Angeloni and Cornet, 2006, (F)(iii)]: $\exists i_0 \in \mathcal{I}$ such that $0 \in int\, Z^{i_0}$.

[Aouani and Cornet, 2009, (FN2)]: $\forall i \in \mathcal{I}, \forall q \in cl\, \mathcal{Q} \cap \langle \bigcup_{i\in\mathcal{I}} Z^i \rangle, q \neq 0, \exists z^i \in Z^i, q \cdot z^i < 0$, where $\mathcal{Q}$ is the set of arbitrage-free prices [1] of assets and $\langle X \rangle$ is the span of X.

[Aouani and Cornet, 2009, 2011, (F2)]: $\forall i \in \mathcal{I}, \forall p \in \mathcal{P}, p_0 = 0, \forall q \in cl\, \mathcal{Q} \cap \langle \bigcup_{i\in\mathcal{I}} Z^i \rangle, q \neq 0, \exists z^i \in Z^i, q \cdot z^i < 0$

Nevertheless, efforts to include dependence of portfolio sets on endogenous variables requires the study of how credit opportunities evolves at odds of changes in those variables. For instance, for exogenous restrictions, there are results that do not require financial survival but impose restrictions on the existence of agents that have additional access to financial markets.

[Cass, Siconolfi, and Villanacci, 2001, (A4)]: For every $j \in \mathcal{J}$, there is some $i \in \mathcal{I}$ such that:

$$z^i_j + g^i_j(p,q) = 0 \quad \text{implies} \quad z^i_j + g^i_j(p,q) + (0, ..., \Delta z_j, ..., 0) = 0,$$

for $\Delta z_j \in \mathbb{R}$. That is, for each asset $j \in \mathcal{J}$, there is an agent $i \in \mathcal{I}$ that is unrestricted in the access to that asset [Cass, Siconolfi, and Villanacci, 2001, Remark 3].

In the following figures it is possible to see graphic examples of exogenous portfolio sets not satisfying financial survival.

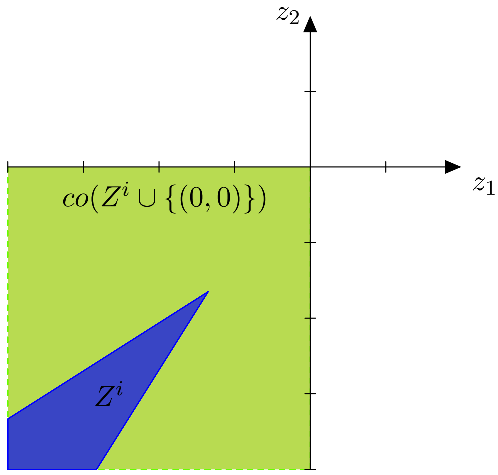

Figure 2: Exogenous Portfolio set Z^i not satisfying financial survival

Indeed, a portfolio set being convex valued and containing the zero vector may implicitly induce financial survival. Indeed, Figure 2 illustrates an extreme example where those

1. We say that the prices $q \in \mathbb{R}^{\mathcal{J}}$ jointly with the asset market (R, Z) is free of arbitrage if there is no $z \in Z$ such that $(-q, R)z^i > 0$ for some $i \in \mathcal{I}$.

conditions induce no credit constraints.

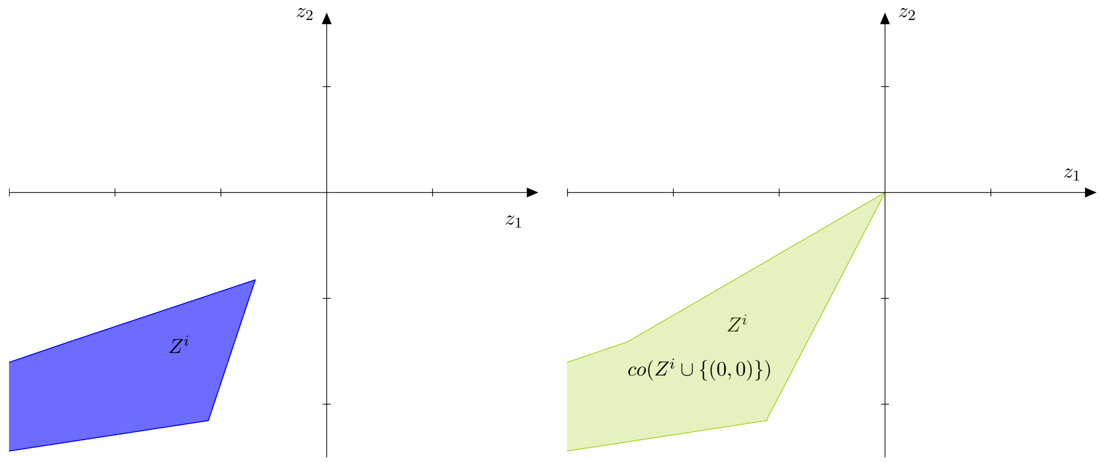

Figure 3: Exogenous Portfolio set

Nevertheless, it is possible to construct examples of convex polyhedral portfolio sets (as the one given in the Figure 3) where the conditions of convexity and zero vector do not imply no credit restrictions, neither financial survival (Figure 4). Note that in this last example, there is exclusion of credit markets (See a general concept given in Definition 2.3.1).

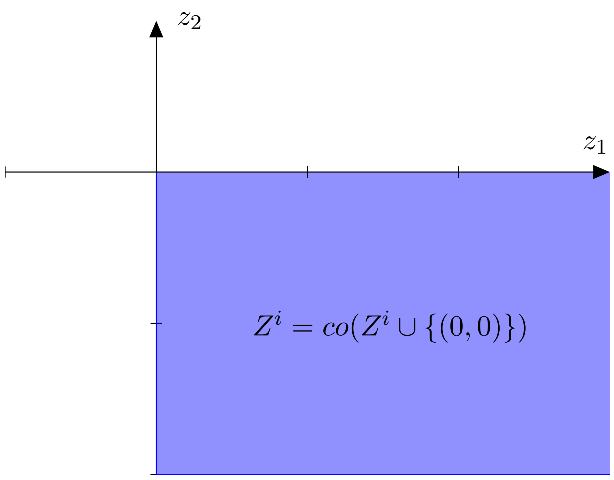

Figure 4: Portfolio set not satisfying financial survival and excluded of asset's 1 credit possibilities.

0.5 Scope and organization of the work

It is not possible to escape from assuming non-redundancy conditions in order to find bounds for allocations. Therefore, leaving asset prices out of a price normalization, it is possible to find upper bounds on asset prices without requiring financial survival. Something that is already present in the literature that takes this approach is through a kind of impatience condition in preferences introduced by Seghir and Torres-Martínez [2011]. It is precisely the argument we give in the Chapter 1 for exogenous portfolio constraints given from asymmetric information.

Extending the scope of the portfolio restrictions we introduce general trading constraints that allow us to encompass purely financial restrictions and consumption constraints (Chapter 2). This configuration, without financial survival, permits the occurrence of market segmentation and, even, exclusion in the participation of financial markets. Something that is specifically precluded by financial survival assumptions. This is the subject of study in the first part of our second chapter (Section 2.3.3).

We introduce Chapter 2 with results that particularize our general framework of trading constraints, i.e. credit constraints (Section 2.2). Then, we present a more general model including investment constraints in Section 2.3. Precisely, in our general model of trading constraints, we provide two different approaches to equilibrium existence. First, introducing a super-replication argument. Second, with a generalization of the impatience hypothesis required in Seghir and Torres-Martínez [2011].

The inclusion of general trading constraints, as they are given in Chapter 2, requires to elaborate on the kind of hypothesis given by [Cass, Siconolfi, and Villanacci, 2001, Assumption (A4)]. First, including trading dependence on endogenous variables and second, because trading constraints we analyze are much more broader than the actual literature encompasses. We also include a result that adds investment restrictions to the credit-restricted model of Seghir and Torres-Martínez [2011] in Section 2.3.4.

Chapter 1

Endogenous Differential Information

We include endogenous differential information in a model with sequential trade and incomplete financial participation. Agents update information through market signals given by commodity prices and asset deliveries. Information acts over admissible strategies and consumption tastes, allowing discontinuities in preferences and choice sets. Therefore, equilibrium may cease to exist. However, internalizing the compatibility between information and consumption through preferences, and without requiring either financial survival assumptions nor fully revealing prices, equilibrium existence can be ensured. [1]

1.1 Introduction

There exists a large literature on competitive equilibrium with asymmetric information, a framework introduced by Radner [1968] who extended the Arrow and Debreu [1954] [2] model by assuming that agents have incomplete and asymmetric information on the future states of nature. In Radner [1968]'s model, agents are not able to improve their initial private information and the model requires information to be consistent with agents' allocations. This is a strong restriction since, it is natural to assume that market signals, summarized by future commodity prices, allow agents to update their initial information. Alternatives to this model were studied in further developments, as for instance Radner [1979], who considered beliefs on the future states of nature and showed the existence of rational expectation equilibrium when future, full-informative prices update agent's initial information (generic equilibrium existence).

Meanwhile, financial imperfections that emerge from regulatory considerations, lack of information or credit risk, induce financial participation constraints. With the aim of studying these situations, a general equilibrium model of incomplete financial markets [3] was extended

1. Cea-Echenique, Hervés-Beloso, and Torres-Martínez [2014] is based on this chapter.
2. Arrow and Debreu [1954] model covers the case of complete information.
3. The incomplete financial markets theory starts with Radner [1972], Dreze [1974] and Hart [1975], which extended Arrow and Debreu [1954] to allow for an incomplete set of financial promises (see for instance, Geanakoplos [1990] and Magill and Quinzii [2008] for surveys of major results in this literature).

to scenarios where agents have personalized access to financial opportunities (see, for instance, the pioneering works of Siconolfi [1989] and Cass [2006]).[4] In this context, Seghir and Torres-Martínez [2011] proposed a model with credit participation constraints, where financial survival conditions are not required and equilibrium existence is proved even when agents may not have access to all credit contracts.

Our aim is to elaborate in this framework, which departs from the model of incomplete markets with *numéraire* assets and differential information[5], in two ways: by considering investment constraints, in contrast to the usual credit restrictions, and allowing an endogenous update of agent's private information. On one hand, when there is incomplete participation in financial markets, for instance due to the lack of information,[6] there is no reason to block the access to the missing information that may be available. Consequently, we will take into account the endogenous information given by market's signals. On the other hand, the fact of incorporating the information given by asset returns avoids inconsistencies between the pattern of financial promises and agents' actions that may arise due to asymmetries on agents' initial information (see Example 4).

As in Dubey, Geanakoplos, and Shubik [1987], we assume that economic activity takes place sequentially in a frame of time which, for simplicity sake, is summarized in two periods. Initially, agents act according to their private information. This produces economic outcomes that could benefit agents with finer information. In this initial period, and due to the lack of information, some group of agents create assets that could avoid some of the restrictions that appear in some models like Radner [1968].[7] Indeed, we maintain the scenario of incomplete financial participation. Thus, the asset deliveries update the involved agents's initial information. Additionally, in the first period, individuals are restricted to trade in a subset of the financial market which is compatible with their own— already updated— information and some agents could benefit again from their finer information. Once the assets are present in the market, agents forecast future commodity prices which, eventually reveal additional information on the future states of nature. Thus, each agent's final information is the result of the update of her initial private information firstly, by the signals given by the initial economic activity and asset returns, and secondly, by market signals given by future spot prices.

Therefore, as in the classical model of rational expectations, Radner [1979], in our model agents' private information can be endogenously improved via market's signals. This frame-

4. Equilibrium models with incomplete financial participation were also studied by Balasko, Cass, and Siconolfi [1990], Polemarchakis and Siconolfi [1997], Angeloni and Cornet [2006], Aouani and Cornet [2009], and Cornet and Ranjan [2011], requiring financial survival assumptions to equilibrium existence. Aouani and Cornet [2011] and Cornet and Gopalan [2010] impose spanning conditions over portfolio sets. Price dependent constraints were addressed by Cass, Siconolfi, and Villanacci [2001] and Carosi, Gori, and Villanacci [2009].

5. Results in the literature of incomplete markets and differential information, as Faias and Moreno-García [2010] analyze the properties of a non informative equilibrium price in the context of real assets.

6. Physical and financial markets providing new information to incomplete informed traders have been previously studied in other contexts, by Polemarchakis and Siconolfi [1993], Rahi [1995], Citanna and Villanacci [2000] and Cornet and De Boisdeffre [2002].

7. For instance, the example of no-trade caused by the lack of information in Correia-da Silva and Hervés-Beloso [2009], does not apply if the obvious assets were to be considered.

work reflects how agents demand consumption plans compatible with their final information. [8]

However, we argue that the primitives of a real economy could be far from being random. Indeed, due to homogeneity of agent's endowments or tastes across states of nature, randomness can be seriously reduced as a consequence of the lack of agents' initial information and the restricted financial participation. This means that, in contrast to Radner [1979], the set of non-fully revealing equilibrium prices could be non-negligible.

As our model explicitly gives room to non-fully revealing prices, we could have lack of continuity of the choice set correspondence due to the fact that agents eventually learn additional information from partially revealing equilibrium prices (see Remark 1 in Section 1.3). This informational discontinuity was already observed by several authors (see Radner [1967] for instance). In order to overcome this lack of continuity, some additional assumptions are needed. For it, we contemplate the situation in which individual's state dependent preferences may depend on the information that spot commodity prices reveal. Although preferences are exogenously given, contingent consumption can depend on the final information available, which may depend on spot prices. Price dependent preferences are studied in the work by Pollak [1977] [9] and in Correia-da Silva and Hervés-Beloso [2008], for a model of preferences for lists of bundles. Indeed, even in the classical expected utility case, agents' objective functions depend on their information and consequently are, in general, implicitly price dependent (we stress this point in Section 1.5).

In this scenario, we define an equilibrium with endogenous differential information and restricted financial participation and show equilibrium existence. The rational expectation equilibrium (REE) is a particular case of our model. When we apply our existence result to the of rational expectations scenario, in order to fulfill the assumptions of our model, we will assume that the agent's state dependent preferences are similar in states that her private information does not distinguish. More precisely, we will assume that the additional information revealed by small differences in prices does not make dramatic changes in the agent's behavior. We may strengthen our assumption requiring that only significative differences in prices produce changes on agent's tastes. This argument is consistent with preferences that take tendencies into account. Therefore, as commodity prices are channels to communicate information, differences in prices are signals of how strong a tendency is realized, i.e. the consumption of a commodity is more exclusive in a state where the price is significantly higher.

The remaining part of the paper is organized as follows: in Section 1.2 we introduce the model and in Section 1.3 we discuss the possibility of making information endogenously compatible. Our main result is stated in Section 1.4. In Section 1.5 we focus on the rational expectations equilibrium model and we also present a non existence example induced by the

8. Correia-da Silva and Hervés-Beloso [2009] showed that compatibility between consumption and information can be endogenized also in another framework, allowing uncertain delivery of commodities and provided that individuals have prudent expectations on market deliveries.

9. Even in economies with perfect and complete information, preferences may be affected by relative prices as a signal of quality, social status, or externalities. In this scenario, equilibrium existence and properties of competitive equilibria were studied by Shafer and Sonnenschein [1975], Greenberg, Shitovitz, and Wieczorek [1979], Balder [2003], Balasko [2003], Cornet and Topuzu [2005] and Noguchi [2009], among others.

lack of continuity in the expected utility. We discuss some examples in Section 1.6, and finally we conclude with some remarks. Proofs are contained in an Appendix, Section 1.7.

1.2 A Financial Model with Endogenous Differential Information

Consider a two period economy without uncertainty in the first period, $t = 0$, and where one state of nature of a finite set S is realized in the second period, $t = 1$. Thus, let $\mathcal{S} = \{0\} \cup S$ be the set of states of nature in the economy, identifying $s = 0$ as the only state of nature in the first period. There is a finite set $\mathcal{L}$ of perfect divisible and non-storable commodities that may be traded at each period in spot markets. We implicitly assume that there are many more states of nature than commodities. [10] Let $p_s = (p_{s,l}; l \in \mathcal{L})$ be the vector of commodity prices at state of nature $s \in \mathcal{S}$ and $p = (p_s; s \in \mathcal{S})$ the commodity prices in the economy. Hereinafter, we fix a bundle $\zeta \in \mathbb{R}^{\mathcal{L}}_{++}$ and normalize unitary prices in such form that $p_s \cdot \zeta = 1, \forall s \in \mathcal{S}$. Thus, the set of commodity prices will be $\mathcal{P} := \{(p_s; s \in \mathcal{S}) \in \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_{+} : p_s \cdot \zeta = 1, \forall s \in \mathcal{S}\}$.

There is a finite set $\mathcal{J}$ of *numéraire* assets indexed to the bundle ζ. [11] Each asset $j \in \mathcal{J}$ is issued at the first period, has a unitary price q_j, and makes promises contingent to the states of nature, $(R_{s,j}\,\zeta; s \in S) \in \mathbb{R}^{\mathcal{L}\times S}_{+} \setminus \{0\}$. Let $q := (q_j; j \in \mathcal{J}) \in \mathbb{R}^{\mathcal{J}}_{+}$. We assume that there are no redundant assets. That is, that the family of vectors $\{(R_{s,j})_{s\in S} : j \in \mathcal{J}\}$ is linearly independent.

There is a finite set of agents, denoted by $\mathcal{I}$. Each individual $i \in \mathcal{I}$ may have incomplete information about the realization of the uncertainty, as she only distinguishes states of nature that are in different elements of a partition $\mathbb{F}^i$ of S, which constitutes her initial private information. Endowments of agent i are given by a $\mathbb{F}^i$-measurable [12] bundle $w^i = (w^i_s; s \in \mathcal{S}) \in \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_{++}$. The only channel that individuals have to improve their private information is through market signals, that are commodity prices and asset deliveries. We also assume that the information required to recognize the realization of the uncertainty can be obtained by the pooling of individuals information, i.e., $\bigvee_{i\in\mathcal{I}} \mathbb{F}^i = \{\{s\}; s \in S\}$. [13]

Exogenous to the model, participation in investment clubs or personalized loan scenarios are captured by the fact that individuals may have limited access to financial contracts. Thus, agent i can trade assets in a subset $\mathcal{J}^i \subseteq \mathcal{J}$. We assume that asset returns are signals that may be used to update information before the trade takes place. Thus, for each $i \in \mathcal{I}$, the

10. Recently, Correia-da Silva and Hervés-Beloso [2014] have shown that when there are not more states of nature than goods, then equilibria of asymmetric information economies are equilibria of associated full information economies generically.

11. Note that, we require the bundle ζ to have strictly positive coordinates whereas the general case of numéraire assets allows for non-negatives coordinates. For instance, in the case of just one commodity in each state, this normalization is standard.

12. Given a partition $\mathbb{F}$ of S, a vector $(v_s; s \in S)$ is $\mathbb{F}$-measurable if $v_s = v_{s'}$ for any pair of states of nature s and s' which belongs to the same element of the partition $\mathbb{F}$.

13. Since preferences will endogenize the information compatibility requirement (see Assumption (A1) below), we do not need to assume that for any $s \in S$ there is $i \in \mathcal{I}$ that distinguishes it, i.e., $\{s\} \in \mathbb{F}^i$. This is a traditional assumption on static general equilibrium models with differential information, and it is used to ensure that (under monotonicity of preferences) the equilibrium price of any contingent commodity contract is strictly positive.

partition $\mathbb{F}^i$ incorporates all the information generated by the return of assets in $\mathcal{J}^i$. That is, for any $j \in \mathcal{J}^i$, the vector $(R_{s,j}; s \in S)$ is $\mathbb{F}^i$-measurable. In addition, although the financial participation can be incomplete, any contract $j \in \mathcal{J}$ can be traded for at least one agent. That is, $\mathcal{J} = \bigcup_{i\in\mathcal{I}} \mathcal{J}^i$.[14]

We assume that the new information available could affect individuals' beliefs about the occurrence of states of nature and her tastes about contingent consumption. The process of adapting agent's private information happens at the same time that each agent adapts her individual preference. Given prices $p \in \mathcal{P}$, let $\tau(p)$ be the partition of S generated by commodity prices. Then, the final private information of agent i is given by the partition $\mathbb{F}^i \vee \tau(p)$. Thus, individual preferences may depend endogenously on the information transmitted by commodity prices. Therefore, each agent $i \in \mathcal{I}$ has a price dependent utility function $V^i : \mathcal{P} \times \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \to \mathbb{R}$

Each agent $i \in \mathcal{I}$ selects her consumption by choosing an informational and budgetary compatible vector $(x_s^i; s \in \mathcal{S}) \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$, implemented through a financial position $z^i = (z_j^i; j \in \mathcal{J}^i) \in \mathbb{R}^{\mathcal{J}^i}$. More precisely, given prices $(p,q) \in \mathbb{P} := \mathcal{P} \times \mathbb{R}_+^{\mathcal{J}}$, the objective of an agent $i \in \mathcal{I}$ is to maximize her utility function $V^i(p,\cdot)$ by choosing a vector in her *choice set*, defined as the collection of vectors $(x^i, z^i) \in \mathbb{E}^i := \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}^{\mathcal{J}^i}$ such that $(x_s^i; s \in S)$ is $\mathbb{F}^i \vee \tau(p)$ -measurable, and

$$p_0 x_0^i + \sum_{j\in\mathcal{J}^i} q_j z_j^i \le p_0 w_0^i, \qquad p_s x_s^i \le p_s w_s^i + \textstyle\sum_{j\in\mathcal{J}^i} R_{s,j} z_j^i,\ \forall s \in S.$$

The collection of vectors $(x^i, z^i) \in \mathbb{E}^i$ that satisfy the budget constraints above is denoted by $B^i(p,q)$, while the collection of vectors $(x^i, z^i) \in \mathbb{E}^i$ for which $(x_s^i; s \in S)$ is $\mathbb{F}^i \vee \tau(p)$-measurable is denoted by $\mathcal{F}^i(p)$. Therefore, given $(p,q) \in \mathbb{P}$, the choice set of agent $i \in \mathcal{I}$ is $B^i(p,q) \cap \mathcal{F}^i(p)$.

Definition. *An equilibrium with endogenous differential information and restricted financial participation is given by prices* $(\overline{p}, \overline{q}) \in \mathbb{P}$ *and allocations* $\left((\overline{x}^i, \overline{z}^i); i \in \mathcal{I}\right) \in \prod_{i\in\mathcal{I}} \mathbb{E}^i$ *such that,*

(i) For any agent $i \in \mathcal{I}$, $(\overline{x}^i, \overline{z}^i) \in \mathbb{E}^i$ *maximizes the utility function* $V^i(\overline{p}, \cdot)$ *among the vectors in the choice set* $B^i(\overline{p}, \overline{q}) \cap \mathcal{F}^i(\overline{p})$.

(ii) The following markets clearing conditions hold,

$$\sum_{i\in\mathcal{I}} \left(\overline{x}_s^i - w_s^i\right) = 0, \quad \forall s \in \mathcal{S}; \qquad \sum_{i\in\mathcal{I}(j)} \overline{z}_j^i = 0,\ \forall j \in \mathcal{J},$$

where, for any asset $j \in \mathcal{J}$, $\mathcal{I}(j) := \{i \in \mathcal{I} : j \in \mathcal{J}^i\}$.

14. Notice that, as in Seghir and Torres-Martínez [2011] we do not impose any kind of financial survival assumption. That is, we do not assume that each agent has access to some amount of credit through any asset $j \in \mathcal{J}$.

1.3 Endogenously Compatible Information

As it was mentioned above, when agents self-restrict their consumption decisions to those that are informational compatible, the presence of endogenous information compromises the continuity of individuals' choice set correspondence and equilibrium may fail to exist. Following, we will illustrate the difficulties that may appear in order to ensure equilibrium existence in a model with endogenous differential information and where agents' consumption plans are restricted by the availability of information.

