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OLYMPICS, EXTERNALITIES AND MATCHING: THREE ESSAYS IN MICROECONOMICS

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[1]This work is based entirely on the publish paper by Contreras, J.L. and Corvalan, A. *Olympic Games: No legacy for sports. Economics Letters, (2014), 122(2), 268-271.*

Preface

This thesis consists of three essays in microeconomics. The first is related to empirical analysis. Particularly, this work study the impact on future performance in Summer Olympic Games for a country which has been host of the event. This study can be seen as another argument in favour of the literature that studies the relationship between economic impact and such events. It is found that the positive effect of being host –measured as medal count– disappears immediately in the next period. This result is robust to a set estimation methods. Given this last the economic impact (due to being host) takes much more importance, since it looks like is the only real benefit of being host.

The other two essays are related to economic theory. One of these works sets a general equilibrium model where negative externalities exist. Specifically, it is considered that private consumption generates public bad which can be mitigated by the production of public goods. However, public goods suffer the free rider problem. Thus, this model not only includes voluntary provision, but also mandatory provision through income taxes. This analysis focused on Pareto allocations which are induced by an extension of the Samuelson condition. Through numerical examples it is shown the effect on welfare of re-distribution of wealth, heterogeneity of preferences and technology shocks when income taxes induce Pareto allocations. A set of numerical examples show similar results to the ones claimed by the neutrality theorem. Also, it is found that technologies that reduces the negative impact of private consumption may reduce the welfare of some agents. Moreover, in some cases a technology change of this kind may reduce social welfare. The latter is a point which has not been considered in the literature so far.

The second theory work deals with the distinction between individuals and coalitions in a general equilibrium framework. So far, the general equilibrium literature makes no distinction between both situations. Under this framework it is shown that this distinction matters. Here the incentives to form coalitions comes from the capacity of reducing rivalry in consumption. The way the reduction acts is not anonymous. That is, the reduction in rivalry is endogenous to the coalition formed. Here, reduction in rivalry can be modelled as externalities on consumption in a more general framework than the literature of household formation has done so far. Through a series of examples it is shown the difficulties to find an stable equilibrium when individuals have many options to form coalitions. Also, numerical examples show that the relationship between coalition formation and inequality is not so clear.

1

Introduction To Essays

1 Olympic Games No Legacy For Sports

Economists are skeptical about the economic benefits of mega-events such as the Summer Olympic Games (Baade and Matheson 2002, Rose and Spiegel 2012, Billings and Holladay 2012). An immediate benefit for the country whose city is hosting the games is an increase in its total medal count (Bernard and Busse 2004, Johnson and Ali 2002). The question of this paper is about the effect of such sport success on posterior competitions. If games produce a "big push" for sports in a country, then future economic benefits will be derived directly from becoming more competitive[1] and indirectly through health and social indicators associated with sports. Indeed, lasting sports success is a strong argument given in favor of hosting. The Legacy Action Plan for London 2012 promised *"making the UK a world-leading sporting nation"* and *"inspiring a new generation of young people"*.

This paper studies the "ex-host effect", defined as the effect of hosting the Summer Olympic Games on the total number of medals in the subsequent games[2]. Does hosting create a positive structural break or the process does reverse to the mean? And in case of a reversion, how lasting is the effect? Hosting involves an advantage in terms of being local, but it also compromises resources in sport training and facilities, which are likely to have a more permanent effect.

We explore a dynamic panel for all Summer Olympic Games during the post war period. When we use the standard specification of Bernard and Busse (2004) with reversion to the mean, we observe that ex-host effects are overestimated in about ten medals by the authors, suggesting that the decay in success is faster than that. In fact, our results indicate that Olympic success on

[1] Sport activities generate as much as 3.0 percent of GDP in OECD countries, being an industry bigger than agriculture and not so far behind manufacturing. In the US, the size of the sports industry was $152 billion in 1995, and supported an additional $259 billion in economic activity (Meek, 1997).

[2] Vagenas and Vlachokyriakou (2012) described as ex-host effect to the medal success of countries that at least once hosted a Summer Olympic Game. Unsurprisingly, this effect is positive given that hosts are typically selected among the more competitive countries.

medals fades away immediately after hosting. Additionally, to disregard the possibility of being capturing an effect previous to the hosting, we compare the ex-host effect with total medals before the countries were elected hosts, showing that there is no significant difference between them.

We confront here the issue that selection of a hosts city is endogenous. If IOC is more likely to award the games to countries with strong athletic programs that outperform their economic size and population size, then estimation will be biased. We follow the strategy proposed by Rose and Spiegel (2012) and Billings and Holladay (2012), who use as counterfactuals the countries whose cities also bid for the Olympics, but unsuccessfully. In all cases we confirm that the "ex-host effect" is null, that is, there is no legacy for sports in Summer Olympic Games.

2 MITIGATION EQUILIBRIUM

This work sets a general equilibrium model with non depletable negative externalities (public bad). At the same time these externalities can be mitigated by public goods produced in the economy.[3] In particular, this model considers: (i.) the source of negative externalities is the aggregate private consumption of the households; and (ii.) every household is myopic with respect to its own effect on negative externalities. That is, every household does not consider the negative impact of his own consumption when choosing an allocation.

Given this framework, compulsory mitigation mechanism appears as a commonly used policy.[4] In this model means the inclusion of income taxes. Thus, we have that income taxes are collected to finance production of public goods. It is also considered that public goods do not generate any kind of negative externalities.[5]

For the one hand, the set-up of this model follows the framework given by the game-theoretic model of Bergstrom, Blume and Varian (1986), but generalized to more than one commodity and more than one public good. But on the other hand, unlike Bergstrom et al (1986), this model takes over the issues of the free rider problem associated to public goods provision.[6] In particular, this model not only considers voluntary contribution, but also considers income taxes to finance production of public goods. Also, unlike the regular literature about public goods,

[3]There is a vast literature on general equilibrium with public goods and/or externalities. See, for instance, Foley (1970); Groves and Ledyard (1977); Shafer and Sonnenschein (1976); Mas-Colell (1980).

[4]See Musgrave (1969)

[5]Or consider that negative externalities associated to public goods are cancelled by its own positive externalities. Further analysis can be made on this topic.

[6]See Musgrave, (1969)

this model do not include prices associated to public goods. This is because, as in Bergstrom et al (1986), all agents contributes commodities to produce public goods using production technologies that are known by all.[7] However, at the equilibrium we can obtain as an outcome the average cost of public goods.

Other possible issue that needs to be clarified –before the model is presented– has to do with the ways of compensate. Under this framework there are no mitigation mechanisms that uses commodities to compensate. Among the reasons to consider this, there is the fact that this mechanisms has to do with re-allocation of private bundles among individuals, which under our assumption implies no reduction on externalities. Another reason is that, even if we assume that there exist commodities capable of compensate,[8] there are relevant aspects to consider before to implement this mechanism: (i.) the technology available to produce private goods (feasibility); and (ii.) the rivalry nature of private goods along with the size of the economy would make it increasingly expensive to mitigate. In a more realistic ground, we claim that compensating a public bad through private commodities might create problems associated to *social fairness*, since different Pareto allocations deliver different compensation allocations.

This model considers that the technology available to produce public goods belongs to all households. Hence, in this model there is no firms trying to maximize profits. Instead, there are technologies producing efficiently according to the resources available. This specific consideration follows the literature of voluntary provision of public goods, specifically, Bergstrom, Blume and Varian (1986). In a more realistic situation, we say that public firms are in charge to produce efficiently all public goods according to some fiscal plan. And these firms are willing to accept any contribution from households to produce public goods.

It is possible -under standard assumptions- prove the existence of an equilibrium defined as *Mitigation equilibrium*. Also, optimality condition is set. This last is an extension of the known *Samuelson Condition*. In this case we obtain that under the existence of externalities the amount of public goods are greater than the optimal amount of public goods according to the former condition. This implies that, when there is a new technology capable of reducing the negative effects of private consumption, there will exist a reduction in the amount of public goods. In the absence of externalities we go back to the original condition.

Finally, considering an Augmented Lagrangian method, it is possible to obtain some numerical examples which are not only *Mitigation equilibriums*, but also Pareto allocations. This

[7]This is a generalization mentioned by Bergstrom, Blume and Varian (1986).

[8]Not generate negative externalities.

allow to analyse the effect on welfare when optimal tax policies are being used. Some of these examples show an equivalent result to the neutrality theorem.[9] In this case, there is no change on voluntary contribution of the households,[10] but given the redistribution of initial wealth we have that mandatory contribution change but leaves all households with the same after-tax wealth. At the same time this new tax profile does not change the total resources available to produce public goods. Therefore, after redistribute initial endowment and consequently change income taxes profile, it is obtained the same Pareto allocation.

Other examples show that *eco-friendly* technology shocks may induce –through an optimal tax policy– a new Pareto superior allocation. However, these examples also show the possibility that a technology shock may induce a new Pareto allocation which is inferior to the original one. Moreover, depending on the type of shock it may also happens that a *eco-friendly* shock may induce a Pareto allocation which is inferior and at the same time generates more externalities than the original situation. However, this last result is due to the specification of the function associated to the generation of externalities and the level of tax burden that the optimal tax policy induce. That is, under this particular case it is not true that for any given level of private consumption a technological change reduces the negative effects. However, when there exist a robust technology shock,[11] we observe that the optimal tax policy induces an allocation that not generate more externalities than before, but still we may observe some welfare losses for some households, or even more, social welfare loss. The welfare losses comes from heterogeneous technology shocks that induce new price equilibriums and income taxes that affect negatively some households. If this negative effect is greater than the improvement on other households we obtain the social loss mentioned before.

3 Coalitions And Endogenous Mixed Goods

This is an exchange economy where individuals can form coalitions motivated by the possibility of reducing rivalry in consumption. Following the idea proposed by Ellickson, Grodal, Scotchmer, and Zame (1999) on the framework of club theory, the way reduction in rivalry acts depend on the coalition formed. That is, the benefits of forming a coalition are not anonymous, instead it depends on the composition of its members.

Under this framework, every coalition will represent their members on the market. Thus, we

[9] See Bergstron, Blume and Varian (1986).

[10] The Nash solutions that numerical examples show are all in the line of not to contribute voluntarily.

[11] That is, for any given amount of private consumption, the technology change will reduce the negative effect of private consumption.

have that each coalition not only considers wealth and preferences of its members in order to determine its demand, but also the reduction in rivalry in consumption that every commodity may have due to coalition formation. Also, given the existence of reduction in rivalry in consumption, coalitions with more than one agent (we call it non-trivial coalitions) not only determine demands over commodities, but also determine the distribution of use of those commodities. Following Musgrave (1969) and Holtermann (1972), the term *distribution of use* instead of *allocations of commodities* is used when we are talking about the allocations that will be imputed to each member of the coalition. The reason comes from the fact that this model treats commodities as private goods from the markets' point of view, but also treats them as mixed goods from the coalition point of view. In other words, they are not necessarily public goods, neither pure private goods inside the coalition.

An important aspect that this model delivers instead of the standard general equilibrium literature is that the latter make no distinction between coalitions (for example, households) and single consumers. Although it seems no relevant, in fact it is, because making a distinction allow inquiries such as: (i.) what to demand on the market; (ii.) how to allocate the usage of the allocation among members; (iii.) formation and stability; and (iv.) the interaction between economic decisions –item (i.)– and social decisions –item (ii.) and item (iii).

It is important to clarify that this model differs from the classical club theory literature (see Berglas, 1976; Ellickson et al, 1999) since, for the one hand, in our framework there is no distinction between private and public goods, but only the reduction in rivalry in consumption in each commodity. Instead, club theory is characterized by individuals shopping for both, club memberships[12] and private consumption. On the other hand, following the arguments presented by Gersbash and Haller (2010) about household formation and club theory, models like the former are equivalent to the latter in the absence of consumption externalities. Here, it is shown that reduction in rivalry in consumption is equivalent to consumption externalities.[13]

Since we follow the work done by Gersbach and Haller (2011), a relevant part of this work deals with the equivalence between a model with non-negative externalities in consumption and a model with reduction in rivalry in consumption. This last makes sense since the literature establishes a close relationship between mixed goods and externalities.[14] First, it is shown that this model satisfy, what Gersbach et al (2011) defined as, *Large Group Advantage*; Which means the existence (at any given price) of a subset of allocations capable to make better-off each

[12]That allow to consume a bundle of local public goods with or without congestion.
[13]In a broader sense than Gersbash and Haller (2011)
[14]See Holtermann (1972)

member of the group. Second, some examples are made in order to set a relation between the two models. It is found that in all cases it is always possible to find a non-negative externality function capable to replicate the equilibrium allocation of a model with reduction in rivalry in consumption. But, when the reduction takes a non-linear form we observe that reduction in rivalry in consumption can also consider cases where all members generate negative externalities. This last is not considered in the existence proposition of the model of Gersbach et al (2011).

Following Ellickson et al (1999) we have that reduction in rivalry consumption is not anonymous. That is, the formation of a particular coalition defines which commodities will be affected by the reduction, and the way this reduction will act. Therefore, when an individual evaluates to be part of a coalition, not only the size of the coalition matters, but also the characteristics of the members. Unlike Ellickson et al (1999) or Gerbash et al (2011), in this framework we model a formation of a group with the only purpose of take advantage of reduction in rivalry. This means that, no matter the coalition to which belong an individual, his preferences over the commodities do not change. However, similar to those authors, this also induce to have preferences over coalitions since reduction in rivalry is endogenous. Also, unlike Gersbash et al (2011) we do not include in this version of the model *Pure Group Externalities*, which means that one can additively separate the pure consumption effect of forming a coalition from the pure group effect. We consider that the framework of this work is more suitable for analysing group formation considering the commonly observed situation where a group of people share the consumption of a particular commodity instead of consuming individually. For example, depending on the composition of a particular coalition, there may be a commodity (house or apartment, for example) used by the coalition *as if* it were a pure public good, or at least a commodity with some degree of reduction in rivalry in consumption. However, we acknowledge that a more general framework must consider group externalities. At last, it is important to mention that this framework considers only the possibility to belong one coalition at the time. Further research on this matter looks interesting.

Finally, through numerical examples, we show the impact on social welfare as well as on inequality (measured as Gini) when coalition formation is allowed. It is shown that under this particular set-up, formation of coalitions always induce a social welfare improvement, but some coalition may suffer a welfare loss. Also, the relation between coalitions and inequality is not so clear. These two cases are due to changes in equilibrium prices induced by the formation of non-trivial coalitions.

2

Olympic Games No Legacy For Sports[1]

1 Abstract

Countries whose cities host the Summer Olympic Games increase significantly their success during the competition. We study whether such effect is lasting or not. We compute the effect of hosting on the total number of medals in the subsequent games. To confront the issue that the selection of the host city is endogenous, we use a natural counterfactual: countries whose cities also bid for the Olympics but were not selected by the International Olympic Committee. In all cases, we find that Olympic success on medals fades away immediately after hosting.

2 Data and Econometric Methods

We study Summer Olympic Games, henceforth the "games", from 1948 to 2012. The election of this period allows comparability with previous works and it provides a large number of countries for each of the games. The base sample is unbalanced since participating nations increased from 59 in 1948 to 204 countries in 2012.

The dependent variable is the medal total count by country, data publicly available from the International Olympic Committee (IOC). Our results are robust to the use of alternative measures of sport success, such as the use of gold medals or the share of medals. The main explanatory variable is "hosting". Cities bid for hosting, and hosts are decided in elections where each non candidate country casts a vote. For the purposes of our paper, "hosting" takes the value one for the country whose city hosted the event. In the period 1948-2012, 14 countries hosted 17 games, with US, UK and Australia hosting twice[2]. We have also data on unsuccessful bidders for the Olympics, which are natural candidates as counterfactuals of actual hosts.

[1] This work is based entirely on the publish paper by Contreras, J.L. and Corvalan, A. *Olympic Games: No legacy for sports. Economics Letters, (2014), 122(2), 268-271.*

[2] Since 1960 there could be only one city candidate per country.

As for controls, we use log of GDP per capita and log of population, both from Madisson (2003), and a dummy index equal to one if the country was a Socialist regime in 1982. Our results are robust to several other measures of these controls.[3]

The baseline specification is the following:

$$m_{i,t} = \alpha\, m_{i,t-1} + \gamma_1\, h_{i,t} + \gamma_2\, h_{i,t-1} + \beta\, x_{i,t} + \theta_i + \theta_t + \varepsilon_{i,t}\,, \tag{1}$$

where $m_{i,t}$ is the total medals count by country i at the games in period t. Hosting $h_{i,t}$ is equal one if country i hosted the games in period t and zero otherwise.[4] Parameters θ_t and θ_i are time and country effects respectively, and $\varepsilon_{i,t}$ is a disturbance term, which we assume heteroskedastic. Estimation uses robust standard errors or clustered by country.

The dynamic specification (1) was studied by Bernard and Busse (2004) with $\gamma_2 = 0$. In that case, the ex-host effect, namely the effect of hosting over the number of medals in the next game, is given by reversion to the mean, i.e. the term $\alpha\,\gamma_1$. We add the term γ_2 to study any additional effect. First, we look for significance in γ_2. Second, we quantify the total ex-host effect as $\alpha\,\gamma_1 + \gamma_2$ and we test whether this term is significantly different from zero or not.[5]

A possible concern about our results is that positive effects on medals emerged even before the hosting period. The election of the hosting city occurs on average 7 years in advance, and so the countries favored could induce a sporting acceleration in the games previous to being host. To consider this possibility, we control for the pre-trend in medals two periods before hosting. We change specification (1) to a static panel, replacing the lag term by the two dummies $h_{i,t+1}$ and $h_{i,t+2}$ that takes the value one for host one and two periods before hosting, respectively.[6] To suppress the lagged term makes interpretation easier and it does not introduce any change in our results.

The static specification is the following:

$$m_{i,t} = \gamma_1\, h_{i,t} + \gamma_2\, h_{i,t-1} + \gamma_3\, h_{i,t+1} + \gamma_4\, h_{i,t+2} + \beta\, x_{i,t} + \theta_i + \theta_t + \varepsilon_{i,t} \tag{2}$$

[3] We used the Penn World Tables for alternative indicator of GDP and population. Instead of Socialism, we tried Soviet countries, or a time-varying index of Socialism, without change in our results. The inclusion of Policy IV for democracy was also used, but it effect vanishes once we control for Socialism.

[4] Data is every four years.

[5] We use non-linear hypothesis tests where the standard errors are computed through the delta method.

[6] For instance, Mexico City hosted the games in 1968 and the host election was done in 1963. Accordingly, we have $h_{i,t+2}$, $h_{i,t+1}$, $h_{i,t}$ and $h_{i,t-1}$ equal to one in 1960, 1964, 1968 and 1972, respectively, and zero otherwise.

In specification (2), $h_{i,t+2}$ accounts for the historical trend of the country and $h_{i,t+1}$ for the effect once it wins the election. The purpose of the exercise is to compare all the dummy coefficients with the historical trend, that is, we provide hypothesis tests for the change of the coefficient compared with $h_{i,t+2}$.

We provide several estimation methods for each specification. First, we consider fixed effects OLS. As the dynamic panel (1) is estimated over a short time period, we also compute the GMM Arellano Bond (1991) estimator using the correction provided by Kiviet (1995).[7] Second, we use a Tobit model in order to confront the issue that a large fraction of countries has no medals in several periods.[8] Our baseline estimator uses pooled regressions with errors clustered by countries. When we move to a fixed effect Tobit estimation, however, the results are inconsistent because of the incidental parameter problem, that is, the fixed effects cannot be omitted through differencing. A practical solution is to parameterize the specific effects, an approach referred as "correlated random effects". For the static case, Chamberlain (1980) proxies the country effects for the average value of the observable independent variables plus a random effect term. Wooldridge (2005) generalizes the method for the dynamic case, adding the initial condition of the lagged dependent variable to the parameterization.[9] Akay (2012) shows that the approximation works well for unbalanced panels of moderately long duration.

3 BASE LINE RESULTS

Table 1 summarizes our results for the specification (1). We provide results for different estimation methods (see description in the Table 1), and for that we use the whole sample of countries participating in the games (columns 1 to 4) as well as a subsample of hosting countries (columns 5 to 8). The last row provides the p-value for the hypothesis test $\alpha\,\gamma_1 + \gamma_2 = 0$.

Table 1 exhibits two interesting findings. First, we observe that γ_2 is negative and significant in all the regressions, justifying its incorporation. A standard mean reversion model, with $\gamma_2 = 0$, overestimates the total medals of an ex-host country. Second, we test whether this negative effect is counteracting the positive lagged effect of hosting or not. We observe that in all columns the simple product $\alpha\,\gamma_1$ gives absolute values close to the negative ex-host coefficient

[7] See also, Bun and Kiviet (2003), and Bruno (2005).

[8] Our data is not actually censored but a case called "corner solution" by the literature, a problem that can also be handled by Tobit.

[9] See Benhabib et al (2013) for the use of these estimators in a panel with similar features. Results also hold for the simpler unconditional FE Tobit that introduces country dummies in the Tobit estimation.

γ_2. In fact, the test does not reject the hypothesis that the aggregate effect $\alpha\,\gamma_1 + \gamma_2$ is zero, suggesting that the reversion to the mean is immediate.[10]

TABLE 1. BASELINE RESULTS[11]

	Sample: All Countries				Sample: Hosting Countries			
	OLS/FE	GMM	Tobit	CorrRe	OLS/FE	GMM	Tobit	CorrRe
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$m_{i,t-1}$	0.550	0.611	0.349	0.252	0.47	0.53	0.64	0.47
	(0.108)	(0.026)	(0.014)	(0.014)	(0.138)	(0.140)	(0.04)	(0.04)
$h_{i,t}$	14.089	14.013	5.518	5.375	14.10	14.02	10.09	10.75
	(2.786)	(1.217)	(1.245)	(0.647)	(2.696)	(2.469)	(2.54)	(1.65)
$h_{i,t-1}$	-5.763	-6.813	-4.160	-3.042	-5.13	-6.09	-7.95	-5.65
	(2.624)	(1.867)	(1.573)	(0.691)	(2.523)	(3.039)	(2.29)	(1.83)
Observations	1253	1107	1253	1253	161	161	161	161
Countries	135	135	135	135	12	12	12	12
	F-test	Chi2	F-test	F-test	F-test	Chi2	F-test	F-test
p-value	0.614	0.291	0.723	0.426	0.659	0.74	0.929	0.951

Next we turn into the issue that sport success may be induced before hosting. The purpose of specification (2) is to compare all the dummy coefficients with the historical trend. Table 2 provides our results. The last three rows in the table describe the p-values for the linear hypothesis test $h_{i,t+1} = h_{i,t+2}$; $h_{i,t} = h_{i,t+2}$; and $h_{i,t-1} = h_{i,t+2}$.

Results in Table 2 confirm that the ex-host effect is null. The p-values do not reject the null hypothesis, but only when the country is local. That is, host effect is significantly different from pre-election trend at 1 percent in all regressions. On the contrary, the effect after the announcement of hosting $h_{i,t+1}$ and the ex-host effect $h_{i,t-1}$ are not significantly different from $h_{i,t+2}$. That is, four years after hosting the games, countries obtain statistically the same number of medals than before being elected as host.[12]

[10]We also check long run effects. In non-reported regressions, we change the dummy $h_{i,t-1}$ for a variable that equals to one in all periods after the first hosting event. We confirm that the ex-host effect is null immediately after the games and it does not turn positive into the future.

[11]Note: dependent variable in Total Medals. Estimators: Fixed Effect OLS with clustered errors in columns (1) and (5); GMM estimates Arellano-Bond in a first stage and provides the Kiviet correction in a second stage, in columns (2) and (6); Tobit with clustered errors in columns (3) and (7); and correlated RE, in columns (4) and (8), uses the Wooldridge approximation for fixed effects, with mean and initial values not reported. All the Tobit regressions report the marginal effects. Standard errors in parenthesis, year dummies and controls (log GDP per capita, Log of Population and Socialist) not reported.

[12]Results in Tables 1 and 2 are robust to the exclusion of boycotted games in 1980 and 1984.

TABLE 2. PRE-TRENDS IN MEDALS[13]

	Sample: All Countries			Sample: Hosting Countries		
	OLS/FE	Tobit	CorrRe	OLS/FE	Tobit	CorrRe
	(1)	(2)	(3)	(4)	(5)	(6)
$h_{i,t+2}$	3.53	5.02	2.00	3.72	4.78	4.25
	(2.526)	(1.679)	(0.748)	(2.870)	(2.55)	(1.76)
$h_{i,t+1}$	6.23	4.53	2.74	5.36	4.86	5.32
	(2.936)	(1.408)	(0.790)	(2.780)	(2.20)	(1.86)
$h_{i,t}$	20.56	10.99	6.02	19.80	16.94	11.94
	(4.937)	(3.197)	(0.765)	(4.617)	(3.88)	(1.79)
$h_{i,t-1}$	6.77	6.28	2.59	5.77	6.74	4.32
	(3.007)	(1.555)	(0.753)	(2.318)	(2.03)	(1.79)
Observations	1321	1321	1122	157	157	149
Countries	135	135	135	12	12	12
	F-test	F-test	F-test	F-test	F-test	F-test
(t+1)						
p-value	0.141	0.336	0.466	0.54	0.95	0.66
(t)						
p-value	0.002	0.001	0.000	0.01	0.00	0.00
(t-1)						
p-value	0.351	0.345	0.579	0.61	0.57	0.98

4 ENDOGENEITY

In the previous section, our statistical model compares hosts with non-hosts for every given year. A question that arises immediately is that countries are not randomly chosen to host the Olympics.[14] One way to get at this issue is to compare the medal patterns of host countries with those that bid unsuccessfully for the games (Rose and Spiegel, 2012 and Billings and Holladay, 2012). The IOC elects host cities in several stages: from all applicant cities, the IOC Executive Board selects a number of applicants to be considered candidate cities, and non candidate members vote to select among them. Our implicit assumption for the use of candidate cities as counterfactuals is that cities that compete for hosting are already self-selected and they are similar in a number of characteristics. As bidders are decided in sequential rounds of voting,

[13]Note: dependent variable in Total Medals. Estimators: Fixed Effect OLS with clustered errors in columns (1) and (4); Tobit with clustered errors in columns (2) and (5); and correlated RE, in columns (3) and (6), uses the Chamberlain approximation for fixed effects, with mean values not reported. All the Tobit regressions report the marginal effects. Standard errors in parenthesis, year dummies and controls (log GDP per capita, Log of Population and Socialist) not reported.

[14]Effectively, the data indicates that host cities come mostly from countries with strong Olympic perfomance.

we select the counterfactual "bidder" as the country whose city lost the final round against the winner hosting city.

For estimation, we construct the variable bidder similar to the variable host: if a city bids unsuccessfully for hosting the games until the last round, we place a one in that games to the country whose city is bidding. We estimate the specification (2), adding the counterfactual bidder at the same four time periods. Table 3 describes our results. The hypothesis tests provided in the last rows of the table compare coefficients for hosts and bidders in each period, in order to investigate whether there are significant departures or not.

Table 3 exhibits all the coefficients for hosts and bidders at different times. Tests for the periods at t+2 and test t+1 show that pre-trends are not significantly different between the two groups,[15] which is consistent with the assumption of using bidders as counterfactuals. In the year of hosting, on the contrary, the local country shows a positive and significant effect as compared to the bidders. Finally, the hypothesis that effects are different at t-1 is rejected in all regressions. Once again, we did not find support to the existence of a positive ex-host effect.