REMARK 1. Fix an agent $i \in \mathcal{I}$ that is not fully informed (i.e., $\mathbb{F}^i \neq \{\{s\}; s \in S\}$), and consider a sequence of commodity prices $\{p_n\}_{n\geq 1} \subset \mathcal{P}$ that converges to $\overline{p}$. Assume that there is a partition $\mathbb{Q}$ strictly finer than $\mathbb{F}^i \vee \tau(\overline{p})$ such that, $\mathbb{Q} = \bigwedge_{n\geq 1} \mathbb{Q}_n$ where $\mathbb{Q}_n := \mathbb{F}^i \vee \tau(p_n)$ for any $n \geq 1$.

Let $z^i = 0$ and $x^i = (w_0^i, (\alpha_s w_s^i; s \in S))$, where $(\alpha_s; s \in S) \in (0,1)^S$ is $\mathbb{Q}$-measurable but not $\mathbb{F}^i$-measurable. Then, independently of $q \in \mathbb{R}_+^{\mathcal{J}}$, the plan (x^i, z^i) belongs to $B^i(p_n, q) \cap \mathcal{F}^i(p_n)$, for any $n \geq 1$. However, $(x^i, z^i) \notin B^i(\overline{p}, q) \cap \mathcal{F}^i(\overline{p})$, since this plan is only $\mathbb{Q}$-measurable. Therefore, the choice set correspondence does not have a closed graph. □

Therefore, to recover the closed graph property of choice sets we will avoid the informational compatibility restriction and we will impose regularity conditions on preferences to ensure that individuals' optimal decisions are compatible with the final information.

Given $(i, p) \in \mathcal{I} \times \mathcal{P}$, let $\mathcal{S}^i(p) = \{a : S \to S : a \text{ is bijective and } a(s) \in A_s^i(p), \forall s \in S\}$, where $A_s^i(p)$ is the element of $\mathbb{F}^i \vee \tau(p)$ that contains the state of nature s. Next, we elaborate on agent i' s behavior when there is no full information. Note that if the states of nature s and s' are in $A_s^i(p)$ then, necessarily $p_s = p_{s'}$, and prices do not deliver new information to agent i in order to update her preferences. The following assumption, which is irrelevant when prices are full informative, states that agent $i \in \mathcal{I}$, being unable to distinguish among states s and s', is indifferent between to consume x_s in state s and x'_s in state s' or to consume x_s in state s' and x'_s in state s.

Moreover, the assumption ensures that, even when an agent may demand any kind of consumption plan in her budget set, at the optimum, her choice will be measurable with respect to her final information.

ASSUMPTION A. *For any agent $i \in \mathcal{I}$, the objective function $V^i : \mathcal{P} \times \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \to \mathbb{R}$ satisfies,*

$$V^i\left(p, x_0^i, \left(x_s^i\right)_{s\in S}\right) = V^i\left(p, x_0^i, \left(x_{a(s)}^i\right)_{s\in S}\right), \quad \forall p \in \mathcal{P}, \forall (x_s^i)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}, \forall a \in \mathcal{S}^i(p).$$

Next result formalizes our arguments,

PROPOSITION 1. *Given prices $(p, q) \in \mathbb{P}$, suppose that $((x_s^i)_{s\in\mathcal{S}}, z^i) \in \mathbb{E}^i$ is an optimal choice for agent $i \in \mathcal{I}$ in her budged set $B^i(p, q)$. If Assumption A holds, and V^i is strictly quasi-concave on consumption, then $(x_s^i)_{s\in S}$ belongs to $\mathcal{F}^i(p)$.*

PROOF. Suppose that states of nature s and s' are in the same element of $\mathbb{F}^i \vee \tau(p)$ and $x_s^i \neq x_{s'}^i$. Since $p \in \mathcal{P}$, we have that $p_s = p_{s'}$. Moreover, as $\mathbb{F}^i$ contains the information revealed by the payments of assets in $\mathcal{J}^i$, $p_s w^i_s + \sum_{j\in\mathcal{J}^i} R_{s,j} z^i_j = p_{s'} w^i_{s'} + \sum_{j\in\mathcal{J}^i} R_{s',j} z^i_j$. Fix $\lambda \in (0,1)$. It follows that $\lambda x^i_s + (1-\lambda)x^i_{s'}$ is budget feasible at both states of nature, s and s'.

Define $c^i := (c^i_k)_{k\in S} \in \mathbb{R}^{\mathcal{L}\times S}_+$ by $c^i_k = x^i_k$ when $k \notin \{s,s'\}$, $c^i_s = c^i_{s'} = \lambda x^i_s + (1-\lambda)x^i_{s'}$. In addition, consider the bijection $\tilde{a} : S \to S$ such that $\tilde{a}(s) = s'$, $\tilde{a}(s') = s$, and $\tilde{a}(k) = k$ for any $k \notin \{s,s'\}$. Since $V^i\left(p, x^i_0, (x^i_k)_{k\in S}\right) = V^i\left(p, x^i_0, (x^i_{\tilde{a}(k)})_{k\in S}\right)$, the strict quasi-concavity of V^i on consumption ensures that $V^i\left(p, x^i_0, (c^i_k)_{k\in S}\right) > V^i\left(p, x^i_0, (x^i_k)_{k\in S}\right)$. Therefore, agent i can improve his utility level choosing the bundle c^i. A contradiction. □

Remark 2. When agents' preferences are represented by the expected utility, Assumption A is fulfilled if state dependent agent's preferences coincide across undistinguishable states (measurability).[15] However, Assumption A is weaker than requiring measurability of preferences (see Example 1). As we already noted, the assumption is inconsequential when prices are full informative.

1.4 Equilibrium Existence

In order to state equilibrium existence, we concentrate in a model that also satisfies the following hypothesis B and C:

Assumption B. *For any agent $i \in \mathcal{I}$, the utility function V^i is continuous, and strictly increasing and strictly quasi-concave on consumption.*

On the other hand, the incomplete participation in financial markets makes difficult to normalize prices and to ensure that budget set correspondences have a non-empty interior, at the same time. For these reasons, and in order to find endogenous upper bounds on asset prices, it will be enough to impose Assumption C on the agents' behavior, which states that a reduction of future consumption (at $t = 1$) can be compensate by rising the consumption in the first period (at $t = 0$.)[16]

Assumption C. *For any agent $i \in \mathcal{I}$, given $\sigma \in (0,1)$ there is a continuous mapping $r_\sigma : \mathcal{P} \times \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+ \to \mathbb{R}^{\mathcal{L}}_+$, that satisfies*

$$V^i(p, x_0 + r_\sigma(p,x), (\sigma x_s; s \in S)) > V^i(p, (x_s; s \in \mathcal{S})), \quad \forall p \in \mathcal{P},\ \forall (x_s; s \in \mathcal{S}) \in \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_{++}.$$

Notice that the Assumption C is satisfied for a great variety of utility functions. It is already satisfied by a monotone utility function that is unbounded in some commodity consumed in the first period. For instance, it is fulfilled by any Cobb-Douglas utility function and by the expected utility with state dependent Cobb-Douglas preferences.

15. In Einy, Moreno, and Shitovitz [2000, 2001], it is assumed that state dependent utilities are equal across states that are undistinguishable with respect to the initial information.

16. This assumption was previously used by Seghir and Torres-Martínez [2011] as an impatience assumption on the agents behavior.

In Seghir and Torres-Martínez [2011], Assumption C is used to find upper bounds for asset prices, however and in contrast to our model, in that paper they have restrictions on credit opportunities only. Therefore, given restrictions on investment as well as on credit opportunities we find upper bounds on asset prices using this Assumption (see Lemma 4 in the Appendix).

THEOREM. *Under Assumptions A, B and C, there exists an equilibrium for the economy with endogenous differential information and incomplete financial participation.*

1.5 The Expected Utility

In order to illustrate our general framework in some particular scenarios, we consider the rational expectations model (REE). For it, we assume that agents' state-dependent preferences may be affected by prices. That is, each agent $i \in \mathcal{I}$ has a utility function, $u_s^i : \mathcal{P} \times \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}$ representing her preferences in each state s, and beliefs about the realization of the states in S denoted by $\pi^i := (\pi_s^i; s \in S) \in \mathbb{R}_{++}^S$ with $\sum_{s \in S} \pi_s^i = 1$. Thus, the objective function of agent i is the expected utility given by:

$$V^i(p, x^i) := \sum_{s \in S} \pi_s^i u_s^i(p, x_0^i, x_s^i).$$

We consider that individuals take optimal decisions regarding all the information available about the realization of the uncertainty.

Let $p \in \mathcal{P}$ the price system and $\mathbb{F}^i(s)$ the initial information of agent $i \in \mathcal{I}$ with respect to state s. The information revealed by prices p is uninformative with respect to the event $\mathbb{F}^i(s)$, if and only if $\mathbb{F}^i(s) = A_s^i(p)$, that is $p_s = p_{s'}$ for all $s' \in \mathbb{F}^i(s)$. In this situation agent i does not possess any information to distinguish among states in $\mathbb{F}^i(s)$. With more generality, agent i is not fully-informed when $A_s^i(p) \neq \{s\}$, that is, she is unable to distinguish different states in $A_s^i(p)$. The argument behind Assumption A states that it is not possible for agent i to valuate differently the same consumption bundle in undistinguishable states. That is, for $s' \in A_s^i(p)$, we require that $u_s^i(p, x_0, \cdot) = u_{s'}^i(p, x_0, \cdot)$ for every $x_0 \in \mathbb{R}_+^{\mathcal{L}}$.

Nevertheless, changes on the information revealed by prices may induce discontinuities on the objective function V^i due to changes in prices. To illustrate this possibility, next example sets an economy that fulfills Assumption A, however, equilibrium fails to exist since there is an agent for whom improving her information matters.

EXAMPLE 1. (NON-EXISTENCE OF EQUILIBRIA)

Consider an economy with two periods and two equiprobable states of nature at the second period denoted by $\{u, d\}$. There are two agents $\{A, B\}$ and two commodities $\{x, y\}$. For simplicity, we assume that there are no financial contracts and no trade at $t = 0$.

Agent A initially recognizes the state of nature realized at $t = 1$. Her state dependent preferences are identical in both states; $u(x, y) = xy$, thus, her objective function is

$$v^A((x_u, y_u), (x_d, y_d)) = \frac{1}{2}x_u y_u + \frac{1}{2}x_d y_d.$$

Agent B does not have information to recognize the state of nature at the second period, i.e., $\mathbb{F}^B = \{\{u, d\}\}$. Her state dependent utilities are $u_u^B(x_u, y_u) = x_u$, and, $u_d^B(x_d, y_d) = x_d y_d$. Therefore, if there is no information via prices to distinguish both states and due to the lack of information, her objective function is the expectation of the average of her state dependent utilities, that is,

$$v^B((x_u, y_u), (x_d, y_d)) = \frac{1}{2}\left(\frac{1}{2}x_u + \frac{1}{2}x_u y_u\right) + \frac{1}{2}\left(\frac{1}{2}x_d + \frac{1}{2}x_d y_d\right).$$

Let $(w_u^A, w_d^A) = ((3, 1), (1, 1))$ and $(w_u^B, w_d^B) = ((1, 1), (1, 1))$ be, respectively, the state dependent endowments.

By monotonicity of objective functions, if equilibrium prices exist, then they are strictly positive. Furthermore, the demand for commodities at prices $(p_u, p_d) \gg 0$ satisfies

$$\left[(x_u^A, y_u^A)\,;\,(x_d^A, y_d^A)\right] = \left[\left(\frac{p_u w_u^A}{2p_{u,x}}, \frac{p_u w_u^A}{2p_{u,y}}\right)\,;\,\left(\frac{p_d w_d^A}{2p_{d,x}}, \frac{p_d w_d^A}{2p_{d,y}}\right)\right].$$

$$\left[(x_u^B, y_u^B)\,;\,(x_d^B, y_d^B)\right] = \begin{cases} \left[\left(\frac{p_u w_u^B + p_{u,y}}{2p_{u,x}}, \frac{p_u w_u^B - p_{u,y}}{2p_{u,y}}\right)\,;\,\left(\frac{p_d w_d^B + p_{d,y}}{2p_{d,x}}, \frac{p_d w_d^B - p_{d,y}}{2p_{d,y}}\right)\right], & \text{when } p_u = p_d; \\ \left[\left(\frac{p_u w_u^B}{p_{u,x}}, 0\right)\,;\,\left(\frac{p_d w_d^B}{2p_{d,x}}, \frac{p_d w_d^B}{2p_{d,y}}\right)\right], & \text{when } p_u \neq p_d. \end{cases}$$

Without loss of generality, we pay particular attention to commodity prices $(p_u, p_d) \gg 0$ satisfying $p_{u,x} + p_{u,y} = 1$ and $p_{d,x} + p_{d,y} = 1$ (i.e., we assume that $\zeta = (1, 1)$). Therefore, if there is an equilibrium in this economy, then we have two possibilities:

(i) Equilibrium prices satisfy $\overline{p}_u \neq \overline{p}_d$. Then, market feasibility implies

$$\begin{aligned} \frac{\overline{p}_u w_u^A}{2\overline{p}_{u,x}} + \frac{\overline{p}_u w_u^B}{\overline{p}_{u,x}} &= w_{u,x}^A + w_{u,x}^B; & \frac{\overline{p}_u w_u^A}{2\overline{p}_{u,y}} + \quad 0 \quad &= w_{u,y}^A + w_{u,y}^B; \\ \frac{\overline{p}_d w_d^A}{2\overline{p}_{d,x}} + \frac{\overline{p}_d w_d^B}{2\overline{p}_{d,x}} &= w_{d,x}^A + w_{d,x}^B; & \frac{\overline{p}_d w_d^A}{2\overline{p}_{d,y}} + \frac{\overline{p}_d w_d^B}{2\overline{p}_{d,y}} &= w_{d,y}^A + w_{d,y}^B. \end{aligned}$$

We conclude that $\overline{p}_u = \overline{p}_d = \left(\frac{1}{2}, \frac{1}{2}\right)$. A contradiction.

(ii) Equilibrium prices satisfy $\overline{p}_u = \overline{p}_d$. Then, from market clearing conditions we obtain that $\overline{p}_u = \left(\frac{3}{7}, \frac{4}{7}\right) \neq \left(\frac{3}{5}, \frac{2}{5}\right) = \overline{p}_d$, which contradicts our assumption. $\square$

Observe that, in the above example, agents update their objective functions regarding all the available information. That is, the objective functions for agent A and B can be written, respectively, as:

$$\begin{aligned} V^A(p,(x_u,y_u),(x_d,y_d)) = &\quad v^A((x_u,y_u),(x_d,y_d)) = \tfrac{1}{2}x_u y_u + \tfrac{1}{2}x_d y_d; \\ V^B\,(\,p\,,(x_u,y_u),(x_d,y_d)) = &\quad v^B\,((x_u,y_u),(x_d,y_d)) \\ &+g(p)\Big(\tfrac{1}{2}\,(x_u + x_d y_d) - v^B\,((x_u,y_u),(x_d,y_d))\Big), \end{aligned}$$

where $g(p)$ is equal to zero when $p_u = p_d$, and it is equal to one in other case. Therefore, in spite of Assumption A to be fulfilled, the example does not accomplishes the conditions of our Theorem since the objective function of agent B is discontinuous as function of prices.

Consider the classical REE scenario, where state-dependent preferences do not depend on prices. Therefore, we simplify our notation with the cost of a little abuse of notation taking $u^i_s(p,x_0,x_s) = u^i_s(x_0,x_s)$. Let $p \in \mathcal{P}$ the prevailing price system and let $i \in \mathcal{I}$ a non-fully informed agent in a state $s \in S$, that is $A^i_s(p) \neq \{s\}$. In order to evaluate her objective function, agent i evaluates her utility in an undistinguished state $s' \in A^i_s(p)$ as a weighted average of the utilities across those undistinguishable states. We denote this average by $u^i_{[s]}(\cdot,\cdot)$ that is equivalent to the expression $\sum_{s'\in\mathbb{F}^i(s)} \pi^i_{s'} u^i(\cdot,\cdot)$.

Thus, if the prevailing price p is informative with respect to state s, for every agent $i \in \mathcal{I}$, the contribution of the consumption plan (x_0,x_s) to the objective function of agent i, in state $s \in S$ is $u^i_s(x_0,x_s)$. On the other hand, if prices are not informative, the contribution is $u^i_{[s]}(x_0,x_s)$. In consequence, the contribution of consumption plan (x_0,x_s) to the objective function of agent i, in state $s \in S$, can be written as follows:

$$v^i_s(p,x_0,x_s) = u^i_{[s]}(x_0,x_s) + g^i(p)(u^i_s(x_0,x_s) - u^i_{[s]}(x_0,x_s))$$

where $g^i(p) = 0$ when $A^i_s(p) \neq \{s\}$ and $g^i(p) = 1$ when $A^i_s(p) = \{s\}$.

This means that, even in the classical REE case, prices are present in each agent's objective function. Moreover, it becomes clear that, as Example 1 shows, the discontinuity of $g^i(p)$ may induce discontinuities in the objective function V^i.

In order to accomplish Assumption A we require additional hypothesis on state dependent utilities that are price (informational) dependent. In this regard, in the works by Einy, Moreno, and Shitovitz [2000, 2001] it is assumed that state dependent utilities (that do not depend on prices) are constant among initially undistinguishable states. That is, for every price $p \in \mathcal{P}$ and state $s' \in \mathbb{F}^i(s)$:

$$u^i_s(p,\cdot) = u^i_{s'}(p,\cdot).$$

This assumption, that implies Assumption A and the continuity of the objective function, is restrictive since precludes the possibility of utility-update when information matters.

With the aim to recover the possibility of utility-update and ensuring the continuity of the objective function, we will assume that, initially, preferences of agent $i \in \mathcal{I}$ in each state $s' \in \mathbb{F}^i(s) = [s]$, are given by $u^i_{[s]}(x_0,x_s)$. Once agent i updates her information via the prevailing price system p, she may smoothly change her preferences to $u^i_s(p,x_0,x_s)$, where $u^i_s(\cdot,\cdot,\cdot)$:

$\mathcal{P} \times \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}_+$ is a continuous function, concave and strictly monotonic on consumption, and such that $u_s^i(p, x_0, x_s) = u_{[s]}^i(x_0, x_s)$ when prices do not add any information, that is, $A_s^i(p) = \mathbb{F}^i(s)$.

Note that,

$$u_s^i(p, x_0, x_s) = u_{[s]}^i(x_0, x_s) + f_s^i(p, x_0, x_s),$$

where $f_s^i : \mathcal{P} \times \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}_+$ is given by $f_s^i(p, x_0, x_s) = u_s^i(p, x_0, x_s) - u_{[s]}^i(x_0, x_s)$.

Contrary to the Example 1, where $f_s^i(\cdot, x_0, x_s)$ is a dichotomic function of prices, we assume that f_s^i is a continuous function that is zero when

$$\max_{s' \in \mathbb{F}^i(s), s' \neq s} \|p_s - p_{s'}\| = 0.$$

Thus, when prices across states initially-undistinguishable for agent $i \in \mathcal{I}$ add new information, then her state dependent utility function is updated in a smooth way. The assumption prevents the transition via the information revealed by prices from being abrupt, as it happens in Example 1. In fact, if the information comes from differences in prices, we can easily argue in favor of the continuity of $u_s^i(\cdot, x_0, x_s)$ as function of prices. Our continuity assumption implies that, the smaller the difference in prices between initially undistinguishable states is, the closer the state utility should be to the uninformed case.

We may strengthen the argument assuming that there is an $\epsilon > 0$ such that $u_s^i(p, x_0, x_s)$ is constant when $\max_{s' \in \mathbb{F}^i(s)} \|p_s - p_{s'}\| \leq \epsilon$. This case represents an scenario where ϵ-differences in prices do not affect at all the state utility. In other words, an $\epsilon > 0$ can be understood as the subjective threshold from which the agent subjectively update her preferences. Observe that, if ϵ is big enough we fall in the case where the state utility functions are not price dependent, as in Einy, Moreno, and Shitovitz [2000, 2001].

1.6 Examples

The following examples offer some insights about price informativeness and information compatibility captured in the model. First, we elaborate on non fully revealing prices. Second, we illustrate the relation between the initial information and the information revealed by the asset that are accessible.

EXAMPLE 2. (NON-INFORMATIVE EQUILIBRIUM)

Consider an economy with two commodities and utility functions given by

$$U^i((x_s; s \in \mathcal{S})) = \sum_{s \in S} \left(x_{0,1}^{\beta} x_{0,2}^{1-\beta} + x_{s,1}^{\beta} x_{s,2}^{1-\beta} \right), \quad \forall i \in \mathcal{I},$$

where $\beta \in (0,1)$ is the same for all agents. Then, Assumptions (A), (B), and (C) hold. Also, first-order conditions of consumer's i problem at state $s \in S$ would imply that, at any equilibrium price $\overline{p}_s$,

$$\frac{\overline{p}_{s,1}}{\overline{p}_{s,2}} = \frac{\beta}{1-\beta} \frac{W_{s,2}}{W_{s,1}},$$

where $W_{s,l} = \sum_{i\in\mathcal{I}} w^i_{s,l}$. Suppose that there is an uninformed agent $i_0 \in \mathcal{I}$ (i.e. $\mathbb{F}^{i_0} = \{S\}$). Then, equilibrium prices are *non-informative*[17] if, and only if, the relative degree of commodity scarcity is constant at the second period, $\frac{W_{s,2}}{W_{s,1}} = \frac{W_{s',2}}{W_{s',1}}$, $\forall (s,s') \in S \times S$, which is a restrictive hypothesis. Thus, for any economy in which this condition does not hold, any equilibrium price will reveal information (at least for the uninformed agent i_0). □

The following proposition illustrates examples of our model in which the equilibrium prices are partially informative. That is, at given prices $p \in \mathcal{P}$, the partition $\mathbb{F}^i \vee \tau(p)$ is finer than $\mathbb{F}^i$. For robust partially revealing examples including ambiguity see Condie and Ganguli [2011].

PROPOSITION 2. *Under Assumptions (A)-(C), assume that there are two commodities l and l' such that, the preferences of any agent $i \in \mathcal{I}$ can be represented by a utility function*

$$V^i(p,x) = \sum_{s\in S} \pi^i_s v^i(p, x^i_0, (x^i_{s,k})_{k\notin\{l,l'\}}, g(x^i_{s,l}, x^i_{s,l'})),$$

where v^i is differentiable and $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a CES function. In addition, suppose that the relative scarcity of commodities l and l' differs across states of nature $(s,s') \in S \times S$, i.e.,

$$\frac{\sum_{i\in\mathcal{I}} w^i_{s,l}}{\sum_{i\in\mathcal{I}} w^i_{s',l}} \neq \frac{\sum_{i\in\mathcal{I}} w^i_{s,l'}}{\sum_{i\in\mathcal{I}} w^i_{s',l'}}.$$

Then, in any equilibrium commodity prices are at least partially informative.