These results confirm the previous ones. Olympic hosting countries increase significantly their success in the games, but that effect is not lasting. We compare the ex-host effect with the pre-trend of medals in the country and with the contemporary success of countries that unsuccessfully bid for hosting. In all cases, Olympic success on medals fades away immediately after hosting.

[15] When we estimate for all sample with the "correlated random effects" the test for period t+1 is not rejected at 10%.

TABLE 3. HOST AND BIDDERS[16]

	Sample: All Countries			Sample: Hosting and Bidders Countries		
	OLS/FE	Tobit	CorrRe	OLS/FE	Tobit	CorrRe
	(1)	(2)	(3)	(4)	(5)	(6)
$h_{i,t+2}$	4.42	4.75	2.18	4.31	6.68	4.48
	(2.539)	(1.684)	(0.752)	(2.781)	(2.94)	(1.73)
$h_{i,t+1}$	5.93	3.87	3.00	5.00	5.97	5.86
	(2.814)	(1.797)	(0.800)	(2.892)	(3.17)	(1.84)
$h_{i,t}$	21.34	11.34	6.50	20.33	19.44	13.02
	(4.808)	(3.122)	(0.770)	(4.680)	(4.67)	(1.74)
$h_{i,t-1}$	8.23	7.59	3.16	6.87	10.58	5.49
	(3.036)	(2.247)	(0.761)	(2.474)	(2.78)	(1.77)
$b_{i,t+2}$	2.50	5.41	1.77	1.75	6.07	3.13
	(5.998)	(1.207)	(0.783)	(5.218)	(2.98)	(1.80)
$b_{i,t+1}$	2.45	5.82	1.05	1.48	6.07	1.84
	(3.084)	(2.412)	(0.816)	(2.536)	(3.10)	(1.89)
$b_{i,t}$	3.01	4.86	1.89	2.89	5.78	3.34
	(3.137)	(1.401)	(0.810)	(2.111)	(1.99)	(1.88)
$b_{i,t-1}$	8.97	9.86	1.72	7.81	13.14	3.05
	(2.761)	(3.687)	(0.797)	(3.401)	(4.55)	(1.87)
Observations	1321	1321	1122	197	197	187
Countries	135	135	135	15	15	15
	F-test	F-test	F-test	F-test	F-test	F-test
(t+2)						
p-value	0.71	0.63	0.69	0.63	0.86	0.57
(t+1)						
p-value	0.24	0.21	0.07	0.16	0.97	0.11
(t)						
p-value	0.00	0.00	0.00	0.00	0.00	0.00
(t-1)						
p-value	0.85	0.37	0.17	0.82	0.52	0.31

5 CONCLUDING REMARKS

This paper shows that host countries of the Olympic Summer Games win significantly more medals than predicted by their size and wealth, but this effect fades away immediately after

[16]Note: dependent variable in Total Medals. Estimators: Fixed Effect OLS with clustered errors in columns (1) and (4); Tobit with clustered errors in columns (2) and (5); and correlated RE, in columns (3) and (6), uses the Chamberlain approximation for fixed effects, with mean values not reported. All the Tobit regressions report the marginal effects. Standard errors in parenthesis, year dummies and controls (log GDP per capita, Log of Population and Socialist) not reported.

hosting. The result open several questions for future research, related to whether the channels through which host countries win additional medals are economic or not. The fleeting nature of the medal count bump suggests that this sort of shock has no transmission mechanism, that is, it may be an athlete fans effect rather than an investment of resources effect.

Bibliography

[1] Akay Alpaslan, “Finite-sample comparison of alternative methods for estimating dynamic panel data models”, Journal of Applied Econometrics, 27(7): 1189-1204, (2012).

[2] Arellano, Manuel, and Stephen Bond. “Some tests of specification for panel data: Monte Carlo evidence and an application to employment equations.” The Review of Economic Studies 58.2 (1991): 277-297.

[3] Baade, R. A., and V. Matheson, “Bidding for the Olympics: Fool’s Gold?”, in Transatlantic Sport, ed. by C. P. Barros, M. Ibrahim, and S. Szymanski, chap. 6, pp. 127-151. Edward Elgar (2002).

[4] Benhabib J., A. Corvalan and M.Spiegel, “Income and Democracy: Evidence from Non Linear Estimations”, Economic Letters, 118(3): 489-492, (2013).

[5] Bernard, A., and Busse, M., “Who Wins the Olympic Games: Economic Resources and Medal Totals”, The Review of Economics and Statistics, Vol. 86, No. 1 (2004), pp. 413-417.

[6] Billings S. and J.Scott Holladay, “Should Cities Go for the Gold? The Long-Term Impacts of Hosting the Olympics”, Economic Inquiry 50(3): 754-772, (2012).

[7] Bruno, Giovanni SF. “Approximating the bias of the LSDV estimator for dynamic unbalanced panel data models.” Economics Letters 87.3 (2005): 361-366.

[8] Bun, Maurice JG, and Jan F. Kiviet. “On the diminishing returns of higher-order terms in asymptotic expansions of bias.” Economics Letters 79.2 (2003): 145-152.

[9] Chamberlain G., “Analysis of covariance with qualitative data”, Review of Economic Studies 47 (1980), pp. 225-238.

[10] Johnson, D. and Ali, A., “A Tale of Two Seasons: Participation and Medal Counts at the Summer and Winter Olympic Games”, Social Science Quarterly, Volume 85, Number 4, (2004).

[11] Kiviet, Jan F. “On bias, inconsistency, and efficiency of various estimators in dynamic panel data models.” Journal of econometrics 68.1 (1995): 53-78.

[12] Maddison, A., The World Economy: Historical Statistics, Paris: OECD, (2003).

[13] Meek, A., “An estimate of the size and supported economic activity on the sports industry in the United States”. Sport Marketing Quarterly, 6 (4), 15-21, (1997).

[14] Rose Andrew and Spiegel Mark, “The Olympic Effect”, The Economic Journal, 121: 652-677, (2012).

[15] Vagenas G. and E. Vlachokyriakou, “Olympic medals and demo-economic factors: Novel predictors, the ex-host effect, the exact role of team size, and the Population-GDP model revisited”, Sport Management Review 15 (2012): 211-217.

[16] Wooldridge Jeffrey M., “Simple Solutions to the Initial Conditions Problem in Dynamic, Non-Linear Panel Data Models with Unobserved Heterogeneity”, Journal of Applied Econometrics 20 (2005): 39-54.

3

Mitigation Economy

1 Abstract

This is a general equilibrium model with provision of public goods and negative externalities. These last are generated by private consumption made by households, and can be mitigated through public goods. Production of public goods depends on voluntary contribution and resources collected through income taxes. Under standard assumptions we prove existence of equilibrium. Also Pareto condition is set. This last is an extension of the Samuelson condition. Some numerical examples are made. It is also shown that "eco-friendly" technology shocks induce an optimal tax policy that may bring a Pareto inferior allocation. Moreover, depending on the kind of shock, it may also induce more negative externalities.

2 An Economy With Public Goods Production And Negative Externalities

Let a *Mitigation Economy* be an exchange economy with transfers and public goods production, in which households consumption generates negative externalities to all agents, $(\mathcal{E}_{G,\xi})$. Let $L = \{1, \ldots, L\}$ be the set of commodities and let $\mathbb{X} \subseteq \mathbb{R}^L_+$ be its consumption space. Also, there exist a finite number of K types of negative externalities, each represented by a non-decreasing function $\xi_j : \mathbb{R}^L_+ \to \mathbb{R}_+$, with $j \in K$. In order to mitigate these externalities public goods are provided through production technologies, which are commonly known to all agents. Public goods production is financed by taxes and voluntary provision.

Commodities can be consumed or to be used to produce public goods. Let $M = \{1, \ldots, M\}$ be the set of public goods and let $\mathbb{M} \subseteq \mathbb{R}^M_+$ be its consumption space. Each $m \in M$ is produced using a given technology. Let $Q_m(x) = \{G_m \in \mathbb{R}_+ : \; G_m - f_m(x) \leq 0\}$ be the set of feasible units of public good m when the inputs are $x \in \mathbb{X}$, where $f_m : \mathbb{R}^L_+ \to R_+$ is the production function associated to public good m. Under this framework, every agent in order to define the allocation of public goods they want, need to estimate how much provision of inputs already exist. Although looks like as a demanding consideration, we claim that transparency acts, public discussions on fiscal budget and donations laws -among others- helps to inform everybody about not only how public goods are produced, but also how are financed. Also, it is worth to mention that this kind of consideration is not new, Bergstrom, Blume and Varian (1986) in their seminal paper mentioned as an extension of their basic model.[1]

Let $H = \{1, \ldots, H\}$ be the set of households. Every $h \in H$ demand commodities and contribute to produce public goods. Every household has endowments of commodities $w_h \in \mathbb{X}$. We denote by $x_h = (x_{h,\ell})_{\ell \in L} \in \mathbb{X}$ the private consumption plan of household h. Similarly, we denote by $G_h = (G_{h,m})_{m \in M} \in \mathbb{M}$ the public goods consumption plan of household h. It is assumed that for every household there is a function $u_h : \mathbb{X} \times \mathbb{M} \to \mathbb{R}_+$ that represents his preferences over those goods. However, private consumption of households, $x_H = \sum_{h \in H} x_h$, generates negative externalities that also affect his welfare. For that, let define $U_h(x, G_h) = u_h(x_h, G_h) - \xi(x)$ as the effective welfare for household h, where $\xi(x) = \sum_{j \in K} \xi_j(x)$. Every $h \in H$ is required to pay an income tax $t_h \geq 0$. In addition, any $h \in H$ may contribute to produce public goods. We denote the household voluntary contribution plan to public good m as $(x_{h,m}, G_{h,m}) \in \mathbb{X} \times \mathbb{M}$, such that $(x_{h,m} + x_{-h,m}, G_{h,m}) \in \text{Gr}\,[Q_m]$, where $x_{h,m} = (x_{h,\ell,m})_{\ell \in L} \in \mathbb{X}$ is the contribution made by h to produce public good m, and $x_{-h,m} = \left(\sum_{j \in H \setminus \{h\}} x_{j,\ell,m} + x_{F,\ell,m} \right)_{\ell \in L} \in \mathbb{X}$ is other's contribution.[2]

There exist a tax structure, $(t_h)_{h \in H} \in \mathbb{R}^H_+$, capable of financing public goods production. Once taxes are collected and contributions are provided[3], a fiscal authority (call it F) uses that resources to implement a production plan,

$$(x_{F,m} + x_{H,m}, G_{F,m}) \in \text{Gr}\,[Q_m], \quad \forall\, m \in M$$

where $x_{F,m} = (x_{F,\ell,m})_{\ell \in L} \in \mathbb{X}$ is the contribution made by the fiscal authority to produce public good m, and $x_{H,m} \in \mathbb{X}$ is the aggregate contribution of households to produce public

[1] See page 31 from *Bergstrom, T., Blume, L., & Varian, H. (1986). On the private provision of public goods. Journal of public economics, 29(1), 25-49.*

[2] Note that the amount of contribution of each household depends on the aggregate contribution of others.

[3] Where other's contribution –households– is defined as,

$$x_{H,m} = \left(\sum_{h \in H} x_{h,\ell,m} \right)_{\ell \in L} \in \mathbb{X}$$

good m. Preferences of the fiscal authority depends on $G_F = (G_{F,m})_{m\in M} \in \mathbb{M}$. It is assumed that there is a function $\Psi_F : \mathbb{M} \to \mathbb{R}_+$ that represents those preferences.

It is worth to mention that, similar to literature associated to public good provision,[4] in this model every agent not only set his own production plan of public goods, but in fact define the equilibrium allocation of all public goods in the economy. This implies that the amount of voluntary contribution of every agent depends on his expectation about the stock of inputs. Also, notice that this set-up does not consider the existence of firms pursuing profits when producing public goods. Instead, what defines the amount of public goods to be produced are provision of inputs, society's preferences and the current technology.

Let define household's *choice set* as,

$$\mathcal{C}_h\left(p, x_{-h,M}\right) = \left\{\left(x_h, (G_{h,m})_{m\in M}, (x_{h,m})_{m\in M}\right) \in \mathbb{X}\times\mathbb{M}\times\mathbb{X}^M : \textit{(C1) and (C2) hold.}\right\};$$

where,

$$p\cdot x_h + p\cdot\sum_{m\in M} x_{h,m} + t_h - p\cdot w_h \le 0 \tag{C1}$$

$$(\,x_{h,m} + x_{-h,m}, G_{h,m}) \in \text{Gr}\left[Q_m\right], \quad \forall m\in M \tag{C2}$$

with, $x_{-h,M} = (x_{-h,m})_{m\in M} \in \mathbb{R}_+^{L\times M}$. Thus, household h chooses an allocation

$$\left(x_h, (G_{h,m})_{m\in M}, (x_{h,m})_{m\in M}\right) \in \mathcal{C}_h\left(p, x_{-h,M}\right),$$

taken as given commodity prices p and others' contribution $x_{-h,M}$, in order to maximize his utility level $u_h\left(x_h, G_h\right)$. Notice that, this setting follows the idea that every household assumes that his own production of negative externalities (through consumption) is negligible relative to the size of the economy. Thus, every h maximize $u_h(x_h, G_h)$ instead of $U_h(x, G_h)$.

Analogously, let define fiscal authority's *choice set* as,

$$\mathcal{C}_F\left(p, x_{H,M}\right) = \left\{\left((G_{F,m})_{m\in M}, (x_{F,m})_{m\in M}\right) \in \mathbb{M}\times\mathbb{X}^M : \textit{(C1') and (C2') hold.}\right\};$$

where,

$$p\cdot\sum_{m\in M} x_{F,m} - \sum_{h\in H} t_h \le 0; \tag{C1'}$$

$$(x_{F,m} + x_{H,m}, G_{F,m}) \in \text{Gr}\left[Q_m\right], \quad \forall m\in M \tag{C2'}$$

[4] See for example, Bergstrom, Blume, and Varian (1986); and Andreoni (1988)

with, $x_{H,M} = (x_{H,m})_{m \in M} \in \mathbb{R}_+^{L \times M}$. Thus, fiscal authority F chooses,

$$\left((G_{F,m})_{m \in M}, (x_{F,m})_{m \in M}\right) \in \mathcal{C}_F(p, x_{H,M}),$$

in order to maximize his utility level $\Psi_F(G_F)$, given commodity prices p, and other's voluntary contribution, $x_{H,M}$.

3 MITIGATION EQUILIBRIA

Given the framework described above, we define a *Mitigation Equilibrium* as a competitive equilibrium for $\mathcal{E}_{G,\xi}$ such that every agent maximize his utility taking as given other's strategies, as well as prices.

DEFINITION 1. A *Mitigation Equilibrium* for the economy $\mathcal{E}_{G,\xi}$ is given by:

1. A vector of prices $\overline{p} \in \mathbb{R}_+^L$;
2. Individual allocations,

$$\left(\overline{x}_h, (\overline{G}_{h,m})_{m \in M}, (\overline{x}_{h,m})_{m \in M}\right) \in \mathbb{X} \times \mathbb{M} \times \mathbb{X}^M, \quad \forall\, h \in H;$$

3. A fiscal plan,

$$\left((\overline{G}_{F,m})_{m \in M}, (\overline{x}_{F,m})_{m \in M}\right) \in \mathbb{M} \times \mathbb{X}^M;$$

such that,

(i) For each $h \in H$, the allocation $\left(\overline{x}_h, (\overline{G}_{h,m})_{m \in M}, (\overline{x}_{h,m})_{m \in M}\right)$ belongs to,

$$\arg\max\left\{u_h(x_h,\, G_h) :\ \left(x_h, (G_{h,m})_{m \in M}, (x_{h,m})_{m \in M}\right) \in \mathcal{C}_h(\overline{p}, \overline{x}_{-h,M})\right\}$$

with $\overline{x}_{-h,M} = (\overline{x}_{-h,m})_{m \in M}$.

(ii) For the fiscal authority F, the allocation $\left((\overline{G}_{F,m})_{m \in M}, (\overline{x}_{F,m})_{m \in M}\right)$ belongs to,

$$\arg\max\left\{\Psi_F(G_F) :\ \left((G_{F,m})_{m \in M}, (x_{F,m})_{m \in M}\right) \in \mathcal{C}_F(\overline{p}, \overline{x}_{H,M})\right\}$$

with $\overline{x}_{H,M} = (\overline{x}_{H,m})_{m \in M}$.

(iii) Market clearing conditions hold,

$$\sum_{h \in H} \left(\overline{x}_h + \sum_{m \in M} \overline{x}_{h,m} \right) + \sum_{m \in M} \overline{x}_{F,m} = \sum_{h \in H} w_h;$$

$$\overline{G}_{j,m} = \overline{G}_m, \quad \forall\, (j,m) \in H \cup \{F\} \times M.$$

where $\overline{G}_m$ stands for the total amount of public good m available in this economy.

A first comment from this equilibrium concept has to do with the kind of utility function that the fiscal planner will have. Until now we have not explicitly mention how preferences of the central planner are correlated with households' preferences over public goods. Second, although it is difficult to be talking about a social welfare function that correctly measures the preferences that members of the society have over public goods. Here, we consider this analysis as a starting point from which we study the effects of mitigation policies on Pareto allocations that consider this conceptual framework.[5] A third comment has to do with the non-existence of a vector prices associated to public goods. Most of the literature associated to public goods deals with the problem of charging a personalized price by unit of public good consumed or an anonymous price as well.[6] Here, without firms producing public goods but rather technologies available that satisfy society's need over public goods, is that we do not need to set any valuation for public goods, instead we can obtain the average cost of producing it as an outcome from the equilibrium.[7] Finally, we explicitly set in each *choice set* other's contribution *as if* it were a mandatory subsidy to be used exclusively on public goods production.[8] The following theorem determine the existence of a *Mitigation Equilibrium*.

[5] A proper extension might consider a model where voting is included, for example.

[6] See Foley (1970) and Groves and Ledyard (1977).

[7] In this sense we have that the average cost of public good at the equilibrium will be $\overline{G}_m/(\overline{p} \cdot \overline{x}_m)$, where $\overline{x}_m$ stands for the optimal inputs allocation.

[8] See Bergstrom, Blume and Varian (1986).

THEOREM 1. *Any economy $\mathcal{E}_{G,\xi}$ that satisfies the following assumptions has a Mitigation Equilibrium;*

(A1) *For any $h \in H$, u_h is continuous, strictly increasing, strictly concave on (x_h, G_h);*

(A2) *The function $\Psi_F(G_F) = \sum_{h\in H} u_h(x_h, G_F)$.*

(A3) *For all $m \in M$ we have that $f_m : \mathbb{R}^L_+ \to R_+$ is a continuous, increasing and concave function, with $f_m(0) = 0$ for all $m \in M$.*

(A4) *For any $h \in H$, $w_h \in \mathbb{R}^L_{++}$;*

(A5) *The income tax $t = \big(t_h\big)_{h\in H} \in \mathbb{R}^H_+ \setminus 0$;*

Proof. The proof follows the regular steps of the equilibrium existence theorem in a generalized game (Debreu, 1952). Thus, given Assumption *(A1)-(A2)* we have that agent's objective functions are continuous and quasi-concave. Following Debreu (1952), we also introduce a new player – an auctioneer who will look for prices at $\Delta = \{p \in \mathbb{R}^L_+ : \big|\big|p\big|\big|_1 = 1\}$ that clears markets– whose objective function will also satisfy continuity and quasi-concavity.

On the other hand, fixing an uniform upper bound for all commodities, considering that $\mathrm{Gr}[Q_m]$ is closed (for all $m \in M$), and Assumption *(A3)*; We are able to prove that the correspondence of admissible strategies for households and Fiscal authority turns out to be continuous, with compact, convex and non-empty values. By construction we have that the auctioneer's strategy set –Δ– will be, by definition, continuous with compact, convex and non-empty values. Thus, Berge's Maximum Theorem ensures that players' best-reply response correspondence are upper hemicontinuous with non-empty, compact and convex values. Therefore, we apply Kakutani's Fixed Point Theorem to the correspondence of optimal strategies in order to find a Cournot-Nash equilibrium for the generalized game associated to the truncated economy. The last part of the demonstration deals with the fact that the Cournot-Nash equilibrium obtained is also a *Mitigation Equilibrium* for the economy $\mathcal{E}_{G,\xi}$. This last is done by contradiction. For that we use Assumptions *(A1)* and *(A2)*. For further details see the appendix. *Q.E.D.*

4 ON THE PARETO ALLOCATION

In this section we set the conditions for the allocations that belong to the Pareto set for $\mathcal{E}_{G,\xi}$. For the rest of the analysis it is assumed that there is an interior solution and utility functions, production technology as well as externalities are differentiable.

Thus, the optimization problem associated to social planner's problem will be,

$$\max \left\{ U_h\left(x_h,\, G\right) : \left((x_h)_{h \in h}, (G_m)_{m \in M}, (x_m)_{m \in M}\right) \in \left(\mathbb{X}_+^H \times \mathbb{M} \times \mathbb{X}_+^M\right) \right\};$$

subject to,

$$\begin{aligned}
(\lambda_j): &\quad u_j\left(x_j,\, G\right) - \xi\left(\sum_{h \in H} x_h\right) \geq V_j, \quad \forall j \in H \setminus h \\
(\mu_\ell): &\quad \sum_{h \in H} x_{h,\ell} + \sum_{m \in M} x_{m,\ell} \leq \sum_{h \in H} w_{h,\ell}, \quad \forall \ell \in L \\
(\rho_m): &\quad G_m \leq f_m\left(x_m\right), \quad \forall m \in M
\end{aligned}$$

For simplicity let ξ_ℓ be the term associated to the marginal disutility that produces private consumption of $\ell \in L$ on every $h \in H$, and let $f_{m,\ell}$ be the term associated to the marginal productivity of $\ell \in L$ on $m \in M$. Analogously, let $u_{h,j}$ be the marginal utility of j on household h.

Hence, the first order conditions are,

$$\begin{aligned}
(1) &\quad u_{h,\ell} - \xi_\ell - \sum_{j \in H \setminus h} \lambda_j\, \xi_\ell - \mu_\ell = 0, \quad \forall\, \ell \in L \\
(2) &\quad \lambda_j\, u_{j,\ell} - \xi_\ell - \sum_{j \in H \setminus h} \lambda_j\, \xi_\ell - \mu_\ell = 0, \quad \forall\, \ell \in L \\
(3) &\quad -\mu_\ell + \rho_m\, f_{m,\ell} = 0, \quad \forall\, (\ell, m) \in L \times M \\
(4) &\quad u_{h,m} + \sum_{j \in H \setminus h} \lambda_j\, u_{j,m} - \rho_m = 0, \quad \forall\, m \in M
\end{aligned}$$

From (1) and (2) we conclude that $\lambda_j = (u_{h,\ell}/u_{j,\ell})$. Solving for (1) we obtain that at the optimum,

$$u_{h,\ell}\left(1 - \sum_{j \in H}\left(\xi_\ell / u_{j,\ell}\right)\right) = \mu_\ell > 0, \quad \forall\, \ell \in L; \tag{5}$$

A first conclusion from these optimality conditions is that at the optimum $\sum_{j \in H}\left(\xi_\ell / u_{j,\ell}\right) \in (0,1), \forall\, \ell \in L$. Now, considering the expression for λ_j and the first order conditions (3) and (4), we obtain the following conditions which are satisfied by any Pareto allocation,

$$f_{m,\ell}\, u_{h,\ell} \left(\sum_{j \in H} \frac{u_{j,m}}{u_{j,\ell}} \right) = \mu_\ell, \quad \forall\, (\ell, m) \in L \times M \tag{6}$$

CLAIM 1. *An allocation* $\left((x_h)_{h \in h}, (G_m)_{m \in M}, (x_m)_{m \in M}\right)$ *in a economy with mitigation through public good provision that satisfy,*

$$\sum_{j \in H} (u_{j,m}/u_{j,\ell}) = \frac{1 - \sum_{j \in H} (\xi_\ell / u_{j,\ell})}{f_{m,\ell}}, \qquad \forall (\ell, m) \in L \times M \tag{7}$$

will be called a Pareto allocation.

Proof. Considering equation (5) and (6) we obtain the expression above. And considering the optimality condition $\sum_{h \in H} (\xi_\ell / u_{h,\ell}) \in (0,1), \forall\, \ell \in L$, we assure that $\sum_{j \in H} (u_{j,m}/u_{j,\ell}) > 0$, $\forall (\ell, m) \in L \times M$. $\square$

Notice that optimality condition (7) is an extension of the well known *Samuelson Condition*. In this case the condition also consider the marginal rate of transformation of consumption to negative externality. The inclusion of this term will induce an equilibrium allocation that bring more public goods than the situation without externalities.

Also considering equation (4) and the expression for λ_j we have that, fixing for any public good m, the marginal rate of substitution between commodity ℓ and ℓ' is equal to the ratio of the marginal rate of substitution of public good m with those commodities,

$$\frac{u_{h,\ell}}{u_{h,\ell'}} = \frac{\sum_{j \in H} \left(u_{j,m}/u_{j,\ell'}\right)}{\sum_{j \in H} (u_{j,m}/u_{j,\ell})}, \quad \forall\, (h, m) \in H \times M \tag{8}$$

Following a similar procedure on (6), we obtain that – for any $\ell \in L$– at the optimum the marginal rate of transformation between public good m and m' is equal to the ratio of the marginal rate of substitution of public good m and m' with commodity ℓ,

$$\frac{f_{m,\ell}}{f_{m',\ell}} = \frac{\sum_{j \in H} \left(u_{j,m'}/u_{j,\ell}\right)}{\sum_{j \in H} (u_{j,m}/u_{j,\ell})}, \quad \forall\, \ell \in L \tag{9}$$

Thus, considering (8) and (9) we conclude that at the optimum,

$$
\frac{f_{m,\ell}}{f_{m,\ell'}} = \frac{f_{m',\ell}}{f_{m',\ell'}}, \quad \forall\,(\ell,\,\ell') \in L,\,(m,\,m') \in M \tag{10}
$$

Finally, considering (7) and (8) we also obtain the following equality at the optimum,

$$
\frac{u_{h,\ell}}{u_{h,\ell'}} = \frac{1 - \sum_{j\in H}\left(\xi_{\ell'}/u_{j,\ell'}\right)}{1 - \sum_{j\in H}\left(\xi_{\ell}/u_{j,\ell}\right)}\frac{f_{m,\ell}}{f_{m,\ell'}}, \quad \forall\,(h,(\ell,\,\ell'),\,m) \in H \times L^2 \times M \tag{11}
$$

Equations (10) and (11) set the expressions for the marginal rate at the optimum. Since we assume that production of public goods does not generate any negative externality the optimality condition is the commonly known equality among the marginal rate of technical substitution. In the case of households we observe that the marginal rate of substitution between commodities must be equal to the marginal rate of technical substitution corrected by the externalities that generates the private consumption of those commodities.

5 Numerical Examples

The following examples are calculated using an Augmented Lagrangian method with general lower-level constraints (See Andreani, Birgin, Martínez and Schuverdt, 2007). This algorithm allow us to calculate all the variables of interest, such as price equilibrium, private allocations, public goods production as well as income tax schedule. Moreover, given the optimality condition defined in the past section, it is possible to calculate an equilibrium which is also Pareto. The latter allows to study the effects of heterogeneity, distribution of wealth, as well as technological changes associated with the production of negative externalities on the Pareto allocations.

Example 1. Let consider an economy with the following characteristics: (i) $|H| = 2$; (ii) $|M| = 1$; and (iii) $|L| = 2$. The utility for household 1 and 2 are represented by the following functions,

$$
u_1\,(x_1,\,G) = (x_{1,1})^{0.3}\;G^{0.7} + (x_{1,2})^{0.2}\,;
$$
$$
u_2\,(x_2,\,G) = (x_{2,2})^{0.3}\;G^{0.7} + (x_{2,1})^{0.2}\;;
$$

where $x_{h,\ell}$ stands for private consumption of commodity ℓ made by household h. The production set is defined as,

$$\text{Gr}[Q] = \left\{(z, G) \in \mathbb{R}^2_+ \times \mathbb{R}_+ : \, G - (z_1)^{0.5} \, (z_2)^{0.5} \leq 0\right\}$$

The welfare function is the one from Assumption *(A2)*. Finally, the externality function is set as,

$$\xi(x) = \left(\frac{1}{2}\right) \left(\sum_{h \in H} x_{h,1}\right)^2 + \left(\frac{1}{2}\right) \left(\sum_{h \in H} x_{h,2}\right)^2$$

The results from Table 1 show outputs from different *Mitigation Equilibrium* at different levels and distribution of wealth.[9] Notice from the first two rows that the re-distribution of wealth not only does not alter the equilibrium prices, but also does not affect private allocations neither welfare. Here, income tax is proportional to initial wealth. So, when individual 1 goes from an initial endowment of $w_1 = (2, 2)$ to $w_1 = (1, 1)$, we observe that the mandatory transfer to finance public good production reduces such that the after tax wealth is the same in both cases. Hence, given that equilibrium prices does not change, the allocation of private consumption and the welfare are the same. This result goes in the line of neutrality theorem of Bergstrome et al (1986), but in this case we have to consider the existence of a tax policy that does not affect the wealth after tax of each household. Following Theorem 1 of Bergstrom et al (1986) this affirmation will be,

"..., the redistribution of wealth among households and Fiscal authority makes that no one loses more after tax income than its original contribution.[10] *After the re-distribution there is a new Nash equilibrium*[11] *in which every consumer change the amount of his voluntary contribution.*[12] *In this new equilibrium each consumer consumes the same amount of the public good and the private good that he did before the re-distribution."*

Therefore, each household makes a contribution and at the new equilibrium each one consumes the same amount of public good and private good that he did before the re-distribution. Therefore, in the case we have that the effect of the redistribution of wealth is cancelled by the optimal tax policy.

[9]Although the optimization problem allow the possibility to contribute voluntarily, all numerical solutions resulted in zero voluntary contribution. In the appendix Table A.1. show the error terms associated to the optimization problems of Table 1.

[10]Which in this case is zero to both households.

[11]The same in our examples.

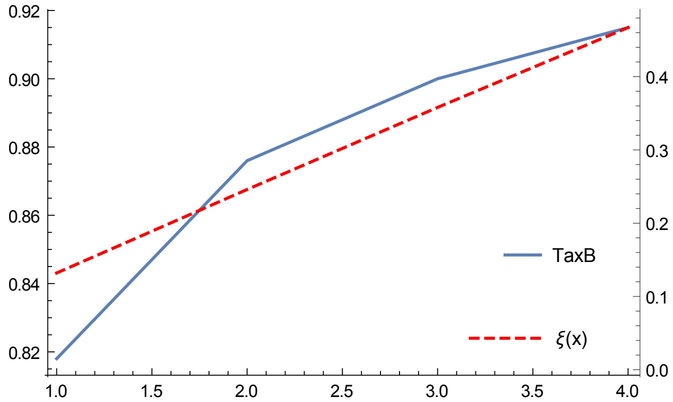
[12]In our case zero

Table 1. Mitigation Equilibriums From Example 1

$(w_{1,1}, w_{1,2})$	$(w_{2,1}, w_{2,2})$	Ψ_F	u_1	u_2	ξ	p_1	t_1	t_2	TS	G
(2,2)	(1,1)	3.67	2.02	2.02	0.19	0.50	1.78	0.78	0.85	2.56
(1,1)	(2,2)	3.67	2.02	2.02	0.19	0.50	0.78	1.78	0.85	2.56
(4,4)	(4,4)	7.20	4.07	4.07	0.47	0.50	3.66	3.66	0.91	7.32
(3,3)	(3,3)	5.84	3.28	3.28	0.36	0.50	2.70	2.70	0.90	5.40
(2,2)	(2,2)	4.41	2.45	2.45	0.25	0.50	1.75	1.75	0.88	3.50
(1,1)	(1,1)	2.89	1.58	1.58	0.13	0.50	0.82	0.82	0.82	1.64

From the last four rows of Table 1 we observe that increments on economy's wealth leads to a greater welfare given the increase in public goods and private consumption. But at the same time we also observe an increase in the generation of externalities. That is, an optimal income tax that efficiently mitigate the effects of externalities does not necessarily reduce the amount of externalities as the economy increases its wealth. Figure 1 shows the relation (from these four rows) between increments on household's wealth and increments of tax burden[13] and externality production. The left vertical axis represent the values for tax burden, and the right-vertical axis represent the amount of externality produced.

Figure 1. Tax Burden & Production of Negative Externality



Notice the level and the increment of tax burden (TaxB), going from 82% to 91% of $\left(p \cdot \sum_{h \in H} w_h\right)$ when there are increments in economy's total wealth. As the economy becomes wealthier the optimal tax policy tend to converge the tax burden at a level above 90%. Although this example has no intention to replicate real values associated to tax burden. It makes a point to discuss, particularly, the real costs associated to the implementation of an optimal mitigation policy. Meanwhile, the pattern associated to the production of externalities becomes increasing

[13]In-house tax burden is similar.

as the economy grows. This last situation results from the increment of after-tax wealth induced by the optimal reaction of the tax policy when economy's wealth increases.

Example 2. Let consider an economy with the following characteristics: (i) $|H| = 2$; (ii) $|M| = 2$; (iii) and $|L|$=1. The preferences for household 1 and 2 are represented by the following functions,

$$u_1(x_{1,1}, G_1, G_2) = x_{1,1}^{(1-\alpha_1-\beta_1)} (G_1)^{\alpha_1} (G_2)^{\beta_1};$$
$$u_2(x_{2,1}, G_1, G_2) = x_{2,1}^{(1-\alpha_2-\beta_2)} (G_1)^{\alpha_2} (G_2)^{\beta_2};$$

where $x_{h,\ell}$ is the private consumption of commodity ℓ made by household h. The endowment distribution is set as $(w_1, w_2) = (2, 1)$. The production sets are defined as,

$$\text{Gr}[Q_1] = \left\{(z_1, G_1) \in \mathbb{R}_+ \times \mathbb{R}_+ : G_1 - (z_1)^{0.5} \leq 0\right\};$$
$$\text{Gr}[Q_2] = \left\{(z_2, G_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : G_2 - (z_2)^{0.75} \leq 0\right\};$$

The preferences of Fiscal authority is the one from Assumption *(A2)*. Finally, the externality function is $\xi(x) := (1/2) \left(\sum_{h\in H} x_{h,1}\right)^2$.

The results show on Table 2[14] are similar to the ones from Table 1. From the first two rows we have that the equivalence to the neutrality theorem still exist. That is, given a re-distribution of wealth that induce a change on the income tax schedule, we observe the same Pareto allocation as before.

Table 2. Mitigation Equilibriums From Example 2

(w_1, w_2)	(α_1, β_1)	(α_2, β_2)	Ψ_F	u_1	u_2	ξ	t_1	t_2	TaxB	G_1	G_2
(2,1)	(0.1,0.2)	(0.3,0.4)	0.91	0.44	0.69	0.11	1.72	0.81	0.84	0.89	1.51
(1,2)	(0.1,0.2)	(0.3,0.4)	0.91	0.44	0.69	0.11	0.72	1.81	0.84	0.89	1.51
(2,1)	(0.3,0.4)	(0.3,0.4)	1.25	0.78	0.64	0.08	1.73	0.86	0.86	0.93	1.51
(2,1)	(0.1,0.2)	(0.1,0.2)	0.55	0.52	0.32	0.14	1.64	0.82	0.82	0.79	1.59

From the last two rows we observe different Pareto allocations induced by different preferences (homogeneous), given a endowment distribution. In this particular case we observe that

[14]The terms of error associated to the optimization problems of Table 2 can be see it in Table A.2. in the appendix.

with households which prefer public goods over private commodity, the tax burden is greater (approximately %4 more) than the case where households prefer private commodity over public goods. This also implies that in the latter there is more externalities generated (almost the double). Also, similarly to all examples of Table 1 we observe that the tax burden is a significant share of total wealth, leaving households a feasible set of private consumption very restricted.

6 Externalities And Exogenous Technological Change

In this section we study the effects on welfare and production of negative externalities when there is an exogenous technological change that reduce the negative impact of private consumption.

It is commonly assumed that a technological shock will improve welfare. This last generally rests on the assumption that there is a reduction on negative externalities. However, under this set-up, technological improvements may induce a reduction on the welfare. Moreover, it may increase the amount of negative externalities produced by the economy.

Under regular assumptions over preferences and production technologies, Table 3 shows examples where technological improvements lead to a reduction on the welfare and at the same time an increment on negative externalities. That is, given a technological improvement, the new optimal tax policy induce a new *Mitigation Equilibrium* which is Pareto inferior[15] and generates more externalities. However, this increase on negative externalities depends exclusively on two aspects: (1.) the particular shock considered to reduce the generation of externalities; and (2.) the level of aggregate private consumption that allow the optimal tax policy. Thus, taking into account these considerations, Table 4 shows the results of another type of technological shock, which in this case satisfy the condition that for any given level of consumption this new technology reduces externalities. The results from this last table show that technology improvements reduce negative externalities, but still may happen a loss of welfare.

To understand why a technological improvement may generates these effects, let consider that all assumptions from the equilibrium definition are satisfied. Moreover, let consider that u_h be twice differentiable. Also, let assume that we are at some equilibrium defined by Definition 1 which is also Pareto, since we assume that the tax policy is optimal.[16] Thus, the welfare function evaluated at the optimum is,

[15]With respect to the original allocation, which is associated to an externality function that generates more negative effects, given a private consumption.

[16]That is, the tax policy induce an allocation that satisfy the Samuelson condition.

$$\sum_{h\in H} U_h\big(\overline{x},\,\overline{G},\,\xi\big(\overline{x},a\big)\big)=\sum_{h\in H} u_h\big(\overline{x}_h,\,\overline{G}\big)-|H|\cdot\xi\big(\overline{x},a\big);$$

where $\overline{x}=\sum_{h\in H}\overline{x}_h$ and $a=(b_\ell,c_\ell)_{\ell\in L}\in\mathbb{R}^L_{++}\times\mathbb{R}^L_{++}$ stands for the productivity parameters associated to the production of negative externalities. Where,

$$\xi(x,a)=\sum_{\ell\in L}\left(\frac{1}{c_\ell}\cdot\left(\sum_{h\in H} x_{h,\ell}\right)^{(1/b_\ell)}\right).$$

Since this analysis considers that Pareto conditions are satisfied, we have that $a\in\mathbb{R}^L_{++}\times\mathbb{R}^L_{++}$ determines the optimal tax income $t=(t_h)_{h\in H}\in\mathbb{R}^H_+$ in some manner. Moreover, we assume that the optimal tax income is a twice differentiable function of the productivity parameters. Thus, without loss of generality, we claim that $a\in\mathbb{R}^L_{++}\times\mathbb{R}^L_{++}$ is an exogenous parameter that also defines the equilibrium allocation. Thus, we can express each variable of interest as a function of these productivity parameters. That is, $\overline{x}_h=\overline{x}_h(a)$ for all $h\in H$; $\overline{G}_m=\overline{G}_m(a)$ for all $m\in M$; and $\overline{x}=\sum_{h\in H}\overline{x}_h(a)=\overline{x}(a)$.

Let consider two kinds of technological change, one induced by a change on $b=(b_\ell)\in\mathbb{R}^L_{++}$ and another one induced by a change on $c=(c_\ell)\in\mathbb{R}^L_{++}$. For simplicity it is considered that only one of these two shocks may happen at one time.[17]

Differentiating the welfare function of any $h\in H$ we have the following expression,

$$dU_h=\sum_{\ell\in L} u_{h,\ell}\,dx_{h,\ell}+\sum_{m\in M} u_{h,m}\,dG_m-d\xi;$$

where $u_{h,j}$ stands for the marginal utility of j to household h. Now, let assume that all changes were induced by one of the technological changes (b or c). Thus,

$$\begin{aligned}
d\xi&=\sum_{j\in L}\left(\frac{\partial\xi}{\partial a_j}+\sum_{\ell\in L}\frac{\partial\xi}{\partial x_\ell}\frac{\partial x_\ell}{\partial a_j}\right)da_j=\sum_{j\in L}\left(\eta_{\xi,a_j}+\sum_{\ell\in L}\eta_{\xi,x_\ell}\cdot\eta_{x_\ell,a_j}\right)\frac{da_j}{a_j}\xi;\\
dG_m&=\sum_{j\in L}\frac{\partial G_m}{\partial a_j}da_j=\sum_{j\in L}\eta_{m,a_j}\,\frac{da_j}{a_j}\,G_m;\\
dx_{h,\ell}&=\sum_{j\in L}\frac{\partial x_{h,\ell}}{\partial a_j}da_j=\sum_{j\in L}\eta_{x_{h,\ell},a_j}\,\frac{da_j}{a_j}\,x_{h,\ell}
\end{aligned}$$

[17] That is, differentiating the productivity parameters we have either $da=(db_\ell)_{\ell\in L}$, or $da=(dc_\ell)_{\ell\in L}$. This only reduce notation and makes no difference in the analysis.

where $\eta_{i,j}$ stands for the elasticity of i with respect to j. Thus, the change on the welfare of any $h \in H$ induced by a technological change can be expressed as,

$$dU_h = \sum_{\ell \in L} u_{h,\ell} \left(\sum_{j \in L} \eta_{x_{h,\ell},a_j} \frac{da_j}{a_j} \right) x_{h,\ell} + \sum_{m \in M} u_{h,m} \left(\sum_{j \in L} \eta_{m,a_j} \frac{da_j}{a_j} \right) G_m \\ - \xi \sum_{j \in L} \left(\eta_{\xi,a_j} + \sum_{\ell \in L} \eta_{\xi,x_\ell} \eta_{x_\ell,a_j} \right) \frac{da_j}{a_j}$$

Notice that the sign of the majority of the terms of the last equation are known. Particularly, we have that: (1.) By Assumption *(A1)*, $((u_{h,\ell})_{\ell \in L}, (u_{h,m})_{m \in M}) \in \mathbb{R}_+^L \times \mathbb{R}_+^M$; and (2.) By definition, $\eta_{\xi,x_\ell} > 0$, for any $\ell \in L$; Notice that in the case of η_{ξ,a_ℓ}, the sign depends on the particular shock (b or c) and in some cases on the private consumption either (x). For example,

$$\frac{\partial \xi(x)}{\partial c_\ell} = -\frac{\xi_\ell(x_\ell)}{c_\ell} < 0, \quad \forall \, \ell \in L$$

where $\xi_\ell(x_\ell)$ is the externality produced by commodity ℓ. But, when the shock comes from b we have that,

$$\frac{\partial \xi(x)}{\partial b_\ell} = -\xi_\ell(x_\ell) \frac{\log [x_\ell]}{(b_\ell)^2} \lesseqgtr 0,$$

whose signs depends on the scale of aggregate private consumption, x.

However, without further assumptions, we do not know the signs of the other terms. That is, we only know that $\left(\eta_{x_{h,\ell},a_j}\right)_{(j,h) \in L \times H} \in \mathbb{R}^{L \times H}$; $\left(\eta_{x_\ell,a_j}\right)_{j \in L} \in \mathbb{R}^L$; and $\left(\eta_{m,a_j}\right)_{(j,m) \in L \times M} \in \mathbb{R}^{L \times M}$.

COROLLARY 1. *Given some technological change that reduces negative externalities on commodity $j \in L$. If,*

$$\eta_{\xi,a_j} + \sum_{\ell \in L} \eta_{\xi,x_\ell} \eta_{x_\ell,a_j} > 0;$$

Then, this technological improvement induce an increase on the amount of externalities. Moreover, it also affects negatively on the welfare of every household.

Since we may presume the signs of two of the three terms, the condition from above rest on the assumption of the sign that takes η_{x_ℓ,a_j}. Hence, if aggregate private consumption of some $\ell \in L$ increases due to a technological improvement that change the optimal tax policy, we could satisfy the inequality from above. Also notice that the increment on the aggregate consumption does not tell if all households increase their consumption. This imply another possible event, which is reflected in the following corollary.

COROLLARY 2. *Given some technological change that reduces negative externalities on some commodities $j \in L$. If for some household $h \in H$ we have that,*

$$\eta_{x_{h,\ell},a_j} < 0; \quad \text{for some } \ell \in L$$

Then, this technological improvement reduce the private consumption of household h in commodity ℓ. Moreover, if the sum over all $j \in L$ still satisfy the inequality, we have that the technological improvement reduce the utility of household h.

A similar conclusion can be obtained for the amount of public good. The following corollary reflect this,

COROLLARY 3. *Given some technological change that reduces negative externalities on some commodities $j \in L$. If for some public good m we have that,*

$$\eta_{m,a_j} < 0;$$

Then, this technological improvement reduce consumption of household h in public good m. Moreover, if the sum over all $j \in L$ still satisfy the inequality, we have that the technological improvement reduce the utility of household h.

Therefore we have that, without further assumptions, technological change may lead to two possible events. Firstly, that the aggregate private consumption increases (generating more externalities); and Secondly, that for some $h \in H$ there exist an welfare loss. Moreover, considering this two possibilities it may happen a social welfare loss. The following example show that under this framework this last is possible.

Example 3. The results shown on Table 3 are obtained following the assumptions made on parameters of Example 1. The only difference here, is that the parameter associated to the production of negative externalities made by commodity 2 change. That is,

$$\xi(x,\alpha) = (1/2)\left(\sum_{h\in h} x_{h,1}\right)^2 + (1/2)\left(\sum_{h\in h} x_{h,2}\right)^{(1/b_2)}$$

with $b_2 \in \{(1/4), (1/3), (1/2), (2/3)\}$.

The results in Table 3[18] show that the technological changes that appear to reduce the negative impact end up providing a worse scenario. In particular we observe that these changes induce equilibriums which are Pareto inferior with respect to previous Pareto allocations that have an externality function that may generates more negative externalities for a given private consumption. Under this particular technological change, we observe that the social welfare loss comes from the reduction on welfare by household 2 and the increment of negative externalities. However, household 1 presents a welfare improvement.

Table 3. Mitigation Equilibriums From Example 3

b_2	Ψ_F	u_1	u_2	ξ	p_1	t_1	t_2	TaxB	G
(1/4)	3.77	1.74	2.28	0.13	0.48	1.87	0.64	0.84	2.52
(1/3)	3.75	1.87	2.19	0.15	0.49	1.84	0.70	0.85	2.54
(1/2)	3.67	2.02	2.02	0.19	0.50	1.78	0.78	0.85	2.56
(2/3)	3.55	2.13	1.85	0.22	0.51	1.74	0.85	0.86	2.58

The channel by which the technological change induce this new Pareto inferior allocation comes from a change in the optimal tax policy. In this case since externality increases more public good is needed according to the (extended) Samuelson condition from the previous section. This implies an increment on fiscal revenue to finance the optimal increment on public goods. However, this increment is not distributed equally among households. The reason comes from the preferences that the households have over commodities. Thus we have, as the technology shock change (going from the first row to the fourth row) the tax burden for household 2 increases more than the reduction of tax burden on household 1.

However, the last results depends exclusively on the kind of functionality associated to the production of externalities. This production technology induces an optimal tax policy with a significant tax burden, which implies a significant reduction on commodities available to private consumption (see appendix, Table A.3.2). Particularly, the aggregate private consumption is always less than 1 in all commodities. That is, private consumption is less than a third of the resources available. Therefore, given the technological change that we explicitly assume,[19]

[18] See Table A.3.1 from the appendix to see the error terms associated to the optimization problem.

[19] The parameter associated to the power of the function.

we observe that even a reduction on private consumption in commodity 2 will increase the production of negative externalities. This last point suggest the importance of considering a particular cost function associated to negative externalities.

Finally, the increment on negative externalities is not only due to the technological change associated to externalities from commodity 2, but also to the increment on private consumption of commodity 1 as a consequence of the continuous optimal reduction in tax burden that household 1 obtains every time there is a change on the production technology of externalities generated by commodity 2.

Example 4. The next example show the impact on the economy of another technological change that reduce the generation of negative externalities. In this case the parametrization is similar to the one from Example 1 and Example 3. However, here the technological change is modelled as changes in the parameter $c = (c_1, c_2)$,

$$\xi(x, \alpha) = \frac{1}{c_1}\left(\sum_{h \in h} x_{h,1}\right)^2 + \frac{1}{c_2}\left(\sum_{h \in h} x_{h,2}\right)^2$$

with $c_1, c_2 \in \{(4/3),\ (2/1),\ (4/1)\}$.

Table 4[20] shows different *Mitigation Equilibriums* associated to different combination of parameters that represent technical change. Firstly, notice that the tax burden still is above 80% of total wealth. However, this burden is reduced when a technology improvement appears. In this case, this also implies a reduction on public goods production. Also, income tax still is proportional to the wealth of each household.[21]

Table 4. Mitigation Equilibriums From Example 4

$((1/c_1),(1/c_2))$	Ψ_F	u_1	u_2	ξ	p_1	t_1	t_2	TaxB	G
(0.75,0.75)	3.51	1.95	1.95	0.20	0.50	1.82	0.82	0.88	2.64
(0.75,0.50)	3.51	1.69	2.21	0.20	0.48	1.89	0.70	0.86	2.59
(0.50,0.75)	3.51	2.21	1.69	0.20	0.52	1.70	0.89	0.86	2.59
(0.50,0.50)	3.67	2.02	2.02	0.19	0.50	1.78	0.78	0.85	2.56
(0.50,0.25)	3.32	1.23	2.47	0.19	0.46	1.97	0.49	0.82	2.46
(0.25,0.50)	3.32	2.47	1.23	0.19	0.54	1.49	0.97	0.82	2.46
(0.25,0.25)	3.92	2.12	2.12	0.16	0.50	1.71	0.71	0.81	2.43

[20] See the appendix Table A.4.1. for the error terms associated to the optimization problem.

[21] Recall that the parameters associated to the endowments are the same as in Example 1 and Exampe 3. That is, $w_1 = (2, 2)$ and $w_2 = (1, 1)$.

Secondly, contrary to Example 3, here a technological change does not increase externalities. However, this technological improvement may also leads to a loss of welfare. Particularly, we observe that when technological change is not parsimonious[22] this induce a change in optimal taxes that affects equilibrium prices making worse-off some members and others better-off. Moreover, in some cases the net effect of this change on welfare leads to a reduction in social welfare. For example, going from the fourth row to the fifth or sixth. Nonetheless, when technological change affects both commodities similarly, we obtain a *Mitigation Equilibrium* which is Pareto superior with respect to the original situation. For example, let consider the technological path going from the first row, to the fourth and finally to the seventh row. This case show a reduction in externalities as well as an improvement on the welfare of all members. On the contrary, consider the technological path that goes from the first row to second (third, resp.) row and finally to the fifth (sixth, resp.) row. In this case we observe that social welfare does not change significantly, but individual welfare does. Particularly, we observe that, given the preferences, some technological changes makes better-off household 2, and in other case makes better-off household 1.

In this case the channel by which the technological change induce the new equilibrium allocation also comes from a change on tax policy. This policy must change in order to maintain the Pareto condition,

$$\sum_{j \in H} \left(\frac{f_{m,\ell}\, u_{j,m} + \xi_\ell}{u_{j,\ell}} \right) = 1. \quad \forall\, (\ell, m) \in L \times M$$

Since in this case externality is reduced we have that the left-hand side of the equality (at the original equilibrium allocations) is less than one. Thus, in order to achieve the condition again, an optimal tax policy must reduce public goods production. This translate into a reduction on tax burden. However, this reduction is not proportional. Instead, increases the tax burden on one household and decreases the tax burden on the other. The way the new tax burden is distributed depends on the type of technological change. For example, going from the first row to the second one implies a reduction in externalities by private consumption of commodity 2. This reduction need to be efficient in terms of reducing at the minimum the loss of public good, given the optimal amount of tax burden that should be reduced. This last means to benefit with the change in the tax policy the agent who values the most the public good in terms of

[22] That is, given an equivalent externality function, technological change is biased to one of the commodities.

commodity 2. In this case, given the functionality and allocations, household 2 satisfy this. Thus, when reducing the whole tax burden the optimal policy increases tax income in household 1 (reducing his welfare) and reduces the tax burden for household 2 (improving his welfare). This re-distribution of tax burden will induce new price equilibriums which leads to a reduction in price of commodity 1. This also affect the combination of inputs used to produce public good. The opposite happens when we go from first row to the third row. In this case there exist an externality reduction in commodity 1. Since household 1 values the most public good in terms of commodity 1, this leads to an increase on after-tax wealth, which increase his consumption in all private goods. (For a detail of how the allocations change see appendix, Table A.2)

7 CONCLUDING REMARKS

This work set a general equilibrium model with private goods and public goods, where the former generates negative externalities to all members and public goods are produced to mitigate this effects. Following the seminal paper of Bergstrom, Blume and Varian (1996), we allow voluntary provision of inputs to be used for production of public goods. But given the very nature of a public good, in the model is also introduced a mandatory provision –through an optimal tax policy– that will finance the production of public goods according to a fiscal plan.

Under standard assumptions over preferences and production technologies the existence of a *Mitigation Equilibrium* is proved. It is also shown the Pareto conditions for such economy. This condition is an extension of the well known *Samuelson Condition* where an extra argument should be introduced, which is associated to the benefits of reducing private consumption in the reduction of negative externalities. From this new extension we conclude that the optimal amount of public goods is greater than the one from the regular optimality condition. Hence, any reduction of externalities is accompanied by a reduction in the amount of public good required.

Some numerical examples are made. It is shown an hybrid version of the Neutrality Theorem of Bergstrom, Blume and Varian (1986), where after a change in the initial wealth the optimal tax induce the same Pareto allocation. Other examples show that the relation between technologies that reduce the negative impact of private consumption and welfare is not so clear. It is shown that if the technological change is parsimonious among the externality function, then there is a welfare improvement to all. However, when the technological change is heterogeneous this leads to an scenario where some households are worse-off and other better-off. Even more, we found that in some cases social welfare can be affected negatively by the technological change.

Bibliography

[1] Andreani, R., Birgin, E. G., Martínez, J. M., & Schuverdt, M. L. (2007). On Augmented Lagrangian methods with general lower-level constraints. SIAM Journal on Optimization, 18(4), 1286-1309.