PROOF. By Contradiction, assume that equilibrium prices are non informative. Therefore, for states of nature s and s' we have $\overline{p}_s = \overline{p}_{s'}$. Since g is a CES function, there exists $\gamma \in (0,1)$ and $\rho \in (-\infty,0) \cup (0,1)$ such that $g(a,b) = (\gamma a^\rho + (1-\gamma)b^\rho)^{\frac{1}{\rho}}$. The marginal rate of substitution between the consumption of commodities l and l' at node $k \in \{s,s'\}$ is given by $\frac{\gamma(\overline{x}^i_{k,l})^{\rho-1}}{(1-\gamma)(\overline{x}^i_{k,l'})^{\rho-1}}$. Thus, Karush-Kuhn-Tucker conditions of agent i's individual problem guarantees that,

$$\overline{x}^i_{k,l} = \left(\frac{(1-\gamma)}{\gamma}\frac{\overline{p}_{k,l}}{\overline{p}_{k,l'}}\right)^{\frac{1}{\rho-1}} \overline{x}^i_{k,l'}.$$

Adding across agents, we obtain that

$$\sum_{i\in\mathcal{I}} w^i_{k,l} = \left(\frac{(1-\gamma)}{\gamma}\frac{\overline{p}_{k,l}}{\overline{p}_{k,l'}}\right)^{\frac{1}{\rho-1}} \sum_{i\in\mathcal{I}} w^i_{k,l'}, \quad k \in \{s,s'\}.$$

Therefore, as $\overline{p}_s = \overline{p}_{s'}$, the relative scarcity of commodities l and l' coincide at nodes s and s', which contradicts our initial assumption. □

17. That is, prices are measurable with respect to the coarse information partition.

EXAMPLE 3. (RESTRICTED PARTICIPATION)

In this example we will illustrate the importance of the compatibility between the initial information and the information revealed by the asset that an agent can trade. For simplicity, consider an economy with only one commodity, three states of nature at $t = 1$, denoted by $\{u, m, d\}$, and two agents who only receive utility for consumption in the second period. Also, they do not have any initial endowment at $t = 0$. Thus, utility functions and endowments are given by

$$U^1(x_u, x_m, x_d) = \sqrt{2}\sqrt{x_u} + \sqrt{x_m} + \sqrt{x_d}, \qquad (w_u^1, w_m^1, w_d^1) = (3, 3, 3);$$

$$U^2(x_u, x_m, x_d) = \sqrt{x_u} + \sqrt{x_m} + \sqrt{2}\sqrt{x_d}, \qquad (w_u^2, w_m^2, w_d^2) = (3, 3, 3).$$

There are two Arrow securities in the economy. One of them has a unitary price q_1 and promises to deliver one unit of the commodity at state of nature $s = u$. The other makes a contingent payment of one unit of the commodity at $s = d$, and is negotiated for a unitary price q_2.

If there is a complete financial participation, the first period budget constraint of agent $i \in \{1, 2\}$ is given by $q_1 z_1^i + q_2 z_2^i = 0$, where z_j^i denotes the position of agent i on asset $j \in \{1, 2\}$.

Assume that unitary prices are given by $\overline{q}_1 = \overline{q}_2 = 1$. Then, the allocations

$$(z_1^1, z_2^1, x_u^1, x_m^1, x_d^1) = (1, -1, 4, 3, 2), \qquad (z_1^2, z_2^2, x_u^2, x_m^2, x_d^2) = (-1, 1, 2, 3, 4).$$

constitute an equilibrium for the economy.

We argue that, if agents are not fully informed—that is, they do not internalize the information revealed by asset payments— the implementation of this equilibrium allocation may not be credible. For instance, assume that $\mathbb{F}^1 = \{\{u\}, \{m, d\}\}$ and $\mathbb{F}^2 = \{\{u, m\}, \{d\}\}$. In order to pay his debt, it is required that agent $i = 1$ observes that the state of nature $s = d$ was realized. This would be impossible to accomplish as commodity prices do not communicate information, and there is no other financial signal which allows recognition between states m and d. Analogously, to pay his debt, it is required that agent $i = 2$ observes some signal that allows him to distinguish between u and m, which is an impossible task to accomplish given the financial structure. □

1.7 Concluding remarks

In this paper we elaborate on a model of competitive market with differential information, where agents have restricted participation in incomplete financial markets. Agents, sequentially, are able to add new information from the traded assets, buy commodities in spot markets and receive the signals given by spot prices allowing them to improve their previous private information.

Our model allows agents to obtain information through the variability of payments in financial markets. Thus, individuals obtain all the information revealed by the awareness conveyed by securities that they can trade. However, there is an incomplete access to financial instruments available in the economy. In order to be consistent with the information transmitted by asset returns and the restrictions that imply the incomplete access to investment opportunities, we need to ensure that equilibrium exists without the requirement of any kind of financial survival restriction. Thus, we extend the model of credit constrained markets of Seghir and Torres-Martínez [2011].

We contemplate an scenario where the agents' final information about the realization of uncertainty needs not to be fully revealing. Moreover, the final information has real effects over the agent's capability to implement heterogeneous preferences across states of nature. Equilibrium existence is obtained without imposing any compatibility requirement between consumption and information. The measurability of optimal bundles is a consequence of the informational-dependent nature of individuals objective functions, since there are no gains for consumption heterogeneity in states of nature that are undistinguishable.

1.8 Appendix: Proof of Theorem 1

To prove equilibrium existence, we first define a generalized game in which agents maximize utility functions in truncated budget sets. Auctioneers choose prices in order to maximize the value of the excess of demand in commodity and financial markets. We prove that this generalized game has a Cournot-Nash equilibrium and also that, when the upper bounds on allocations are high enough, any equilibrium of the generalized game will be an equilibrium of our economy.

The generalized game $\mathcal{G}(Q,X,\Theta)$. Given any vector $(Q,X,\Theta)\in\mathbb{R}^3$, we define a game characterized by the following set of players and strategies.

Set of players. There is a finite set of players constituted by,

(i) The set of agents of the economy, $\mathcal{I}$.

(ii) An auctioneer, $h(s)$, for each $s\in\mathcal{S}$.

We denote the set of players by $H=\mathcal{I}\cup H(\mathcal{S})$ where $H(\mathcal{S}):=\{h(s):s\in\mathcal{S}\}$.

Sets of strategies. Given $\overline{W}:=\max_{(s,l)\in\mathcal{S}\times\mathcal{L}}\sum_{i\in\mathcal{I}}w^i_{s,l}$, define for any $i\in\mathcal{I}$,

$$K^i(X,\Theta)=[0,X]^{\mathcal{L}}\times[0,2\overline{W}]^{S\times\mathcal{L}}\times[-\Theta,\Theta]^{\mathcal{J}^i},$$

and, for any $s\in\mathcal{S}$, let $\mathcal{P}_s=\{p\in\mathbb{R}^{\mathcal{L}}_+:p\cdot\zeta=1\}$. The set of strategies for the players in the generalized game, $(\overline{\Gamma}^h;h\in H)$, are given by,

(i) For each $h\in\mathcal{I}$, $\overline{\Gamma}^h=K^h(X,\Theta)$.

(ii) For $h=h(0)$, $\overline{\Gamma}^h=\mathcal{P}_0\times[0,Q]^{\#\mathcal{J}}$

(iii) For $h=h(s)$, with $s\in S$, $\overline{\Gamma}^h=\mathcal{P}_s$.

For simplicity, let $\eta^h=\left(x^h,z^h\right)\in\overline{\Gamma}^h$ be a generic vector of strategies for a player $h\in\mathcal{I}$; (p_0,q) will denote a generic strategy for the player $h(0)$; and p_s a generic strategy for a player $h(s)$, with $s\in S$. Finally, let $\overline{\Gamma}=\prod_{h\in H}\overline{\Gamma}^h$ be the space of strategies of $\mathcal{G}(Q,X,\Theta)$. A generic element of $\overline{\Gamma}$ is denoted by (p,q,η), where $\eta:=\left(\eta^h;h\in\mathcal{I}\right)$ is a generic element of $\prod_{i\in\mathcal{I}}\overline{\Gamma}^i$.

Admissible strategies. Strategies effectively chosen for players depend on the actions taken by other players, through a correspondence of admissible strategies $\phi^h:\overline{\Gamma}_{-h}\twoheadrightarrow\overline{\Gamma}^h$, where $\overline{\Gamma}_{-h}=\prod_{h'\neq h}\overline{\Gamma}^{h'}$. Let $(p,q,\eta)_{-h}$ be a generic element of $\overline{\Gamma}_{-h}$. We suppose that,

(i) If $h\in\mathcal{I}$, $\phi^h\left[(p,q,\eta)_{-h}\right]=B^h(p,q)\cap\overline{\Gamma}^h$.

(ii) If $h\in H(\mathcal{S})$, $\phi^h\left[(p,q,\eta)_{-h}\right]=\overline{\Gamma}^h$.

Objective functions. Each player is also characterized by an objective function $F^h:\overline{\Gamma}^h\times\overline{\Gamma}_{-h}\to\mathbb{R}_+$. We assume that,

(i) When $h\in\mathcal{I}$ and $\eta^h=\left(x^h,z^h\right)\in\overline{\Gamma}^h$, then $F^h\left(\eta^h;(p,q,\eta)_{-h}\right)=V^i\left(p,x^h\right)$.

(ii) If $h = h(0)$ and $(p,q) \in \overline{\Gamma}^h$, then

$$F^h\left((p_0,q);(p,q,\eta)_{-h}\right) := p_0 \sum_{i\in\mathcal{I}} \left(x_0^i - w_0^i\right) + \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}^i} q_j z_j^i.$$

(iii) If $h(s) \in H(\mathcal{S}) \setminus \{h(0)\}$ and $p_s \in \overline{\Gamma}^h$, then $F^h\left(p_s;(p,q,\eta)_{-h}\right) := p_s \sum_{i\in\mathcal{I}} (x_s^i - w_s^i)$.

We define the correspondence of optimal strategies for each $h \in H$, $\Psi^h : \overline{\Gamma}_{-h} \twoheadrightarrow \overline{\Gamma}^h$ as

$$\Psi^h\left((p,q,\eta)_{-h}\right) := \underset{y\in\phi^h\left((p,q,\eta)_{-h}\right)}{\arg\max} F^h\left(y;(p,q,\eta)_{-h}\right).$$

Finally, let $\Psi : \overline{\Gamma} \twoheadrightarrow \overline{\Gamma}$ be the correspondence of optimal game response, which is given by $\Psi(p,q,\eta) = \prod_{h\in H} \Psi^h\left((p,q,\eta)_{-h}\right)$.

DEFINITION. *A Cournot-Nash equilibrium for the generalized game $\mathcal{G}(Q,X,\Theta)$ is given by a strategy profile $(\overline{p},\overline{q},\overline{\eta}) \in \overline{\Gamma}$ such that, $(\overline{p},\overline{q},\overline{\eta}) \in \Psi(\overline{p},\overline{q},\overline{\eta})$.*

In order to prove the existence of equilibrium in the generalized game, we need some properties of the admissible strategy correspondence which the following lemma provides.

LEMMA 1. *For any $h \in H$, ϕ^h is continuous and has non-empty, compact, and convex values.*

PROOF. For each player $h \in H(\mathcal{S})$, the correspondence of admissible strategies is constant and, therefore, it is continuous and non-empty. Also, by definition, its values are compact and convex.

On the other hand, for each player $h \in \mathcal{I}$, it follows from the definition of the budget set, that the correspondence of admissible strategies ϕ^h has non-empty, compact and convex values. Since the graph of this correspondence is closed, we obtain upper hemicontinuity. To assure the lower hemicontinuity of ϕ^h, we consider the correspondence $\mathring{\phi}^h((p,q,\eta)_{-h}) := int_{K^h(X,\Theta)} B^h(p,q)$, which associates to a vector of commodity and asset prices the set of allocations in $K^h(X,\Theta)$ that satisfy all the budget restrictions of agent h as strict inequalities. Note that, this correspondence has non-empty values and open graph. Therefore, it is lower hemicontinuous. We know that the closure of $\mathring{\phi}^h((p,q,\eta)_{-h})$, which is equal to $\phi^h((p,q,\eta)_{-h})$, is also lower hemicontinuous. Therefore, correspondences of admissible strategies $(\phi^h; h \in \mathcal{I})$ are continuous. □

LEMMA 2. *Under (A) and (B), the set of Cournot-Nash equilibria of $\mathcal{G}(Q,X,\Theta)$ is non-empty.*

PROOF. By Assumption (A) and (B), each objective function in the game is continuous in all variables and quasi-concave in its own strategy. Also, the sets of strategies are non-empty, compact and convex. By Lemma 1, admissible correspondence is continuous with non-empty, convex and compact-values. Thus, we can apply Berge's Maximum Theorem to assure that, for each player $h \in H$ the correspondence of optimal strategies, Ψ^h, is upper hemicontinuous with non-empty, convex and compact values. Therefore, the correspondence Ψ has closed graph with non-empty, compact and convex values. Applying Kakutani's Fixed Point Theorem to Ψ we conclude the proof.

□

We will prove that, for vectors $(Q, X, \Theta) \in \mathbb{R}^3_+$ for which coordinates are high enough, any equilibrium of the generalized game is an equilibrium for our economy. However, we need to previously find endogenous upper bounds for equilibrium variables.

LEMMA 3. *For each $s \in S$, fix a vector $(p_s, w_s, x_s) \in \mathcal{P}_s \times \mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{\mathcal{L}}_+$, with $x_s < \overline{W}$. Then, there exists $A > 0$ such that, any vectors $(\kappa_j; j \in \mathcal{J}) \in \mathbb{R}^{\mathcal{J}}$ satisfying*

$$p_s x_s = p_s w_s + \sum_{j \in \mathcal{J}} R_{s,j} \kappa_j, \ \forall s \in S;$$

belongs on $[-A, A]^{\#\mathcal{J}+1}$. Also, A only depends on $((\overline{W}, w_s, R_{s,j}); (s,j) \in S \times \mathcal{J})$.

PROOF. Note that, as S (respectively, $\mathcal{J}$) is a finite set, by abusing of the notation and identifying it with $\{1, \dots, S\}$ (respectively, $\{1, \dots, J\}$) we can rewrite the conditions in the statement of the Lemma in a matricial form:

$$\begin{bmatrix} p_1(x_1 - w_1) \\ \vdots \\ p_S(x_S - w_S) \end{bmatrix} = \begin{bmatrix} R_{1,1} & \cdots & R_{1,J} \\ \vdots & \ddots & \vdots \\ R_{S,1} & \cdots & R_{S,J} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_J \end{bmatrix}$$

Since there are no redundant assets in the economy, we have that $J \leq S$. Moreover, we can find a non-singular sub-matrix of dimension $J \times J$. Specifically, we may assume, without loss of generality, that this matrix is given by

$$B = \begin{vmatrix} R_{1,1} & \cdots & R_{1,J} \\ \vdots & \vdots & \vdots \\ R_{J,1} & \cdots & R_{J,J} \end{vmatrix}.$$

Thus, we have that

$$\begin{bmatrix} p_1(x_1 - w_1) \\ \vdots \\ p_J(x_J - w_J) \end{bmatrix} = B \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_J \end{bmatrix}$$

By Cramer's Rule,

$$\kappa_j = \frac{det(B(y,j))}{\det(B)}, \ \forall j \in \{1, \dots, J\},$$

where $y = (p_1(x_1 - w_1), \dots, p_J(x_J - w_J))$ and $B(y,j)$ is the matrix obtained by change, in the matrix B, the j-ith column for the vector y. Since (i) the determinant is a continuous function; (ii) the vector y depends continuously of $((p_s, x_s); s \in S)$; and (iii) vectors $((p_s, x_s, w_s); s \in S)$ are in a compact space, it follows that vector $(k_j; j \in \mathcal{J})$ is bounded, independently of the value of $((p_s, x_s, w_s); s \in S)$. Thus, there exists $A > 0$ which satisfies the conditions of the lemma and depends on $((\overline{W}, w_s, R_{s,j}); (s,j) \in S \times \mathcal{J})$. □

Following the notation of the previous lemma, let $\overline{\Theta} := 2A$.

The next two lemmas are used to prove that equilibrium asset prices of the generalized game are uniformly bounded. For convenience of notations, let $W_0 = (W_{0,l}; l \in \mathcal{L})$ be the vector of aggregated physical resources at $t = 0$, where $W_{0,l} := \sum_{i\in\mathcal{I}} w^i_{0,l}$.

LEMMA 4. *Under Assumptions (B) and (C), fix $(\overline{p}, \overline{q}) \in \mathcal{P} \times \mathbb{R}^{\mathcal{J}}_{+}$ and suppose that, for any agent $i \in \mathcal{I}$, there is an optimal solution $(\overline{x}^i, \overline{z}^i) \in \overline{\Gamma}^i$ for his individual problem such that, $\overline{x}^i_0 \leq W_0$ and $\overline{x}^i_{s,l} \leq 2\overline{W}$, $\forall (s,l) \in S \times \mathcal{L}$. Then, there exists $\overline{Q} > 0$, independent of prices, such that $\max_{j\in\mathcal{J}} \overline{q}_j < \overline{Q}$.*

PROOF. Fix $j \in \mathcal{J}$. Suppose that an agent $i \in \mathcal{I}$ for which $j \in \mathcal{J}^i$ borrows a quantity $\tilde{z}_j > 0$ of asset j such that $R_{s,j}\tilde{z}_j \leq \mu := \frac{\min_{(k,l,h)\in S\times\mathcal{L}\times\mathcal{I}} w^h_{k,l}}{2}$, for any $s \in S$.[18] This position on asset j reports a quantity of resources which allow agent i to consume at the first period the bundle $w^i_0 + (\overline{q}_j\tilde{z}_j)\zeta$ and, therefore,

$$V^i\left(\overline{p}, w^i_0 + (\overline{q}_j\tilde{z}_j)\zeta, (0.5w^i_s; s \in S)\right) \leq V^i(\overline{p}, \overline{x}^i) < V^i\left(\overline{p}, W_0, (2\overline{W}(1,\dots,1))_{s\in S}\right).$$

On the other hand, Assumption (C) guarantees that there exists $\overline{r}(\overline{p}) \in \mathbb{R}^{\mathcal{L}}_{+}$ such that,

$$V^i\left(\overline{p}, W_0, (2\overline{W}(1,\dots,1))_{s\in S}\right) < V^i(\overline{p}, w^i_0 + \overline{r}(\overline{p}), (0.5w^i_s; s \in S)).$$

Indeed, following the notation of Assumption (C), the inequality above follows from

$$\overline{r}(\overline{p}) = \overline{r}_{\tilde{\sigma}}\left(\overline{p}, (W_0, (2\overline{W}(1,\dots,1))_{s\in S})\right) + W_0 - w^i_0 \in \mathbb{R}^{\mathcal{L}}_{+},$$

where $\tilde{\sigma} \in (0,1)$ is chosen to satisfiy $2\overline{W}\tilde{\sigma} < \mu$.

We conclude that,

$$\overline{q}_j < Q_j(\overline{p}) := \frac{\|\overline{r}(\overline{p})\|_\Sigma}{\tilde{z}_j\|\zeta\|_\Sigma}.$$

Moreover, the upper bound $Q_j(p)$ is well defined for any $p \in \mathcal{P}$ and, it follows from Assumption (C), that it varies continuously with commodity prices. Thus, the function $Q : \mathcal{P} \to \mathbb{R}$ defined by $Q(p) = \max_{j\in\mathcal{J}} Q_j(p)$ is continuous. Since $\mathcal{P}$ is compact, we conclude that there exists $\overline{Q} > 0$ such that, $\max_{j\in\mathcal{J}} \overline{q}_j < \overline{Q}$. □

We define $\overline{X} = 2(1 + \overline{Q})\overline{W}$.

Note that, for any $X > \overline{X}$ and $Q > \overline{Q}$, in the associated generalized game $\mathcal{G}(Q, X, \Theta)$ any player $h \in \mathcal{I}$ may demand in the first period the bundle used in the proof of Lemma 4. Thus, in this type of generalized game, the existence of an optimal plan satisfying the condi-

18. Notice that, by definition, $\tilde{z}_j$ depends only on primitive parameters of the economy (endowments and unitary financial payments).

tions of lemma above will imply that the unitary prices of assets are bounded from above by $\overline{Q}$.

The existence of equilibria in our economy is a consequence of the following result.

LEMMA 5. *Under Assumptions (A), (B) and (C), if* $(Q, X, \Theta) \gg (\overline{Q}, \overline{X}, \overline{\Theta})$*, then every Cournot-Nash equilibrium for* $\mathcal{G}(Q, X, \Theta)$ *is an equilibrium of the original economy.*

PROOF. Let $(\overline{p}, \overline{q}, (\overline{\eta}^i; i \in \mathcal{I}))$, where $\overline{\eta}^i = (\overline{x}^i, \overline{z}^i) \in \overline{\Gamma}^i$, be a equilibrium for the generalized game $\mathcal{G}(Q, X, \Theta)$, with $(Q, X, \Theta) \gg (\overline{Q}, \overline{X}, \overline{\Theta})$.

Step I: Market feasibility. Aggregating agent's first period budget constraints we have,

$$\overline{p}_0 \sum_{i \in \mathcal{I}} \left(\overline{x}_0^i - w_0^i \right) + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}^i} \overline{q}_j \overline{z}_j^i \leq 0.$$

It follows that, if $\sum_{i \in \mathcal{I}} \left(\overline{x}_{0,l}^i - w_{0,l}^i \right) > 0$, then the auctioneer $h(0)$ will choose the greater price for this good, $\overline{p}_l = 1$, and zero prices for the other goods and assets, making his objective function positive, which contradicts the inequality above. Therefore, $\sum_{i \in \mathcal{I}} \overline{x}_0^i \leq \sum_{i \in \mathcal{I}} w_0^i < W_0$. Analogously, if $\sum_{i \in \mathcal{I}(j)} \overline{z}_j^i > 0$, then the auctioneer $h(0)$ would choose the maximum price possible for this asset, i.e. $\overline{q}_j = Q > \overline{Q}$, which is a contradiction with the result of Lemma 4. Thus, for any $j \in \mathcal{J}$, $\sum_{i \in \mathcal{I}(j)} \overline{z}_j^i \leq 0$.

Since first period consumption is bounded from above by the aggregate endowment, which is less than X, it follows that budget constraints at $t = 0$ are satisfied with equality. Hence, the auctioneer $h(0)$ has an optimal value equal to zero. As a consequence, if $\sum_{i \in \mathcal{I}} \left(\overline{x}_{0,l}^i - w_{0,l}^i \right) < 0$, the auctioneer $h(0)$ would choose a zero price for the good l, a contradiction with the strictly monotonicity of preferences (Assumption (B)). Therefore, $\sum_{i \in \mathcal{I}} \overline{x}_0^i = W_0$. Furthermore, if $\sum_{i \in \mathcal{I}(j)} \overline{z}_j^i < 0$, the auctioneer would choose $\overline{q}_j = 0$, a contradiction with the strictly monotonicity of preferences. Then, market feasibility conditions hold at $t = 0$ in both physical and financial markets.

Using the market feasibility of $\left((\overline{x}^i, \overline{z}^i) ; i \in \mathcal{I} \right)$ at $t = 0$, and aggregating budget constraints at $s \in S$, we obtain that $\overline{p}_s \sum_{i \in \mathcal{I}} (\overline{x}_s^i - w_s^i) \leq 0$. Therefore, analogous arguments to those made above ensure that $\sum_{i \in \mathcal{I}} (\overline{x}_s^i - w_s^i) \leq 0$. This last property guarantees that budget constraints are satisfied as an equality in the state of nature s. Finally, if $\sum_{i \in \mathcal{I}} \left(\overline{x}_{s,l}^i - w_{s,l}^i \right) < 0$, then the auctioneer $h(s)$ would choose a zero price for the good $l \in \mathcal{L}$, which contradicts individual optimality under strictly monotonic preferences. We conclude that market feasibility also holds at each state of nature $s \in S$.

Step II. Optimality of individual allocations. Since market feasibility holds in physical markets, it follows that $\overline{x}_{0,l}^i < X$ and $\overline{x}_{s,l}^i < 2\overline{W}$, for any $(i, s, l) \in \mathcal{I} \times S \times \mathcal{L}$. Using Lemma 3, we have that for any $i \in \mathcal{I}$ and $j \in \mathcal{J}^i$, $|z_j^i| < \Theta$. Thus, for any $i \in \mathcal{I}$, $\overline{\eta}^i$ belongs on the interior of $K^i(X, \Theta)$.

Suppose that there exists another vector $\eta^i \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}^{\mathcal{J}^i}$ such that $V^i(\overline{p}, \eta^i) > V^i(\overline{p}, \overline{\eta}^i)$. Since for $\lambda \in (0,1)$ sufficiently small, $\eta^i(\lambda) := \lambda\eta^i + (1-\lambda)\overline{\eta}^i \in K^i(X, \Theta)$, the strictly concavity of $V^i(\overline{p}, \cdot)$ implies that $V^i(\overline{p}, \eta^i(\lambda)) > V^i(\overline{p}, \overline{\eta}^i)$, a contradiction with the optimality of $\overline{\eta}^i \in \overline{\Gamma}^i$. Therefore, for any $\eta^i \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}^{\mathcal{J}^i}$, $V^i(\overline{p}, \eta^i) \leq V^i(\overline{p}, \overline{\eta}^i)$, which proves the optimality of $\overline{\eta}^i \in B^i(\overline{p}, \overline{q})$ among the elementss in the agent i's budget set. Notice that, as was proved in Section 3, informational compatibility of consumption vectors follows from Assumption (A).[19] □

19. See the Proposition in Section 3.

Chapter 2

Trading Constraints

We build a general equilibrium model where agents are subject to endogenous trading constraints, making the access to financial trade dependent on prices and consumption decisions. Thus, in a framework compatible with the existence of endogenous financial segmentation and credit markets' exclusion, we provide several results of equilibrium existence. In the first family of results, we assume individuals can *super-replicate* financial payments buying durable commodities and/or investing in assets that give liquidity to all agents. In the second family, we suppose that there are agents with monotonic preferences which may compensate with increments in present demand the losses of well-being generated by reductions of future consumption. [1]

2.1 Introduction

The differentiated access to commodity or asset markets endogenously emerges due to regulatory or institutional considerations. As a consequence, several kinds of trading restrictions are observed in financial markets: margin calls, collateral requirements, short-sale constraints, consumption quotas or income-based access to funding, among others. With the aim of understanding the effects of those restrictions in competitive markets, a vast literature of general equilibrium has been developed. That research has given consideration to models where financial trade is restricted by fixed, price-dependent, or consumption-dependent portfolio constraints. Nevertheless, channels connecting prices with both portfolio constraints and consumption possibilities, have not thoroughly been addressed by the literature. The objective of this paper is to contribute in this direction.

The existence of competitive equilibria was deeply studied in incomplete markets models where agents are subject to exogenous portfolio constraints. The case of portfolio restrictions determined by linear equality constraints is addressed by Balasko, Cass, and Siconolfi [1990] for nominal assets, and by Polemarchakis and Siconolfi [1997] for real assets. When portfolio restrictions are determined by convex and closed sets containing zero, the case of nominal or numéraire assets is studied by Cass [1984, 2006], Siconolfi [1989], Cass, Siconolfi, and Villanacci [2001], Martins-da Rocha and Triki [2005], Won and Hahn [2007, 2012], Aouani

1. Cea-Echenique and Torres-Martínez [2014] is based on this chapter.

and Cornet [2009, 2011], and Cornet and Gopalan [2010]. In the same context, the case of real assets is analyzed by Radner [1972], Angeloni and Cornet [2006], and Aouani and Cornet [2011]. In general terms, these authors prove equilibrium existence under non-redundancy hypotheses over financial structures and/or financial survival requirements. Under these assumptions, individuals' allocations and asset prices can be endogenously bounded without inducing frictions in the model.

There are also several results that include price-dependent portfolio constraints in nominal or real assets markets. These models assume that financial constraints are determined by a finite number of inequalities, and use differentiable techniques to ensure the existence of equilibrium and to analyze its stability and local-uniqueness. In this context, equilibrium existence is addressed by Carosi, Gori, and Villanacci [2009] for numéraire asset markets with portfolio constraints, by Gori, Pireddu, and Villanacci [2013] for numéraire and real asset markets with borrowing constraints, and by Hoelle, Pireddu, and Villanacci [2012] for real asset markets with wealth-dependent credit limits.

In addition, Seghir and Torres-Martínez [2011] propose a model where trading constraints restrict the access to debt in terms of first-period consumption. Financial survival conditions are not required, and the relationship between financial access and individual consumption allows to include financial practices as collateralized borrowing. In order to prove equilibrium existence, they assume individuals may compensate with increments in present demand the losses on well-being generated by reductions of future consumption.

2.2 Credit Constraints

In this section,[2] we analyze the existence of equilibria in a two period economy with incomplete financial markets, where agents are subject to price-dependent credit constraints that affect the access to commodities and financial contracts. Furthermore, we make the financial segmentation compatible with the existence of a competitive equilibrium. Our approach is general enough to be compatible with incomplete market economies where there exist wealth-dependent and/or investment-dependent credit access, borrowing constraints precluding bankruptcy, commodity options with deposit requirements, and assets backed by physical collateral.

One of the main difficulties to ensure equilibrium existence with restricted participation is to ensure that asset prices can be bounded without compromising the continuity of individual's demands. To overcome this difficulty, some authors impose *financial survival conditions*, assuming that every agent has access to resources by short-selling any financial contract (see Angeloni and Cornet [2006], Won and Hahn [2007], and Aouani and Cornet [2009, 2011]).

Notwithstanding, as we want to include financial market segmentation, we need to follow alternative approaches to establish bounds for asset prices. For this reason, we focus on financial markets where assets payments can be *super-replicated* either by the deliveries of non-perishable commodities or by the promises of assets that all agents can short-sale. Thus,

2. Cea-Echenique and Torres-Martínez [2014] is based on this section.

there are three scenarios where equilibrium existence is guaranteed: (i) when individuals have access to borrow resources through a financial contract that makes positive payments at all states of nature where other assets pay; (ii) when all assets are real and promises are measured in units of non-perishable commodities; and (iii) when default is allowed and debts are backed by physical collateral.

Our model is described in the next section. In Section 2.2.2 we characterize trading constraints and Sections 2.2.3 and 2.2.4 are devoted to state our main results, whose proof are given in the Appendix.

2.2.1 A Model with Credit Constraints

We consider a two-period economy with uncertainty about the realization of a state of nature in the second period, which belongs to a finite set S. Let $\mathcal{S} = \{0\} \cup S$ be the set of states of nature in the economy, where $s = 0$ denotes the unique state at the first period.

There is a finite set $\mathcal{L}$ of perfectly divisible commodities, which are subject to transformation between periods and that can be traded in spot markets at prices $p = (p_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$. We model the transformation of commodities between periods by linear technologies $(Y_s)_{s\in S}$. Thus, a bundle $y \in \mathbb{R}_+^{\mathcal{L}}$ demanded at the first period is transformed, after its consumption and the realization of a state of nature $s \in S$, into the bundle $Y_s y \in \mathbb{R}_+^{\mathcal{L}}$.

There is a finite set $\mathcal{J}$ of financial contracts available for trade at the first period that make promises contingent to the realization of uncertainty. Let $q = (q_j)_{j\in\mathcal{J}} \in \mathbb{R}_+^{\mathcal{J}}$ be the vector of asset prices and denote by $R_j(p) = (R_{s,j}(p))_{s\in S} \in \mathbb{R}_+^{S}$ the vector of payments associated to asset $j \in \mathcal{J}$.[3]

For notation convenience, let $\mathbb{P} := \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}_+^{\mathcal{J}}$ be the space of commodity and asset prices, and let $\mathbb{E} := \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}^{\mathcal{J}}$ be the space of consumption and portfolio allocations.

There is a finite set $\mathcal{I}$ of consumers that trade assets in order to smooth their consumption. Each agent $i \in \mathcal{I}$ has a utility function $V^i : \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \to \mathbb{R}$ and endowments $w^i = (w^i_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$.

Each individual i is subject to trading constraints, which are determined by a correspondences $\Phi^i : \mathbb{P} \twoheadrightarrow \mathbb{E}$. Notice that, agents may be subject to endogenous borrowing constraints, as the access to liquidity can depends on prices, investment and consumption. We assume that there are no restrictions on investment, i.e., $\Phi^i(p,q) + \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times \mathbb{R}_+^{\mathcal{J}} \subseteq \Phi^i(p,q),\ \forall (p,q) \in \mathbb{P}$, $\forall i \in \mathcal{I}$.[4]

Given prices $(p,q) \in \mathbb{P}$, each agent i chooses a consumption bundle $x^i = (x^i_s)_{s\in\mathcal{S}}$ and a portfolio $z^i = (z^i_j)_{j\in\mathcal{J}}$ in her choice set $C^i(p,q)$, which is characterized by the set of allocations $(x^i, z^i) \in \mathbb{E}$ satisfying the trading constraint $(x^i, z^i) \in \Phi^i(p,q)$ and the following budget restrictions:

$$p_0 \cdot x^i_0 + q \cdot z^i \leq p_0 \cdot w^i_0; \qquad p_s \cdot x^i_s \leq p_s \cdot (w^i_s + Y_s x^i_0) + \sum_{j\in\mathcal{J}} R_{s,j}(p) z^i_j, \quad \forall s \in S.$$

3. Our financial structure is general enough to be compatible with several types of assets. For instance, to include a nominal asset j it is sufficient to assume that there is $(N_{s,j})_{s\in S} \in \mathbb{R}_+^{S}$ such that $R_{s,j} \equiv N_{s,j}, \forall s \in S$. To include a real asset k we can define payments $R_{s,k}(p) = p_s \cdot A_{s,k}$, $\forall s \in S$, where $(A_{s,k})_{s\in S} \in \mathbb{R}_+^{\mathcal{L}\times S}$.

4. At the cost of additional complexity, we include price-dependent investment constraints in Section 2.3.

DEFINITION 1. *A vector* $((\overline{p},\overline{q}),(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}}) \in \mathbb{P}\times\mathbb{E}^{\mathcal{I}}$ *is a competitive equilibrium for the economy with endogenous trading constraints when the following conditions hold:*

(i) Each agent $i\in\mathcal{I}$ *maximizes her preferences,* $(\overline{x}^i,\overline{z}^i) \in \underset{(x^i,z^i)\in C^i(\overline{p},\overline{q})}{\arg\max} V^i(x^i)$.

(ii) Individuals' plans are market feasible,

$$\sum_{i\in\mathcal{I}}(\overline{x}_0^i,(\overline{x}_s^i)_{s\in S},\overline{z}^i) = \sum_{i\in\mathcal{I}}(w_0^i,(w_s^i+Y_s w_0^i)_{s\in S},0).$$

Our objective is to determine conditions that make price-dependent trading constraints $\{\Phi^i\}_{i\in\mathcal{I}}$ compatible with equilibrium existence, even in the presence of *credit market segmentation*, in the sense of the following definition.

DEFINITION 2. *There exists credit market segmentation in the economy with endogenous trading constraints when there are contracts that not all agents can short-sale, i.e.,*

$$\{j\in\mathcal{J}: \exists i\in\mathcal{I},\ (x^i,z^i)\in\Phi^i(p,q) \implies z_j^i\geq 0,\ \forall (p,q)\in\mathbb{P}\}\neq\emptyset.$$

Notice that, the existence of credit market segmentation is incompatible with *financial survival*, which requires that independent of prices all agents have access to some amount of liquidity by selling endowments or financial contracts, i.e.,

$$\bigcap_{(p,q)\in\mathbb{P}\setminus\{0\}}\{i\in\mathcal{I}:\ \exists (x^i,z^i)\in\Phi^i(p,q),\ p_0\cdot w_0^i - q\cdot z^i > 0\} = \mathcal{I}.$$

The following examples illustrate the generality of our approach to restricted participation. In particular, the first three examples never satisfy financial survival.

EXAMPLE 1 (INCOME-BASED CREDIT ACCESS)

Given $(p,q)\in\mathbb{P}$ and $i\in\mathcal{I}$, assume that $(x^i,z^i)\in\Phi^i(p,q) \implies z_j^i \in [\min\{p_0\cdot(\tau - w_0^i),0\},+\infty)$, where $\tau\in\mathbb{R}_+^{\mathcal{L}}$. Then, agent i can short-sale asset j if and only if the value of her first period endowment is greater than the threshold $p_0\tau$. That is, j is a credit line available for high income agents. Alternatively, if we suppose that $(x^i,z^i)\in\Phi^i(p,q) \implies z_k^i\in[\min\{p_0\cdot(w_0^i-\tau),0\},+\infty)$, then only low-income agents can short-sale asset k. □

EXAMPLE 2 (EXCLUSIVE CREDIT LINES)

We can consider the case where the access to credit depends on the amount of investment in some financial contracts. That is, there exists $j\in\mathcal{J}$ and $\mathcal{J}'\subset\mathcal{J}\setminus\{j\}$ such that,

$$(x^i,z^i)\in\Phi^i(p,q) \qquad \implies \qquad z_j^i\in\left[\min\left\{K-\sum_{k\in\mathcal{J}'}q_k z_k^i,0\right\},+\infty\right).$$

Hence, only investors that expend an amount greater than K in assets belonging to $\mathcal{J}'$ have access to short-sale the financial contract j. □

Example 3 (Commodity Options)

Let $j \in \mathcal{J}$ be a financial contract such that, for every $(p,q) \in \mathbb{P}$,

$$R_{s,j}(p) = \max\{Y_s y - K, 0\},\ \forall s \in S; \qquad (x^i, z^i) \in \Phi^i(p,q) \implies \kappa\, p_0 \cdot y + \min\{z^i_j, 0\} \geq 0,$$

where $y \in \mathbb{R}^{\mathcal{L}}_{+}$, $K > 0$ and $\kappa \in [0,1)$. Then, j is a commodity option that gives the right to buy in the second period, at a strike price K, the bundle obtained by the transformation of y through time. To short-sell this option, agents are required to buy a portion κ of y as guarantee. □

Example 4 (Debt Constraints)

If there is $\kappa \in (0,1)$ such that, for any $(p,q) \in \mathbb{P}$ and for some $i \in \mathcal{I}$,

$$(x^i, z^i) \in \Phi^i(p,q) \qquad \Longrightarrow \qquad \kappa p_s \cdot (w^i_s + Y_s x^i_0) + \sum_{j \in \mathcal{J}} R_{s,j}(p) \min\{z^i_j, 0\} \geq 0,\ \forall s \in S,$$

then agent i's trading constraints ensure that her debt is not greater than an exogenously-fixed portion of physical-resources' value. Notice that, if a portion $\rho > \kappa$ of physical resources can be garnished in case of bankruptcy, the above restriction ensures that i honors her commitments.

□

2.2.2 Assumptions

Assumption (A1)

(i) For any agent $i \in \mathcal{I}$, V^i is continuous, strictly increasing and strictly quasi-concave. [5]

(ii) For any agent $i \in \mathcal{I}$, $(W^i_s)_{s \in \mathcal{S}} := (w^i_0, (w^i_s + Y_s w^i_0)_{s \in S}) \gg 0$.

(iii) Asset payments are continuous functions of prices with $R_j(p) \neq 0$, $\forall j \in \mathcal{J}$, $\forall p \gg 0$.

Assumption (A2)

The correspondences $\{\Phi^i\}_{i \in \mathcal{I}}$ are lower hemicontinuous with closed graph and convex values. In addition, agents are not burden to trade assets, i.e. $(0,0) \in \bigcap_{(p,q) \in \mathbb{P}} \Phi^i(p,q)$, $\forall i \in \mathcal{I}$.

Under Assumptions (A1)-(A2) individuals' choice set correspondences vary continuously with prices and, therefore, they do not compromise the continuity of individual demands (see Lemma 1 in the Appendix).

Notice that, as a particular case of our framework, we can have trading constraints that are independent of prices. That is, for every agent $i \in \mathcal{I}$, $\Phi^i(p,q) = \mathbb{R}^{\mathcal{L} \times \mathcal{S}}_{+} \times Z^i$, $\forall (p,q) \in \mathbb{P}$, where $Z^i + \mathbb{R}^{\mathcal{J}}_{+} \subseteq Z^i$. In this situation, Assumption (A2) is satisfied if and only if $\{Z^i\}_{i \in \mathcal{I}}$ are closed and convex sets containing zero. Alternatively, we can consider price-dependent

5. Strictly quasi-concavity of V^i requires that $V^i(\lambda x^i + (1-\lambda) y^i) > \min\{V^i(x^i), V^i(y^i)\}$ when $V^i(x^i) \neq V^i(y^i)$.

borrowing constraints by assuming that $\Phi^i(p,q) = \{(x^i,z^i) \in \mathbb{E} : z^i + g^i_k(p,q) \geq 0, \ \forall k \in \{1,\ldots,m_i\}\}$, $\forall((p,q),i) \in \mathbb{P} \times \mathcal{I}$, where $g^i_k : \mathbb{P} \to \mathbb{R}^{\mathcal{J}}_+$, $\forall k \in \{1,\ldots,m_i\}$. In this context, (A2) holds if and only if $\{g^i_k\}_{1\leq k \leq m_i}$ are continuous functions for every $i \in \mathcal{I}$.

Restrictions on trading constraints are also imposed by assumptions over the correspondence of *attainable allocations* $\Omega : \mathbb{P} \twoheadrightarrow \mathbb{E}^{\mathcal{I}}$, defined as the set-valued mapping that associates prices with market feasible allocations satisfying individuals' budget and trading constraints, i.e.,

$$\Omega(p,q) := \left\{((x^i,z^i))_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} C^i(p,q) : \sum_{i\in\mathcal{I}}(x^i,z^i) = \sum_{i\in\mathcal{I}}((W^i_s)_{s\in\mathcal{S}},0)\right\}.$$

ASSUMPTION (A3)
For every compact set $\mathbb{P}' \subseteq \mathbb{P}$, $\bigcup_{(p,q)\in\mathbb{P}' :\, (p,q)\gg 0} \Omega(p,q)$ *is bounded.*

This assumption holds when $\mathcal{J}$ is composed by non-redundant nominal assets, by collateralized assets, or when agents are subject to exogenous short-sale constraints—i.e., when for every $i \in \mathcal{I}$ there exists $m \in \mathbb{R}^{\mathcal{J}}_+$ such that $(x^i,z^i) \in \Phi^i(p,q) \implies z^i \geq -m$, $\forall(p,q) \in \mathbb{P}$.

2.2.3 Equilibrium Existence

Our first result ensures the compatibility between equilibrium and credit segmentation when there is an asset that all agents can short-sale,[6] and whose payments are positive whenever one of the remaining contracts has non-trivial promises. Thus, for instance, there is equilibria when all agents have access to borrowing through a risk-free asset.

THEOREM 1. *Under Assumptions (A1)-(A3), assume that all agents can short-sale an asset* j *such that* $(R_{s,j}(p))_{s\in S_j} \gg 0$, $\forall p \in \mathbb{R}^{\mathcal{L}}_+ \times (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$, *where* $S_j := \{s \in S : \exists k \neq j, \ R_{s,k} \neq 0\}$. *Then, there is a competitive equilibrium.*

To ensure the equilibrium existence it is crucial to find endogenous bounds for individual allocations and prices. Individual allocations can be bounded as a consequence of Assumption (A3). Thus, the main difficulty to prove Theorem 1 is to bound prices without induce frictions.

When all agents can short-sale an asset, we can normalize it price jointly with first-period commodity prices without compromising the continuity of individual's demand correspondences. In addition, if this asset has positive payments when remaining contracts make promises, then using a long position on it we can *super-replicate* deliveries of any other financial contract (see Lemma 3 in the Appendix). Thus, by a non-arbitrage argument, we can prove that asset prices are endogenously bounded from above.

Let $\mathcal{L}^* := \{l \in L : Y_s(l,l) > 0, \ \forall s \in \bigcup_j S_j\}$ be the set of commodities that not-fully depreciate at the states of nature where assets make non-trivial promises. The following result guarantees equilibrium existence in real asset markets with credit segmentation.

6. Notice that, all agents can short-sale an asset j if for every $(p,q) \in \mathbb{P}$ there is $\delta > 0$ such that $-\delta\vec{e}_j \in \bigcap_{i\in\mathcal{I}} \Phi^i(p,q)$, where $\vec{e}_j \in \mathbb{E}$ is the individual allocation composed by just one unit of asset j.

THEOREM 2. *Under Assumptions (A1)-(A3), there is a competitive equilibrium if assets are real and promises are given in terms of commodities in $\mathcal{L}^*$.*

A particular case where the requirements above hold is when asset payments are given by bundles of non-perishable commodities.[7] It follows that, under the conditions of Theorem 2, financial contracts' deliveries can be super-replicated by a bundle of commodities. This implies that asset prices can be bounded from above endogenously and without induce frictions on the economy.

2.2.4 Restricted Participation in Collateralized Asset Markets

Our model is general enough to be compatible with the inclusion of default and non-recourse collateralized assets.[8] That is, we can extend the model of Geanakoplos and Zame [2013] to include financial market segmentation and price-dependent trading constraints.

More precisely, assume that each $j \in \mathcal{J}$ is characterized by a pair $(C_j, (D_{s,j}(p))_{s\in S})$, where $C_j = (C_{j,l})_{l\in\mathcal{L}} \in \mathbb{R}^{\mathcal{L}}_+ \setminus \{0\}$ is the collateral guarantee, and $(D_{s,j}(p))_{s\in S} \in \mathbb{R}^S_+$ are the state contingent promises. Borrowers are required to pledge the associated collateral, i.e., for any $(x^i, z^i) \in \Phi^i(p, q)$ we have that $x^i_0 \geq \sum_{j\in\mathcal{J}} C_j \max\{-z^i_j, 0\}$. In addition, as the only enforcement in case of default is the seizure of collateral guarantees, borrowers give strategic default, delivering the minimum between collateral value and promises, i.e., $R_{s,j}(p) := \min\{D_{s,j}(p), p_s Y_s C_j\}$, $\forall s \in S$.

Since payments associated to non-recourse collateralized loans can be super-replicated by the collateral bundle, we can endogenously bound asset prices by the collateral cost (see Appendix). This allow us to prove equilibrium existence without requiring neither financial survival conditions nor the credit access to a risk-free asset.

THEOREM 3. *Under Assumptions (A1)-(A3) there is an equilibrium in collateralized asset markets.*

Notice that Theorems 2 and 3 are compatible with an extreme form of financial segmentation: the exclusion of some agents from credit markets.[9]

2.2.5 Remarks

As we want to include financial segmentation, our results of equilibrium existence do not rely on financial survival conditions. Thus, we need to determine endogenous upper bounds for financial prices without induce frictions on the economy. Based on the idea that

7. A commodity $l \in \mathcal{L}$ is non-perishable when $Y_s(l, l) > 0$, $\forall s \in S$.

8. Non-recourse collateralized assets are promises backed by collateral such that the only payment enforcement mechanism in case of default is given by the seizure of collateral guarantees. Notice that, in the absence of payment enforcement mechanisms over collateral repossession, the monotonicity of preferences guarantees that borrowers of a collateralized loan always deliver the minimum between promises and collateral values. Therefore, lenders that finance these loans perfectly foresight the payments that they will receive. Hence, as in Geanakoplos and Zame [2013], we can capture with a same financial contract both the collateralized line of credit and the collateralized loan obligation (CLO) that passthrough the payments made by borrowers.

9. See Definition 3 and 4 when investment constraints are considered.

asset prices can be bounded when their promises can be *super-replicated* at a finite cost, we propose three ways to ensure the existence of a competitive equilibrium: allowing individuals to have access to borrow resources through a contract that makes positive payments at states of nature where remaining assets make payments; assuming that assets make real promises in terms of non-perishable commodities; or including default burdening borrowers to pledge physical collateral.

2.3 General trading constraints

In this section, we analyze the existence of equilibria in a two period economy with incomplete financial markets, where agents are subject to price-dependent trading constraints that affect the access to commodities and financial contracts. Furthermore, we make the financial segmentation and exclusion of debt markets compatible with the existence of a competitive equilibrium. Our approach is general enough to be compatible with incomplete market economies where there exist wealth-dependent financial access, investment-dependent credit access, borrowing constraints precluding bankruptcy, security exchanges, commodity options with deposit requirements, and/or assets that are backed by financial collateral.