[2] Andreoni, J. (1988). Privately provided public goods in a large economy: the limits of altruism. Journal of Public Economics, 35(1), 57-73.

[3] Bergstrom, T., Blume, L., & Varian, H. (1986). On the private provision of public goods. Journal of public economics, 29(1), 25-49.

[4] Debreu, G. (1952). A social equilibrium existence theorem. Proceedings of the National Academy of Sciences of the United States of America, 38(10), 886.

[5] Foley, D. K. (1970). Lindahl's solution and the core of an economy with public goods. Econometrica: Journal of the Econometric Society, 66-72.

[6] Groves, T., & Ledyard, J. (1977). Optimal allocation of public goods: A solution to the" free rider" problem. Econometrica: Journal of the Econometric Society, 783-809.

[7] Mas-Colell, A. (1980). Efficiency and decentralization in the pure theory of public goods. The Quarterly Journal of Economics, 626-641.

[8] Musgrave, R. A. (1969). Provision for Social Goods. In Julius Margolis and H. Guitton (eds.), Public Economics: An analysis of Public Production and Con sumption and their Relations to the Private Sectors. London: Macmillan.

[9] Samuelson, P. A. (1954). The pure theory of public expenditure. The review of economics and statistics, 387-389.

4

Coalitions And Endogenous Mixed Goods

1 Abstract

This is a general equilibrium model in which people can form coalitions. Coalitions with more than one member are called non-trivial coalitions. The latter formations may reduce rivalry in consumption among members. The kind of reduction depends exclusively on the people that belong to the coalition. That is, here there is no anonymity since the benefits of belonging a coalition depend on the characteristics of the people that belong to it.

Under this framework coalitions not only demand commodities on the markets, but also determine the distributions of use of the commodities among members. This last characteristics differs from previous literature associated to club theory and household formation, where the effect of belonging to a coalition is described through the change on preferences due to the formation of the club/household,[1] and not to by the allocation of use of the commodities as in this model.

It is shown the existence of a particular equilibrium, which is first described by Gersbach and Haller (2011), where people that belong to non-trivial coalitions are better-off than when alone. Some examples are made in order to show the difficulties that arise when we look for a more general equilibrium concept. Some refinements are shown as a possible extension to find an stable coalition structure.

It is shown that a model with endogenous reduction in rivalry in consumption satisfy the *Large Group Advantage* defined by Gersbach et al (2011). Also some examples show the relation between the two models. It is shown that under our framework we can relax some assumptions made by Gersbach et al (2011) in order to find an equilibrium. In particular we do not need to have a member without imposing any negative externality.

[1]See Ellickson et al (1999) and Gersbach et al (2011).

Finally, through numerical examples some analysis over social welfare and inequality (measured as Gini) are made when coalition formation is allowed. It is shown that given this set-up, the social welfare always improves. Still there could exist some coalitions that lose welfare. The relation with inequality is not so clear. The reason for this two results comes from the fact that coalition formation induces new price equilibriums that affect differently the welfare of each coalition and the expenditure of each individual.

2 A Model For Coalition Formation With Endogenous Mixed Goods

This is an exchange economy where individuals can form coalitions –from now on we call non-trivial coalition to any coalition with more than one member– and determine together what and how much to trade in the market as well as how to distribute the use of commodities among members. Under this framework the incentives to form a non-trivial coalition comes from the implicit assumption that this formation reduces rivalry in consumption.[2]

Let $L = \{1, \ldots, L\}$ be the set of commodities. The consumption space is $\mathbb{R}^L_+$ and we denote by $\Delta = \left\{p \in \mathbb{R}^L_+ : \sum_\ell p_\ell = 1\right\}$ the set of commodity prices at which goods will be traded among coalitions.

Let $I = \{1, \ldots, I\}$ be the set of individuals that belong to the economy. Under this framework every individual will evaluates his current welfare against others possible situations –going from a non-trivial to a trivial coalition, for example. We denote by $U_i : \mathbb{R}^L_+ \to \mathbb{R}$ the utility function that represents the preferences over commodities of individual $i \in I$. Let $w_i \in \mathbb{R}^L_+$ be his private endowment.

Let $H = \{h \subseteq I : h \neq \emptyset\}$ be the set of all possible coalitions that can be formed in the economy, and denote by $N(h)$ the number of members that belong to coalition $h \in H$. Also, let $H_i = \{h \in H : i \in h\}$ be the set of all coalitions to which individual i belong. From the definition of H we can define $\mathcal{H}$ as the set of distributions of individuals into coalitions. Moreover, we call any partition A of I as a coalition structure, such that $A \in \mathcal{H}$. We say that $A \in \mathcal{H}$ is a non-trivial structure when $N(h) > 1$, for at least some $h \in A$. Furthermore, let $A^+ = \{h \in A : N(h) > 1\}$. Finally, every coalition $h \in H$ has associated an initial endowment consisting on the aggregate wealth of the coalition, $w_h = \sum_{i \in h} w_i$.

[2]Not necessarily all commodities.

In order to allow reduction in rivalry in consumption among individuals of the same non-trivial coalition, we define $\theta_h(y)$ as the set of allocations $(z_i)_{i\in h} \in \mathbb{R}_+^{L\times h}$ that represent the distribution of use obtained by members of h when coalition demand is $y \in \mathbb{R}_+^L$.[3] Thus, it can be observed reduction in rivalry in consumption when,

$$(z_i)_{i\in h} \in \theta_h(y) \wedge \left(\sum_{i\in h} z_i - y\right) \in \mathbb{R}_+^L \setminus \{0\}$$

When coalition is trivial, that is $N(h) = 1$, we have that $\big(z_i - y_i\big) \in -\mathbb{R}_+^L$, for all $z_i \in \theta_{\{i\}}(y_i)$.

When trivial coalition exists, we define for simplicity $\Phi_{\{i\}}(p) = \big\{y_i \in \mathbb{R}_+^L : p\cdot y_i \leq p\cdot w_i\big\}$ as the regular choice set used in the literature.[4] On the other hand, let

$$\Phi_h(p) = \{(y_h, (z_i)_{i\in h}) \in \mathrm{Gr}[\theta_h] : p\cdot y_h \leq p\cdot w_h\},$$

be the choice set of non-trivial coalition h. Thus, any coalition $h \in A$ chooses allocations on $\Phi_h(p)$ in order to maximize his welfare.

Under this framework every individual evaluates the possibility to leave the current coalition in order to improve his own welfare. These outside options are determined exogenously. The feasibility of the outside options reflect the fact that there are some coalitions to which individual i cannot belong, for example, due to social or geographical reasons. However, we claim that all individuals, no matter the coalition to which belong, will always have the option to go alone.

3 ASSUMPTIONS OF THE MODEL

The following are the assumptions of a model with coalition formation and endogenous mixed goods. We can separate these into assumptions associated to individuals and assumptions associated to the allocation set. As one may notice, some of the assumptions are commonly used in the literature, but others are intrinsically to the model. This last reflects the rationale associated to reducing rivalry in consumption.

[3] The term use fits well the idea of this model rather than consumption. See Musgrave (1969) and Holtermann (1972).

[4] Equivalently, we may define,

$$\Phi_{\{i\}}(p) = \big\{(y_i, z_i) \in \mathrm{Gr}[\theta_{\{i\}}] : p\cdot y_i \leq p\cdot w_i\big\},$$

but we think is more confusing and not provide much more to the analysis.

First, Assumption *(A1)* is commonly used in the literature as well as *(A2)(i)*. In the case of *(A1)* we impose concavity considering the fact that this model describe an economy where agents that go to the markets not necessarily represent one individual only.[5]

ASSUMPTION (A1)

For any $i \in I$, the function $U_i : \mathbb{R}^L_+ \to \mathbb{R}$ is continuous, strictly increasing, and strictly concave.

Assumption *(A2)(ii)* defines regular properties associated to the strategy set of each coalition. That is, the limit point of the distributions of use must be feasible (closed) and any convex combination between two feasible distribution of use must be also feasible (convex). The requirement imposed in Assumption *(A2)(iii)* has to do with the fact that no matter the distribution of use that the coalition chooses, no member can use more than the physical demand obtained by the coalition.[6] At the same time, Assumption *(A2)(iv)* says that even allocations which on the aggregate are less than physical demand, belongs to the θ_h. Notice that this last case incorporates the event where a non-trivial coalition gives to his members an allocation equivalent to their demands as if they were alone. Assumption *(A2)(v)* introduces a *type of* Minkowski summation that represents the benefits of pooling of resources.

ASSUMPTION (A2)

For any coalition $h \in H$,

(i) Physical endowments satisfies $w_i \gg 0,\ \forall i \in h$.

(ii) $\theta_h : \mathbb{R}^L_+ \twoheadrightarrow \mathbb{R}^{L\times h}_+$ has closed and convex graph.

(iii) For every $(z_i)_{i\in h} \in \theta_h(y)$ we have that $z_i \leq y$, $\forall i \in h$.

(iv) If $(z_i)_{i\in h} \in \mathbb{R}^{L\times h}_+$ satisfies $\sum_{i\in h} z_i \leq y$, then $(z_i)_{i\in h} \in \theta_h(y)$.

(v) For any $y_1, y_2 \in \mathbb{R}^L_+$ we have that $\theta_h(y_1) + \theta_h(y_2) \subseteq \theta_h(y_1 + y_2)$.

The items associated to Assumption *(A3)* deals with the attributes necessary to generate incentives to form non-trivial coalitions. Particularly, in order to prove existence in more than an standard exchange economy where all coalitions being trivial, we impose specific characteristics on the strategy set of non-trivial coalitions. Specifically, Assumption *(A3)(i)* describe the reduction in rivalry in consumption that some non-trivial coalitions must have. Thus, we say that a non-trivial coalition is capable of reducing rivalry in consumption when some allocations that belong to Gr$[\theta_h]$ gives (in the aggregate) more to its members than the actual physical demand.

[5]If we relax the assumption to quasi-concavity this may not assure us that the welfare function associated to some non-trivial coalition will satisfy quasi-concavity. For example, the sum of quasi-concave functions is not necessarily quasi-concave.

[6]This is a distinction with respect to the literature of externalities, since this last does not impose any restriction over the effect that others can do to an individual.

Assumption *(A3)(ii)* is equivalent to the well known Inada condition, which we need to prove equilibrium existence. Assumption *(A3)(iii)* along with *(A1)* helps to determine the equivalence between the truncated and regular individual's problem when $i \in h$ evaluates the possibility to go alone. Finally, Assumption *(A3)(iv)* sets the standard properties that we need for the welfare function in order to prove existence when there are non-trivial coalitions.

ASSUMPTION (A3)

There is an $h \in H$ such that:

(i) For some $(y, (z_i)_{i\in h}) \in \mathrm{Gr}[\theta_h]$ we have $(N(h)z_i - y) \in \mathbb{R}^L_+ \setminus \{0\}$, $\forall i \in h$.

(ii) For any agent $i \in h$, $U_i(w_i) > U_i(x),\ \forall x \in \mathbb{R}^L_+ \setminus \mathbb{R}^L_{++}$.

(iii) There exists $i \in h$ such that,

$$\forall v \in \mathbb{R}^L_+ \setminus \{0\},\ \exists\, a_v > 0:\ U_i(w_i + a_v \cdot v) > U_i\left(\sum_{j\in I} w_j\right).$$

(iv) The welfare function $\mathcal{W}_h : \mathbb{R}^h_+ \to \mathbb{R}_+$ is continuous, strictly increasing and strictly quasi-concave on $(U_i)_{i\in h}$.

4 EQUILIBRIUM FOR AN ECONOMY WITH COALITION FORMATION AND ENDOGENOUS MIXED GOODS

In the following we define an equilibrium concept in the context where individuals can form non-trivial coalitions and obtain some benefit of it. This first definition considers (exclusively) a particular outside option for every member of a non-trivial coalition. Following Gersbach and Haller (2011) we define for this economy with endogenous mixed goods (call it $\mathcal{E}_\eta$) a competitive equilibrium with free exit (CEFE) as a competitive equilibrium where no individual belonging to a non-trivial coalition will exercise his option to go alone.

DEFINITION 1.

A competitive equilibrium with free exit (CEFE) for an economy $\mathcal{E}_\eta$ with a coalition structure $A \in \mathcal{H}$, is a vector of prices $\overline{p} \in \Delta$, and allocations $(\overline{y}_h, (\overline{z}_i)_{i\in h})_{h\in A} \in \prod_{h\in A}\left(\mathbb{R}^L_+ \times \mathbb{R}^{L\times h}_+\right)$ such that:

(i.) For every $h \in A$, the pair $(\overline{y}_h, (\overline{z}_i)_{i\in h})$ belongs to $\Phi_h(\overline{p})$ and there is no other pair $(y_h, (z_i)_{i\in h})$ belonging to $\Phi_h(\overline{p})$ for which $(U_i(z_i) - U_i(\overline{z}_i))_{i\in h} \in \mathbb{R}^h_+ \setminus \{0\}$.

(ii.) Demands are market feasible, i.e., $\sum_{h \in A} \overline{y}_h = \sum_{h \in A} w_h$.

(iii.) If $A^+ \neq \{\emptyset\}$*, we have that for any* $h \in A^+$, $U_i(x) < U_i(\overline{z}_i)$, $\forall i \in h$, $\forall x \in \Phi_{\{i\}}(\overline{p})$.

From Definition 1 we can say that item (i.) represents an extension of the rationality associated with the optimal allocation chosen by an agent. First, under this setting there are not only individuals demanding commodities, but also some non-trivial coalitions that demand commodities according to some collective rule.[7] Second, all agents not only determine the private allocation of commodities, but also each coalition determines the use among its members, also according to some collective rule.[8] It is important to mention that item (i.) does not consider any specific functional form for non-trivial coalitions. Hence, any continuous and concave Paretian social welfare function for non-trivial coalition h will be enough. Item (ii.) is the regular condition of market clearing. Item (iii.) reflects a voluntary participation (membership) constraint for all $i \in h$, with $h \in A^+$. The following proposition establishes the existence of this type of equilibrium.

PROPOSITION 1. *Given Assumptions (A1)-(A3), there exist a non-trivial coalition structure* $A \in \mathcal{H}$ *for which there is a (CEFE).*

SKETCH OF THE PROOF. The proof follows the regular steps of the equilibrium existence theorem in a generalized game (Debreu, 1952). Thus, given Assumptions *(A1)* and *(A3)(iv)* we have that all coalitions' objective functions are continuous and quasi-concave. Following Debreu (1952), we also introduce a new player – an auctioneer who will look for prices at Δ that clears markets– whose objective function will also satisfy continuity and quasi-concavity.

On the other hand, fixing an uniform upper bound for all commodities[9] and given Assumptions *(A1)*, *(A2)(i)*, *(A2)(iii)*, *(A3)(ii)* and *(A3)(i)*; We are able to prove that the correspondence of admissible strategies for a non-trivial coalition capable of reducing rivalry in consumption turns out to be continuous, with compact, convex and non-empty values. When coalitions are trivial we use Assumption *(A2)(i)* in order to prove that their correspondence are continuous, with compact, convex and non-empty values. The auctioneer's strategy set –Δ– will be, by definition, continuous with compact, convex and non-empty values. Thus, Berge's Maximum Theorem ensures that players' best-reply response correspondence are upper hemicontinuous with non-empty, compact and convex values. Therefore, we apply Kakutani's Fixed Point

[7] See Chiapori (1988, 1992).

[8] This last extension can be associated to Berglas (1976) and Konishi (2010) where they model congestion in clubs.

[9] Market clearing condition set this bound.

Theorem to the correspondence of optimal strategies in order to find a Cournot-Nash equilibrium for the generalized game associated to the truncated economy. The last part of the proof deals with the fact that the Cournot-Nash equilibrium obtained is also a competitive equilibrium with free exit (CEFE) for the economy $\mathcal{E}_\eta$. This last is done by contradiction. For that we use Assumptions *(A1)*, *(A2)(i)*, and *(A3)(iii)*. The complete proof is in the appendix. *Q.E.D.*

From Definition 1 is important to notice that the participation constraint -item (iii.)- is just one of many possible outside options. In a more general setting we claim that every individual $i \in I$ will have an specific subset of H_i from which evaluates his permanence in the current coalition. Hence, given an initial coalition structure $A \in \mathcal{H}$, let define $G_{i,h} \subseteq H_i$, as the set of all feasible outside options when $i \in h$, with $h \in A$. We assume that $\{i\} \in G_{i,h}$ for all $i \in h$, with $N(h) > 1$. That is, every member of a non-trivial coalition must have (at least) as an outside option the possibility to go alone. For simplicity we consider for the rest of the paper that all non-trivial coalition structure $A \in \mathcal{H}$ satisfy Assumptions *(A1)-(A3)*. Therefore, by Proposition 1 we have that for every non-trivial structure $A \in \mathcal{H}$ there exist a competitive equilibrium with free exit (CEFE).

Thus, for any non-trivial structure $A \in \mathcal{H}$, exist an allocation,

$$\left(\overline{p};\, (\overline{y}_h, (\overline{z}_i)_{i\in h})_{h\in A}\right) \in \Delta \times \prod_{h\in A} \left(\mathbb{R}^L_+ \times \mathbb{R}^{L\times h}_+\right);$$

which is a (CEFE). Moreover, given an outside option profile $\mathcal{G} = (G_{i,h})_{i\in I}$ we say that individual $i \in I$ can benefit from exit non-trivial coalition $h \in A$, *if and only if*, there exist some $g \in G_{i,h}$ for which the following condition holds,

$$U_j(\widetilde{z}_j) > U_j(\overline{z}_j),\, \forall\, j \in g; \tag{$\star$}$$

with, $\left(\widetilde{y}_g,\, (\widetilde{z}_j)_{j\in g}\right) \in \Phi_g(\overline{p})$ such that there is no other pair $(y_g, (z_i)_{i\in g})$ belonging to $\Phi_g(\overline{p})$ for which $(U_i(z_i) - U_i(\widetilde{z}_i))_{i\in g} \in \mathbb{R}^g_+ \setminus \{0\}$.

Notice that the allocation $\left(\widetilde{y}_g,\, (\widetilde{z}_j)_{j\in g}\right)$ *looks like* a (CEFE), but is not necessarily true because $\overline{p}$ might not hold when the new coalition is effectively formed.

Hence, from this condition we can establish a definition of equilibrium more general than Definition 1. Particularly, this new concept consider any other possible option –besides to go alone– such as to join an existent coalition or forming a new one. Moreover, this equilibrium concept considers not all options (H_i), but instead, the feasible ones when $i \in h$, $(G_{i,h})$. This consideration goes in the line that for some individuals there are some forbidden coalitions.[10] For example, in reality we have that geographic distance and social punishment may restraint the set of feasible coalitions. The following definition describe an equilibrium with reduction in rivalry in consumption in more general terms.

DEFINITION 2.
A $\mathcal{G}$-competitive equilibrium with free coalition formation (CEFC-$\mathcal{G}$) for an economy $\mathcal{E}_\eta$ with a coalition structure $A \in \mathcal{H}$ and outside option profile $\mathcal{G}$, is a vector of prices $\overline{p} \in \Delta$ and allocations $((\overline{y}_h, (\overline{z}_i)_{i\in h})_{h\in A} \in \prod_{h\in A}\left(\mathbb{R}^L_+ \times \mathbb{R}^{L\times h}_+\right)$ such that:

(i) For every $h \in A$, the pair $(\overline{y}_h, (\overline{z}_i)_{i\in h})$ belongs to $\Phi_h(\overline{p})$ and there is no other $(y_h, (z_i)_{i\in h})$ belonging to $\Phi_h(\overline{p})$ for which $(U_i(z_i) - U_i(\overline{z}_i))_{i\in h} \in \mathbb{R}^h_+ \setminus \{0\}$.

(ii) Demands are market feasible, i.e., $\sum_{h\in A} \overline{y}_h = \sum_{h\in A} w_h$.

(iii) If $A^+ \neq \{\emptyset\}$. Then, for any $h \in A^+$ and for all $i \in h$, $\nexists\ g \in G_{i,h}$ such that condition ($\star$) holds.

Similar to Definition 1 we have that item (i.) reflects individual rationality as well as rationality of a multi-member coalition; while item (ii.) reflects market clearing condition. However, as one may notice, item (iii.) is a more general condition than Definition 1. In particular, we claim that a (CEFE) is an specific case of (CEFC-$\mathcal{G}$) where, given a $A \in \mathcal{H}$, we have that for all members of any non-trivial coalition $h \in A^+$, the set $G_{i,h} = \big\{\{i\}\big\}$. Moreover, even when the outside option profile includes more options than going alone for some $i \in I$, we have that a (CEFC-$\mathcal{G}$) will satisfy the conditions of Definition 1 when Assumptions *(A1)-(A3)* hold. An important implication of this discussion is reflected in the next corollary.

COROLLARY 1. *Any $\mathcal{G}$-competitive equilibrium with free coalition formation (CEFC-$\mathcal{G}$) for an economy $\mathcal{E}_\eta$ is a competitive equilibrium with free exit (CEFE), but not necessarily the other way around.*

Although we have already proved the existence of a (CEFE) under Assumptions *(A1)-(A3)*. The existence of a (CEFE) that it is also a (CEFC-$\mathcal{G}$) depends not only from characteristics

[10]See Ellickson et al. (1999) for similarities. Where in their terms an agent can belong to a club only if the description of that club type includes one or more members with his/her external characteristics. In our case external characteristics could be geographical distance (or not), ethnicity (or not), social status (or not), educational status (or not), marital status (or not), gender preferences (or not), and so on.

associated to reduction in rivalry, but also on prices, wealth, the specification of the Paretian welfare function, and even $\mathcal{G}$. The following examples illustrate some of these points.

Example 0. Let consider an economy $\mathcal{E}_\eta$ with a unique commodity and price $p = 1$. Let $I = \{\alpha,\, \beta,\, \gamma\}$ and $w_h = |h|$, for all $h \in H$. This last satisfy Assumption *(A2)(i)*. Preferences satisfy Assumption *(A1)*, *(A3)(ii)* and *(A3)(iv)*. We also claim that for any $i \in h$, with $h \in A^+$, Assumption *(A3)(iii)* is satisfied. We set a continuous Paretian social welfare function $\mathcal{W}_h$, for all $h \in A^+$. In general terms we only allow to form non-trivial coalitions with individual α. The following are different sets of private use associated to the formation of non-trivial coalitions with α,[11]

$$\theta_\beta(y) \equiv \theta_\beta = \left\{(z_i)_{i\in\{\alpha,\beta\}} \in \mathbb{R}_+^{1\times 2} \,:\, (z_\alpha)^3 + (z_\beta)^3 \le (y)^3\right\};$$
$$\theta_\gamma(y) \equiv \theta_\gamma = \left\{(z_i)_{i\in\{\alpha,\gamma\}} \in \mathbb{R}_+^{1\times 2} \,:\, (z_\alpha)^2 + (z_\gamma)^2 \le (y)^2\right\};$$

Then, by definition of θ_β we have that it is possible to find a $\left(y,\, (z_i)_{i\in\{\alpha,\beta\}}\right) \in \mathrm{Gr}[\theta_\beta]$ capable to satisfy Assumption *(A3)(i)*. The same will happen with θ_γ.

Then, given the coalition structure $A_0 = \big\{\{\beta\}, \{\alpha,\gamma\}\big\}$, we claim that any allocation,

$$\Big(p;\ \big((1,1), (2, (\overline{z}_i)_{i\in\{\alpha,\gamma\}})\big)\Big) \in \Delta \times \prod_{h\in A}\left(\mathbb{R}_+ \times \mathbb{R}_+^{1\times h}\right),$$

with $(\overline{z}_i)_{i\in\{\alpha,\gamma\}} \in \theta_\gamma$ such that,

$$U_i(\overline{z}_i) \ge U_i(1) + \delta_i, \quad \text{with } \delta_i > 0,\ \forall\, i \in \{\alpha,\gamma\};$$

is a competitive equilibrium with free exit (CEFE).[12] Moreover, it is also a (CEFC-$\mathcal{G}$) when $G_{\alpha,\{\alpha,\gamma\}} = \big\{\{\alpha\}\big\}$. In fact, notice that the set,

$$\mathcal{D}_{\{\alpha,\gamma\}} = \left\{(\delta_i)_{i\in\{\alpha,\gamma\}} \in \mathbb{R}^2_{++} : \big(U_\alpha^{-1}\big(U_\alpha(1)+\delta_\alpha\big)\big)^2 + \big(U_\gamma^{-1}\big(U_\gamma(1)+\delta_\gamma\big)\big)^2 \le \big(\big|\{\alpha,\gamma\}\big|\big)^2\right\};$$

is non-empty, which implies that there exist at least an allocation $\left(U_i^{-1}\left(U_i(1)+\delta_i\right)\right)_{i\in\{\alpha,\gamma\}} \in \theta_\gamma$ such that the participation constraint is satisfied. This is true since $2 < \big(\big|\{\alpha,\gamma\}\big|\big)^2$. Hence, for some $(\delta_\alpha, \delta_\gamma) \gg 0$ –small enough– the condition of the set $\mathcal{D}_{\{\alpha,\gamma\}}$ is met. Thus, we have that any $(\overline{z}_\alpha, \overline{z}_\gamma) = (\sqrt{2+\pi}, \sqrt{2-\pi})$, with,

[11] Notice that all sets satisfy Assumption *(A2)(ii)-(A2)(v)*.

[12] See the appendix for an specific definition δ_i.

$$\big(U_\alpha^{-1}\big(U_\alpha(1)+\delta_\alpha\big)\big)^2-2\le\pi\le 2-\big(U_\gamma^{-1}\big(U_\gamma(1)+\delta_\gamma\big)\big)^2,\quad \forall(\delta_i)_{i\in\{\alpha,\gamma\}}\in\mathcal{D}_{\{\alpha,\gamma\}}$$

will satisfy the previous participation constraints. Also, since there no exist a $g\in G_{\alpha,\{\alpha,\gamma\}}$ such that condition $(\star)$ holds, we have that the allocation is a (CEFE) as well as (CEFC-$\mathcal{G}$). However, when $G_{\alpha,\{\alpha,\gamma\}}=\big\{\{\alpha\},\{\alpha,\beta\}\big\}$, we claim that there exist a $g\in G_{\alpha,\{\alpha,\gamma\}}$ such that condition $(\star)$ might holds. Specifically, when $g=\{\alpha,\beta\}$ we have that,

$$\mathcal{D}_{\{\alpha,\beta\}}=\left\{(\delta_i)_{i\in\{\alpha,\beta\}}\in\mathbb{R}^2_{++}:\big(U_\alpha^{-1}\big(U_\alpha(\sqrt{2+\pi})+\delta_\alpha\big)\big)^3+\big(U_\beta^{-1}\big(U_\beta(1)+\delta_\beta\big)\big)^3\le\big(\big|\{\alpha,\beta\}\big|\big)^3\right\};$$

is non-empty. Indeed, when $(\delta_\alpha,\delta_\beta)=0$, we have that $(2+\pi)^{3/2}+1\le\big(\big|\{\alpha,\beta\}\big|\big)^3$. Thus, considering the upper bound of π[13] and fixing $\delta_\gamma=0$, it is true that $3^{3/2}+1<8$. Thus, for some $(\widetilde{\delta}_\alpha,\delta_\alpha,\delta_\beta,\delta_\gamma)\gg 0$ –small enough– $\mathcal{D}_{\{\alpha,\beta\}}\neq\emptyset$, with $(\widetilde{\delta}_\alpha,\delta_\gamma)\in\mathcal{D}_{\{\alpha,\gamma\}}$. Therefore, we have that any allocation $(\widetilde{z}_\alpha,\widetilde{z}_\beta)=(\sqrt[3]{4+\mu},\sqrt[3]{4-\mu})$, with,

$$\big(U_\alpha^{-1}\big(U_\alpha(\sqrt{2+\pi})+\delta_\alpha\big)\big)^3-4\le\mu\le 4-\Big(U_\beta^{-1}\big(U_\beta(1)+\delta_\beta\big)\Big)^3,\quad \forall(\delta_i)_{i\in\{\alpha,\beta\}}\in\mathcal{D}_{\{\alpha,\beta\}};$$

will satisfy condition $(\star)$. Therefore, the allocation,

$$\Big(p;\ \big((1,1),(2,(\overline{z}_i)_{i\in\{\alpha,\gamma\}})\big)\Big)\in\Delta\times\prod_{h\in A}\Big(\mathbb{R}_+\times\mathbb{R}_+^{1\times h}\Big),$$

with, $\Big(U_\alpha(\overline{z}_\alpha)\ge U_\alpha(1)+\widetilde{\delta}_\alpha\Big)$ and $\Big(U_\gamma(\overline{z}_\gamma)\ge U_\gamma(1)+\delta_\gamma\Big)$, is not a (CEFC-$\mathcal{G}$). However, still satisfy the conditions of Definition 1. Thus, still is a (CEFE).