Two results of equilibrium existence are developed. First, we prove that a competitive equilibrium exists when individuals can *super-replicate* financial payments buying durable commodities and investing in assets that give liquidity to all agents (Theorem 4). As particular cases, we obtain results of equilibrium existence in markets where financial survival conditions hold or where assets are backed by physical collateral. Secondly, we prove that there is an equilibrium when there are agents with monotonic preferences that can increase their present demand to compensate any loss of utility generated by a reduction on future consumption (Theorem 5). In particular, we extend the model and the results of Seghir and Torres-Martínez [2011] to be able to allow price-dependent trading constraints that affect the access to both debt and investment.

A generalization of model given in the previous section, including investment constraints is described in the next section. In Section 2.3.2 we characterize trading constraints and, more precisely for the case of super-replication in Section 2.3.3. We state our main result of the chapter in Section 2.3.3. Section 2.3.4 develops the framework when impatience on agents' preferences is considered and states our second main result. Finally, Section 2.4 concludes. The proofs of our results are given in an Appendix, Section 2.5.

2.3.1 A model with credit and investment constraints

We take as given the model developed in Section 2.2.1, but generalizing the trading constraints to be determined by the correspondence $\Phi^i : \mathbb{P} \twoheadrightarrow \mathbb{E}$ for each $i \in \mathcal{I}$. That is, in this section we are giving room to investment constraints.

One of our objectives is to determine conditions that make price-dependent trading constraints $\{\Phi^i\}_{i\in\mathcal{I}}$ compatible with equilibrium existence. Another one is to have within our findings equilibrium existence results for economies where *financial market segmentation* and *exclusion of credit markets* is observed. The former is a generalization of the concept stated in Definition 2 including investment segmentation, the latter is given when there are agents without access to liquidity through financial contracts.

Definition 3. *There exists financial market segmentation when there are contracts that not all agents can trade, i.e.,*

$$\{j \in \mathcal{J} : \exists i \in \mathcal{I},\ \ (x^i, z^i) \in \Phi^i(p,q) \implies z^i_j = 0,\ \forall (p,q) \in \mathbb{P}\} \neq \emptyset.$$

DEFINITION 4. *There exists exclusion of credit markets when there are agents without access to liquidity through financial contracts, i.e.,*

$$\{i \in \mathcal{I} : \ (x^i, z^i) \in \Phi^i(p,q) \implies z^i \geq 0, \ \forall (p,q) \in \mathbb{P}\} \neq \emptyset.$$

We impose the following assumptions about agents' characteristics and financial payments:

ASSUMPTION (A1')
For any $i \in \mathcal{I}$, the following properties hold:
(i) V^i is continuous and strictly quasi-concave. [10]
(ii) $W^i = (W^i_s)_{s\in\mathcal{S}} := (w^i_0, (w^i_s + Y_s w^i_0)_{s\in S}) \gg 0$.
(iii) Asset payments are continuous functions of prices with $R_j(p) \neq 0$, $\forall j \in \mathcal{J}$, $\forall p \gg 0$.
(iv) V^i is strictly increasing in at least one commodity at any state of nature.

Furthermore, for each $(s,l) \in \mathcal{S} \times \mathcal{L}$, there is an agent whose utility function is strictly increasing in commodity l at state of nature s.

The requirements imposed in Assumption (A1') are classical. Assumption (A1')(iii) guarantees that asset payments do not compromise the continuity of choice set correspondences. Also, financial payments are assumed non-trivial when commodity prices are strictly positive.

2.3.2 Basic Assumptions on Trading Constraints

In this section we introduce the basic assumptions over trading constraints. We depart with hypotheses that ensure that the well behavior of choice sets is not affected by trading constraints. To shorten notations, given $j \in \mathcal{J}$, let $\widehat{e}_j \in \mathbb{E}$ be the plan composed by one unit of investment in j.

ASSUMPTION (A2')
(i) For any agent $i \in \mathcal{I}$, Φ^i is lower hemicontinuous with closed graph and convex values.
(ii) For any $i \in \mathcal{I}$ and $(p,q) \in \mathbb{P}$ the following properties hold:
(a) If $(x^i, z^i) \in \Phi^i(p,q)$, then $(y^i, z^i) \in \Phi^i(p,q)$, $\forall y^i \geq x^i$. Also, $(0,0) \in \Phi^i(p,q)$.
(b) For every $j \in \mathcal{J}$ there is an agent $h \in \mathcal{I}$ such that $\Phi^h(p,q) + \widehat{e}_j \subseteq \Phi^h(p,q)$.

Under Assumption (A2')(i) trading constraints do not compromise the continuity or the convexity of choice set correspondences. Moreover, agents are not required to trade financial contracts if they want to demand a portion of initial endowments or increase consumption departing from a trading feasible allocation (Assumption (A2')(ii)(a)). Assumption (A2')(ii)(b)

10. Strong quasi-concavity of V^i requires that $V^i(\lambda x^i + (1-\lambda) y^i) > \min\{V^i(x^i), V^i(y^i)\}$ when $V^i(x^i) \neq V^i(y^i)$.

requires that for any financial contract there is at least one agent that can increase her long position on it.[11]

Example 5 (Exogenous Portfolio Constraints)

Assume that, for every $i \in \mathcal{I}$, $\Phi^i(p,q) = \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times Z^i$, $\forall (p,q) \in \mathbb{P}$, where $Z^i \subseteq \mathbb{R}^{\mathcal{J}}$. Then, Assumption (A2')(i) is satisfied if and only if $\{Z^i\}_{i\in\mathcal{I}}$ are closed and convex sets. Assumption (A2')(ii)(a) holds if and only if $0 \in \bigcap_{i\in\mathcal{I}} Z^i$. Assumption (A2')(ii)(b) holds if and only if, for each asset j there is an agent i such that $Z^i + \vec{e}_j \subseteq Z^i$, where $\vec{e}_j$ is the j-th canonical vector of $\mathbb{R}^{\mathcal{J}}$.

Notice that, (A2') is satisfied when $\{Z^i\}_{i\in\mathcal{I}}$ are linear spaces and $\{\vec{e}_j\}_{j\in\mathcal{J}} \subset \bigcup_{i\in\mathcal{I}} Z^i$. Also, if trading constraints only restrict the access to credit (i.e., $Z^i + \mathbb{R}_+^{\mathcal{J}} \subseteq Z^i$, $\forall i \in \mathcal{I}$), then (A2') hold if and only if $\{Z^i\}_{i\in\mathcal{I}}$ are closed and convex sets containing zero. □

Example 6 (Price-Dependent Borrowing Constraints)

Assume that, for every $i \in \mathcal{I}$,

$$\Phi^i(p,q) = \{(x^i, z^i) \in \mathbb{E} : \, z^i + g_k^i(p,q) \geq 0, \ \forall k \in \{1,\dots,m_i\}\}, \quad \forall (p,q) \in \mathbb{P},$$

where $m_i \in \mathbb{N}$ and $g_k^i = (g_{k,j}^i) : \mathbb{P} \to \mathbb{R}_+^{\mathcal{J}}$, $\forall k \in \{1,\dots,m_i\}$. In this context, Assumption (A2') is satisfied if and only if $\{g_k^i\}_{1\leq k\leq m_i}$ are continuous for every $i \in \mathcal{I}$. □

2.3.3 Super Replication

The first approach to equilibrium existence we take is given by a super replication argument. Essentially, the set of attainable allocations is bounded in the sense of (A3) given in Section 2.2.2. In order to find endogenous bounds on asset prices, we generalize the argument given when only credit constraints are considered (see for instance the proof of Theorem 1).

Bounds on Attainable Allocations

In our results of equilibrium existence for general trading constraints, we require sets of attainable allocations to be uniformly bounded in the sense of (A3) or, in section 2.3.4, (A3').

Bounds on Asset Prices

One of the main steps in any proof of equilibrium existence is to ensure that endogenous variables can be bounded without inducing frictions over individual demand correspondences. Under Assumption (A3) we can obtain natural upper bounds for individual allocations. However, it is also necessary to ensure that prices can be bounded. With this objective, some authors impose *financial survival conditions*, assuming that every agent has access to resources by short-selling any financial contract (see Angeloni and Cornet [2006], Won and Hahn [2007], and Aouani and Cornet [2009, 2011]). Notwithstanding, as we want to include

11. Under Assumption (A2')(i), (A2')(ii)(a) is equivalent to require that: $\forall j \in \mathcal{J}$, $\exists i \in \mathcal{I}:$ $\Phi^i(p,q) + \delta\widehat{e}_j \subseteq \Phi^i(p,q)$, $\forall \delta > 0$.

financial market segmentation, we need to follow alternative approaches to establish bounds for asset prices.

Before discussing these alternatives, we introduce some concepts.

DEFINITION 5. *A financial contract $j \in \mathcal{J}$ is an ultimate source of liquidity when, given $(p,q) \in \mathbb{P}$ with $p_0 = 0$, there exists $(\theta(p,q), \zeta(p,q)) \in (0,1) \times (0,1)$ such that, each agent i can short-sell $\zeta(p,q)$ units of asset j in order to demand the bundle $((1+\theta(p,q))W_0^i, ((1-\theta(p,q))W_s^i)_{s\in S})$.*

Thus, agents have access to liquidity even when they cannot obtain resources by selling physical endowments. It follows that, under Assumptions (A2'), an ultimate source of liquidity is a contract that any agent can short-sale in order to make trading-feasible a small increment on current consumption in exchange of a reduction on future demand. For notation convenience, let $\mathcal{J}_u$ be the (possibly empty) maximal subset of $\mathcal{J}$ composed by contracts that are ultimate sources of liquidity.

ASSUMPTION (A4)
(i) Given $j \in \mathcal{J}_u$, for every $i \in \mathcal{I}$ we have that $\Phi^i(p,q) + \widehat{e}_j \subseteq \Phi^i(p,q)$, $\forall (p,q) \in \mathbb{P}$.
(ii) Given $j \notin \mathcal{J}_u$, for every $i \in \mathcal{I}$ and $(x^i, z^i) \in \Phi^i(p,q)$,

$$(x^i, z^i) - \delta \widehat{e}_j \in \Phi^i(p,q),\ \forall \delta \in [0, \max\{z^i_j, 0\}], \quad \forall (p,q) \in \mathbb{P}.$$

Assumption (A4)(i) requires that all agents have access to invest in each asset belonging to $\mathcal{J}_u$, while (A4)(ii) holds if and only if long positions for assets in $\mathcal{J} \setminus \mathcal{J}_u$ can be reduced without compromising the trading feasibility of allocations.

We affirm that, under Assumptions (A1')-(A2'), (A3)-(A4), if $\mathcal{J}_u$ satisfies the super-replication property defined below, then there are endogenous bounds for asset prices.

DEFINITION 6. *Agents can super-replicate financial payments investing in contracts $\mathcal{J}_u$ and buying commodities when for any compact set $\mathbb{P}_1 \subset (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$ there exists $(\widehat{x}, (\widehat{z}_k)_{k\in\mathcal{J}_u}) \geq 0$ such that,*

$$\sum_{j\notin\mathcal{J}_u} R_{s,j}(p_s) > 0 \quad \Longrightarrow \quad \sum_{j\notin\mathcal{J}_u} R_{s,j}(p_s) < p_s Y_s \widehat{x} + \sum_{k\in\mathcal{J}_u} R_{s,k}(p_s)\widehat{z}_k, \qquad \forall s \in S,\ \forall (p_s)_{s\in S} \in \mathbb{P}_1.$$

Intuitively, if agents can super-replicate financial payments investing in contracts $\mathcal{J}_u$ and buying commodities, then the price of any traded contract $j \notin \mathcal{J}_u$ can be bounded from above in terms of $(p_0, (q_k)_{k\in\mathcal{J}_u})$. In addition, since all agents have access to some amount of credit through any $k \in \mathcal{J}_u$, it is possible to normalize prices $(p_0, (q_k)_{k\in\mathcal{J}_u})$ without inducing frictions on individual demand correspondences (see Lemma 3B for detailed arguments).

Notice that, the continuity of assets payments (Assumption (A1')(iii)) ensures that any contract j satisfying $(R_{s,j}(p_s))_{s\in S} \gg 0$, $\forall (p_s)_{s\in S} \in (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$ super-replicates the payments of the remaining assets, e.g., when j is a risk-free nominal asset, i.e., $R_{s,j} \equiv 1$, $\forall s \in S$.

Examples of Trading Constraints

In this section we present some examples of trading constraints allowing: wealth-dependent financial access, investment-dependent credit access, debt constraints precluding bankruptcy, security exchanges, commodity options with deposit requirements, and assets that are backed by physical or financial collateral.

Example 8 (Income-Based Financial Access)

Given $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}$, assume there exists an asset j such that,

$$(x^i, z^i) \in \Phi^i(p,q) \qquad \Longrightarrow \qquad z^i_j \in [\min\{p_0 \cdot (\tau_1 - w^i_0), 0\}, \max\{p_0 \cdot (w^i_0 - \tau_2), 0\}],$$

where $\tau_1, \tau_2 \in \mathbb{R}^{\mathcal{L}}_+$. Then, agent i can short-sale asset j if and only if the value of her first period endowment is greater than $p_0\tau_1$. Analogously, she can invest in asset j if and only if her first period endowment is greater than $p_0\tau_2$. That is, j can only be traded by high income agents.

If we suppose that for some $k \in \mathcal{J}$, $(x^i, z^i) \in \Phi^i(p,q) \implies z^i_k \in [\min\{p_0 \cdot (w^i_0 - \tau_1), 0\}, +\infty]$, then all agents can invest on k, but only low-income agents can short-sale it. □

Example 9 (Security Exchanges)

Suppose that we split the sets of agents and financial contracts such that,

$$\mathcal{I} = \bigcup_{r=1}^{a} \mathcal{I}_r, \qquad \mathcal{J} = \bigcup_{r=1}^{b} \mathcal{J}_r,$$

and assume that for every $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}_r$,

$$(x^i, z^i) \in \Phi^i(p,q) \qquad \Longrightarrow \qquad \begin{cases} z^i_j \geq 0, & \forall j \in G_+(\mathcal{I}_r); \\ z^i_j \leq 0, & \forall j \in G_-(\mathcal{I}_r); \\ z^i_j = 0, & \forall j \notin G_+(\mathcal{I}_r) \cup G_-(\mathcal{I}_r), \end{cases}$$

where $G_+, G_- : \{\mathcal{I}_1, \dots, \mathcal{I}_a\} \twoheadrightarrow \{\mathcal{J}_1, \dots, \mathcal{J}_b\}$ are non-empty valued correspondences.

Then, we obtain a structure of *exchanges*, $\{\mathcal{J}_1, \dots, \mathcal{J}_b\}$, where an agent $i \in I_r$ can only short-sale assets that are available in the exchanges belonging to $G_-(\mathcal{I}_r)$, whereas she can only invest in assets traded in exchanges belonging to $G_+(\mathcal{I}_r)$. Notice that the markets of debt and investment are not necessarily segmented, as $G_+(\mathcal{I}_r)$ and $G_-(\mathcal{I}_r)$ are not required to be disjoint. Also, by Assumption (A4)(i), if $j \in \mathcal{J}_u$, then $j \in \bigcap_{r=1}^{a} (G_+(\mathcal{I}_r) \cap G_-(\mathcal{I}_r))$.

Since the same agent can participate in several exchanges—because G_+ and G_- are not necessarily singled-valued—we obtain a model of exchanges with heterogeneous participation, multi-membership, and price-dependent trading constraints. [12] □

12. Faias and Luque [2013] address an equilibrium model with exchanges where individual preferences satisfy the kind of impatience condition imposed by Seghir and Torres-Martínez [2011]. Different to the example above, they allow cross listing and transactions fees.

EXAMPLE 10 (COLLATERALIZED ASSETS WITH FINANCIAL COLLATERAL)

We can include non-recourse collateralized assets.[13] Indeed, a collateralized contract j can be characterized by a pair $(C_j, (D_{s,j}(p_s))_{s\in S})$, where $C_j = (C_{j,l})_{l\in\mathcal{L}} \in \mathbb{R}^{\mathcal{L}}_+ \setminus \{0\}$ is the collateral guarantee, and $(D_{s,j}(p_s))_{s\in S} \in \mathbb{R}^S_+$ are the state contingent promises, which determine payments $R_{s,j}(p_s) = \min\{D_{s,j}(p_s), p_s Y_s C_j\}$, $\forall s \in S$. Since borrowers are required to pledge the associated collateral, we assume that, given $(x^i, z^i) \in \Phi^i(p, q)$, the following properties hold

$$x^i_0 + C_j z^i_j \geq 0, \quad \text{and} \qquad ((x^i_0 - \alpha C_j, (x^i_s)_{s\in S}), z^i) + \alpha \widehat{e}_j \in \Phi^i(p, q), \ \forall \alpha \in [0, -\min\{z^i_j, 0\}].$$

Thus, individual consumption plans include the required collateral guarantees and any reduction in short positions reduces the requirements of collateral. That is, there is no *cross-collateralization* of payments, i.e., several loans backed by the same collateral. Notice that, payments associated to non-recourse collateralized loans can be super-replicated by the collateral bundle.

To include assets backed by financial collateral, we can assume that there are $j, k \in \mathcal{J}$ such that, given $(p, q) \in \mathbb{P}$ and $i \in \mathcal{I}$, for any $s \in S$ we have that $R_{s,j} = \min\{T_{s,j}(p_s), R_{s,k}(p_s)\}$ and

$$(x^i, z^i) \in \Phi^i(p, q) \qquad \Longrightarrow \quad \exists (\theta^i, \varphi^i) \in \mathbb{R}^{\mathcal{J}}_+ \times \mathbb{R}^{\mathcal{J}}_+ : \quad \theta^i_k \geq \varphi^i_j \ \wedge \ z^i = \theta^i - \varphi^i.$$

where $T_{s,j} : \mathbb{R}^{\mathcal{L}}_+ \to \mathbb{R}_+$. Then, each unit of asset j promises to deliver an amount $T_{s,j}(p_s)$ at a state of nature s, and it is backed by one unit of financial contract k in case of default.[14] □

Equilibrium Existence

The following result ensures the compatibility between equilibrium and markets segmentation.

THEOREM 4. *Under Assumptions (A1')-(A2'), and (A3)-(A4), if agents can super-replicate financial payments investing in assets in $\mathcal{J}_u$ and buying commodities, then there exists a competitive equilibrium.*

The super-replication property trivially holds when there is an ultimate source of liquidity with strictly positive payments, when $\mathcal{J} = \mathcal{J}_u$ or when there is a bundle of commodities that

13. In the absence of payment enforcement mechanisms over collateral repossession, the monotonicity of preferences guarantees that borrowers of a collateralized loan always deliver the minimum between promises and collateral values. Therefore, lenders that finance these loans perfectly foresight the payments that they will receive. Hence, as in Geanakoplos and Zame [2013], we can capture with a same financial contract both the collateralized line of credit and the collateralized loan obligation (CLO) that passthrough the payments made by borrowers.

14. Notice that, as k is used as financial collateral, the investment in it may not be reduced without affecting the trading feasibility. Thus, under the conditions of Theorem 1, $k \in \mathcal{J}_u$.

super-replicates financial payments. Thus, departing from Theorem 1, we can obtain the following results.

COROLLARY 1. *Under Assumptions (A1')-(A2'), and (A3)-(A4), if there exists* $j \in \mathcal{J}_u$ *such that* $(R_{s,j}(p_s))_{s\in S} \gg 0$, $\forall (p_s)_{s\in S} \in (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$, *then there exists a competitive equilibrium.*

COROLLARY 2. *Under Assumptions (A1')-(A2') and (A3), there is a competitive equilibrium if all assets are ultimate sources of liquidity (i.e., financial survival holds).*

COROLLARY 3. *Under Assumptions (A1')-(A2'), and (A3)-(A4)(ii), there is a competitive equilibrium if one of the following conditions is satisfied:*

(i) all assets are backed by physical collateral;

(ii) all assets are real and claims are measure in units of non-perishable commodities.

In particular, we extend Geanakoplos and Zame [2013] to include financial market segmentation and price-dependent trading constraints. Notice that, the results above are compatible with the exclusion of some agents from credit markets only if $\mathcal{J}_u = \emptyset$.

2.3.4 Impatience

In this section we present a different approach to find endogenous bounds for asset prices. We use an argument that was introduced by Seghir and Torres-Martínez [2011] in the case of credit constraints.

Bounds on Attainable Allocations

In order to dispense of replication arguments to ensure equilibrium existence, we have to strengthen the requirements of Assumption (A3) to bound the set of attainable allocations.

ASSUMPTION (A3')
If the projection of $\mathbb{P}' \subseteq \mathbb{P}$ *on* $\mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+$ *is a compact set, then*

$$\bigcup_{(p,q)\in\mathbb{P}':\,(p,q)\gg 0} \Omega(p,q) \qquad \textit{is bounded.}$$

Assumption (A3'), which is stronger than (A3), holds when $\mathcal{J}$ is composed by non-redundant nominal assets, by collateralized assets, or when agents are subject to exogenous short-sale constraints—i.e., when for every $i \in \mathcal{I}$ there exists $m \in \mathbb{R}^{\mathcal{J}}_+$ such that $(x^i, z^i) \in \Phi^i(p,q) \implies z^i \geq -m$, $\forall (p,q) \in \mathbb{P}$.

The existence of upper bounds on attainable allocations is directly related with the non-redundancy of the financial structure. That is, the non-existence of unbounded sequences of trading admissible portfolios that do not generate commitments.

To formalize this relationship, given $(p,q)\in\mathbb{P}$ and $i\in\mathcal{I}$, define

$$\begin{aligned}\mathcal{A}_0^i(p,q) &:= \Big\{z^i\in\mathbb{R}^{\mathcal{J}}\setminus\{0\}:\ q\cdot z^i=0\ \wedge\ R(p)z^i=0\ \wedge\ (W^i,\delta z^i)\in\Phi^i(p,q),\ \forall\delta>0\Big\},\\ \mathcal{A}_1^i(p,q) &:= \Big\{z^i\in\mathbb{R}^{\mathcal{J}}\setminus\{0\}:\ R(p)z^i=0\ \wedge\ (W^i,\delta z^i)\in\Phi^i(p,q),\ \forall\delta>0\Big\}.\end{aligned}$$

We focus our attention on two non-redundancy conditions, which are defined for every non-empty set $\mathbb{P}'\subseteq\mathbb{P}$. The first one, is a generalization of the requirement imposed by [Siconolfi, 1989, Assumption (A5)] in nominal asset markets with exogenous portfolio constraints,

$$(\mathrm{NR}_1(\mathbb{P}')) \qquad \bigcup_{i\in\mathcal{I}}\mathcal{A}_1^i(p,q)=\emptyset, \qquad \forall(p,q)\in\mathbb{P}'.$$ [15]

The second one, avoids the existence of unbounded sequences of trading admissible portfolios that do not implement transfers of wealth among states of nature, i.e.,

$$(\mathrm{NR}_0(\mathbb{P}')) \qquad \bigcup_{i\in\mathcal{I}}\mathcal{A}_0^i(p,q)=\emptyset, \qquad \forall(p,q)\in\mathbb{P}'.$$

Since $\mathcal{A}_0^i(p,q)\subseteq\mathcal{A}_1^i(p,q)$, for every non-empty set $\mathbb{P}'\subseteq\mathbb{P}$, $\mathrm{NR}_0(\mathbb{P}')$ is weaker than $\mathrm{NR}_1(\mathbb{P}')$.

PROPOSITION 1. *Under Assumptions (A1')-(A2'), for every $\mathbb{P}'\subseteq\mathbb{P}$ non-empty and compact,*

$$\mathrm{NR}_0(\mathbb{P}') \implies \bigcup_{(p,q)\in\mathbb{P}'}\Omega(p,q)\ \textit{is bounded} \implies 0\notin\bigcup_{(p,q)\in\mathbb{P}'}\ \bigcup_{\mathcal{I}'\subseteq\mathcal{I}}\ \sum_{i\in\mathcal{I}'}\mathcal{A}_0^i(p,q).$$

Thus, each non-redundancy condition, $\mathrm{NR}_1(\mathbb{P}')$ or $\mathrm{NR}_0(\mathbb{P}')$, guarantees that Assumption (A3) holds. Furthermore, as the following example illustrates, when assets are nominal and trading constraints are exogenous, Assumption (A3) is weaker than the traditional non-redundancy hypothesis imposed by Siconolfi [1989].

EXAMPLE 7. Consider an economy with exogenous trading constraints and nominal assets. There are three agents $\mathcal{I}=\{1,2,3\}$ and two assets $\mathcal{J}=\{1,2\}$, which have identical payments satisfying $N_{1,1}=N_{1,2}=1$ and $(N_{s,j})_{s\neq1}=0,\ \forall j\in\mathcal{J}$. Also, there is $m>0$ such that, $Z^1=[-m,+\infty)\times\{0\}$, $Z^2=\{0\}\times[-m,+\infty)$, and $Z^3=[-m,+\infty)\times(-\infty,0]$. Then, Assumption (A3') holds, although $\Big\{z\in\mathbb{R}^{\mathcal{J}}\setminus\{0\}:\ Nz=0\ \wedge\ \delta z\in Z^3,\ \forall\delta>0\Big\}\neq\emptyset$. □

15. Assume that assets are nominal, i.e., $R(p)\equiv N$, and that trading constraints are given by exogenous portfolio restrictions, i.e., for every agent i there is a set $Z^i\subseteq\mathbb{R}^{\mathcal{J}}$ such that $\Phi^i(p,q)=\mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}\times Z^i,\ \forall(p,q)\in\mathbb{P}$. Then, $(\mathrm{NR}_1(\mathbb{P}'))$ holds if and only if the following non-redundancy condition imposed by Siconolfi (1987) holds,

$$\bigcup_{i\in\mathcal{I}}\left\{z^i\in\mathbb{R}^{\mathcal{J}}\setminus\{0\}:\ Nz^i=0\ \wedge\ \delta z^i\in Z^i,\ \forall\delta>0\right\}=\emptyset.$$

Bounds on Asset Prices

An alternative to obtain upper bounds for asset prices is to have individuals whose preferences satisfy a kind of impatience condition.

Assumption (A5)
There is a non-empty subset of agents $\mathcal{I}^* \subseteq \mathcal{I}$ with monotonic preferences such that:
(i) Given $i \in \mathcal{I}^*$ and $(\rho, x^i) \in (0,1) \times \mathbb{R}_{++}^{\mathcal{L}\times\mathcal{S}}$, there exists $\tau^i(\rho, x^i) \in \mathbb{R}_+^{\mathcal{L}}$ such that,

$$V^i\left(x_0^i + \tau^i(\rho, x^i), (\rho\, x_s^i)_{s\in S}\right) > V^i(x^i).$$

(ii) Let $j \notin \mathcal{J}_u$, $\exists i \in \mathcal{I}^*$ and $z^i \in \mathbb{R}_+^{\mathcal{J}}$ with $z_j^i > 0$ and $-(0, z^i) \in \Phi^i(p,q)$, $\forall (p,q) \in \mathbb{P}$.

Assumption (A5)(i) holds independently of the representation of preferences, and was introduced by Seghir and Torres-Martínez [2011] to analyze equilibrium existence in a model with borrowing constraints depending on first-period consumption. Intuitively, it requires the existence of agents that, in terms of preferences, can compensate any loss in utility associated with a reduction in future demand with an increment of present consumption. In particular, Assumption (A5) is satisfied when there is an agent h such that V^h is unbounded on first period consumption and, independent of prices $(p,q) \in \mathbb{P}$, the zero vector belongs to the interior of $\Phi^h(p,q)$.

In this context, the main idea behind the existence of upper bounds for asset prices is as follows: consider an agent $i \in \mathcal{I}^*$ such that, at prices $(p,q) \in \mathbb{P}$, her optimal consumption allocation is market feasible. Suppose that, as an alternative to her optimal strategy, she decides to make a promise on an asset $j \notin \mathcal{J}_u$ using the borrowed resources to increase first period consumption. Also, assume that this promise can be paid with her future endowments. As a consequence of (A5), if the new strategy generates a high enough liquidity, then she will ensure a utility level greater than the one associated to aggregated endowments. Thus, q_j needs to be bounded (see Lemma 3C for detailed arguments).

Since one of our results of equilibrium existence is related with Seghir and Torres-Martínez [2011], it is interesting to discuss our assumptions when we restrict the attention to that framework.

Example 11 (Consumption-Dependent Borrowing Constraints)
Suppose that trading constrains are independent of prices and determine restrictions on borrowing and first-period consumption. Thus, given $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}$, we assume that

$$\Phi^i(p,q) = \{(x^i, z^i) \in \mathbb{E} : \ \exists(\theta^i, \varphi^i) \in \mathbb{R}_+^{\mathcal{J}} \times \mathbb{R}_+^{\mathcal{J}}, \quad \varphi^i \in \Psi^i(x_0^i) \ \ \wedge \ \ z^i = \theta^i - \varphi^i \ \},$$

where $\Psi^i = (\Psi_j^i) : \mathbb{R}_+^{\mathcal{L}} \twoheadrightarrow \mathbb{R}_+^{\mathcal{J}}$. In this context, Assumption (A2')(i) holds if and only if $\{\Psi^i\}_{i\in\mathcal{I}}$ have a closed and convex graph. Assumption (A2')(ii)(a) holds if and only if, for every agent i, $0 \in \Psi^i(x_0^i)$ and $\Psi^i(x_0^i) \subseteq \Psi^i(y_0^i)$, $\forall y_0^i \geq x_0^i$. This last property implies that, to ensure Assumption (A3) or (A3'), it is sufficient to require that $\{\Psi^i\}_{i\in\mathcal{I}}$ have bounded values.

Since trading constraints only affect short-sales, Assumptions (A2')(ii)(b) and Assumption (A4) always holds. Assumption (A5)(ii) holds if and only if there exists $\delta > 0$ such that $\delta(1, \ldots, 1) \in \sum_{i\in\mathcal{I}^*} (\Psi^i_j(0))_{j\notin\mathcal{J}_u}$.

Therefore the hypotheses of the main result in Seghir and Torres-Martínez [2011] imply that Assumptions (A1')(A3)', and (A4)-(A5) hold. □

Equilibrium Existence

However, as the following result shows, even without requiring financial payments to be super-replicated by physical markets it is possible to guarantee equilibrium existence in a model that allows exclusion of credit markets.

THEOREM 5. *Under Assumptions (A1')-(A3') and (A4)-(A5) there exists a competitive equilibrium for the economy with endogenous trading constraints.*

This result extends Seghir and Torres-Martínez [2011] in order to include price-dependent trading constraints and investment restrictions. It also guarantees that their main result holds under weaker assumptions. In fact, we only impose the impatience condition on a subset of agents. More importantly, they assume that sets of trading admissible short-sales are compact, an hypothesis that is stronger than Assumption (A3'). [16]

Recently, Pérez-Fernández [2013] also extends the results of Seghir and Torres-Martínez [2011] including price-dependent trading constraints in an environment with non-ordered preferences. In his model, the relationship between investment and debt is more general than ours, because Assumption (A5)(ii) does not necessarily hold. However, as in Seghir and Torres-Martínez [2011], it is assumed that correspondences of trading admissible allocations have compact values.

As was pointed out above, Assumption (A5)(i) holds when there are agents whose preferences can be represented by utility functions that are unbounded in first-period consumption. However, as the following result shows, when preferences can be represented by separable utility functions, we can ensure the existence of equilibrium without imposing Assumption (A5)(i).

COROLLARY 4. *Under Assumptions (A1')-(A3'), assume that there exists a non-empty set of agents $\mathcal{I}^\diamond \subseteq \mathcal{I}$ with monotonic preferences that satisfy:*
(i) For every $i \in \mathcal{I}^\diamond$, there is a commodity $l \in \mathcal{L}$ such that,

$$V^i(x^i) = v^i_l(x^i_{0,l}) + v^i((x^i_{0,r})_{r\neq l}, (x^i_s)_{s\in S}),$$

where the function $v^i_l : \mathbb{R}_+ \to \mathbb{R}$ is concave. [17]

16. See Example 11 for a detailed comparison between assumptions in the two models.
17. It is implicitly assumed that $\{v^i_l, v^i\}$ are such that V^i satisfies the properties of monotonicity, continuity, and quasi-concavity required by Assumption (A1'). In particular, to ensure the strong quasi-concavity V^i it is sufficient to assume that $\{v^i_l, v^i\}$ are strictly concave functions.

(ii) For any financial contract $j \notin \mathcal{J}_u$*, there is an agent* $i \in \mathcal{I}^\diamond$ *and a portfolio* $z^i \in \mathbb{R}^{\mathcal{J}}_+$ *such that*

$$z^i_j > 0 \text{ and } -(0, z^i) \in \Phi^i(p, q),\ \forall (p, q) \in \mathbb{P}.$$ [18]

Then, there exists a competitive equilibrium for the economy with endogenous trading constraints.

Essentially, when there are agents whose utility functions satisfy the separability condition above, we can construct an auxiliary economy where these individuals have preferences that satisfies Assumption (A5)(i). Thus, we obtain equilibrium existence by applying Theorem 5 and showing that any equilibrium of the auxiliary economy is an equilibrium of the original economy.[19]

2.4 Concluding Remarks

In this chapter we extend the theory of general equilibrium with incomplete financial markets to include price-dependent trading constraints that restrict both consumption alternatives and admissible portfolios. Our approach is general enough to incorporate several types of dependencies between prices, consumption, and financial access. For instance, the access to liquidity may depend on individuals income, the short-sale of derivatives may require the deposit of margins, and borrowers could be required to pledge physical and/or financial collateral.

Hence, we propose two ways to ensure the existence of a competitive equilibrium, based on either the super-replication of promises (Theorems 1 to 4) or a kind of agents' impatience (Theorem 5). The super-replication property holds when there is a risk-free nominal asset with unrestricted investment and such that all agents can sell it. The impatience condition holds when utility functions are unbounded in the first period consumption.

18. That is, the set $\mathcal{I}^\diamond$ satisfies the requirements of Assumption (A5)(ii).
19. This corollary is inspired by a result of [Moreno-García and Torres-Martínez, 2012, Corollary 2] for equilibrium existence in infinite horizon incomplete markets economies.

2.5 Appendix

Proof of Theorem 1.

Denote by $\mathcal{J}_a \subseteq \mathcal{J}$ the non-empty set of financial contracts j such that:
(i) $(R_{s,j}(p))_{s\in S_j} \gg 0$, $\forall p \in \mathbb{R}^{\mathcal{L}}_+ \times (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$; and
(ii) $\forall (p,q) \in \mathbb{P}$, $\exists \delta > 0: \ -\delta \vec{e}_j \in \bigcap_{i\in\mathcal{I}} \Phi^i(p,q)$,
where $\vec{e}_j \in \mathbb{E}$ is an individual allocation composed by just one unit of asset j. Let $\mathcal{J}_b := \mathcal{J}\setminus\mathcal{J}_a$.

Given $M \in \mathbb{N}$, consider the set of normalized prices

$$\mathbb{P}(M) := \mathcal{P}_0 \times [0,M]^{\mathcal{J}_b} \times \mathcal{P}_1^S,$$

where $\mathcal{P}_0 := \{y \in \mathbb{R}^{\mathcal{L}\cup\mathcal{J}_a}_+ : \|y\|_\Sigma = 1\}$, and $\mathcal{P}_1 := \{y \in \mathbb{R}^{\mathcal{L}}_+ : \|y\|_\Sigma = 1\}$.[20] Note that, a typical element of $\mathbb{P}(M)$ is of the form $(p,q) = ((p_0,(q_k)_{k\in\mathcal{J}_a}),(q_j)_{j\in\mathcal{J}_b},(p_s)_{s\in S})$. When $(p,q) \in \mathbb{P}(M)$, the commodity price $p = (p_0,(p_s)_{s\in S})$ belongs to $\mathcal{P} := \{y \in \mathbb{R}^{\mathcal{L}}_+ : \|y\|_\Sigma \leq 1\} \times \mathcal{P}_1^S$.

LEMMA 1. *Under Assumptions (A1)(iii) and (A2), for every agent $i \in \mathcal{I}$ the choice set correspondence $C^i : \mathbb{P}(M) \twoheadrightarrow \mathbb{E}$ is lower hemicontinuous with closed graph and non-empty and convex values.*

PROOF. Fix $i \in \mathcal{I}$. Since for every $(p,q) \in \mathbb{P}$ the allocation $((W^i_s)_{s\in\mathcal{S}},0) \in C^i(p,q)$, C^i is non-empty valued. Assumption (A2) implies that C^i has convex values and closed graph. To prove that C^i is lower hemicontinuous, let $\mathring{C}^i : \mathbb{P}(M) \twoheadrightarrow \mathbb{E}$ be the correspondence that associates to each $(p,q) \in \mathbb{P}(M)$ the set of allocations $(x^i,z^i) \in C^i(p,q)$ satisfying budget constraints with strict inequalities. We affirm that $\mathring{C}^i$ is lower hemicontinuous and has non-empty values. Since C^i is the closure of $\mathring{C}^i$, these properties imply that C^i is lower hemicontinuous (see Border (1985, 11.19(c))). Thus, to obtain the results it is sufficient to ensure the claimed properties for $\mathring{C}^i$.

Claim A. $\mathring{C}^i$ has non-empty values. Fix $(\mu_0,\mu_1) \in (0,1)\times(0,1)$ such that $\mu_0 > \mu_1$. It follows from Assumption (A2) that $((\mu_0 W^i_0,(\mu_1 W^i_s)_{s\in S}),0) \in \Phi^i(p,q)$ for all $(p,q) \in \mathbb{P}(M)$. Notice that, for any $(p,q) \in \mathbb{P}(M)$ with $p_0 \neq 0$ we have that $((\mu_0 W^i_0,(\mu_1 W^i_s)_{s\in S}),0) \in \mathring{C}^i(p,q)$.

Thus, fix $(p,q) \in \mathbb{P}(M)$ such that $p_0 = 0$ and, therefore $(q_j)_{j\in\mathcal{J}_a} \neq 0$. By definition, for every $j \in \mathcal{J}_a$ there exists $\delta_j(p,q) \in (0,1)$ such that $\delta_j \vec{e}_j \in \Phi^i(p,q)$ for all $\delta_j \in (0,\delta_j(p,q))$. Since Φ^i has convex values, we conclude that there exists $\delta(p,q) > 0$ such that

$$((W^i_0,(0.5W^i_s)_{s\in S}),0) - \delta \sum_{j\in\mathcal{J}_a} \vec{e}_j \in \Phi^i(p,q),$$

for every $\delta \in (0,\delta(p,q))$.[21] Furthermore, assume that $\delta \in (0,\delta(p,q))$ satisfies

$$\delta \sum_{k\in\mathcal{J}_a} \max_{(\tilde{p},s)\in\mathcal{P}\times S} R_{s,k}(\tilde{p}) < 0.5 \min_{(s,l)\in S\times\mathcal{L}} W^i_{s,l},$$

20. Trading constraints are not necessarily homogeneous of degree zero in prices. Consequently, the normalization of prices may induce a selection of equilibria.
21. It is sufficient to consider $\delta(p,q) := \min_{j\in\mathcal{J}_a} \ \delta_j(p,q)/\#\mathcal{J}_a$.

then promises can be honored with the resources that became available after the consumption of $0.5W_s^i$. Therefore, under these requirements, we have that

$$((W_0^i, (0.5W_s^i)_{s\in S}), 0) - \delta \sum_{j\in\mathcal{J}_a} \vec{e}_j \in \mathring{C}^i(p,q).$$

Claim B. $\mathring{C}^i$ is lower hemicontinuous. Fix $(p,q) \in \mathbb{P}(M)$ and $(x^i, z^i) \in \mathring{C}^i(p,q)$. Given a sequence $\{(p_n, q_n)\}_{n\in\mathbb{N}} \subset \mathbb{P}(M)$ converging to (p,q), the lower hemicontinuity of Φ^i (Assumption (A2)) ensures that there is $\{(x^i(n), z^i(n))\}_{n\in\mathbb{N}} \subset \mathbb{E}$ converging to (x^i, z^i) such that $(x^i(n), z^i(n)) \in \Phi^i(p_n, q_n)$, $\forall n \in \mathbb{N}$. Thus, for $n \in \mathbb{N}$ large enough, $(x^i(n), z^i(n)) \in \mathring{C}^i(p_n, q_n)$. It follows from the sequential characterization of hemicontinuity that $\mathring{C}^i$ is lower hemicontinuous (see Border (1985, 11.11(b))). □

DEFINITION. *Payments of assets in $\mathcal{J}_b$ can be super-replicated if there is $(\widehat{x}_0, \widehat{z}) \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}_a}$ such that,*

$$\sum_{j\in\mathcal{J}_b} R_{s,j}(p) > 0 \quad \Longrightarrow \quad \sum_{j\in\mathcal{J}_b} R_{s,j}(p) < p_s \cdot Y_s\widehat{x}_0 + \sum_{k\in\mathcal{J}_a} R_{s,k}(p)\widehat{z}_k, \qquad \forall (p,s) \in \mathcal{P}\times S.$$

Since $\mathcal{J}_a \neq \emptyset$, payments of assets in $\mathcal{J}_b$ can be super-replicated by choosing $\widehat{x}_0 = 0$ and

$$\widehat{z}_k = \max_{(p,s)\in\mathcal{P}\times S} \left(\sum_{j\in\mathcal{J}_b} R_{s,j}(p)/R_{s,k}(p) \right), \qquad \forall k \in \mathcal{J}_a.$$

Define

$$\overline{Q} := \max\left\{1,\ \|\widehat{x}_0\|_\Sigma + \max_{k\in\mathcal{J}_a} \widehat{z}_k \right\}; \qquad \overline{\Omega} := 2 \sup_{(p,q)\in\mathbb{P}(\overline{Q}):\,(p,q)\gg 0}\ \sup_{(x^i,z^i)_{i\in\mathcal{I}}\in\Omega(p,q)}\ \sum_{i\in\mathcal{I}} \|z^i\|_\Sigma.$$

Notice that, Assumption (A3) guarantees that $\overline{\Omega}$ is finite.

Given $(p,q) \in \mathbb{P}(M)$, for any $i \in \mathcal{I}$ we consider the truncated choice set $C^i(p,q) \cap \mathbb{K}$, where

$$\mathbb{K} := \left[0, 2\overline{W}\right]^{\mathcal{L}\times\mathcal{S}} \times \left[-\overline{\Omega},\ \#\mathcal{I}\ \overline{\Omega}\right]^{\mathcal{J}},$$

$$\overline{W} := \left(\#J\,\#\mathcal{I}\ \overline{\Omega} + \sum_{(s,l)\in\mathcal{S}\times\mathcal{L}} \sum_{i\in\mathcal{I}} W_{s,l}^i\right) \left(1 + \max_{(p,s)\in\mathcal{P}\times S} \sum_{j\in\mathcal{J}} R_{s,j}(p)\right).$$

Let $\Psi_M : \mathbb{P}(M) \times \mathbb{K}^{\mathcal{I}} \twoheadrightarrow \mathbb{P}(M) \times \mathbb{K}^{\mathcal{I}}$ be the correspondence given by

$$\Psi_M(p,q,(x^i,z^i)_{i\in\mathcal{I}}) = \phi_{0,M}((x_0^i, z^i)_{i\in\mathcal{I}}) \times \prod_{s\in S} \phi_s((x_s^i)_{i\in\mathcal{I}}) \times \prod_{i\in\mathcal{I}} \phi^i(p,q),$$

where

$$\begin{aligned}\phi_{0,M}((x_0^i,z^i)_{i\in\mathcal{I}}) &:= \operatorname*{arg\,max}_{(p_0,q)\in\mathcal{P}_0\times[0,M]^{\mathcal{J}_b}} \quad p_0\cdot\sum_{i\in\mathcal{I}}(x_0^i-w_0^i)+q\cdot\sum_{i\in\mathcal{I}}z^i;\\ \phi_s((x_s^i)_{i\in\mathcal{I}}) &:= \operatorname*{arg\,max}_{p_s\in\mathcal{P}_1} \quad p_s\cdot\sum_{i\in\mathcal{I}}(x_s^i-W_s^i), \quad \forall s\in S;\\ \phi^i(p,q) &:= \operatorname*{arg\,max}_{(x^i,z^i)\in C^i(p,q)\cap\mathbb{K}} \quad V^i(x^i), \quad \forall i\in\mathcal{I}.\end{aligned}$$

LEMMA 2. *Under Assumptions (A1)-(A3), Ψ_M has a non-empty set of fixed points.*

PROOF. By Kakutani's Fixed Point Theorem, it is sufficient to to prove that Ψ_M has a closed graph with non-empty and convex values. Since $\mathbb{P}(M)$ is non-empty, convex and compact, Berge's Maximum Theorem establishes that $\{\phi_{0,M},\{\phi_s\}_{s\in S}\}$ have a closed graph with non-empty and convex values.

It remains to prove that the same properties hold for $\{\phi^i\}_{i\in\mathcal{I}}$. Given $i\in\mathcal{I}$, Lemma 1 implies that C^i has a closed graph with non-empty and convex values. Since $\mathbb{K}$ is compact and convex and $((W_s^i)_{s\in\mathcal{S}},0)\in\mathbb{K}$, it follows that $(p,q)\in\mathbb{P}(M)\twoheadrightarrow C^i(p,q)\cap\mathbb{K}$ has a closed graph and non-empty, compact, and convex values. The proof of Lemma 1 also ensures that C^i is lower hemicontinuous and $((W_s^i)_{s\in\mathcal{S}},0)\in C^i(p,q)\cap\operatorname{int}(\mathbb{K})$. As $(p,q)\in\mathbb{P}(M)\twoheadrightarrow\operatorname{int}(\mathbb{K})$ has open graph, it follows that $(p,q)\in\mathbb{P}(M)\twoheadrightarrow C^i(p,q)\cap\operatorname{int}(\mathbb{K})$ is lower hemicontinuous (see Border (1985, 11.21(c))). Therefore, $(p,q)\in\mathbb{P}(M)\twoheadrightarrow C^i(p,q)\cap\mathbb{K}$ is lower hemicontinuous too (see Border (1985, 11.19(c))). Berge's Maximum Theorem and the continuity and quasi-concavity of V^i guarantees that ϕ^i satisfies the required properties. □

LEMMA 3. *Under Assumptions (A1)-(A3), assume that financial payments of assets in $\mathcal{J}_b$ can be super-replicated. Let $(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}})$ be a fixed point of Ψ_M such that $\overline{p}\gg 0$ and*

$$\sum_{i\in\mathcal{I}}\overline{z}_k^i\le 0,\ \ \forall k\in\mathcal{J}_a; \qquad \sum_{i\in\mathcal{I}}\overline{x}_{s,l}^i<2\overline{W},\ \ \forall(s,l)\in\mathcal{S}\times\mathcal{L}.$$

Then, for any $j\in\mathcal{J}_b$ we have that,

$$\overline{q}_j>0 \ \ \wedge\ \ \sum_{i\in\mathcal{I}}\overline{z}_j^i>0 \ \Longrightarrow\ \overline{q}_j\le\overline{Q}\,.$$

PROOF. Let $j\in\mathcal{J}_b$ such that $\overline{q}_j>0$ and $\sum_{i\in\mathcal{I}}\overline{z}_j^i>0$. Due to $\overline{p}\gg 0$, it follows from (A1)(iii) that $R_j(\overline{p})\neq 0$. Hence, there exists $a\in S$ such that $\sum_{r\in\mathcal{J}_b}R_{a,r}(\overline{p})>0$. Since financial payments of assets in $\mathcal{J}_b$ can be super-replicated, it follows that

$$\sum_{r\in\mathcal{J}_b}R_{a,r}(\overline{p})<\overline{p}_a\cdot Y_a\widehat{x}_0+\sum_{k\in\mathcal{J}_a}R_{a,k}(\overline{p})\widehat{z}_k\le\overline{p}_a\cdot Y_a\widehat{x}_0+\left(\max_{k\in\mathcal{J}_a}\widehat{z}_k\right)\sum_{k\in\mathcal{J}_a}R_{a,k}(\overline{p}).$$ [22]

22. Although $\widehat{x}_0=0$, it appears in our argument because we refer to this Lemma in the proof of Theorems 2 and 3.