On the other hand, given the coalition structure $A_1=\big\{\{\gamma\},\{\alpha,\beta\}\big\}$, we claim that the allocation,

$$\Big(p;\ \big((1,1),(2,(\overline{z}_i)_{i\in\{\alpha,\beta\}})\big)\Big)\in\Delta\times\prod_{h\in A}\Big(\mathbb{R}_+\times\mathbb{R}_+^{1\times h}\Big),$$

[13] $\pi=2-\big(U_\gamma^{-1}\big(U_\gamma(1)+\delta_\gamma\big)\big)^2$.

with $(\overline{z}_i)_{i\in\{\alpha,\beta\}} \in \theta_\beta$ such that,

$$U_i(\overline{z}_i) \geq U_i(1) + \delta_i, \quad \text{with } \delta_i > 0, \forall\, i \in \{\alpha, \beta\};$$

is a competitive equilibrium with free exit (CEFE).[14] However, we claim that is not necessarily a (CEFC-$\mathcal{G}$) when $G_{\alpha,\{\alpha,\beta\}} = \big\{\{\alpha\}, \{\alpha,\gamma\}\big\}$. Indeed, first notice that the set,

$$\mathcal{D}_{\{\alpha,\beta\}} = \Big\{(\delta_i)_{i\in\{\alpha,\beta\}} \in \mathbb{R}^2_{++} : \big(U_\alpha^{-1}\big(U_\alpha(1)+\delta_\alpha\big)\big)^3 + \big(U_\beta^{-1}\big(U_\beta(1)+\delta_\beta\big)\big)^3 \leq \big(\big|\{\alpha,\beta\}\big|\big)^3\Big\};$$

is non-empty since $2 < \big(\big|\{\alpha,\beta\}\big|\big)^3$, when $(\delta_\alpha, \delta_\beta) = 0$, thus there exist some $(\delta_\alpha, \delta_\beta) \gg 0$ –small enough– such that $\mathcal{D}_{\{\alpha,\beta\}} \neq \emptyset$. Thus, we have that any $(\overline{z}_\alpha, \overline{z}_\beta) = (\sqrt[3]{4+\pi}, \sqrt[3]{4-\pi})$, with,

$$\big(U_\alpha^{-1}\big(U_\alpha(1)+\delta_\alpha\big)\big)^3 - 4 \leq \pi \leq 4 - \Big(U_\beta^{-1}\big(U_\beta(1)+\delta_\beta\big)\Big)^3, \quad \forall (\delta_i)_{i\in\{\alpha,\beta\}} \in \mathcal{D}_{\{\alpha,\beta\}}$$

will satisfy the participation constraint. Therefore, the allocation is a (CEFE). However, following a similar procedure as before, we claim that the allocation is not necessarily (CEFC-$\mathcal{G}$) since there is a $g \in G_{\alpha,\{\alpha,\beta\}}$ such that condition $(\star)$ may hold. In particular, we have that associated to coalition $g = \{\alpha,\gamma\}$ there exist an allocation $(\widetilde{z}_i)_{i\in\{\alpha,\gamma\}} \in \theta_\gamma$ capable to satisfy condition $(\star)$.[15] Indeed, first notice that the set,

$$\mathcal{D}_{\{\alpha,\gamma\}} = \Big\{(\delta_i)_{i\in\{\alpha,\gamma\}} \in \mathbb{R}^2_{++} : \big(U_\alpha^{-1}\big(U_\alpha(\sqrt[3]{4+\pi}) + \delta_\alpha\big)\big)^2 + \Big(U_\gamma^{-1}\big(U_\gamma(1)+\delta_\gamma\big)\Big)^2 \leq \big(\big|\{\alpha,\gamma\}\big|\big)^2\Big\};$$

is non-empty since $(4+\pi)^{2/3} + 1 < 4$, when $(\delta_\alpha, \delta_\gamma) = 0$. Then, considering the lower bound of π, we can express the inequality as $\Big(U_\alpha^{-1}\big(U_\alpha(1) + \widetilde{\delta}_\alpha\big)\Big)^2 + 1 < 4$. Thus, we claim that there will always exist some $(\widetilde{\delta}_\alpha, \delta_\alpha, \delta_\beta, \delta_\gamma) \gg 0$ –small enough– such that $\mathcal{D}_{\{\alpha,\gamma\}} \neq \emptyset$, with $(\widetilde{\delta}_\alpha, \delta_\beta) \in \mathcal{D}_{\{\alpha,\beta\}}$. Thus the allocation that consider the partition $\{\{\gamma\},\{\alpha,\beta\}\}$ will not be a (CEFC-$\mathcal{G}$). Moreover, we claim that there exist an allocation with a partition $\{\{\beta\},\{\alpha,\gamma\}\}$ which is a (CEFC-$\mathcal{G}$). However, considering now the upper bound of π and fixing $(\delta_\alpha, \delta_\gamma) = 0$, we have that the following inequality

[14] See the appendix for an specific definition δ_i.

[15] But at the same time there exist another usufruct profile $(\widehat{z}_i)_{i\in\{\alpha,\gamma\}} \in \theta_\gamma$ not capable to satisfy condition $(\star)$. Which allocation will be used to evaluates his permanence on the current coalition will also depend on the welfare function.

$$\left(8-\left(U_\beta^{-1}\left(U_\beta(1)+\widetilde{\delta}_\beta\right)\right)^3\right)^{2/3}+1<4$$

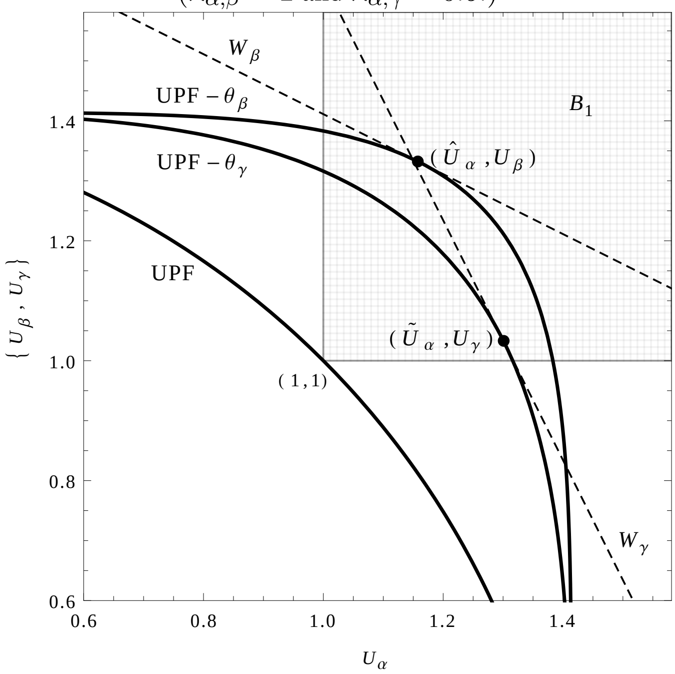
is a sufficient condition to prove the existence of $(\widetilde{\delta}_\alpha, \delta_\alpha, \widetilde{\delta}_\beta, \delta_\gamma) \gg 0$, such that there exist a (CEFC-$\mathcal{G}$) with partition $\{\{\beta\}, \{\alpha, \gamma\}\}$.[16] However, notice that for some $\widetilde{\delta}_\beta > 0$ –small enough– the previous inequality will not be satisfied. Indeed, notice that when $\widetilde{\delta}_\beta = 0$ we have that $7^{2/3} + 1 \not< 4$. Therefore, for some $(\widetilde{\delta}_\alpha, \delta_\alpha, \widetilde{\delta}_\beta, \delta_\gamma) \gg 0$ the inequality does not hold. That is, for small extra gains of forming non-trivial coalition –$(\widetilde{\delta}_\alpha, \delta_\alpha, \widetilde{\delta}_\beta, \delta_\gamma)$– It might happen that there is no payoff that makes agent α to leave coalition $\{\alpha, \beta\}$ in order to form coalition $\{\alpha, \gamma\}$. Thus, the allocation that consider the partition $\{\{\gamma\}, \{\alpha, \beta\}\}$ will be a (CEFC-$\mathcal{G}$).

The next figure illustrate the idea of the previous example. In this case we set an specific functional form $U_i = \sqrt{z_i}$, for all $i \in I$.

Figure 1.

$W_j = U_\alpha + \lambda_{\alpha,j}\, U_j$; with, j=$\beta, \gamma$.

($\lambda_{\alpha,\beta} = 2$ and $\lambda_{\alpha,\gamma} = 0.5$.)



Under this set-up we observe that the existence of a (CEFE) is assured. However, (CEFC-$\mathcal{G}$) is not. In this particular case we have that, given the parametrization, α prefers to form a coalition with γ instead of β. Even if, potentially, there is allocations that makes α better off. The frontier UPF is merely illustrative and represent the utility possibility frontier when there is

[16]Where $(\widetilde{\delta}_\alpha, \widetilde{\delta}_\beta) \in \mathcal{D}_{\{\alpha,\beta\}}$ and $(\delta_\alpha, \delta_\gamma) \in \mathcal{D}_{\{\alpha,\gamma\}}$.

no reduction on rivalry in consumption for a non-trivial coalition. Instead, the frontier UPF-θ_β and UPF-θ_γ represent the utility possibility frontier when there is usufruct associated to the formation of coalition $\{\alpha, \beta\}$ and $\{\alpha, \gamma\}$, respectively. In this case, given that all two-member coalitions associated to individual α are feasible, α will prefer to be part of coalition $\{\alpha, \gamma\}$ and obtain an allocation such that obtains an utility $\widetilde{U}_\alpha$, which is greater than the one obtained when forming part of coalition $\{\alpha, \beta\}$, $\widehat{U}_\alpha$.[17] Notice that both coalitions belong to a (CEFE), but in this case only when coalition $\{\alpha, \gamma\}$ is formed we obtain a (CEFC-$\mathcal{G}$).

From this example we conclude that in order to study the stability of non-trivial coalition structures $A \in \mathcal{H}$, it is more comprehensive to use individuals' preference order over feasible coalitions. Second, without further assumptions, we cannot determine anything about the existence proposed by Definition 2. Thus, following the seminal work of Gale and Shapley (1962), we ask if for any pattern of preferences of individuals over coalitions it is possible to find an stable coalition structure. In our case the answer will depend on the feasible options to form coalitions. In this sense, the marriage problem presented by Gale et al (1962) is a particular case of our model, where feasible options are associated to match a pair male and female only. In the following example we will observe that, when the set of feasible options is perturbed, a coalition can go from an stable to an unstable situation.

Example 1. Following Example 3 of Gale et al (1962), we show that under some specific outside option profile $(\mathcal{G})$, there is no stable coalition structure. For that, let consider a similar structure as Example 0. That is, there is one commodity satisfying Assumption *(A2)(i)*, $I = \{\alpha, \beta, \gamma\}$, assumptions of utility functions are met, and associated to any feasible non-trivial coalition h, Gr$[\theta_h]$ satisfy Assumptions *(A2)-(A3)*.[18] Here we only allow non-trivial coalitions formed by two members only. In this case the reason for instability comes from the redundancy given by the preference order over coalitions,

$$\begin{aligned}
\alpha: &\quad h_{\alpha,\beta} \succ_\alpha h_{\alpha,\gamma} \succ_\alpha h_\alpha \\
\beta: &\quad h_{\beta,\gamma} \succ_\beta h_{\beta,\alpha} \succ_\beta h_\beta \\
\gamma: &\quad h_{\gamma,\alpha} \succ_\gamma h_{\gamma,\beta} \succ_\gamma h_\gamma
\end{aligned}$$

[17] Even if $\theta_\gamma(y) \subset \theta_\beta(y)$, for all $y \in \mathbb{R}_+$.

[18] From now on we consider that all individuals know the payoff associated to every non-trivial coalition that might belong. In reality we have that people eventually will know the chances and the payoff of every feasible coalition as time goes by and builds up his social network.

where: (1.) $h_i = \{i\}$, for $i \in I$; (2.) $h_{i,j} = h_{j,i} = \{i,j\}$, when $i,j \in I$. For example, given the initial coalition structure $A_0 = \{\{\alpha\},\{\beta,\gamma\}\}$, we have that individual α has the incentive to form a new non-trivial coalition with β, since is feasible. But the coalition $\{\alpha,\beta\}$ does not satisfy condition $(\star)$. This last is because β already belong to his first best option. However, coalition $\{\alpha,\gamma\}$ is feasible and satisfy $(\star)$, so the new coalition structure will be $A_1 = \{\{\alpha,\gamma\},\{\beta\}\}$. Following the same procedure for β, we will have the following coalition $A_2 = \{\{\alpha,\beta\},\{\gamma\}\}$. Then, when γ is alone, we will have that the next coalition structure will be $A_3 = A_0$. Thus, there is no stable coalition. However, following the algorithm proposed by Gale et al (1962), if we allow to form only non-trivial coalitions with α as a member, coalition $\{\{\alpha,\beta\},\{\gamma\}\}$ is a stable non-trivial coalition. Thus, associated to this structure, we claim that there is an allocation which is (CEFE) as well as (CEFC-$\mathcal{G}$). Analogously, we have that the non-trivial coalition $\{\{\alpha,\gamma\},\{\beta\}\}$ is not stable, and for that we claim that it is a (CEFE) but not a (CEFC-$\mathcal{G}$).

What if now it is possible to form coalition $h_I = \{\alpha,\beta,\gamma\}$? For that, let consider the following preference order,

$$\begin{aligned}
\alpha: &\quad h_I \succ_\alpha h_{\alpha,\beta} \succ_\alpha h_{\alpha,\gamma} \succ_\alpha h_\alpha \\
\beta: &\quad h_{\beta,\gamma} \succ_\beta h_{\beta,\alpha} \succ_\beta h_I \succ_\beta h_\beta \\
\gamma: &\quad h_I \succ_\gamma h_{\gamma,\alpha} \succ_\gamma h_{\gamma,\beta} \succ_\gamma h_\gamma
\end{aligned}$$

In this case –you may check– we have one stable coalition structure $\{I\}$. However, the existence of coalition structure $\{I\}$ depends on the strategy that takes β when alone. That is, if β always prefers to form a new coalition exclusively with α –which satisfy $(\star)$– we go back to the redundancy presented above. Instead, if β chooses to form coalition $\{\alpha,\beta,\gamma\}$, which also satisfy condition $(\star)$, this coalition will be stable. Hence, Which one will β choose when alone? A priori there is no straight answer.

On the one hand, if we consider a dynamic process with memoryless agents[19] such that every time when β is alone, he chooses to form coalition $\{\alpha,\beta\}$. Then there is no stable coalition.

On the other hand, let consider a repeated infinitely game where at every period, given the current coalition structure, individuals decide to stay or to form a new coalition. Thus, let consider an initial coalition structure $A_0 = \{\{\alpha,\beta\},\{\gamma\}\}$ an a discount factor $\delta \in (0,1)$. If

[19] Or discount factor zero.

β, when alone, always decide to form a coalition with α (exclusively), we have the following payoff associated to β,

$$\widehat{\pi}_\beta = U_\beta(z_\beta(A_0)) + \delta\, U_\beta(z_\beta(A_1)) + \delta^2\, U_\beta(z_\beta(A_2)) + \delta^3\, U_\beta(z_\beta(A_0)) + \ldots$$

where $z_\beta(A_j)$ and $U_\beta(z_\beta(A_j))$ represents β's allocation –and consequently his utility– when coalition structure is A_j. In this case we have that $A_1 = \{\{\beta,\gamma\},\{\alpha\}\}$ and $A_2 = \{\{\alpha,\gamma\},\{\beta\}\}$, respectively. Thus, given the redundancy of this strategy profile, we have that β's payoff can be expressed as,

$$\widehat{\pi}_\beta = \frac{U_\beta(z_\beta(A_0)) + \delta\, U_\beta(z_\beta(A_1)) + \delta^2\, U_\beta(z_\beta(A_2))}{1-\delta^3}.$$

If β chooses to form coalition $\{I\}$ when alone, we have the following payoff for β,

$$\widetilde{\pi}_\beta = U_\beta(z_\beta(A_0)) + \delta\, U_\beta(z_\beta(A_1)) + \delta^2\, U_\beta(z_\beta(A_2)) + \frac{\delta^3}{1-\delta}\, U_\beta(z_\beta(I)).$$

Thus, the decision of whether β chooses to be part of $\{\alpha,\beta\}$ or $\{I\}$ will depend on,

$$\frac{U_\beta(z_\beta(A_0)) + \delta\, U_\beta(z_\beta(A_1)) + \delta^2\, U_\beta(z_\beta(A_2))}{1-\delta^3} = \widehat{\pi}_\beta \lesseqgtr \frac{1}{1-\delta}\, U_\beta(z_\beta(I)).$$

Thus, if the right-hand side of the inequality is greater than the left-hand side of the inequality, we can also claim that there is exist a (CEFE), which at the same time it is a (CEFC-$\mathcal{G}$).[20]

5 An Economy With Non-negative Consumption Externalities And Coalition Formation

The literature associated to externalities, public goods, and in a more general aspect mixed goods[21], are closely linked. For example, many externalities produced by commodities have the

[20] For further analysis on this matter it is important to consider the kind of punishment that the former coalition might apply to a member when he go to another coalition in the next period. For example, Tit for tat, Tit for two tats and Grim trigger.

[21] Goods that are not entirely private neither public. See Samuelson ans Musgrave (1969).

characteristics of public goods or mixed goods.[22] For that, we consider relevant to describe an economy –following closely the work done by Gersbach and Haller (2011)– with non-negative consumption externalities associated to formation of coalitions (call it $\mathcal{E}_\xi$). That is, individuals belonging to a common coalition may generates (exclusively) positives externalities on other members of that coalition through their own private consumption.

Thus, we define $x_i \in \mathbb{R}^L_+$ as the allocation of private consumption of individual $i \in I$. Hence, we denote $x_h = (x_i)_{i \in h} \in \mathbb{R}^{L \times h}_+$ as the private consumption for coalition h. Also, it is assumed that each individual $i \in I$ has a utility representation $U_i : \mathcal{X}_i \to \mathbb{R}$, where, $\mathcal{X}_i = \left\{ x_h \in \mathbb{R}^{L \times h}_+ \,:\, h \in H_i \right\}$.[23]

For any coalition h, we denote its expenditure on consumption plan x_h at prices $p \in \Delta$ as $p \cdot \left(\sum_{i \in h} x_i \right)$. Thus, the choice set of any particular coalition h is defined as,[24]

$$B_h(p) = \left\{ x_h \in \mathbb{R}^{L \times h}_+ \,:\, p \cdot \sum_{i \in h} x_i \leq p \cdot \sum_{i \in h} w_i \right\}$$

Thus, any coalition h will choose allocations in $B_h(p)$ in order to maximize his welfare. Hence, given this framework, we define similarly to the previous equilibriums concepts the (CEFE) for an economy $\mathcal{E}_\xi$.

DEFINITION 3.

A competitive equilibrium with free exit (CEFE) for an economy $\mathcal{E}_\xi$ with coalition structure $A \in \mathcal{H}$, is a vector of prices $\overline{p} \in \Delta$, and allocations $(\overline{x}_h)_{h \in A} \in \mathbb{R}^{L \times I}_+$ such that:

(i.) For every $h \in A$, the allocation $\overline{x}_h$ belongs to $B_h(\overline{p})$ and there is no other y_h belonging to $B_h(\overline{p})$ for which $\left(U_i(y_h) - U_i(\overline{x}_h) \right)_{i \in h} \in \mathbb{R}^h_+ \setminus \{0\}$.

(ii.) Demands are market feasible, i.e., $\sum_{i \in I} \overline{x}_i = \sum_{h \in A} w_h$.

(iii.) If $A^+ \neq \{\emptyset\}$, we have that for any $h \in A^+$, $U_i\left(x\right) < U_i\left(\overline{x}_h\right)$, $\forall i \in h$, $\forall x \in B_{\{i\}}(\overline{p})$.

In order to determine if in this economy there exist at least a (CEFE) further assumptions are needed. For that, we consider some regular assumptions already defined. Specifically, we

[22] For example, pollution is a public bad that can affect differently to people, since it may depend on the distance between the person and the place of origin.

pollution is a public bad that can affect differently to people depending on the distance that this person's place of origin.

[23] Remember that H_i is the set of all coalitions that member i might belong.

[24] When $h = \{i\}$, we have that, $B_{\{i\}}(p) = \left\{ x_i \in \mathbb{R}^L_+ \,:\, p \cdot x_i \leq p \cdot w_i \right\}$.

consider Assumption *(A1)* as well as Assumption *(A2)(i)*.

Now, following Gersbach et al (2011), we need to describe the advantages against trivial coalitions when forming a non-trivial coalition $h \in A$ with non-negative consumption externalities. According to the authors, this results in the existence of a maximizer[25] for any $i \in I$ and certain properties associated to the set $X_h(p) \subset B_h(p) \cap K^{L \times h}$.[26] Where, for any given price, $X_h(p)$ is the set capable of improving the situation of each member with respect to its outside option (going alone).

LARGE GROUP ADVANTAGE. *A non-trivial coalition h has large group advantage, if:*

(i.) Every $i \in h$ has a utility maximizer $\widehat{x}_i(p)$ and achieves indirect utility $v_i(p) = U_i(\widehat{x}_i(p))$, when trading individually from the endowment w_i at prices $p \in Int[\Delta]$.

(ii.) For every price system $p \in \Delta$ there exist a set $X_h(p) \subseteq B_h(p) \cap K^h$ with the following properties:

(ii.a) $X_h(p)$ is non-empty, compact and convex.

(ii.b) $X_h(p)$ depends continuously on p.

(ii.c) There exist threshold $\delta_i(p) \geq 0$ for $p \in Int[\Delta]$ and $i \in h$ such that for $p \in Int[\Delta]$ and $x_h \in B_h(p) \cap K^h$:

$$x_h \in X_h(p) \Leftrightarrow U_i(x_h) - v_i(p) \geq \delta_i(p), \quad \forall\, i \in h$$

Then, considering this description of advantage for a non-trivial coalition, we have the following proposition by Gersbach et al (2011).

PROPOSITION 2. *Given Assumptions (A1), (A2)(i) and household structure $A \in \mathcal{H}$, with some h being non-trivial with large group advantage (LGA), and a member $j \in h$ is not imposing any negative consumption externalities on other coalition members. Then, there exist an allocation $\big(p; (x_i)_{i \in I}\big)$ which is a (CEFE).*

SKETCH OF THE PROOF. The proof consider the properties established over $X_h(p)$ that imply that the coalition's demand is non-empty, convex, compact and by Berge's Maximum Theorem

[25]Which is satisfied given Assumptions *(A1)* and *(A2)(i)*.

[26]Where $K = [0, 2\,k]^L$, with $k > \big|\sum_{i \in I} w_i\big|_{+\infty}$.

also upper hemicontinuous. Following the regular assumptions, it is also obtained that the individuals demand has no-empty, convex and compact values and by Berge's Maximum Theorem upper hemicontinuous too. The assumption over a member $j \in h$ that does not impose any negative externalities, along with Assumption *(A1)*, imply budget exhaustion. To prove the existence of a price system $p \in \Delta$ such that clears markets Gersbach and Haller (2011) uses the excess demand function. From the regular assumptions over preferences, it is concluded that the maximizer at the individual level will be the same for the truncated and the non-truncated set. Therefore, the excess demand will be zero with $p \gg 0$.

The crucial part of the proof associated to the non-trivial coalition comes from the fact that $\overline{x}_h$ is an efficient choice on $B_h(\overline{p})$ and at the same time –at the going prices– nobody want to leaves the coalition to go alone. For that GH initially show, by means of contradiction, that if $\overline{x}_h$ maximizes the welfare function $\mathcal{W}_h$ on the truncated set $B_h(\overline{p}) \cap K$, subject to the inequality of item (iii.), this same allocation will also be the one that maximizes $B_h(\overline{p}) \cap K$ without further conditions. The following steps that implies going from $B_h(\overline{p}) \cap K$ to $B_h(\overline{p})$ are the regularly uses in the literature, which also is by means of contradiction. Finally, the fact that no member of the non-trivial coalition wants to go alone comes from the fact that $\overline{x}_h \in X_h(p)$. Hence, no individual wants to exit the non-trivial coalition. See Proposition 2 of Gersbach and Haller (2011) for a complete detail of the proof. *Q.E.D.*

6 RELATION BETWEEN $\mathcal{E}_\eta$ AND $\mathcal{E}_\xi$

The aim of this section is to analyse the relation between the two models described before. For that we check if our assumptions satisfy the *Large Group Advantage* assumptions defined by Gersbach et al (2011). First, unlike the authors, in our framework the utility function has no *Pure Group Externality*.[27] However, we will see that reduction on rivalry in consumption over commodities of non trivial-coalitions could be also interpreted as consumption externalities.[28] Thus, analogously to Gersbach et al (2011), we define the following set,

$$\Lambda_h(p) = \left\{ z_h = (z_i)_{i \in h} \in \mathbb{R}_+^{L \times h} \;:\; \left((1.)\ \exists\ y \in \mathbb{R}_+^L,\ p \cdot y \leq p \cdot \sum_{i \in h} w_i \right) \wedge \left((2.)\ z_h \in \theta_h(y) \right) \right\};$$

[27]That is, the coalition effect is additively separable from the utility given by private consumption.