We affirm that,

$$\overline{q}_j \quad \leq \quad \overline{p}_0 \cdot \widehat{x}_0 + \left(\max_{k \in \mathcal{J}_a} \widehat{z}_k\right) \sum_{k \in \mathcal{J}_a} \overline{q}_k.$$

Let i be an agent that invests in asset j. If the inequality above does not hold, then there is $\varepsilon > 0$ such that, i can reduce her long position on asset j in $\varepsilon \overline{z}_j^i$ units, change her first-period consumption to $\overline{x}_0^i + \varepsilon \overline{z}_j^i \widehat{x}$, and increase in $(\max_{r \in \mathcal{J}_a} \widehat{z}_r) \varepsilon \overline{z}_j^i$ units the investment in each $k \in \mathcal{J}_a$.[23] With this strategy, i changes her wealth at state of nature $s \in S$ by

$$\left(\overline{p}_s \cdot Y_s \widehat{x}_0 + \left(\max_{k \in \mathcal{J}_a} \widehat{z}_k\right) \sum_{k \in \mathcal{J}_a} R_{s,k}(\overline{p}) - R_{s,j}(\overline{p})\right) \varepsilon \overline{z}_j^i \geq 0,$$

where the last inequality follows because of super-replication and holds as strict inequality for $s = a$. This contradicts the optimality of $(\overline{x}^i, \overline{z}^i)$ on $C^i(\overline{p}, \overline{q}) \cap \mathbb{K}$. We conclude that $\overline{q}_j \leq \overline{Q}$. □

LEMMA 4. *Under Assumptions (A1)-(A3), assume that assume that financial payments of assets in $\mathcal{J}_b$ can be super-replicated. Then, for any $M > \overline{Q}$ the fixed points of Ψ_M are competitive equilibria.*

PROOF. Given $M > \overline{Q}$, let $(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i \in \mathcal{I}})$ be a fixed point of Ψ_M. Adding first period budget constraints across agents, the definition of $\phi_{0,M}$ guarantees that,

$$p_0 \cdot \sum_{i \in \mathcal{I}} (\overline{x}_0^i - w_0^i) + q \cdot \sum_{i \in \mathcal{I}} \overline{z}^i \leq \overline{p}_0 \cdot \sum_{i \in \mathcal{I}} (\overline{x}_0^i - w_0^i) + \overline{q} \cdot \sum_{i \in \mathcal{I}} \overline{z}^i \leq 0, \quad \forall\, (p_0, q) \in \mathcal{P}_0 \times [0, M]^{\mathcal{J}_b}.$$

Hence,

$$\sum_{i \in \mathcal{I}} (\overline{x}_0^i - w_0^i) \leq 0, \qquad \sum_{i \in \mathcal{I}} \overline{z}_k^i \leq 0,\ \forall k \in \mathcal{J}_a,$$

and $\overline{q}_j = M$ for every $j \in \mathcal{J}_b$ such that $\sum_{i \in \mathcal{I}} \overline{z}_j^i > 0$. Furthermore, adding individual budget constraints at any state of nature in the second period, the definition of $\mathbb{K}$ guarantees that,

$$p_s \cdot \sum_{i \in \mathcal{I}} (\overline{x}_s^i - W_s^i) \leq \overline{p}_s \cdot \sum_{i \in \mathcal{I}} (\overline{x}_s^i - W_s^i) \leq \overline{W}, \quad \forall p_s \in \mathcal{P}_1, \forall s \in S.$$

We obtain that $\sum_{i \in \mathcal{I}} \overline{x}_{s,l}^i < 2\overline{W}$, $\forall (s,l) \in \mathcal{S} \times \mathcal{L}$, which implies that $\overline{p} \gg 0$. In another case, Assumptions (A1) and (A2) guarantee that at least one agent can improve her utility by increasing her consumption without additional costs. A contradiction to the optimality of plans $(\overline{x}^i, \overline{z}^i)_{i \in \mathcal{I}}$.

The strict positivity of commodity prices has several consequences. First, by Assumption (A1)(iii) asset promises are non-trivial and the strict monotonicity of preferences (Assumption (A1)(i)) jointly with the absence of restrictions on investment ensure that asset prices are strictly positive. Second, as $(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i \in \mathcal{I}})$ satisfies the hypotheses of Lemma 3 and $M > \overline{Q}$, we obtain that $\sum_{i \in \mathcal{I}} \overline{z}^i \leq 0$. Third, Assumption (A1)(i) guarantees that budget constraints are satisfied by equality.

23. As the new strategy needs to be on $\mathbb{K}$, the value of ε may depend on $(\overline{q}_j, \overline{x}_0^i, (\overline{z}_k^i)_{k \in \mathcal{J}_a}, \overline{z}_j^i)$.

We conclude that,

$$(\overline{p},\overline{q}) \in \mathbb{P}(\overline{Q}), \qquad (\overline{p},\overline{q}) \gg 0, \qquad \sum_{i\in\mathcal{I}}(\overline{x}^i - (W^i_s)_{s\in\mathcal{S}}) = 0, \qquad \sum_{i\in\mathcal{I}} \overline{z}^i = 0,$$

and Assumption (A3) implies that $(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}} \in \Omega(\overline{p},\overline{q}) \cap \mathrm{int}(\mathbb{K})$.[24]

As for any $i \in \mathcal{I}$ the allocation $(\overline{x}^i,\overline{z}^i)$ belongs to $C^i(\overline{p},\overline{q}) \cap \mathrm{int}(\mathbb{K})$, given $(x^i,z^i) \in C^i(\overline{p},\overline{q})$ with $x^i \neq \overline{x}^i$ there exists $\lambda \in (0,1)$ such that $\lambda(\overline{x}^i,\overline{z}^i) + (1-\lambda)(x^i,z^i) \in C^i(\overline{p},\overline{q}) \cap \mathbb{K}$. The strongly quasi-concavity of utility functions (Assumption (A1)(i)) implies that,

$$V^i(\lambda(\overline{x}^i,\overline{z}^i) + (1-\lambda)(x^i,z^i)) > \min\{V^i(\overline{x}^i), V^i(x^i)\}.$$

Since $(\overline{x}^i,\overline{z}^i) \in \phi^i(\overline{p},\overline{q})$, we obtain that $V^i(x^i) < V^i(\overline{x}^i)$. Thus, $(\overline{x}^i,\overline{z}^i)$ is an optimal choice for agent i in $C^i(\overline{p},\overline{q})$, which concludes the proof. □

Proof of Theorem 2.

To prove this result we can follow identical arguments to those made in the proof of Theorem 1. In fact, although $\mathcal{J}_a$ can be an empty-set, it is sufficient to ensure that financial payments of assets in $\mathcal{J}_b$ can be super-replicated.

Since assets are real and payments are given in terms of commodities in $\mathcal{L}^*$, for any $j \in \mathcal{J}_b$ there is $A_{s,j} : \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+ \to \mathbb{R}^{\mathcal{L}}_+$ such that $R_{s,j}(p) = p_s \cdot A_{s,j}(p)$ and $(A_{s,j})_l = 0$, $\forall l \notin \mathcal{L}^*$. Thus, we can super-replicate financial payments of asset in $\mathcal{J}_b$ by choosing $(\widehat{z}_k)_{k\in\mathcal{J}_a} = 0$ and $\widehat{x}_0 = a(1,\ldots,1) \gg 0$ such that the following condition holds,

$$\max_{l\in\mathcal{L}^*} \max_{(p,s)\in\mathcal{P}\times\widehat{S}} \sum_{j\in\mathcal{J}_b} (A_{s,j}(p))_l < a \min_{(s,l)\in\widehat{S}\times\mathcal{L}^*} Y_s(l,l),$$

where $\widehat{S} := \bigcup_{j\in\mathcal{J}} S_j$ are the states of nature in which assets make promises. □

Proof of Theorem 3.

Analogous to the proof of previous results, it is sufficient to ensure that financial payments of assets in $\mathcal{J}_b$ can be super-replicated. Since assets are backed by physical collateral, it is possible to super-replicate financial payments by choosing $(\widehat{z}_k)_{k\in\mathcal{J}_a} = 0$ and $\widehat{x}_0 = \sum_{j\in\mathcal{J}} C_j$. □

Proof of Proposition 1

Let $\mathbb{P}' \subseteq \mathbb{P}$ be a non-empty and compact set.

Assume that there is an unbounded sequence $\{(x^i_n,z^i_n)_{i\in\mathcal{I}}\}_{n\in\mathbb{N}} \in \bigcup_{(p,q)\in\mathbb{P}'} \Omega(p,q)$. Then, there exists a sequence $\{(p_n,q_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{P}'$ such that, $(x^i_n,z^i_n) \in \Omega(p_n,q_n)$, $\forall n \in \mathbb{N}$. Also, Assumption (A2')(ii)(a) ensures that, for every n and i, $(W,z^i_n) \in \Phi^i(p_n,q_n)$, where $W =$

24. Assumption (A3) is only required to ensure that $\left[(\overline{p},\overline{q}) \gg 0 \ \wedge \ (\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}} \in \Omega(\overline{p},\overline{q})\right] \Longrightarrow (\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}} \in \mathrm{int}(\mathbb{K})$. Notice that, if we change the upper bound $\overline{\Omega}$ for an arbitrary positive number in the definition of $\mathbb{K}$, then all the other arguments in the proof of Theorem 1 still hold.

$(W_s)_{s\in\mathcal{S}} := \sum_{i\in\mathcal{I}} W^i$. Hence, for some agent h there is an unbounded subsequence $\{z^h_{n_k}\}_{k\in\mathbb{N}} \subseteq \{z^h_n\}_{n\in\mathbb{N}}$ such that, for every $k\in\mathbb{N}$, $z^h_{n_k}\neq 0$, $\|z^h_{n_k}\|_\Sigma \leq \|z^h_{n_{k+1}}\|_\Sigma$, and $(W, z^h_{n_k}) \in \Phi^h(p_{n_k}, q_{n_k})$. Let $((\tilde{p},\tilde{q}),\tilde{z}^h)$ be a cluster point of $\{(p_{n_k},q_{n_k}), z^h_{n_k}/\|z^h_{n_k}\|_\Sigma\}_{k\in\mathbb{N}}$.

We affirm that, $R(\tilde{p})\tilde{z}^h = 0$ and $\tilde{q}\cdot\tilde{z}^h = 0$. First, if there is an state of nature $s\in S$ such that $\sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_s)\tilde{z}^h_j < 0$, then $\delta_0\sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_s)\tilde{z}^h_j < -2\tilde{p}_s\cdot W_s$, for some $\delta_0>0$. This implies that, for $k\in\mathbb{N}$ large enough, $\delta_0\sum_{j\in\mathcal{J}} R_{s,j}(p_{n_k,s})z^h_{n_k,j}/\|z^h_{n_k}\|_\Sigma < -2p_{n_k,s}\cdot W_s$. Since $\lim_k \|z^h_{n_k}\|_\Sigma = +\infty$, it follows that for k large enough $\sum_{j\in\mathcal{J}} R_{s,j}(p_{n_k,s})z^h_{n_k,j} < -2p_{n_k,s}\cdot W_s$, a contradiction with $(x^i_{n_k}, z^i_{n_k})_{i\in\mathcal{I}} \in \Omega(p_{n_k},q_{n_k})$. Second, if there is $s\in S$ such that $\sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_s)\tilde{z}^h_j > 0$, then $\delta_1\sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_s)\tilde{z}^h_j > 2(\#\mathcal{I}-1)\tilde{p}_s\cdot W_s$, for some $\delta_1>0$. Hence, for $k\in\mathbb{N}$ large enough, we have that $\sum_{j\in\mathcal{J}} R_{s,j}(p_{n_k,s})z^h_{n_k,j} > 2(\#\mathcal{I}-1)\,p_{n_k,s}\cdot W_s$. Due to $\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} R_{s,j}(p_{n_k,s})z^i_{n_k,j} = 0$, there exists $h'\neq h$ such that $\sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_{n_k,s})z^{h'}_{n_k,j} < -2\,\tilde{p}_{n_k,s}\cdot W_s$, a contradiction with $(x^i_{n_k}, z^i_{n_k})_{i\in\mathcal{I}}\in\Omega(p_{n_k},q_{n_k})$. The property $\tilde{q}\cdot\tilde{z}^h = 0$ follows by analogous arguments.

In addition, $(W^h,\delta\tilde{z}^h)\in\Phi^h(\tilde{p},\tilde{q})$ for every $\delta>0$. Indeed, given $\delta>0$ there exists $k(\delta)\in\mathbb{N}$ such that $\|z^h_{n_k}\|_\Sigma\geq\delta, \forall k\geq k(\delta)$. Hence, as Φ^h has convex values and $(W,0)\in\Phi^h(p_{n_k},q_{n_k})$ for every $k\in\mathbb{N}$, it follows that $(W,\delta z^h_{n_k}/\|z^h_{n_k}\|_\Sigma)\in\Phi^h(p_{n_k},q_{n_k})$ for any $k\geq k(\delta)$, which in turn implies that $(W,\delta\tilde{z}^h)\in\Phi^h(\tilde{p},\tilde{q})$. Furthermore, Assumption (A1') guarantees that there is $\sigma\in(0,1)$ such that $\sigma W \ll W^h$. As for every $\delta>0$ we have that $(1-\sigma)(0,0)+\sigma(W,\delta\tilde{z}^h/\sigma)\in\Phi^h(\tilde{p},\tilde{q})$, the property follows from Assumption (A2')(ii)(a). Therefore, $\tilde{z}^h\in\mathcal{A}^h_0(\tilde{p},\tilde{q})$, which implies that $\bigcup_{(p,q)\in\mathbb{P}'}\bigcup_{i\in\mathcal{I}}\mathcal{A}^i_0(p,q)\neq\emptyset$. This concludes the proof of the first implication.

Notice that, if there is $(p,q)\in\mathbb{P}'$ and $\mathcal{I}'\subseteq\mathcal{I}$ such that $0\in\sum_{i\in\mathcal{I}'}\mathcal{A}^i_0(p,q)$, then for every $i\in\mathcal{I}'$ there is $z^i\in\mathbb{R}^{\mathcal{J}}\setminus\{0\}$ such that $q\cdot z^i=0$, $R(p)z^i=0$, and $(W^i,\delta z^i)\in\Phi^i(p,q)$, $\forall\delta>0$, with $\sum_{i\in\mathcal{I}'} z^i = 0$. We conclude that, $\sum_{i\in\mathcal{I}'}\delta z^i = 0$ and $(W^i,\delta z^i)\in C^i(p,q)$, $\forall i\in\mathcal{I}'$, $\forall\delta>0$. Hence, $\Omega(p,q)$ is unbounded. □

Proof of Theorem 4

Given $M\in\mathbb{N}$, consider the space $\mathbb{P}(M)$ of normalized prices, which is defined by

$$(p,q)\equiv((p_0,(q_k)_{k\in\mathcal{J}_u}),(q_j)_{j\in\mathcal{J}\setminus\mathcal{J}_u},(p_s)_{s\in S})\in\mathbb{P}(M):=\mathcal{P}_0\times[0,M]^{\mathcal{J}\setminus\mathcal{J}_u}\times\mathcal{P}_1^S,$$

where $\mathcal{P}_0 := \{y\in\mathbb{R}^{\mathcal{L}\cup\mathcal{J}_u}_+ : \|y\|_\Sigma = 1\}$, and $\mathcal{P}_1 := \{y\in\mathbb{R}^{\mathcal{L}}_+ : \|y\|_\Sigma = 1\}$.[25]

Lemma 1B. *Under Assumptions (A1')(iii), (A2')(i) and (A2')(ii)(a), for every agent $i\in\mathcal{I}$ the choice set correspondence $C^i:\mathbb{P}(M)\twoheadrightarrow\mathbb{E}$ is lower hemicontinuous with closed graph and non-empty and convex values.*

Proof. Fix $i\in\mathcal{I}$. Assumption (A2')(ii)(a) ensures that for every $(p,q)\in\mathbb{P}$ the allocation $(W^i,0)\in C^i(p,q)$, which implies that C^i is non-empty valued. Assumptions (A1')(iii) and (A2')(i) imply that C^i has convex values and closed graph. To prove that C^i is lower hemicontinuous, let $\mathring{C}^i:\mathbb{P}(M)\twoheadrightarrow\mathbb{E}$ be the correspondence that associates to each $(p,q)\in\mathbb{P}(M)$

25. Trading constraints are not necessarily homogeneous of degree zero in prices. Consequently, the normalization of prices may induce a selection of equilibria.

the set of allocations $(x^i, z^i) \in C^i(p,q)$ satisfying budget constraints with strict inequalities. We affirm that $\mathring{C}^i$ is lower hemicontinuous and has non-empty values. Since C^i is the closure of $\mathring{C}^i$, these properties imply that C^i is lower hemicontinuous (see Border (1985, 11.19(c))).

Thus, we close the proof ensuring the claimed properties for $\mathring{C}^i$.

To prove that $\mathring{C}^i$ has non-empty values, fix $(\mu_0, \mu_1) \in (0,1) \times (0,1)$ such that $\mu_0 > \mu_1$. It follows from Assumption (A2')(ii)(a) that $((\mu_0 W_0^i, (\mu_1 W_s^i)_{s\in S}), 0) \in \Phi^i(p,q)$ for all $(p,q) \in \mathbb{P}(M)$.

Notice that, for any $(p,q) \in \mathbb{P}(M)$ with $p_0 \neq 0$ we have that $((\mu_0 W_0^i, (\mu_1 W_s^i)_{s\in S}), 0) \in \mathring{C}^i(p,q)$. Thus, fix $(p,q) \in \mathbb{P}(M)$ such that $p_0 = 0$. Since Φ^i has convex values, it follows from Assumption (A1')(iii) that there exists $\lambda \in (0,1)$, high enough, such that

$$(\widetilde{x}^i, \widetilde{z}^i) := \lambda\left((\mu_0 W_0^i, (\mu_1 W_s^i)_{s\in S}), 0\right) + \frac{(1-\lambda)}{\max\{\#\mathcal{J}_u, 1\}} \sum_{k\in\mathcal{J}_u} \left[\Big(W^i + \theta_k(p,q)(W_0^i, -(W_s^i)_{s\in S})), 0\Big) - \zeta_k(p,q)\,\widehat{e}_k\right] \in \Phi^i(p,q);$$

$$\lambda\mu_0 + \sum_{k\in\mathcal{J}_u} \frac{(1-\lambda)}{\max\{\#\mathcal{J}_u, 1\}}(1+\theta_k(p,q)) < 1;$$

$$\frac{(1-\lambda)}{\max\{\#\mathcal{J}_u, 1\}} \sum_{k\in\mathcal{J}_u} \zeta_k(p,q) \max_{(\tilde{p},\tilde{q})\in\mathbb{P}(M)} \max_{s\in S} R_{s,k}(\tilde{p}_s) < \frac{\lambda(\mu_0-\mu_1)}{2} \min_{i\in\mathcal{I}} \min_{(s,l)\in S\times\mathcal{L}} W_{s,l}^i;$$

where $(\theta_k, \zeta_k)_{k\in\mathcal{J}_u}$ are the functions that guarantee that contracts in $\mathcal{J}_u$ are ultimate sources of liquidity (see Definition 5). Notice that, the first condition above ensures that $(\widetilde{x}^i, \widetilde{z}^i)$ is trading feasible at prices (p,q), the second requirement implies that $\widetilde{x}_0^i \ll w_0^i$, and the last inequality guarantees that, at each state of nature $s \in S$, debts can be paid with the resources that became available after the consumption of $\widetilde{x}_s^i$. Thus, the definition of $\mathbb{P}(M)$ guarantees that $(\widetilde{x}^i, \widetilde{z}^i) \in \mathring{C}^i(p,q)$. Hence, $\mathring{C}^i$ has non-empty values. [26]

To prove that $\mathring{C}^i$ is lower hemicontinuous, fix $(p,q) \in \mathbb{P}(M)$ and $(x^i, z^i) \in \mathring{C}^i(p,q)$. Given a sequence $\{(p_n, q_n)\}_{n\in\mathbb{N}} \subset \mathbb{P}(M)$ that converges to (p,q), the lower hemicontinuity of Φ^i (Assumption (A2')(i)) ensures that there exists a sequence $\{(x^i(n), z^i(n))\}_{n\in\mathbb{N}} \subset \mathbb{E}$ converging to (x^i, z^i) such that $(x^i(n), z^i(n)) \in \Phi^i(p_n, q_n)$, $\forall n \in \mathbb{N}$. Thus, for $n \in \mathbb{N}$ large enough, $(x^i(n), z^i(n)) \in \mathring{C}^i(p_n, q_n)$. It follows from the sequential characterization of hemicontinuity that $\mathring{C}^i$ is lower hemicontinuous (see Border (1985, 11.11(b))). □

For notation convenience, let $(\widehat{x}, (\widehat{z}_k)_{k\in\mathcal{J}_u})$ be an allocation that allows agents to super-replicate financial payments when second period commodity prices belong to $\mathcal{P}_1^S$. Also, define

$$\overline{Q} := \max\left\{1,\ \|\widehat{x}\|_\Sigma + \max_{k\in\mathcal{J}_u} \widehat{z}_k\right\};$$

$$\overline{\Omega} := 2 \sup_{(p,q)\in\mathbb{P}(\overline{Q}):\,(p,q)\gg 0}\ \sup_{(x^i,z^i)_{i\in\mathcal{I}}\in\Omega(p,q)} \sum_{i\in\mathcal{I}} \|z^i\|_\Sigma.$$

Notice that, Assumption (A3) guarantees that $\overline{\Omega}$ is finite.

26. Dividing by $\max\{\#\mathcal{J}_u, 1\}$ we ensure that the arguments above still hold when $\mathcal{J}_u$ is an empty set.

Given $(p,q) \in \mathbb{P}(M)$, for any $i \in \mathcal{I}$ we consider the truncated choice set $C^i(p,q) \cap \mathbb{K}$, where

$$\mathbb{K} := \left[0, 2\overline{W}\right]^{\mathcal{L}\times\mathcal{S}} \times \left[-\overline{\Omega},\, \#\mathcal{I}\,\overline{\Omega}\right]^{\mathcal{J}},$$

$$\overline{W} := \left(\#\mathcal{J}\,\#\mathcal{I}\,\overline{\Omega} + \sum_{(s,l)\in\mathcal{S}\times\mathcal{L}} \sum_{i\in\mathcal{I}} W^i_{s,l}\right)\left(1 + \max_{s\in S}\max_{p_s\in\mathcal{P}_1}\sum_{j\in\mathcal{J}} R_{s,j}(p_s)\right).$$

Let $\Psi_M : \mathbb{P}(M)\times\mathbb{K}^{\mathcal{I}} \twoheadrightarrow \mathbb{P}(M)\times\mathbb{K}^{\mathcal{I}}$ be the correspondence given by

$$\Psi_M(p,q,(x^i,z^i)_{i\in\mathcal{I}}) = \phi_{0,M}((x^i_0,z^i)_{i\in\mathcal{I}}) \times \prod_{s\in S}\phi_s((x^i_s)_{i\in\mathcal{I}}) \times \prod_{i\in\mathcal{I}}\phi^i(p,q),$$

where

$$\begin{aligned}
\phi_{0,M}((x^i_0,z^i)_{i\in\mathcal{I}}) &:= \operatorname*{arg\,max}_{(p_0,q)\in\mathcal{P}_0\times[0,M]^{\mathcal{J}\setminus\mathcal{J}_u}} p_0\cdot\sum_{i\in\mathcal{I}}(x^i_0 - w^i_0) + q\cdot\sum_{i\in\mathcal{I}} z^i;\\
\phi_s((x^i_s)_{i\in\mathcal{I}}) &:= \operatorname*{arg\,max}_{p_s\in\mathcal{P}_1} p_s\cdot\sum_{i\in\mathcal{I}}(x^i_s - W^i_s), \quad \forall s\in S;\\
\phi^i(p,q) &:= \operatorname*{arg\,max}_{(x^i,z^i)\in C^i(p,q)\cap\mathbb{K}} V^i(x^i), \quad \forall i\in\mathcal{I}.
\end{aligned}$$

Lemma 2B. *Under Assumptions (A1')-(A2') and (A3), Ψ_M has a non-empty set of fixed points.*

Proof. By Kakutani's Fixed Point Theorem, it is sufficient to to prove that Ψ_M has a closed graph with non-empty and convex values. Since $\mathbb{P}(M)$ is non-empty, convex and compact, Berge's Maximum Theorem establishes that $\{\phi_{0,M}, \{\phi_s\}_{s\in S}\}$ have a closed graph with non-empty and convex values.