[28]See Musgrave (1969).

where z_h is the distribution of use among members of the non-trivial coalition when there exist reduction in rivalry in consumption. Then, we define $X_h(p) \subseteq \Lambda_h(p) \cap K^{L\times h}$, where $p \in \Delta$ and in our case $K = \big[0\,, 2 \cdot \Omega_z\big]$ such that,[29] for all $z_h \in X_h(p)$ the following condition is satisfied: (3.) $U_i(z_i) - v_i(p) \geq \delta_i(p) > 0$, $\forall\, i \in h$. Where, $v_i(p) = \max\big\{U_i(x) : x \in [0, 2\cdot\Omega_y]^L \;\wedge\; p\cdot x \leq p \cdot w_i\big\}$.[30]

CLAIM 1. *Any non-trivial coalition $h \in H$ satisfying Asumptions (A1)-(A3) will satisfy the large group advantage (LGA) of Gersbach and Haller (2011).*

Proof. Considering Assumption *(A1)* and *(A2)(i)* along with the Extreme Value Theorem,[31] we know that item (i.) of LGA is satisfied. Therefore only remains to show whether our assumptions imply that $X_h(p)$ satisfy item (ii.) of LGA. For that notice $X_h(p)$ has compact co-domain. Now, given a convergent sequence $\big\{p_n,\, (z_{i,n})_{i\in h}\big\}_{n\in\mathbb{N}} \subset \mathrm{Gr}[X_h]$, we have that,

$$\left((1.)\ \exists\ y \in \mathbb{R}^L_+,\ p_n \cdot y \leq p_n \cdot \sum_{i\in h} w_i\right);$$
$$\Big((2.)\ (z_{i,n})_{i\in h} \in \theta_h(y)\Big);$$
$$\Big((3.)\ U_i(z_{i,n}) - v_i(p_n) \geq \delta_i(p_n) > 0\Big).$$

From our Assumptions *(A1)-(A3)* we can set a $\delta_i(p_n) > 0$ as a continuous and strictly positive function. Particularly, considering Assumption *(A3)(i)* we can always find a pair $(\widetilde{y}_h,\, (\widetilde{z}_i)_{i\in h})$ that belongs to Gr $[\theta_h]$ such that,[32]

$$\delta_i(p_n) = \mu\left(\ U_i\Big(\widehat{z}_i(p_n) + \lambda_h(p_n)(N(h)\widetilde{z}_i - \widetilde{y}_h)\Big) - v_i(p_n)\ \right);$$

where: (i.) $\mu > 0$; (ii.) $\widehat{z}_i(p) = \arg\max\big\{U_i(x) : x \in [0, 2\cdot\Omega_y]^L \;\wedge\; p\cdot x \leq p\cdot w_i\big\}$; and (iii.) $\lambda_h : \Delta \to (0,1)$ and continuous.

Thus, by Assumptions *(A1)* and *(A2)(i)* along with Berge's Maximum Theorem we claim that continuity of $(v_i,\, U_i, \delta_i)$ is assured. This implies that condition (1.), (2.), and (3.) on

[29]Where, $\Omega_z > \big|I\big| \cdot \big|\big|\sum_{i\in I} w_i\big|\big|_1$.
[30]Where, $\Omega_y > \big|I\big| \cdot \big|\big|\sum_{i\in I} w_i\big|\big|_1$.
[31]See Weierstrass Extreme Value Theorem.
[32]See the existence proof in the appendix for more detail.

$X_h(p)$ are satisfied, for all $n \in \mathbb{N}$. Moreover, since Δ and K are closed sets, we have that $\left(\lim_{n\to+\infty} p_n, \lim_{n\to+\infty}(z_{i,n})_{i\in h}\right) \in \text{Gr}[X_h]$. Thus $X_h(p)$ is upper hemicontinuous.

On the other hand, considering assumptions *(A2)(i)*, *(A2)(iii)*, and *(A3)(i.)*, we set (1.), (2.), and (3.) with strict inequality. Hence, we have that Int$[X_h(p)] \neq \{\emptyset\}$, for any $p \in \Delta$. Also, given $\epsilon > 0$, we have that for any $p \in B_\epsilon(p')$, with $p' \in \Delta$, we always can find an $x \in B_\epsilon(x') \cap \text{Int}[X_h(p)]$, with $x' \in \text{Int}[X_h(p')]$. Therefore, Int$[X_h]$ as well as X_h are lower hemicontinuous.[33] Thus $X_h : \Delta \twoheadrightarrow \Lambda_h(p) \cap K^{L\times h}$ is continuous. At last, considering Assumption *(A1)* and Assumption *(A2)(ii)*, we have that $X_h(p)$ has convex values. $\square$

Thus, given that our model satisfy LGA assumptions, it only remains to know whether the models are equivalent or not in terms of equilibrium allocations and welfare. One way to consider this equivalence is to translate the model of coalition formation in terms of a model with non-negative externalities. For that we need to set a relation between allocations of use $z_h = (z_i)_{i\in h} \in \theta_h\left(\sum_{i\in h} x_i\right)$ and distributions of commodities $x_h = (x_i)_{i\in h} \in B_h(p)$. In particular, we are interested on relations between $\left\{x_h \in \mathbb{R}_+^{L\times h} : p\cdot \sum_{i\in h} x_i = p\cdot w_h\right\} \subset B_h(p)$ and $\left\{z_h \in \mathbb{R}_+^{L\times h} : z_h \in \text{bd}\left[\theta_h\left(\sum_{i\in h} x_i\right)\right]\right\} \subset \theta_h\left(\sum_{i\in h} x_i\right)$, since the rest of allocations are negligible from the point of view of optimality.

It is not hard to realize that there are many continuous functions that can take a $x_h \in \text{bd}\left[B_h(p)\right]$ to an element of bd $\left[\theta_h\left(\sum_{i\in h} x_i\right)\right]$. However, it is more demanding to have a function that takes an $x_h \in \text{bd}\left[B_h(p)\right]$, and evaluated at any distribution $\left(\sum_{i\in h} \widehat{x}_i\right) = \left(\sum_{i\in h} x_i\right)$ is equivalent to an element of bd $\left[\theta_h\left(\sum_{i\in h} x_i\right)\right]$. Moreover, we say that a function is fully capable when replicate all elements of bd $\left[\theta_h\left(\sum_{i\in h} x_i\right)\right]$ by not changing the coalition demand, but the distribution of that demand. Also, this function should be neutral to rescaling. Since any change on equilibrium prices may induce a new coalition demand, and we want that the same function replicates the elements of that set, but with a different demand level. If not, we say that the function only partially replicate the effect of the reduction on rivalry. These conditions assure us that the optimal allocation $(\overline{z}_i)_{i\in h}$ will always have an equivalent allocation $(f_{i,h}(\overline{x}_h))_{i\in h}$, no matter the coalition demand.

The following example show that under the same coalition structure $A \in \mathcal{H}$, the same wealth distribution $(w_i)_{i\in I}$, and the same preferences $\left((\mathcal{W}_h)_{h\in A}, (U_i)_{i\in I}\right)$, both models have equivalent outputs in terms of price equilibrium, coalition's demands as well as welfare. How-

[33]Let $\Gamma : X \twoheadrightarrow Y$ be a correspondence, with $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Then, if the correspondence Int$[\Gamma]$ is lower hemicontinuos, then Γ is lower hemicontinuos too.

ever, as the next example show, there are models of non-negative externalities that are partially equivalent, and others that are completely equivalent to a model with reduction in rivalry in consumption.

Example 2. let consider the following optimization problem that represents the problem of the unique non-trivial coalition $h = \{1, 2\}$ in a model of coalition formation with endogenous reduction rivalry,

$$\max \left\{ \sum_{i=1}^{2} \sigma_i \, (z_{i,1})^{\gamma} (z_{i,2})^{1-\gamma} : \; (y_h, (z_i)_{i \in h}) \in \text{Gr}\,[\theta_h] \right\}$$

subject to,

(i.) $(p,\, 1-p) \cdot \left(y_h - \sum_{i=1}^{2} w_i \right) \leq 0$;

(ii.) $\sum_{i=1}^{2} z_{i,\ell} \leq \eta_\ell \, y_{h,\ell}$ for all $\ell \in L$;

(iii.) $z_{i,\ell} \leq y_{h,\ell}$, for all $(i, \ell) \in h \times L$;

(iv.) $-(z_{i,1})^{\gamma} (z_{i,2})^{1-\gamma} + v_i(p) + \mu \nu_i(p) \leq 0$, for all $i \in h$.

with: (1.) $\gamma \in (0,1)$; (2.) $\eta_\ell \in [1,2) \; \forall \ell \in L$, with $\eta_\ell > 1$ for some $\ell \in L$; (3.) $v_i(p) := \max \left\{ U_i(x) : \, x \in \Phi_{\{i\}}(p) \right\}$; (4.) $\mu > 0$; and (5.) $\nu_i(p) := U_i \left(\widehat{z}_i(p) + \lambda_h(p) \left(N(h) \tilde{z}_i - \tilde{y}_h \right) \right)$.[34]

Thus, the optimal solutions –when $\sigma_i = \sigma$ for $i \in \{1, 2\}$– are given by,[35]

$$\begin{aligned}
\overline{y}_{h,1} &= \gamma \, \frac{(p,\, 1-p) \cdot w_h}{p} \\
\overline{y}_{h,2} &= (1-\gamma) \, \frac{(p,\, 1-p) \cdot w_h}{(1-p)} \\
\overline{z}_{1,1} &= \overline{\pi}_1 \, \overline{y}_{h,1} \\
\overline{z}_{1,2} &= \overline{\pi}_2 \, \overline{y}_{h,2} \\
\overline{z}_{2,1} &= (\eta_1 - \overline{\pi}_1) \, \overline{y}_{h,1} \\
\overline{z}_{2,2} &= (\eta_2 - \overline{\pi}_2) \, \overline{y}_{h,2} \\
\frac{\overline{\pi}_2}{\overline{\pi}_1} &= \frac{\eta_2}{\eta_1}, \quad \text{with, } \overline{\pi}_\ell \in [\eta_\ell - 1, 1] \, \forall \ell \in L.
\end{aligned}$$

[34] With, $\widehat{z}_i(p) \in \arg\max \{ U_i(x) : p \cdot (x - w_i) \leq 0 \}$; $\lambda_h : \Delta \to (0,1)$ a continuous function; and $(\tilde{y}_h, (\tilde{z}_i)_{i \in h})$ satisfy Assumption *(A3)(i)*.

[35] See the appendix for details.

where π_ℓ can be see it as the proportion of use of commodity ℓ given to memeber 1.

First, from the solutions notice that coalition's demand is independent of the distribution of use. This result is important for the example since this implies that no matter the distribution of use we impose the price equilibrium will be the same. Secondly, there are infinite combinations of $(\overline{z}_i)_{i\in h}$ that brings the same welfare to the coalition, while the last condition holds. Therefore, given Assumptions *(A1)-(A3)*, in terms of equilibrium concept we obtain an equilibrium price $\overline{p}$, demands that clear markets $(\overline{y}_h)_{h\in A}$ and a set of different distribution of use $\Gamma_h(\overline{y}_h)$. Where,

$$(\overline{z}_i)_{i\in h} \in \Gamma_h(\overline{y}_h) = \arg\max \left\{ \mathcal{W}_h(z_h) : (z_i)_{i\in h} \in \theta_h(\overline{y}_h) \right\}$$

On the other hand, considering the optimality conditions[36], we set a generic linear continuous function with parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2) \geq 0$ that represents a non-negative effect of coalition's allocation $(x_i)_{i\in h}$ to all members. Thus,

$$\frac{\beta_1 x_{2,1} + x_{1,1}}{\beta_2 x_{2,2} + x_{1,2}} = \frac{\overline{z}_{1,1}}{\overline{z}_{1,2}} = \left(\frac{\eta_1}{\eta_2}\right)\left(\frac{\gamma\,(1-p)}{(1-\gamma)\,p}\right) = \frac{\overline{z}_{2,1}}{\overline{z}_{2,2}} = \frac{a_1 x_{1,1} + x_{2,1}}{\alpha_2 x_{1,2} + x_{2,2}}$$

where $x_{j,\ell}$ stands for the private consumption of commodity ℓ made by member j. By imposing the condition $x_{1,1} + x_{2,1} = (\gamma/p)$ and $x_{1,2} + x_{2,2} = (1-\gamma)/(1-p)$, we will be looking for functions that induce allocations that do not change coalition's demand.[37] Therefore, solving for the pair $(x_{2,1}, x_{2,2})$ we obtain for the left-hand side of the equality and the right-hand side of the equality, respectively,

$$x_{2,1} = \left[\left(\frac{\eta_1}{\eta_2}\right) - 1\right] \frac{\gamma}{p\,(\beta_1 - 1)} + \left(\frac{\eta_1}{\eta_2}\right)\left(\frac{\gamma\,(1-p)}{(1-\gamma)\,p}\right)\left(\frac{\beta_2 - 1}{\beta_1 - 1}\right) x_{2,2} \tag{E1}$$

$$x_{2,1} = \left[\left(\frac{\eta_1}{\eta_2}\right)\alpha_2 - \alpha_1\right] \frac{\gamma}{p\,(1 - \alpha_1)} + \left(\frac{\eta_1}{\eta_2}\right)\left(\frac{\gamma\,(1-p)}{(1-\gamma)\,p}\right)\left(\frac{1 - \alpha_2}{1 - \alpha_1}\right) x_{2,2} \tag{E2}$$

Notice from these last two equations that $\beta_1 \neq 1$ nor $\alpha_1 \neq 1$. However, considering that this function should take a private allocation to the space of distribution of use,[38] Assumption *(A2)(iii.)* must be also satisfied. This implies that $\beta_1, \beta_2 \in [0, 1)$ and $\alpha_1, \alpha_2 \in [0, 1)$. Finally,

[36] Interior solutions are considered.
[37] Hence, neither price equilibrium.
[38] That is $f_h \in \mathbf{C}^1_B\left(B_h(p),\, \theta_h\left(\sum_{i\in h} x_i\right)\right)$.

considering the monotonicity of preferences and the inequality associated to distribution of use,[39] we can solve the following equations,

$$x_{2,1} = \frac{\eta_1\,\gamma}{(1+\beta_1)\,p} - \left(\frac{1+\alpha_1}{1+\beta_1}\right) x_{1,1} \tag{E3}$$

$$x_{1,2} = \frac{\eta_2\,(1-\gamma)}{(1+\alpha_2)\,(1-p)} - \left(\frac{1+\beta_2}{1+\alpha_2}\right) x_{2,2} \tag{E4}$$

For $w_h = (1,1)$, $\overline{p} = 0.5$, $\eta_1 = 1$, $\eta_2 = 1.2$, and $\gamma = 0.5$; Figure 2 shows *(E1)-(E4)*, when the parameters associated to the non-negative externality function are the one described in the figure. Although there are two intersections, the only that matters in order to find an optimal allocation is the associated to *(E1)* and *(E2)*. The others two, *(E3)* and *(E4)*, show the other allocations as a consequence of the optimality conditions.

Figure 2.

$(\alpha_1 = 0, \alpha_2 = 0.16, \beta_1 = 0, \beta_2 = 0.39)$

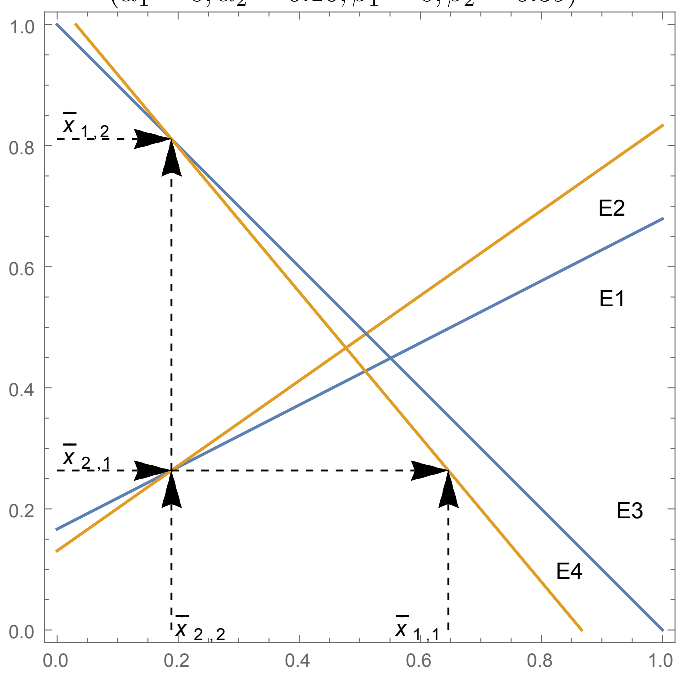


Figure 2 show us that the optimality condition only occurs once, when $(\overline{x}_{2,1}, \overline{x}_{2,2})$. In consequence, in terms of a model of non-negative externalities consumption, when the externality is represented by parametrization $(\alpha_1 = 0, \alpha_2 = 0.16, \beta_1 = 0, \beta_2 = 0.39)$, there is a unique allocation that brings the same welfare[40] as the one obtained by this model. Any other

[39] Recall $\sum_{i=1}^{2} z_{i,\ell} \leq \eta_\ell\, y_{h,\ell}$ for all $\ell \in L$.

[40] We have that $\mathcal{W}_h = 0.547$.

distribution of the allocation of commodities will bring a completely different effect to the ones associated to bd $\left[\theta_h\left(\sum_{i\in h} x_i\right)\right]$.

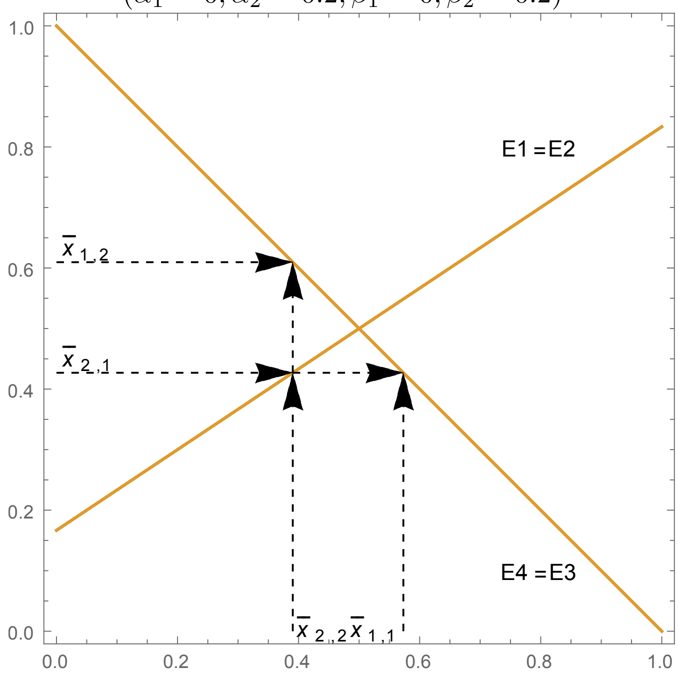
Therefore, this externality function is capable to represent only one of the many possible equilibrium allocations that the model with coalition formation has. Putting it another way, the intersection between the set of elements that brings this externality function intersect with only one element of the boundary of $\theta_h\left(\sum_{i\in h} x_i\right)$, any other allocation $(x_i^*)_{i\in h}$, with $\sum_{i\in h} x_i^* = \sum_{i\in h} x_i$, brings another different effect with respect to any $(z_i)_{i\in h} \in$ bd $\left[\theta_h\left(\sum_{i\in h} x_i\right)\right]$.

The next figure show another externality function which is capable of replicate all possible equilibrium allocations for any possible coalition demand. That is, at the given prices $\overline{p}$ there exist infinite combinations of $(x_i)_{i\in h}$ that leaves the same welfare and demands to all coalitions.

In this case the linearity associated to the set $\theta_h[y_h]$ makes easy to find that particular function. Particularly, the optimal values associated to this type of solution are $\alpha_1 = \beta_1 = (\eta_1 - 1)$ and $\alpha_2 = \beta_2 = (\eta_2 - 1)$.

Figure 3.

$(\alpha_1 = 0, \alpha_2 = 0.2, \beta_1 = 0, \beta_2 = 0.2)$



Therefore, we have two families of uniformly bounded linear operators. One, associated to linear operators that satisfy *(E1)-(E2)* at least in one point, and the other one, consisting in a unique function that satisfy *(E1)-(E2)* in all pairs $(x_{2,1}, x_{2,2})$.

REMARK 1. *It is important to remember that there are more non-negative externalities from consumption that the model does not capture since violates Assumption (A2).*

However, where we can observe a significant difference between both models comes from the versatility that allow non-linear rivalry reduction. This means that for some cases of rivalry reduction there is no non-negative externality function capable of fully replicate it. Moreover, in terms of assumptions of the models, reduction in rivalry in consumption allow the existence of zones of negative externalities in consumption. The next example summarize this point.

Example 3. Let consider an economy with one non-trivial coalition consisting on two individuals, with one commodity such that, $y_h = w_h$, for any $h \in A$. Here we only show that the conditions that an externality function must satisfy in order to be capable of fully replicate the rivalry reduction are associated not only to non-negative externalities, but also with negative externalities.

The set of allocations of use is defined as,

$$\theta_h(y_h) = \left\{(z_i)_{i\in h} \in \mathbb{R}^2_+ : \, A_1 \; (z_1)^2 + A_2 \; (z_2)^2 \leq (y_h)^2\right\}$$

with $(A_1, A_2) \geq 1$.

In this case we normalize price to $\overline{p} = 1$ and coalition demand is w_h, for any $h \in A$. Hence, we are looking for externality functions $f_h(x_h) = (f_{h,i}(x_h))_{i\in h}$ that are capable to satisfy,

$$A_1 \; (f_{h,1}(x_h))^2 + A_2 \; (f_{h,2}(x_h))^2 = (w_h)^2,$$

Hence in order to have an externality function capable to fully replicate all elements of bd $[\theta_h \, (w_h)]$, it is sufficient to find a function that satisfy the condition above for all $\sum_{i\in h} x_i = w_h$. That is, a function that takes a particular distribution of coalition demand to the boundary of the set $\theta_h(y_h)$,

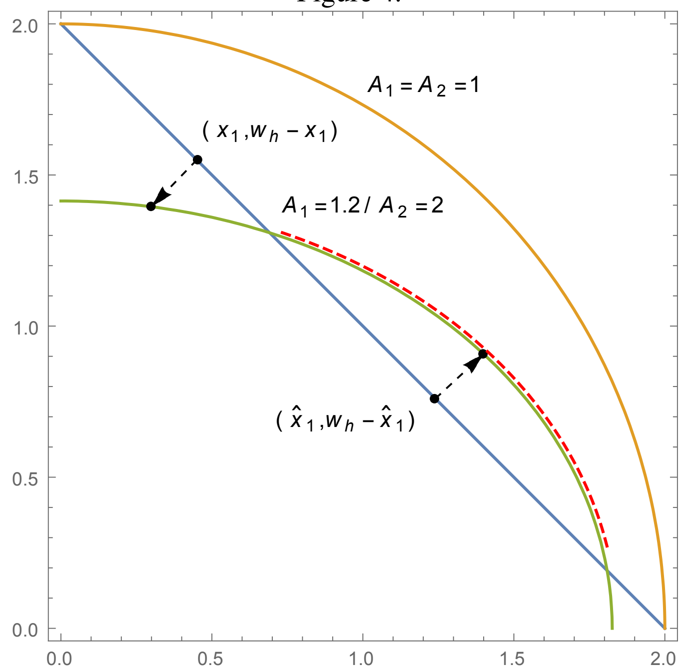
$$A_1 \; (f_{h,1}(x_1, w_h - x_1))^2 + A_2 \; (f_{h,2}(x_1, w_h - x_1))^2 = (w_h)^2, \quad \forall \, x_1 \in [0, w_h]$$

with $(f_{h,i})_{i\in h}$ satisfying the conditions of Assumption *(A2)*. Moreover, the function not only requires continuity, but also some robustness in the sense of,

$$\text{If } \widetilde{x}_i \geq \widehat{x}_i \Rightarrow f_{h,i}(\widetilde{x}_i, w_h - \widetilde{x}_i) \geq f_{h,i}(\widehat{x}_i, w_h - \widehat{x}_i), \quad \forall i \in h$$

If we set $A_1 = 1$ and $A_2 = 1$ there exist a non-negative externality function capable of fully replicate rivalry reduction.[41] However, when $(A_1, A_2) \geq 1$ –with one with strict inequality– the externality function that is capable of fully replicate the rivalry reduction has commodity allocations that makes that all members generate negative consumption externalities. Figure 4 represent this point.

Figure 4.



Firstly, notice that the set Gr $[\theta_h]$ satisfy Assumptions *(A2)-(A3)*. Secondly, in order to fully replicate the rivalry reduction when $(A_1, A_2) \geq 1$, a private allocation, such as $(x_1, w_h - x_1)$, should use a transformation that represent a negative consumption externality generated by all members. However, there is another allocation, which is optimal, and can be represented by a non-negative externality function. For example, the one associated to private allocation $(\widehat{x}_1, w_h - \widehat{x}_1)$. Finally, the red dashed line represent a non-negative externality function that partially replicate the rivalry reduction. This last is an example of equivalence in outcomes.

From this example we conclude that a model with coalition formation and reduction in rivalry in consumption is less restrictive than a model with household formation and non-negative

[41] Some numerical approximations can be made in order to approximate the effect of reduction in rivalry in terms of private allocation $(\overline{x}_i)_{i\in h}$. For example, projection methods (see Judd (1997)).

externalities in consumption. This is because, in terms of the latter, the former requires to have a compact convex subset of private allocations where non-negative externalities exist. Instead, the latter requires that at least there exist a member that not generate any negative externality. In terms of Proposition 2 of Gersbach et al (2011),

''..., a member $j \in h$ whose preferences are strictly monotonic in own consumption and who is not imposing any negative consumption externalities on other household members".

REMARK 2. *Notice if the condition of robustness is not imposed we may have externality functions that satisfy Proposition 2 and at the same time fully replicate the rivalry reduction.*

7 IMPACT ON WELFARE AND INEQUALITY

The aim of this section is to analyse the impact on welfare as well as on inequality when coalition formation is allowed. Here we conclude that, although coalition formation may induce allocations that brings a higher social welfare –measured as $\sum_{i \in I} U_i$, this is not necessarily true for all $h \in A$. The reason comes from the relation that exist between social structure and prices. Hence, at a given coalition structure there will exist equilibrium prices that may lead to a reduction on welfare for some coalition $h \in A$ with respect to an economy without coalition formation.

Here inequality is studied in terms of Gini index with respect to an economy Arrow-Debreu. However, it is important to notice that when non-trivial coalitions are allowed, there is no individual wealth to be used to construct a Gini index. Also, using the personal contribution of each member in order to obtain a measure of inequality does not make any modification in the extent of inequality. Hence, in order to capture the effect of coalition formation on inequality, in this framework each member of a non-trivial coalition is associated to the minimum expenditure (at the going prices) he needs to achieve the same welfare level when coalition is formed. Thus, if we observe a reduction in inequality by the formation of a non-trivial coalition, we can consider the extra expenditure[42] as a measure of gain (or savings), since in terms of inequality, forming a non-trivial coalition is equivalent to give each member a certain amount of wealth.

[42]Given Assumptions *(A1)-(A3)*, we know that each member of a non-trivial coalition will obtain an allocation that gives him a welfare strictly greater than the one obtained when alone. Hence, the expenditure associated to that particular welfare level should be greater than the wealth of the member.