It remains to prove that the same properties hold for $\{\phi^i\}_{i\in\mathcal{I}}$. Given $i\in\mathcal{I}$, Lemma 1 implies that C^i has a closed graph with non-empty and convex values. Since $\mathbb{K}$ is compact and convex and $(W^i,0)\in\mathbb{K}$, it follows that $(p,q)\in\mathbb{P}(M)\twoheadrightarrow C^i(p,q)\cap\mathbb{K}$ has a closed graph and non-empty, compact, and convex values. The proof of Lemma 1B also ensures that C^i is lower hemicontinuous and $(W^i,0)\in C^i(p,q)\cap\operatorname{int}(\mathbb{K})$. As $(p,q)\in\mathbb{P}(M)\twoheadrightarrow\operatorname{int}(\mathbb{K})$ has open graph, it follows that $(p,q)\in\mathbb{P}(M)\twoheadrightarrow C^i(p,q)\cap\operatorname{int}(\mathbb{K})$ is lower hemicontinuous (see Border (1985, 11.21(c))). Therefore, $(p,q)\in\mathbb{P}(M)\twoheadrightarrow C^i(p,q)\cap\mathbb{K}$ is lower hemicontinuous too (see Border (1985, 11.19(c))). Berge's Maximum Theorem and the continuity and quasi-concavity of V^i guarantees that ϕ^i satisfies the required properties. □

Lemma 3B. *Under Assumptions (A1')-(A2'), (A3) and (A4), assume that agents can super-replicate financial payments investing in assets $\mathcal{J}_u$ and buying commodities. Let $(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}})$ be a fixed point of Ψ_M such that $\overline{p}\gg 0$ and*

$$\sum_{i\in\mathcal{I}}\overline{z}^i_k \le 0,\ \ \forall k\in\mathcal{J}_u; \qquad \sum_{i\in\mathcal{I}}\overline{x}^i_{s,l} < 2\overline{W},\ \ \forall(s,l)\in\mathcal{S}\times\mathcal{L}.$$

Then, for any $j \notin \mathcal{J}_u$ we have that,

$$\overline{q}_j > 0 \quad \wedge \quad \sum_{i \in \mathcal{I}} \overline{z}^i_j > 0 \quad \Longrightarrow \quad \overline{q}_j \leq \overline{Q}\,.$$

PROOF. Let $j \notin \mathcal{J}_u$ such that $\overline{q}_j > 0$ and $\sum_{i\in\mathcal{I}} \overline{z}^i_j > 0$. Due to $\overline{p} \gg 0$, it follows from (A1')(iii) that $(R_{s,j}(\overline{p}_s))_{s\in S} \neq 0$. Hence, there exists $a \in S$ such that $\sum_{r\notin\mathcal{J}_u} R_{a,r}(\overline{p}_a) > 0$. Since financial payments can be super-replicated by investments in $\mathcal{L} \cup \mathcal{J}_u$, it follows from Definition 6 that

$$\sum_{r\notin\mathcal{J}_u} R_{a,r}(\overline{p}_a) < \overline{p}_a Y_a \widehat{x} + \sum_{k\in\mathcal{J}_u} R_{a,k}(\overline{p}_a)\widehat{z}_k \leq \overline{p}_a Y_a \widehat{x} + \left(\max_{k\in\mathcal{J}_u} \widehat{z}_k\right) \sum_{k\in\mathcal{J}_u} R_{a,k}(\overline{p}_a).$$

We affirm that,

$$\overline{q}_j \quad \leq \quad \overline{p}_0 \widehat{x} + \left(\max_{k\in\mathcal{J}_u} \widehat{z}_k\right) \sum_{k\in\mathcal{J}_u} \overline{q}_k.$$

Let i be an agent that invests in asset j. If the inequality above does not hold, then there is $\varepsilon > 0$ such that, i can reduce her long position on asset j in $\varepsilon\overline{z}^i_j$ units, change her first-period consumption to $\overline{x}^i_0 + \varepsilon\overline{z}^i_j\widehat{x}$, and increase in $(\max_{r\in\mathcal{J}_u} \widehat{z}_r)\varepsilon\overline{z}^i_j$ units the investment in each $k \in \mathcal{J}_u$.[27] With this strategy, i changes her wealth at state of nature $s \in S$ by

$$\left(\overline{p}_s Y_s \widehat{x} + \left(\max_{k\in\mathcal{J}_u} \widehat{z}_k\right) \sum_{k\in\mathcal{J}_u} R_{s,k}(\overline{p}_s) - R_{s,j}(\overline{p}_s)\right) \varepsilon\overline{z}^i_j \geq 0,$$

where the last inequality follows from Definition 6 and holds as strict inequality for $s = a$. This contradicts the optimality of $(\overline{x}^i, \overline{z}^i)$ on $C^i(\overline{p}, \overline{q}) \cap \mathbb{K}$. We conclude that $\overline{q}_j \leq \overline{Q}$. □

LEMMA 4B. *Under Assumptions (A1')-(A2'), and (A3)-(A4), assume that agents can super-replicate financial payments investing in assets $\mathcal{J}_u$ and buying commodities. Then, for any $M > \overline{Q}$ the fixed points of Ψ_M are competitive equilibria.*

PROOF. Given $M > \overline{Q}$, let $(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}})$ be a fixed point of Ψ_M. Adding first period budget constraints across agents, the definition of $\phi_{0,M}$ guarantees that,

$$p_0 \cdot \sum_{i\in\mathcal{I}} (\overline{x}^i_0 - w^i_0) + q \cdot \sum_{i\in\mathcal{I}} \overline{z}^i \leq \overline{p}_0 \cdot \sum_{i\in\mathcal{I}} (\overline{x}^i_0 - w^i_0) + \overline{q} \cdot \sum_{i\in\mathcal{I}} \overline{z}^i \leq 0, \quad \forall\, (p_0, q) \in \mathcal{P}_0 \times [0, M]^{\mathcal{J}\setminus\mathcal{J}_u}.$$

Hence,

$$\sum_{i\in\mathcal{I}} (\overline{x}^i_0 - w^i_0) \leq 0, \qquad \sum_{i\in\mathcal{I}} \overline{z}^i_k \leq 0, \;\; \forall k \in \mathcal{J}_u,$$

and $\overline{q}_j = M$ for every $j \notin \mathcal{J}_u$ such that $\sum_{i\in\mathcal{I}} \overline{z}^i_j > 0$. Furthermore, adding individual budget

27. As the new strategy needs to be on $\mathbb{K}$, the value of ε may depend on $(\overline{q}_j, \overline{x}^i_0, (\overline{z}^i_k)_{k\in\mathcal{J}_u}, \overline{z}^i_j)$.

constraints at any state of nature in the second period, the definition of $\mathbb{K}$ guarantees that,

$$p_s \cdot \sum_{i\in\mathcal{I}}(\overline{x}_s^i - W_s^i) \leq \overline{p}_s \cdot \sum_{i\in\mathcal{I}}(\overline{x}_s^i - W_s^i) \leq \overline{W}, \quad \forall p_s \in \mathcal{P}_1, \forall s \in S.$$

We obtain that $\sum_{i\in\mathcal{I}} \overline{x}_{s,l}^i < 2\overline{W}$, $\forall (s,l) \in \mathcal{S}\times\mathcal{L}$, which implies that $\overline{p} \gg 0$. In another case, Assumptions (A1') and (A2')(ii)(a) guarantee that at least one agent can improve her utility by increasing her consumption without additional costs. A contradiction to the optimality of plans $(\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}$.

The strict positivity of commodity prices has several consequences. First, by Assumption (A1')(iii) asset promises are non-trivial and Assumptions (A1') and (A2')(ii)(b) ensure that asset prices are strictly positive. Second, as $(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}})$ satisfies the hypotheses of Lemma 3B and $M > \overline{Q}$, we obtain that $\sum_{i\in\mathcal{I}} \overline{z}^i \leq 0$. Third, Assumption (A1') guarantees that budget constraints are satisfied by equality.

We conclude that,

$$(\overline{p}, \overline{q}) \in \mathbb{P}(\overline{Q}), \qquad (\overline{p}, \overline{q}) \gg 0, \qquad \sum_{i\in\mathcal{I}}(\overline{x}^i - W^i) = 0, \qquad \sum_{i\in\mathcal{I}} \overline{z}^i = 0,$$

and Assumption (A3) implies that $(\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}} \in \Omega(\overline{p}, \overline{q}) \cap \mathrm{int}(\mathbb{K})$. [28]

As for any $i \in \mathcal{I}$ the allocation $(\overline{x}^i, \overline{z}^i)$ belongs to $C^i(\overline{p}, \overline{q}) \cap \mathrm{int}(\mathbb{K})$, given $(x^i, z^i) \in C^i(\overline{p}, \overline{q})$ with $x^i \neq \overline{x}^i$ there exists $\lambda \in (0,1)$ such that $\lambda(\overline{x}^i, \overline{z}^i) + (1-\lambda)(x^i, z^i) \in C^i(\overline{p}, \overline{q}) \cap \mathbb{K}$. The strict quasi-concavity of utility functions (Assumption (A1')) implies that,

$$V^i(\lambda(\overline{x}^i, \overline{z}^i) + (1-\lambda)(x^i, z^i)) > \min\{V^i(\overline{x}^i), V^i(x^i)\}.$$

Since $(\overline{x}^i, \overline{z}^i) \in \phi^i(\overline{p}, \overline{q})$, we obtain that $V^i(x^i) < V^i(\overline{x}^i)$. Thus, $(\overline{x}^i, \overline{z}^i)$ is an optimal choice for agent i in $C^i(\overline{p}, \overline{q})$, which concludes the proof. □

Proof of Theorem 4 Corollaries'

PROOF OF COROLLARY 1. Notice that, given a compact set $\mathbb{P}_1 \subset (\mathbb{R}_+^{\mathcal{L}} \setminus \{0\})^S$, the allocation

$$(\widehat{x}, (\widehat{z}_k)_{k\in\mathcal{J}_u}) := \max_{p\in\mathbb{P}_1} \sum_{s\in S} \left(\sum_{k\notin\mathcal{J}_u} R_{s,k}(p_s)/R_{s,j}(p_s) \right) \widehat{e}_j$$

satisfies the conditions of Definition 6. Thus, agents can super-replicate financial payments just by investing in asset j. □.

PROOF OF COROLLARY 2. Since all assets are ultimate sources of liquidity, the results of Lemma 3B are not necessary to ensure equilibrium existence. Thus, Assumption (A4) can be dispensed. □.

28. Assumption (A5)(i) is only required to ensure that $\left[(\overline{p}, \overline{q}) \gg 0 \ \wedge \ (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}} \in \Omega(\overline{p}, \overline{q})\right] \Longrightarrow (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}} \in \mathrm{int}(\mathbb{K})$. Notice that, if we change the upper bound $\overline{\Omega}$ for an arbitrary positive number in the definition of $\mathbb{K}$, then all the other arguments in the proof of Theorem 4 still hold.

PROOF OF COROLLARY 3. Since assets are either backed by physical collateral or have payments measured in units of a non-perishable commodity (i.e., a commodity $l \in \mathcal{L}$ such that $Y_s(l,l) > 0$, $\forall s \in S$), it follows that agents can super-replicate the financial payments by buying commodities. Thus, Lemma 3B can be ensured without assuming that assets in $\mathcal{J}_u$ have unrestricted investment (Assumption (A4)(i)). $\square$

Proof of Theorem 5

Let $\mathcal{P} = \{(p_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} : \|p_s\|_\Sigma \leq 1, \forall s \in \mathcal{S}\}$.

Given $N > 0$, define

$$\widehat{\mathbb{K}}(N) := \left[0, 2\widehat{W} + N\right]^{\mathcal{L}\times\mathcal{S}} \times \left[-\widehat{\Omega}, \#\mathcal{I}\,\widehat{\Omega}\right]^{\mathcal{J}},$$

where

$$\widehat{W} := \left(\#J\,\#\mathcal{I}\,\widehat{\Omega} + \sum_{(s,l)\in\mathcal{S}\times\mathcal{L}} \sum_{i\in\mathcal{I}} W^i_{s,l}\right)\left(1 + \max_{s\in S}\max_{p_s\in\mathcal{P}_1}\sum_{j\in\mathcal{J}} R_{s,j}(p_s)\right),$$

$$\widehat{\Omega} := 2 \sup_{(p,q)\in\mathcal{P}\times\mathbb{R}_+^{\mathcal{J}}:\,(p,q)\gg 0}\ \sup_{(x^i,z^i)_{i\in\mathcal{I}}\in\Omega(p,q)} \sum_{i\in\mathcal{I}} \|z^i\|_\Sigma.$$

Notice that, Assumption (A3') guarantees that $\widehat{\Omega}$ is finite.

Let $\Psi_{(M,N)}$ be the correspondence obtained by replacing $\mathbb{K}$ by $\widehat{\mathbb{K}}(N)$ in the definition of Ψ_M. Hence, identical arguments to those given in the proof of Theorem 4 ensure that, under Assumptions (A1')-(A3'), and even with $\mathcal{J}_u = \emptyset$, the results of Lemmata 1B and 2B still hold. Thus, for each $(M,N) \gg 0$, the correspondence $\Psi_{(M,N)}$ has a non-empty set of fixed points.

Our objective is to ensure that, for (M,N) large enough and by analogous arguments to those used in the proof of Lemma 4B, the fixed points of $\Psi_{(M,N)}$ are competitive equilibria for our economy. Therefore, we need to determine upper bounds for prices $(\overline{q}_j)_{j\notin\mathcal{J}_u}$.

LEMMA 3C. *Under Assumptions (A1')-(A3') and (A5), let $(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}})$ be a fixed point of $\Psi_{(M,N)}$ satisfying $\overline{x}^i_{s,l} < 2\widehat{W}$, $\forall (s,l) \in \mathcal{S}\times\mathcal{L}$. Then, there is $\widehat{Q} > 0$ such that, for N large enough, $\overline{q}_j \leq \widehat{Q}$, $\forall j \notin \mathcal{J}_u$.*

PROOF. For any $i \in \mathcal{I}^*$, let $\rho^i \in (0,1)$ such that $2\widehat{W}\rho^i = 0.25 \min_{l\in\mathcal{L}} W^i_{s,l}$. Hence, Assumption (A3) and the monotonicity of preferences imply that,

$$V^i(\overline{x}^i) \leq V^i(2\widehat{W}(1,\dots,1)) < V^i\left(2\widehat{W}(1,\dots,1) + \tau^i(\rho^i, 2\widehat{W}), \left(\frac{W^i_s}{2}\right)_{s\in S}\right).$$

Fix $j \notin \mathcal{J}_u$ and $i = i(j) \in \mathcal{I}^*$ satisfying Assumption (A5)(ii). Then, there is $z^i \geq 0$ such that $z^i_j > 0$ and $-(0,z^i) \in \Phi^i(p,q)$, $\forall (p,q) \in \mathbb{P}$. Since Φ^i has convex values and $(0,0) \in \Phi^i(\overline{p},\overline{q})$, it follows that $-(0,\varepsilon z^i) \in \Phi^i(\overline{p},\overline{q})$, $\forall \varepsilon \in [0,1]$. Also, Assumption (A1')(iii) ensures that there is $\varepsilon^i \in (0,1)$ such that, for any state of nature $s \in S$, $\varepsilon^i \max_{p_s\in\mathcal{P}_1} \sum_{k\in\mathcal{J}} R_{s,k}(p_s)\, z^i_k < \left(\min_{l\in\mathcal{L}} W^i_{s,l}\right)/2$.

Then, for each $N > \widehat{N} := \max_{i \in \mathcal{I}^*} \|\tau^i(\rho^i, 2\widehat{W})\|_\Sigma$ we have that

$$\left(\left(2\widehat{W}(1,\dots,1) + \tau^i(\rho^i, 2\widehat{W}), \left(\frac{W_s^i}{2} \right)_{s\in\mathcal{S}} \right), -\varepsilon^i z^i \right) \in \Phi^i(\overline{p}, \overline{q}) \cap \widehat{\mathbb{K}}(N).$$

Consequently, as $(\overline{x}^i, \overline{z}^i)$ is an optimal choice for agent i in $C^i(\overline{p}, \overline{q}) \cap \widehat{\mathbb{K}}(N)$, it follows that

$$2\widehat{W}\|\overline{p}_0\|_\Sigma + \overline{p}_0 \cdot (\tau^i(\rho^i, 2\widehat{W}) - w_0^i) > \varepsilon^i \, \overline{q} \cdot z^i \geq \varepsilon^i \overline{q}_j z_j^i,$$

which implies that $\overline{q}_j \leq (2\widehat{W} + \widehat{N})/(\varepsilon^i z_j^i)$. Since $i = i(j)$ was fixed, we can consider

$$\widehat{Q} := \max_{j \notin \mathcal{J}_u} \frac{2\widehat{W} + \widehat{N}}{\varepsilon^{i(j)} z_j^{i(j)}}. \qquad \square$$

LEMMA 4C. *Under Assumptions (A1')-(A3') and (A5), fix $(M, N) \gg (\widehat{Q}, \widehat{N})$. Then, each fixed point of $\Psi_{(M,N)}$ is a competitive equilibria.*

This result follows from analogous arguments to those made in the proof of Lemma 4B.

Proof of Corollary 4

For each agent $i \in \mathcal{I}^\diamond$, let $\widetilde{V}^i : \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \to \mathbb{R}$ be the function defined by

$$\widetilde{V}^i(x^i) = \left(v_{l(i)}^i \left(\min\left\{ x_{0,l(i)}^i, 2W_{0,l(i)} \right\} \right) + \rho^i \max\left\{ x_{0,l(i)}^i - 2W_{0,l(i)},\, 0 \right\} \right) + v^i((x_{0,r}^i)_{r\neq l(i)},\, (x_s^i)_{s\in S}),$$

where $l(i) \in \mathcal{L}$ is the commodity that satisfies the condition (i) of Corollary 4, $W_{0,l(i)} = \sum_{h\in\mathcal{I}} w_{0,l(i)}^h$, and $\rho^i \in \partial v_{l(i)}^i(2W_{0,l(i)})$.[29]

Consider an economy where we replace $\{V^i\}_{i\in\mathcal{I}^\diamond}$ with $\{\widetilde{V}^i\}_{i\in\mathcal{I}^\diamond}$. Then, Assumption (A5)(i) holds. Hence, Theorem 5 guarantees that there exists a competitive equilibrium $((\overline{p}, \overline{q}), (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}) \in \mathbb{P} \times \mathbb{E}^{\mathcal{I}}$ for this auxiliary economy. To conclude the proof it is sufficient to ensure that the allocation $(\overline{x}^i, \overline{z}^i)$ satisfies $V^i(\overline{x}^i) \geq V^i(x^i)$, $\forall i \in \mathcal{I}^\diamond$, $\forall (x^i, z^i) \in C^i(\overline{p}, \overline{q})$.

Suppose, by contradiction, that there exists $i \in \mathcal{I}^\diamond$ and $(x^i, z^i) \in C^i(\overline{p}, \overline{q})$ such that $V^i(x^i) > V^i(\overline{x}^i)$. The market feasibility of consumption allocations ensures that, for every $l \in \mathcal{L}$, $\overline{x}_{0,l}^i < 2W_{0,l}$. Therefore, $\widetilde{V}^i(\overline{x}^i) = V^i(\overline{x}^i)$ and there exists $\lambda \in (0,1)$ such that $\lambda\overline{x}_{0,l}^i + (1-\lambda)x_{0,l}^i < 2W_{0,l}, \forall l \in \mathcal{L}$.

Since $\lambda(\overline{x}^i, \overline{z}^i) + (1-\lambda)(x^i, z^i) \in C^i(\overline{p}, \overline{q})$, we conclude that,

$$\widetilde{V}^i(\overline{x}^i) = V^i(\overline{x}^i) = \min\{V^i(\overline{x}^i), V^i(x^i)\} < V^i(\lambda\overline{x}^i + (1-\lambda)x^i) = \widetilde{V}^i(\lambda\overline{x}^i + (1-\lambda)x^i).$$

This contradicts the optimality of $(\overline{x}^i, \overline{z}^i)$ for agent i in the auxiliary economy. $\square$

29. As customary, $\partial v_{l(i)}^i(x) := \{\rho \in \mathbb{R} : v_{l(i)}^i(y) - v_{l(i)}^i(x) \leq \rho(y-x),\, \forall y \geq 0\}$ denotes the super-differential of $v_{l(i)}^i$ at point x. Notice that, as $W_0 = (W_{0,l})_{l\in\mathcal{L}} \gg 0$, the monotonicity and concavity of $v_{l(i)}^i$ ensure that $\partial v_{l(i)}^i(2W_{0,l(i)})$ is a non-empty subset of $\mathbb{R}_+$.

Chapter 3

Conclusions

The results developed in this work contribute to understand the role of financial restrictions in two-period general equilibrium models with incomplete markets. First, analyzing financial restrictions generated by the existence of asymmetric information. Second, introducing segmentation and exclusion in financial markets in a model with endogenous trading constraints that define consumption and portfolio sets.

There are some directions to continue the research. As noted in Chapter 1, the continuity in the utility functions is not easy to justify when endogenous update is considered. The lack of such an important property for equilibrium existence is actually an active area of research, mainly in game theory. In addition, there is room to analyze the effect that changes in information may produce in terms of welfare, i.e. the value of information.

Furthermore, general environments as the one presented in Chapter 2 allow some interesting cases that are pointed in our examples. Among them, of particular interest is the one given by the segmentation and exclusion in exchange structures (Example 9). Precisely, when such form of restrictions are considered, asset pricing is not an straight-forward development neither the equilibrium computation. Those are areas we continuously explore to develop.

Given that equilibrium existence is ensured, another possible extension is related with the welfare analysis when the financial structure changes. In this line, it is possible study the relation between financial structures and some cooperative arguments behind. For instance, the endogenous formation of groups that arise or affect the structures in equilibrium.

As a complement of this, we do not give here an stability analysis. For this purpose, it is required to analyze how the set of equilibria changes to small perturbations in exogenous parameters that characterizes agents and, therefore, economies. The most used concept in the literature is the regularity of equilibrium. Their cost is differentiability. Nevertheless, another concept of stability can be studied without requiring it. For instance, essential stability as introduced by Fort [1950] for fixed points of correspondences. Indeed, it is also matter of today's and future research to give a general parameterization of economies. That is, analyze stability in different dimensions out of the classical explored in the literature, i.e. endowments and preferences. For instance, stability with respect to perturbations on the financial structure.

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