Therefore, given a household structure $A \in \mathcal{H}$ and Assumptions *(A1)-(A3)*, we claim that there is an (CEFE), defined as $(\overline{p}, (\overline{y}_h, (\overline{z}_i)_{i\in h})_{h\in A}) \in \Delta \times \prod_{h\in A} \left(\mathbb{R}^L_+ \times \mathbb{R}^{L\times N(h)}_+\right)$, such that for all $i \in I$ there exists,

$$e_i(\overline{p}) = \min\left\{\overline{p}\cdot x_i : U_i(x_i) \geq U_i(\overline{z}_i), \text{ with } x_i \in \mathbb{R}^L_+\right\}$$

Thus, after calculate every minimum expenditure for all individulas, we construct the Gini index following Sen (1997),

$$G = \frac{I+1}{I} - \frac{2}{n^2\,\overline{e}} \sum_{i\in I} (I+1-i)\, e_i;$$

where, $\overline{e} = (1/I)\sum_{i\in I} e_i$ and the elements of the vector $\vec{e} = \{e_1,\, e_2,\, e_3, \ldots,\, e_I\}$ are arranged in non-decreasing order.

The results on Table 1 show the impact of coalition formation in welfare and inequality on an economy with the following parametrization.[43]

1. One trivial coalition $\{\gamma\}$ and another non-trivial coalition $h = \{\alpha,\, \beta\}$.
2. Every member has preferences over both commodities represented by,

   $$U_i = (z_{i,1})^{\delta_i}\,(z_{i,2})^{1-\delta_i}, \quad \forall\, i \in I.$$

   where, $\delta_\alpha = 0.6$; $\delta_\beta = 0.4$; $\delta_\gamma = 0.25$.
3. Commodity one has no usufruct but commodity two does. Where,

   $$\sum_{i\in\{\alpha,\beta\}} z_{i,\ell} = \eta_\ell\, y_{h,\ell},$$

   with $\eta_1 = 1$ and $\eta_2 = 1.5$.
4. The social wealth is $\sum_{i\in I} w_i = (4, 4)$.

Every column on Table 1 represents an equilibrium of Definition 1 as well as an equilibrium in an economy Arrow-Debreu. We compare, for different distributions of social wealth, the impact on welfare and inequality induced by the optimal allocations of these economies. The results show a robust pattern with respect to social welfare, but there is no clear relation when

[43] We use an Augmented Lagrangian algorithm from Andreani, R., Birgin, E. G., Martínez, J. M., and Schuverdt, M. L., (2007), to compute all these examples.

inequality is studied. For the latter more considerations are needed, in particular the initial social distribution of wealth, individual preferences, as well as the parameters associated to reduction in rivalry.

The last three rows represent the share of wealth that we impute initially to each individual. The rest of the rows are the different outputs calculated by the *Augmented Lagrangian Method*. The last three rows represent the effective gains when non-trivial coalition is formed. It is effective, since recall that the participation constraint depends on the equilibrium price induced by the formation of the coalition. However, if formation is not allow the equilibrium prices will be different, hence, attaining a different welfare.

Table 1: Impact on Welfare and Inequality.

	Equilibriums						
	(I)	(II)	(III)	(IV)	(V)	(VI)	(VII)
Gini							
Arrow-Debreu	0.167	0.000	0.083	0.167	0.200	0.167	0.133
Coalition Formation	0.149	0.086	0.165	0.146	0.174	0.171	0.049
Welfare							
Arrow-Debreu	4.182	4.178	4.165	4.186	4.167	4.134	4.213
Coalition Formation	4.655	4.800	4.866	4.847	4.884	4.867	4.740
Price equilibrium $\ell = 1$							
Arrow-Debreu (p_0)	0.375	0.417	0.438	0.463	0.470	0.413	0.420
Coalition Formation (p_1)	0.386	0.407	0.428	0.449	0.456	0.425	0.401
Individual Effective Gain							
$U_\alpha(p_1) - v_\alpha(p_0)$	0.453	0.034	0.045	0.042	0.046	0.513	0.034
$U_\beta(p_1) - v_\beta(p_0)$	0.005	0.596	0.664	0.631	0.680	0.211	0.505
$U_\gamma(p_1) - v_\gamma(p_0)$	0.012	-0.009	-0.008	-0.013	-0.010	0.009	-0.013
Shares of Social Wealth							
s_α	0.250	0.330	0.375	0.500	0.500	0.250	0.400
s_β	0.250	0.330	0.375	0.250	0.300	0.500	0.200
s_γ	0.500	0.330	0.250	0.250	0.200	0.250	0.400
$\|\|\epsilon\|\|_1$	2.66E-007	2.67E-007	1.25E-007	1.85E-007	3.40E-007	2.43E-007	2.29E-007

$||\epsilon||_1 = \sum_{i=1}^{N} |\epsilon_i|$, where ϵ_i is the i-th error term associated to the optimization problem. Here, $N = 15$.

A first comment from Table 1 would be that, under this parametrization, coalition formation will always lead to a greater social welfare. However, from the rows associated to the individual effective gains, we observe that for some coalition $h \in A$ the formation of non-trivial groups causes a welfare loss. In particular, trivial coalition $\{\gamma\}$ reduces his welfare when coalition $\{\alpha, \beta\}$ is formed. Still, in these examples this loss is always offset by the welfare gains of the non-trivial coalition. These losses of welfare are explained by the different equilibrium prices achieved by both economies. For example, column (IV) show that, when non-trivial coalition is formed, trivial coalition $\{\gamma\}$ faces equilibrium prices which makes more expensive the commodity who prefer the most. Thus, given that the only difference between the two economies is the formation of coalitions, we can infer that the change in equilibrium prices is induced by changes in demand due to the formation of such groups. That is, reduction in rivalry in consumption induces to the non-trivial coalition to demand differently with respect to the aggregated individual demands of its members.[44]

It is also clear that coalition formation not necessarily induces a reduction on inequality since, for the one hand, the expenditure imputed to each member tend to be unequal; and for the other hand, the net effect also depends on the initial social wealth distribution.

Notice that when inequality is reduced, the difference in expenditure between the two models can be see it as the expenditure saved on reduction of inequality by forming a certain coalition structure. On the contrary case, when inequality worsens, the increment on expenditure can be see it as the amount that should be imputed in an Arrow-Debreu economy to obtain the same welfare obtained in a economy where coalition formation is allowed. Also notice the particular case presented in column (VI), where there is a welfare improvement to all individuals with respect to the Arrow-Debreu situation, but still inequality worsens. This last is due to the combination of social wealth distribution and preferences over commodities.

The unequal expenditure distribution depends on a combination of factors, among them, the share of wealth of each agent, preferences and the way that reduction in rivalry acts. For example, column (I) shows that the greater improvement among members of the non-trivial coalition goes to agent α, which prefers commodity 1 (without reduction in rivalry) over 2 (with reduction in rivalry). This is because: (1.) The greatest share of wealth belongs to whom prefers commodity 2 over commodity 1 and hence $p_2 > p_1$; (2.) Inside the coalition, wealth is distributed equally; and (3.) The reduction in rivalry over commodity 2 reduce the expenditure needed for the non-trivial coalition. And so this spending goes to commodity 1 (since is cheaper) making better off agent α.

However, when there is a economy such as the one from column (IV) or column (VI), most of the total extra expenditure goes to the member with less wealth. Nonetheless, given heterogeneity among members, similar distribution inside coalition such as from equilibrium

[44] However, although the reduction in the rivalry in consumption induces a change in demand, the welfare function also affects such assignments.

(IV) and equilibrium (VI) induce opposite effects on inequality. This last effect is due to preferences and initial individual wealth. In one case agent β, who prefer commodity 2 over commodity 1, obtain more gains when forming the coalition since his initial wealth. This translate into a greater demand for commodity 2 which leads to an increase in p_2, making worse-off agent γ, who can not form any coalition. On the contrary, when agent α is the one with less wealth among members, this translate into a greater demand for commodity 1 by the non-trivial coalition, hence the increase in p_1 with respect to the Arrow-Debreu situation. This last makes better-off agent γ by the same reasons mentioned before.

8 CONCLUDING REMARKS

A general equilibrium model that allow coalition formation among individuals is proposed. The incentives to form non-trivial coalition comes from the possibility of reducing rivalry in consumption. The way of reduction in rivalry acts may depend on preference among members as well as the wealth that non-trivial coalition has. Here, as in Ellickson et al (1999), the effect of forming a coalition is endogenous to the group, or not anonymous. That is, it does matter who belongs to the coalition.

Under regular assumptions over preferences of individuals as well as specific assumptions over the set that represents the effect on reduction in rivalry, there exist an equilibrium with free exit. This equilibrium concept –that we take from Gersbach et al (2011)– determine the existence of a coalition structure where a non-trivial coalition exist. Moreover, this equilibrium concept determine that a non-trivial coalition is stable with respect to the possibility that members prefer to go alone. However, some examples are made in order to show that stability with respect to other non-trivial coalitions depends on more demanding assumptions besides reduction in rivalry in consumption. Therefore, under the assumptions considered in this model, the equilibrium concept will exist but we can not say anything further with respect to stability. Considering this last, an example takes the game theoretic approach to find an stable coalition structure under strategies associated to Folk theorem.

It is also shown the relation between a model of coalition formation and a model of non-negative externalities presented by Gersbach and Haller (2011). It is proved that the set of distributions of use satisfy, what Gersbash et al (2011) call, *Large Group Advantage*. This property determines that, for any price in the simplex, there exist a compact, convex and non-empty subset from the allocation set that makes every member better-off than its option to go alone. Leaving aside other aspects that this authors considered for externalities (specifically, pure groups externalities), some examples are made in order to show that some non-negative externalities can bring an equivalent equilibrium allocation with the same welfare outputs, but not necessarily all the equilibrium allocations. For the latter to happens, it is required (at least) that there exist a function capable to take any coalition demand –satisfying $p \cdot \left(\sum_{i\in h} x_i - w_h\right) = 0$– and be able to replicate all the allocations that belong to bd $\left[\theta_h\left(\sum_{i\in h} x_i\right)\right]$. This last could be possible since the possibility to re-allocate private consumption among members –without changing coalition demand– still induce changes in members' welfare. Thus, an economy with non-negative externalities with an specific function that fully replicate those allocations of bd $\left[\theta_h\left(\sum_{i\in h} x_i\right)\right]$ (for any $\sum_{i\in h} x_i$) trough the possibility to re-allocate private consumption of members, is considered an an equivalent economy with coalition formation in terms of that the welfare and the equilibriums allocations are the same. An example show externality functions that partially or fully replicate the rivalry reduction. However, with non-linear rivalry reduction there exist externality functions that fully replicate rivalry reduction in which there are private allocation that generates negative externalities in consumption by all members. This last is not considered in a model with household formation and non-negative externalities. Hence, endogenous rivalry reduction represent a more general setting.

It is also studied the effect that the formation of coalitions have over welfare and inequality. The numerical examples show that, given the assumptions (specifically, Assumption *(A3)*), there always exist a social welfare improvement. However, some coalitions may have a welfare loss when coalitions are formed, but this loss is always offset by the gains of forming a non-trivial coalition. The loss on welfare by some coalitions, it is due to a change on equilibrium prices, which are induced by the formation of non-trivial coalitions. The effect over inequality is unknown since depends not only in reduction in rivalry, but also from the social distribution of wealth and preferences. So, it may happen that formation of non-trivial coalition increase inequality even if inside coalition all members are better off *as if* all have more resources to spend on consumption.

Bibliography

[1] Andreani, R., Birgin, E. G., Martínez, J. M., & Schuverdt, M. L. (2007). On Augmented Lagrangian methods with general lower-level constraints. SIAM Journal on Optimization, 18(4), 1286-1309.

[2] Becker, G. S. (1974). A theory of marriage. In Economics of the family: Marriage, children, and human capital (pp. 299-351). UMI.

[3] Berglas, E. (1976). On the theory of clubs. American Economic Review, 66(2), 116-21.

[4] Chiappori, P. A. (1988). Rational household labor supply. Econometrica: Journal of the Econometric Society, 63-90.

[5] Chiappori, P. A. (1992). Collective labor supply and welfare. Journal of political Economy, 437-467.

[6] Debreu, G. (1952). A social equilibrium existence theorem. Proceedings of the National Academy of Sciences of the United States of America, 38(10), 886.

[7] Ellickson, B., Grodal, B., Scotchmer, S., & Zame, W. R. (1999). Clubs and the Market. Econometrica, 67(5), 1185-1217.

[8] Gale, D., & Shapley, L. S. (1962). College admissions and the stability of marriage. American Mathematical Monthly, 9-15.

[9] Gersbach, H., & Haller, H. (2011). Groups, collective decisions and markets. Journal of Economic Theory, 146(1), 275-299.

[10] Gersbach, H., & Haller, H. (2010). Club theory and household formation. Journal of Mathematical Economics, 46(5), 715-724.

[11] Holtermann, S. E. (1972). Externalities and public goods. Economica, 78-87.

[12] Konishi, H. (2010). Efficient Mixed Clubs: Nonlinear-Pricing Equilibria With Entrepreneurial Managers. Japanese Economic Review, 61(1), 35-63.

[13] Musgrave, R. A. (1969). Provision for Social Goods. In Julius Margolis and H. Guitton (eds.), Public Economics: An analysis of Public Production and Con sumption and their Relations to the Private Sectors. London: Macmillan.

[14] Samuelson, P. A. (1954). The pure theory of public expenditure. The review of economics and statistics, 387-389.

[15] Sen, A. K., & Foster, J. E. (Eds.). (1997). On economic inequality. Oxford University Press.

A

APPENDIX MITIGATION EQUILIBRIUM

Proof of Theorem 1.

Step 1. Notice that *(A3)-(A4)* ensure that allocations,

$$(x_h, (G_{h,m})_{m\in M}, (x_{h,m})_{m\in M})_{h\in H} \in \left(\mathbb{X} \times \mathbb{M} \times \mathbb{X}^M\right)^H$$

and,

$$((G_{F,m})_{m\in M}, (x_{F,m})_{m\in M}) \in \left(\mathbb{M} \times \mathbb{X}^M\right)$$

that satisfy market clearing conditions of equilibrium definition (item (iii)) is bounded. Thus, we set an uniform upper bound $\Omega > 0$ such that, any feasible allocation that satisfies market clearing conditions is less than $\Omega(1, \ldots, 1) \in \left(\mathbb{X} \times \mathbb{M} \times \mathbb{X}^M\right)^H \times \left(\mathbb{M} \times \mathbb{X}^M\right)$. Let $\mathbb{E}_1(\Omega) = [0, 2\,\Omega]^L$, $\mathbb{E}_2(\Omega) = [0, 2\,\Omega]^M$, and $\mathbb{E}_3(\Omega) = \mathbb{E}_1(\Omega)^M$. Finally, let $\mathbb{E}(\Omega) = \mathbb{E}_1(\Omega) \times \mathbb{E}_2(\Omega) \times \mathbb{E}_3(\Omega)$ be the set of allocations of any $h \in H$ whose coordinates are lower than or equal to $2\,\Omega$. Analogously, let $\mathbb{G}_1(\Omega) = \mathbb{E}_2(\Omega)$ and $\mathbb{G}_2(\Omega) = \mathbb{E}_3(\Omega)$. Hence, $\mathbb{G}(\Omega) = \mathbb{G}_1(\Omega) \times \mathbb{G}_2(\Omega)$ be the set of allocations of the fiscal authority F whose coordinates are lower than or equal to $2\,\Omega$.

Step 2. Let define the price simplex as $\Delta = \left\{v \in \mathbb{R}^L_+ : \sum_{\ell\in L} v_\ell = 1\right\}$. Consider a generalized game $\mathcal{G}(\Omega, t)$, with $\#H + 2$ players, such that:

1. Given $p \in \Delta$, and other's allocation,

   $$\Big((x_j, (G_{j,m})_{m\in M}, (x_{j,m})_{m\in M})_{j\in H\setminus\{h\}}, (G_{F,m})_{m\in M}, (x_{\tau,m})_{m\in M}\Big) \in \mathbb{E}(\Omega)^{H-1} \times \mathbb{G}(\Omega),$$

   each household $h \in H$ maximizes his objective function u_h in the truncated choice set $\mathcal{C}_h(p, x_{-h,M}) \cap \mathbb{E}(\Omega)$.

2. Given $p \in \Delta$, and other's allocation,

$$(x_h, (G_{h,m})_{m \in M}, (x_{h,m})_{m \in M})_{h \in H} \in \mathbb{E}(\Omega)^H,$$

a fiscal authority F maximizes Ψ_F in the truncated choice set $\mathcal{C}_F(p, x_{H,M}) \cap \mathbb{G}(\Omega)$.

3. Given other's allocations,

$$\left((x_h, (G_{h,m})_{m \in M}, (x_{h,m})_{m \in M})_{h \in h}, (G_{F,m})_{m \in M}, (x_{F,m})_{m \in M}\right) \in \mathbb{E}(\Omega)^H \times \mathbb{G}(\Omega),$$

there exists an auctioneer a_0 that chooses $p \in \Delta$ in order to maximize,

$$p \cdot \left(\sum_{h \in H} \left(x_h + \sum_{m \in M} x_{h,m} \right) + \sum_{m \in M} x_{\tau,m} - \sum_{h \in H} w_h \right).$$

Therefore, a *Cournot-Nash equilibrium* for $\mathcal{G}(\Omega, t)$ is given by a vector

$$\left[(x_h, (G_{h,m})_{m \in M}, (x_{h,m})_{m \in M})_{h \in H}, (G_{\tau,m})_{m \in M}, (x_{\tau,m})_{m \in M}, p\right] \in \mathbb{E}(\Omega)^H \times \mathbb{G}(\Omega) \times \Delta$$

which simultaneously solves the optimal problem of each player in the generalized game.

Claim 1. The correspondences

$$\mathcal{C}_h(p, x_{-h,M}) \cap \mathbb{E}(\Omega) : \Delta \times \mathbb{E}_3(\Omega)^{H-1} \times \mathbb{G}_2(\Omega) \twoheadrightarrow \mathbb{E}(\Omega)$$

and

$$\mathcal{C}_F(p) \cap \mathbb{G}(\Omega) : \Delta \times \mathbb{E}_3(\Omega)^H \twoheadrightarrow \mathbb{G}(\Omega)$$

are continuous and have compact, convex and non-empty values.

Proof. From Step 1 we know that $\mathcal{C}_h(p, x_{-h,M}) \cap \mathbb{E}(\Omega)$ has compact co-domain. Also, considering *(A3)* and *(A4)*, $\mathcal{C}_h(p, x_{-h,M}) \cap \mathbb{E}(\Omega)$ has the closed graph property. Hence, $\mathcal{C}_h(p, x_{-h,M}) \cap \mathbb{E}(\Omega)$ is upper hemicontinuous with closed values. Notice that $\mathcal{C}_h(p, x_{-h,M}) \cap \mathbb{E}(\Omega)$ also has convex and non-empty values. Lower hemicontinuity comes from the fact that Int $[\mathcal{C}_h(p, x_{-h,M}) \cap \mathbb{E}(\Omega)]$ has non-empty values –*(A4)*– and has the open graph property. Therefore $\mathcal{C}_h(p, x_{-h,M}) \cap \mathbb{E}(\Omega)$ is a continuous correspondence with compact, convex and non-empty values. Following similar steps we have that $\mathcal{C}_F(p, x_{H,M}) \cap \mathbb{G}(\Omega)$ has compact domain as well as compact co-domain. Considering *(A3)* and *(A5)* we assure that $\mathcal{C}_F(p, x_{H,M}) \cap \mathbb{G}(\Omega)$ also has convex and non-empty values. Upper hemicontinuity comes from the closed graph property and the compact co-domain. Lower hemicontinuity comes from *(A5)* and the open graph property of Int $[\mathcal{C}_F(p, x_{H,M}) \cap \mathbb{G}(\Omega)]$. Therefore, $\mathcal{C}_F(p, x_{H,M}) \cap \mathbb{G}(\Omega)$ is continuous with compact, convex and non-empty values. □

Claim 2. There exists a Cournot-Nash equilibrium for $\mathcal{G}(\Omega, t)$.

Proof. The objective functions of households, the auctioneer and the fiscal authority are continuous and quasi-concave on their own strategies. Notice that auctioneers' correspondence of admissible strategies is constant, hence, it is continuous and has compact, convex and non-empty values. Therefore, considering Claim 1, we have that for every player in this game the correspondence of admissible strategies is continuous and has compact, convex and non empty values. Therefore, Berge's Maximum Theorem ensures that, for any player, the best-reply correspondence is upper hemicontinuous and has non-empty, compact and closed values. In addition, the set $\mathbb{E}(\Omega)^H \times \mathbb{G}(\Omega) \times \Delta$ is non-empty, compact and convex. Hence, we can apply Kakutani's Fixed Point Theorem to the correspondence of optimal strategies in order to find a Cournot-Nash equilibrium for our generalized game. □

Step 3. Now, to prove that an equilibrium for the economy can be obtained as a Cournot-Nash equilibria of the generalized game $\mathcal{G}(\Omega, t)$ let assume that,

$$\left[\left(\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M}\right)_{h\in H}, (\overline{G}_{F,m})_{m\in M}, (\overline{x}_{F,m})_{m\in M}, \overline{p}\right] \in \mathbb{E}(\Omega)^H \times \mathbb{G}(\Omega) \times \Delta$$

is a Cournot-Nash equilibrium for $\mathcal{G}(\Omega, t)$. Thus, for every $h \in H$,

$$\left(\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M}\right) \in \mathcal{C}_h(\overline{p}, \overline{x}_{-h,M}) \cap \mathbb{E}(\Omega)$$

Also for F we have that $\left(\overline{G}_{F,m}, \overline{x}_{F,m}\right)_{m\in M} \in \mathcal{C}_F(\overline{p}, \overline{x}_{H,M}) \cap \mathbb{G}(\Omega)$. Therefore, adding the budget constraints across households and considering the fiscal budget constraints, we obtain that,

$$\overline{p}\left(\sum_{h\in H}\left(\overline{x}_h + \sum_{m\in M}\overline{x}_{h,m}\right) + \sum_{m\in M}\overline{x}_{F,m} - \sum_{h\in H} w_h\right) \leq 0.$$

The optimality decision of the auctioneer a_0 implies that,

$$\sum_{h\in H}\left(\overline{x}_h + \sum_{m\in M}\overline{x}_{h,m}\right) + \sum_{m\in M}\overline{x}_{F,m} - \sum_{h\in H} w_h \leq 0$$

which ensures that for any $h \in H$, $\left(\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M}\right) \leq \Omega(1,\dots,1)$, and for the fiscal authority F we have that $\left(\overline{G}_{F,m}, \overline{x}_{F,m}\right)_{m\in M} \leq \Omega(1,\dots,1)$. Thus,

$$\left((\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M})\right)_{h\in H}$$

and,

$$\left(\overline{G}_{F,m}, \overline{x}_{F,m}\right)_{m\in M}$$

are allocations in the interior of $\mathbb{E}(\Omega)$ and $\mathbb{G}(\Omega)$ (respectively), i.e., the upper bounds on agents' allocations are not binding. As a consequence of the monotonicity *–(A1)* and *(A2)–* feasibility constraints hold with equality for all $h \in H$ and F, with $\overline{p} \gg 0$. That is, item (iii) of equilibrium definition holds.

Therefore, to ensure that

$$\left[\left(\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M}\right)_{h\in H}, \left(\overline{G}_{F,m}, \overline{x}_{F,m}\right)_{m\in M}, \overline{p}\right] \in \mathbb{E}(\Omega)^H \times \mathbb{G}(\Omega) \times \Delta$$

is a competitive equilibrium for $\mathcal{E}_{G,\xi}$, it remains to prove that for any $h \in H$ the allocation $(\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M})$ is an optimal choice on $\mathcal{C}_h(\overline{p}, \overline{x}_{-h,M})$, as well as for the fiscal authority F the allocation $(\overline{G}_{F,m}, \overline{x}_{F,m})_{m\in M}$ is an optimal choice on $\mathcal{C}_F\left(\overline{p}, \overline{x}_{H,M}\right)$.

Thus, for any $h \in H$, let define $\overline{s}_h = (\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M})$ as an optimal strategy for $h \in H$ in the truncated choice set $\mathcal{C}_h(\overline{p}, \overline{x}_{-h,M}) \cap \mathbb{E}(\Omega)$. By contradiction let assume that there is another allocation $\widetilde{s}_h = (\widetilde{x}_h, (\widetilde{G}_{h,m})_{m\in M}, (\widetilde{x}_{h,m})_{m\in M}) \in \mathcal{C}_h(\overline{p}, \overline{x}_{-h,M})$ such that,

$$u_h\left(\widetilde{x}_h, \widetilde{G}_h\right) > u_h\left(\overline{x}_h, \overline{G}_h\right).$$

By feasibility condition $(\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M}) \in \text{Int}\left[\mathcal{C}_h(\overline{p}, \overline{x}_{-h,M}) \cap \mathbb{E}(\Omega)\right]$ and since objective functions are strictly quasi-concave, we can set a $\lambda \in (0,1)$ such that,

$$u_h\left(\widehat{x}_h, \widehat{G}_h\right) > u_h\left(\overline{x}_h, \overline{G}_h\right).$$

where $(\widehat{x}_h, (\widehat{G}_{h,m})_{m\in M}, (\widehat{x}_{h,m})_{m\in M}) := \lambda\overline{s}_h + (1-\lambda)\widetilde{s}_h \in \mathcal{C}_h(\overline{p}, \overline{x}_{-h,M}) \cap \mathbb{E}(\Omega)$. This last property contradicts the fact that $(\overline{x}_h, (\overline{G}_{h,m})_{m\in M}, (\overline{x}_{h,m})_{m\in M})$ solves the optimal problem of player $h \in H$ on $\mathcal{G}(\Omega, t)$. Hence, condition stated in item (i) of equilibrium definition holds.

Following the same procedure for the fiscal authority F, $(\overline{G}_{F,m}, \overline{x}_{F,m})_{m\in M}$ is an optimal choice in the truncated choice set $\mathcal{C}_F(\overline{p}, \overline{x}_{H,M}) \cap \mathbb{G}(\Omega)$. Now, by contradiction define the allocation $(\widetilde{G}_{F,m},\widetilde{x}_{F,m})_{m\in M}\in\mathcal{C}_F\left(\overline{p},\overline{x}_{H,M}\right)$ such that,

$$\Psi\left(\widetilde{G}_F\right)>\Psi\left(\overline{G}_F\right)$$

By feasibility we know that $(\overline{G}_{F,m},\overline{x}_{F,m})_{m\in M}\in \text{Int}\left[\mathcal{C}_F\left(\overline{p},\overline{x}_{H,M}\right)\cap\mathbb{G}(\Omega)\right]$ and since objective functions are strictly quasi-concave, we can set a $\lambda\in(0,1)$ such that,

$$\Psi\left(\widehat{G}_F\right)>\Psi\left(\overline{G}_F\right)$$

where, $(\widehat{G}_{F,m},\widehat{x}_{F,m})_{m\in M}:=\lambda(\overline{G}_{F,m},\overline{x}_{F,m})_{m\in M}+(1-\lambda)(\widetilde{G}_{F,m},\widetilde{x}_{F,m})_{m\in M}\in\mathcal{C}_F\left(\overline{p},\overline{x}_{H,M}\right)\cap\mathbb{E}(\Omega)$. This last contradicts the fact that $(\overline{G}_{F,m},\overline{x}_{F,m})_{m\in M}$ solves the maximization problem of the fiscal authority on $\mathcal{G}(\Omega,t)$. Hence, condition stated in item (ii) of equilibrium definition holds.

Therefore $\left[\left(\overline{x}_h,(\overline{G}_{h,m})_{m\in M},(\overline{x}_{h,m})_{m\in M}\right)_{h\in H},\left(\overline{G}_{F,m},\overline{x}_{F,m}\right)_{m\in M},\overline{p}\right]$ is an equilibrium for $\mathcal{E}_{G,\xi}$. *Q.E.D*

Error Terms.

Table A.1. Error Terms Example 1

$(w_{1,1}, w_{1,2})$	$(w_{2,1}, w_{2,2})$	ϵ_1	ϵ_2	ϵ_3
(2,2)	(1,1)	-1.19E-009	-1.72E-009	1.35E-006
(1,1)	(2,2)	-7.47E-010	-6.62E-010	2.78E-006
(4,4)	(4,4)	-5.96E-009	4.28E-009	1.86E-006
(3,3)	(3,3)	-1.73E-010	7.47E-010	4.82E-009
(2,2)	(2,2)	-5.72E-009	2.22E-010	5.86E-008
(1,1)	(1,1)	3.68E-010	-1.18E-009	6.59E-008

Note1: $\epsilon_{\ell,m}$ is the error term associated to the Pareto condition associated to the pair (ℓ, m), with $\ell = \{1, 2\}$ and $m = \{1\}$.

Note2: $\epsilon_3 = \sum_{n=1}^{N} |\varepsilon_n|$, where ε_n is the nth-error term associated to the first order conditions. Here $N = 27$.

Table A.2. Error Terms Example 2

(w_1, w_2)	(α_1, β_1)	(α_2, β_2)	$\epsilon_{1,1}$	$\epsilon_{1,2}$	ϵ_3
(2,1)	(0.1,0.2)	(0.3,0.4)	-1.09E-007	8.57E-008	4.73E-006
(1,2)	(0.1,0.2)	(0.3,0.4)	6.89E-009	-1.12E-008	7.24E-007
(2,1)	(0.3,0.4)	(0.3,0.4)	-5.39E-008	1.33E-008	7.91E-007
(2,1)	(0.1,0.2)	(0.1,0.2)	-1.03E-009	-1.66E-008	5.92E-007

Note1: $\epsilon_{\ell,m}$ is the error term associated to the Pareto condition associated to the pair (ℓ, m), with $\ell = \{1\}$ and $m = \{1, 2\}$.

Note2: $\epsilon_3 = \sum_{n=1}^{N} |\varepsilon_n|$, where ε_n is the nth-error term associated to the first order conditions. Here $N = 20$.

Table A.3.1. Error Terms Example 3

b_2	ϵ_1	ϵ_2	ϵ_3
(1/4)	1.88E-009	-1.07E-009	5.18E-008
(1/3)	-4.00E-010	5.35E-010	2.23E-007
(1/2)	-1.19E-009	-1.72E-009	1.35E-006
(2/3)	1.03E-008	-2.46E-008	8.37E-007

Note1: $\epsilon_{\ell,m}$ is the error term associated to the Pareto condition associated to the pair (ℓ, m), with $\ell = \{1, 2\}$ and $m = \{1\}$.

Note2: $\epsilon_3 = \sum_{n=1}^{N} |\varepsilon_n|$, where ε_n is the nth-error term associated to the first order conditions. Here $N = 27$.

Table A.3.2. Allocations From Example 3

b_2	$x_{1,1}$	$x_{1,2}$	$x_{2,1}$	$x_{2,2}$	$z_{3,1}$	$z_{3,2}$	ξ_1	ξ_2	ξ
(1/4)	0.20	0.06	0.17	0.53	2.63	2.41	0.068	0.059	0.127
(1/3)	0.25	0.07	0.14	0.45	2.61	2.47	0.078	0.073	0.151
(1/2)	0.33	0.10	0.10	0.33	2.56	2.56	0.095	0.095	0.189
(2/3)	0.40	0.12	0.07	0.24	2.53	2.64	0.111	0.109	0.220

Table A.4.1. Error Terms Example 4

$((1/c_1), (1/c_2))$	ϵ_1	ϵ_2	ϵ_3
(0.75,0.75)	1.22E-009	-1.46E-009	2.47E-006
(0.75,0.50)	1.59E-010	-1.83E-010	1.48E-006
(0.50,0.75)	-1.97E-010	2.29E-010	5.64E-007
(0.50,0.50)	-1.19E-009	-1.72E-009	1.35E-006
(0.50,0.25)	8.14E-011	-9.37E-011	1.83E-006
(0.25,0.50)	-9.32E-011	8.20E-011	7.23E-007
(0.25,0.25)	1.94E-010	-6.41E-010	2.89E-006

Note1: $\epsilon_{\ell,m}$ is the error term associated to the Pareto condition associated to the pair (ℓ, m), with $\ell = \{1, 2\}$ and $m = \{1\}$.

Note2: $\epsilon_3 = \sum_{n=1}^{N} |\varepsilon_n|$, where ε_n is the nth-error term associated to the first order conditions. Here $N = 27$.

Table A.4.2. Allocations From Example 4

(c_1, c_2)	$x_{1,1}$	$x_{1,2}$	$x_{2,1}$	$x_{2,2}$	$z_{3,1}$	$z_{3,2}$	ξ_1	ξ_2	ξ
(0.75,0.75)	0.28	0.08	0.08	0.28	2.64	2.64	0.10	0.10	0.20
(0.75,0.50)	0.17	0.05	0.14	0.45	2.69	2.50	0.07	0.13	0.20
(0.50,0.75)	0.45	0.14	0.05	0.17	2.50	2.69	0.13	0.07	0.20
(0.50,0.50)	0.33	0.10	0.10	0.33	2.56	2.56	0.09	0.09	0.19
(0.50,0.25)	0.05	0.02	0.26	0.73	2.69	2.26	0.05	0.14	0.19
(0.25,0.50)	0.73	0.26	0.02	0.05	2.26	2.69	0.14	0.05	0.19
(0.25,0.25)	0.44	0.13	0.13	0.44	2.43	2.43	0.08	0.08	0.16

Note1: $z_{3,1}$ and $z_{3,2}$ are demands for commodity 1 and 2 by the Fiscal authority.

B

APPENDIX COALITIONS AND ENDOGENOUS MIXED GOODS

PROOF OF PROPOSITION 1. Fix a non-trivial coalition structure $A \in \mathcal{H}$ in which any $h \in A^+$ satisfies *(A3)*.

Step 1. Let $y = (y_h)_{h\in A} \in \mathbb{R}_+^{L\times A}$ be an allocation that satisfies market clearing condition (ii.) of equilibrium definition. Then $y = (y_h)_{h\in A}$ is bounded and together with *(A2)(ii)* and *(A2)(iii)* ensure that $\Theta(y) := \prod_{h\in A} \theta_h(y_h)$ is a compact set. Therefore, there exists $\Omega := (\Omega_y, \Omega_z) \in \mathbb{R}^2_{++}$ such that, for any $(y_h)_{h\in A} \in \mathbb{R}_+^{L\times A}$ satisfying condition (ii) of Definition 1, and for every $((z_i)_{i\in h})_{h\in A} \in \Theta(y)$, we have that $(y_h, (z_i)_{i\in h}) \leq N(h)(\Omega_y(1,\dots,1), \Omega_z(1,\dots,1))$, $\forall h \in A$.

Step 2. Fix $h \in A^+$. Let $\mathbb{E}^h(\Omega)$ be the set of allocations $(y_h, (z_i)_{i\in h}) \in \mathbb{R}_+^L \times \mathbb{R}_+^{L\times h}$ that are bounded from above by $2N(h)(\Omega_y(1,\dots,1), \Omega_z(1,\dots,1))$. It follows that, $\mathbb{E}^h(\Omega)$ is non-empty, compact, and convex. In addition, *(A2)(i)*, *(A2)(ii)*, and *(A2)(iii)* guarantee that the correspondence $p \twoheadrightarrow \Phi_h(p) \cap \mathbb{E}^h(\Omega)$ is continuous with compact, convex and non-empty values.

Given $\kappa \in \mathbb{N}$ and $i \in h$, let $\widehat{z}_{i,\kappa} : \Delta \to [0,\kappa]^L$ be the private optimal decision of agent i, i.e., $\widehat{z}_{i,\kappa}(p) := \arg\max \left\{U_i(x) : x \in [0,\kappa]^L \ \wedge \ p\cdot x \leq p\cdot w_i\right\}$. It follows from *(A1)*, *(A2)(ii)* and Berge's Maximum Theorem that $(\widehat{z}_{i,\kappa})_{i\in h}$ are continuous functions. In addition, *(A3)(ii)* implies that $\widehat{z}_{i,\kappa}(p) \gg 0$, $\forall i \in h$, $\forall p \in \Delta$.

Fix an allocation $(\widetilde{y}_h, (\widetilde{z}_i)_{i\in h}) \in \mathbb{E}^h(\Omega)$ that satisfies *(A3)(i)*.[1] Define the continuous function $\lambda_{h,\kappa} : \Delta \to (0,1)$ such that $\widehat{z}_{i,\kappa}(p) - \lambda_{h,\kappa}(p)\widetilde{y}_h \gg 0,\ \forall i \in h$.[2] Thus, *(A2)(iv)* implies

[1]The convexity of the graph of θ_h ensures that this allocation can be always taken inside $\mathbb{E}^h(\Omega)$.

[2]Notice that *(A3)(iii)* guarantees that $\widehat{z}_{i,\kappa}(p) \gg 0$. Thus, we can fix $\lambda_{h,\kappa}(p) = \frac{\min_{i\in h}\min_{\ell\in L}\widehat{z}_{i,\kappa,\ell}(p)}{2\max_{\ell\in L}\widetilde{y}_\ell}$.

that,

$$\left(\widehat{z}_{i,\kappa}(p) - \lambda_{h,\kappa}(p)\widetilde{y}_h\right)_{i\in h} \in \theta_h\left(\sum_{i\in h}\widehat{z}_{i,\kappa}(p) - \lambda_{h,\kappa}(p)N(h)\widetilde{y}_h\right).$$

On the other hand, convexity of $\mathrm{Gr}[\theta_h]$ guarantees that,

$$\left(\lambda_{h,\kappa}(p)N(h)\widetilde{z}_i\right)_{i\in h} \in \theta_h\left(\lambda_{h,\kappa}(p)N(h)\widetilde{y}_h\right),$$

Thus by *(A2)(v)* we have that,

$$\left(\sum_{i\in h}\widehat{z}_{i,\kappa}(p),\ (\widehat{z}_{i,\kappa}(p) + \lambda_{h,\kappa}(p)(N(h)\widetilde{z}_i - \widetilde{y}_h))_{i\in h}\right) \in \Phi_h(p).$$

Step 3. Given $(\kappa, i) \in \mathbb{N}\times I$, let $v_{i,\kappa} : \Delta \to \mathbb{R}_+$ be the function defined by

$$v_{i,\kappa}(p) = \max_{\{x\in[0,\kappa]^L:\, p\cdot x \le p\cdot w_i\}} U_i(x).$$

It follows from Berge's Maximun Theorem that for any $\kappa \in \mathbb{N}$ the mappings $(v_{i,\kappa})_{i\in I}$ are well defined and continuous.

For every $h \in A^+$, let $X_{h,\kappa} : \Delta \twoheadrightarrow \mathbb{R}^L_+ \times \mathbb{R}^{L\times h}_+$ be the correspondence that associates to every $p \in \Delta$ the set of allocations $(y_h, (z_i)_{i\in h}) \in \Phi_h(p)$ such that,

$$U_i(z_i) - v_{i,\kappa}(p) \ge \mu\left(U_i\big(\widehat{z}_{i,\kappa}(p) + \lambda_{h,\kappa}(p)(N(h)\tilde{z}_i - \tilde{y}_h)\big) - v_{i,\kappa}(p)\right),\ \forall i \in h,$$

with $\mu > 0$.

Notice that as a consequence of *(A1)*, *(A2)(i)*, *(A3)(i)*, and Berge's Maximun Theorem, the right-hand side of the inequality above is strictly positive and continuously depends on $p \in \Delta$.

CLAIM 1. *Given $\kappa \in \mathbb{N}$, for every $h \in A^+$, the correspondence $X_{h,\kappa} : \Delta \to \mathbb{R}^L_+ \times \mathbb{R}^{L\times h}_+$ is closed, and lower hemicontinuous, with non-empty and convex values.*

PROOF. Fix $h \in A^+$. Since $(U_i)_{i\in h}$ are quasi-concave functions, it follows that $X_{h,\kappa}$ has convex values. The continuity of functions $(v_{i,\kappa}, U_i, \widehat{z}_{i,\kappa})_{i\in h}$ ensure that $X_{h,\kappa}$ has closed graph.

The continuity of functions $(v_{i,\kappa}, U_i, \widehat{z}_{i,\kappa})_{i\in h}$ along with *(A2)(i)*, *(A2)(iii)*, and *(A3)(i)*, ensure that the correspondence $\mathring{X}_{h,\kappa}$ that associates to any $p \in \Delta$ to the interior of $X_{h,\kappa}(p)$

(relative to Gr$[\theta_h]$) has non-empty values and open graph.[3] Hence, $\mathring{X}_{h,\kappa}$ is lower hemicontinuous. Therefore, $\overline{\mathring{X}_{h,\kappa}} = X_{h,\kappa}$ is lower hemicontinuous too. $\square$

Now, consider a generalized game $\mathcal{G}_A(\Omega, \kappa)$ characterized by:

1. Given $p \in \Delta$, each coalition $h \in A^+$ maximizes the function $W_h\left((U_i(z_i))_{i\in h}\right) \equiv W_h\left((z_i)_{i\in h}\right)$ by choosing an allocation $(y_h, (z_i)_{i\in h}) \in X_{h,\kappa}(p) \cap \mathbb{E}^h(\Omega)$.

2. Given $p \in \Delta$, each agent $h \in A \setminus A^+$ maximizes the function U_h by choosing an allocation $y_h \in [0, \kappa]^L$ such that $p \cdot y_h \leq p \cdot w_h$.

3. Given allocations $(y_h, (z_i)_{i\in h})_{h\in A^+} \in \prod_{h\in A^+} \mathbb{E}^h(\Omega)$ and $((y_h)_{h\in A\setminus A^+}) \in [0, \kappa]^{L\times(A\setminus A^+)}$, there exists an auctioneer that chooses prices $p \in \Delta$ in order to maximize $p\cdot \sum_{h\in A} (y_h - w_h)$.

A *Cournot-Nash equilibrium* for $\mathcal{G}_A(\Omega, \kappa)$ is given by a vector

$$(p, (y_h, (z_i)_{i\in h})_{h\in A^+}, (y_h)_{h\in A\setminus A^+}) \in \Delta \times \prod_{h\in A^+} \mathbb{E}^h(\Omega) \times [0, \kappa]^{L\times(A\setminus A^+)}$$

which simultaneously solves the optimal problem of each player in the generalized game.

CLAIM 2. *For any $\kappa \in \mathbb{N}$, there exists a Cournot-Nash equilibrium for $\mathcal{G}_A(\Omega, \kappa)$.*

PROOF. It follows from Assumptions *(A1)-(A3)(iv)* and the linearity of auctioneer's function that players' objective functions are continuous and quasi-concave in their own strategies. It follows from Claim 1 that, for every coalition $h \in A^+$ the correspondence of admissible strategies $p \twoheadrightarrow X_{h,\kappa}(p) \cap \mathbb{E}^h(\Omega)$ is continuous with compact, convex and non-empty values. Assumption *(A2)(i)* guarantees that for every coalition $h \in A \setminus A^+$ the correspondence of admissible strategies $p \twoheadrightarrow \{y_h \in [0, \kappa]^L : p \cdot y_h \leq p \cdot w_h\}$ is continuous and has non-empty, compact, and convex values. The auctioneer correspondence of admissible strategies is continuous with compact, convex, and non empty values. Therefore, Berge's Maximum Theorem ensures that players' best-reply correspondences are upper hemicontinuous with non-empty, compact and convex values. Applying Kakutani's Fixed Point Theorem to the correspondence of optimal strategies we can find a Cournot-Nash equilibrium for the generalized game $\mathcal{G}_A(\Omega, \kappa)$. $\square$

CLAIM 3. *For any $\kappa \in \mathbb{N}$ –big enough– any Cournot-Nash equilibrium for the generalized game $\mathcal{G}_A(\Omega, \kappa)$ is a competitive equilibrium with free exit for the coalition structure $A \in \mathcal{H}$.*

[3] For every $\alpha \in (0, 1)$ high enough, $\left(\alpha \sum_{i\in h} \widehat{z}_{i,\kappa}(p), (\alpha\widehat{z}_{i,\kappa}(p) + \alpha\lambda_{h,\kappa}(p)(N(h)\widetilde{z}_i - \widetilde{y}_h))_{i\in h}\right) \in \mathring{X}_{h,\kappa}(p)$.

PROOF. Let,

$$(\overline{p},(\overline{y}_h,(\overline{z}_i)_{i\in h})_{h\in A^+},(\overline{y}_h)_{h\in A\setminus A^+})\in\Delta\times\prod_{h\in A^+}\mathbb{E}^h(\Omega)\times[0,\kappa]^{L\times(A\setminus A^+)}$$

be a Cournot-Nash equilibrium for $\mathcal{G}_A(\Omega,\kappa)$. Adding budget set constraints across coalitions, the optimality of auctioneer decisions ensures that $\sum_{h\in A}(\overline{y}_h-w_h)\leq 0$. This property guarantees that for any $h\in A^+$, $\left(\overline{y}_h,\,\left(\overline{z}_i\right)_{i\in h}\right)\in\text{Int}\left[\mathbb{E}^h(\Omega)\right]$, and for any $h\in A\setminus A^+$, $\overline{y}_h\in\text{Int}\left[\left[0,\kappa\right]^L\right]$, when κ is big enough. By Assumptions *(A1)-(A3)(iv)* we have that each coalition's budget constraints hold with equality and $\overline{p}\gg 0$. Thus, $\sum_{h\in A}(\overline{y}_h-w_h)\leq 0$, and $\overline{p}\cdot\sum_{h\in A}(\overline{y}_h-w_h)=0$. Therefore, we conclude that market clearing condition (ii) of Definition 1 holds.

Notice that, for every $h\in A^+$ the allocation $(\overline{y}_h,\,\left(\overline{z}_i\right)_{i\in h})$ that maximizes $W_h\left((z_i)_{i\in h}\right)$ in the set $X_{h,\kappa}(\overline{p})\cap\mathbb{E}^h(\Omega)$ it is also a Pareto optimal allocation in $\Phi_h(\overline{p})\cap\mathbb{E}^h(\Omega)$. By contradiction let assume that there is another allocation $(y_h,(z_i)_{i\in h})\in\Phi_h(\overline{p})\cap\mathbb{E}^h(\Omega)$ for which $(U_i(z_i)-U_i(\overline{z}_i))_{i\in h}\in\mathbb{R}^h_+\setminus\{0\}$. Then the optimality of $(\overline{y}_h,\,\left(\overline{z}_i\right)_{i\in h})$ guarantees that $(y_h,(z_i)_{i\in h})\notin X_{h,\kappa}(\overline{p})\cap\mathbb{E}^h(\Omega)$. Hence, for at least one agent $i\in h$ we have that,

$$U_i(z_i)-v_{i,\kappa}(\overline{p})<\mu\,\left(U_i(\widehat{z}_{i,\kappa}(\overline{p})+\lambda_{h,\kappa}(p)(N(h)\tilde{z}_i-\tilde{y}_h))-v_{i,\kappa}(\overline{p})\right)\leq U_i(\overline{z}_i)-v_{i,\kappa}(\overline{p}),$$

where the right-hand side of the inequality comes from the fact that $(\overline{y}_h,\,\left(\overline{z}_i\right)_{i\in h})\in X_{h,\kappa}(\overline{p})\cap\mathbb{E}^h(\Omega)$. The latter contradicts the assumption of the existence of a Pareto improvement when we change from $(\overline{y}_h,\,\left(\overline{z}_i\right)_{i\in h})$ to $(y_h,(z_i)_{i\in h})$.

Furthermore, $(\overline{y}_h,\,\left(\overline{z}_i\right)_{i\in h})$ is a Pareto optimal allocation in $\Phi_h(\overline{p})$. Otherwise, there is another allocation $(y_h,(z_i)_{i\in h})\in\Phi_h(\overline{p})$ such that $(U_i(z_i)-U_i(\overline{z}_i))_{i\in h}\in\mathbb{R}^h_+\setminus\{0\}$. Thus, $W_h\left((z_i)_{i\in h}\right)>W_h\left((\overline{z}_i)_{i\in h}\right)$. By Assumptions *(A1)* and *(A3)(iv)* we have that for any $\sigma\in(0,1)$,

$$W_h\left((\sigma\overline{z}_i+(1-\sigma)z_i)_{i\in h}\right)>W_h\left((\overline{z}_i)_{i\in h}\right).$$

Since $(\overline{y}_h,\,(\overline{z}_i)_{i\in h})\in X_{h,\kappa}(\overline{p})\cap\text{Int}[\mathbb{E}^h(\Omega)]$, we have that,

$$\sigma(\overline{y}_h,\,(\overline{z}_i)_{i\in h})+(1-\sigma)(y_h,\,(z_i)_{i\in h})\in X_{h,\kappa}(\overline{p})\cap\mathbb{E}^h(\Omega),$$

for $\sigma \in (0,1)$ high enough. The latter contradicts the fact that $(\overline{y}_h, (\overline{z}_i)_{i\in h})$ maximizes $W_h\left((z_i)_{i\in h}\right)$ on the set $X_{h,\kappa}(\overline{p}) \cap \mathbb{E}^h(\Omega)$. Therefore, requirements imposed in Definition 1 item (i) holds.

Fix $h \in A^+$. Since for every agent $i \in h$ we have that

$$U_i(\overline{z}_i) - v_{i,\kappa}(\overline{p}) \geq \mu\, \left(U_i(\widehat{z}_{i,\kappa}(\overline{p}) + \lambda_{h,\kappa}(\overline{p})(N(h)\tilde{z}_i - \tilde{y}_h)) - v_{i,\kappa}(\overline{p})\right) > 0,$$

it follows from assumption *(A2)(iii)* that,

$$U_i(\widehat{z}_{i,\kappa}(\overline{p})) = v_{i,\kappa}(\overline{p}) < U_i(\overline{z}_i) \leq U_i(\overline{y}_h) \leq U_i\left(\sum_{j\in I} w_j\right), \qquad \forall i \in h.$$

On the other hand *(A1)*, *(A2)(i)* and *(A3)(iii)* guarantees that for some agent $i_0 \in h$ there is $\lambda_{i_0} \in (0,1)$ and $\overline{a}(i_0) := (\overline{a}_\ell(i_0))_{\ell\in L} \gg 0$ such that,

$$U_{i_0}(\lambda_{i_0}\, w_{i_0} + \overline{a}_\ell(i_0)\vec{e}_\ell) > U_{i_0}\left(\sum_{j\in I} w_j\right), \;\; \forall\, \ell \in L.$$

Therefore, for any $\kappa > \left|\left|\, (w_{i_0,\ell} + \overline{a}_\ell(i_0))_{\ell\in L}\, \right|\right|_{+\infty}$, allocations $(\lambda_{i_0} w_{i_0,\ell} + \overline{a}_\ell(i_0)\vec{e}_\ell)$, for any $\ell \in L$, cannot be privately affordable by agent i_0 at prices $\overline{p}$ and, hence,

$$\overline{p}_\ell \geq \Delta_\ell(i_0) := (1-\lambda_{i_0})\, \frac{\min\limits_{\ell'\in L} w_{i_0,\ell'}}{\overline{a}_\ell(i_0)} > 0, \qquad \forall\, \ell \in L,$$

where $(\Delta_\ell(i_0))_{\ell\in L}$ is independent of κ.

Hence, given $\Phi_{\{i\}}(\overline{p}) \cap [0,\kappa]^L$, the private consumption of any agent i on commodity $l \in L$ at prices $\overline{p}$ is lower than $\frac{2\max_{i\in I} \|w_i\|_\Sigma}{\Delta_l(i_0)}$. This condition and Assumption *(A1)* imply that, when κ is high enough, for any agent i the solution of the truncated individual problem $\widehat{z}_{i,\kappa}(\overline{p})$ coincides with the bundle that maximizes $U_i(y_i)$ in the set $\Phi_{\{i\}}(\overline{p}) = \{y_i \in \mathbb{R}^L_+ : \overline{p}\cdot y_i \leq \overline{p}\cdot w_i\}$.

That is, for κ high enough,

$$\max_{\{y_i\in\mathbb{R}^L_+ : \overline{p}\cdot y_i \leq \overline{p}\cdot w_i\}} U_i(y_i) = v_{i,\kappa}(\overline{p}) < U_i(\overline{z}_i), \quad \forall i \in h, \forall h \in A^+.$$

Which ensures that conditions of Definition 1(iii) hold. *Q.E.D.*

Optimization Problem.

First, considering Assumptions *(A1)-(A2)*, the welfare function can be reinterpreted as,

$$\max\left\{\sigma_1\,(\pi_1\,y_{h,1})^{\alpha}(\pi_2\,y_{h,2})^{1-\alpha}+\sigma_2\,((\eta_1-\pi_1)\,y_{h,1})^{\alpha}((\eta_2-\pi_2)\,y_{h,2})^{1-\alpha}\,:\,(y_h,(\pi_\ell)_{\ell\in L})\in\mathbb{R}^2_+\times[0,1]^2\right\}$$

subject to,

$(\lambda_h)\; p\cdot\left(y_h-\sum_{i=1}^{2}w_i\right)\leq 0$, with $i\in h$;

$(\gamma_\ell)\;\pi_\ell-1\leq 0$, for all $\ell\in L$;

$(\delta_\ell)\;\eta_\ell-\pi_\ell-1\leq 0$, for all $\ell\in L$;

$(r_1)\; -(\pi_1 y_{h,1})^{\alpha}(\pi_2 y_{h,2})^{1-\alpha}+v_1(p)+\mu\nu_1(p)\leq 0;$

$(r_2)\; -((\eta_1-\pi_1)y_{h,1})^{\alpha}((\eta_2-\pi_2)y_{h,2})^{1-\alpha}+v_2(p)+\mu\nu_2(p)\leq 0$

with λ_h, $(\gamma_\ell)_{\ell\in L}$, $(\delta_\ell)_{\ell\in L}$, and $(r_i)_{i\in h}$ are the respective Lagrange multipliers. The first order conditions associated to this problem will be,

$$\alpha\,y_{h,1}\,\left((\sigma_1+r_1)x^{1-\alpha}-(\sigma_2+r_2)q^{1-\alpha}\right)-\gamma_1+\delta_1=0$$

$$(1-\alpha)\,y_{h,2}\,\left((\sigma_1+r_1)x^{-\alpha}-(\sigma_2+r_2)q^{-\alpha}\right)-\gamma_2+\delta_2=0$$

$$\alpha\,\left((\sigma_1+r_1)x^{1-\alpha}\pi_1+(\sigma_2+r_2)q^{1-\alpha}(\eta_1-\pi_1)\right)-\lambda_h\,p_1=0$$

$$(1-\alpha)\,\left((\sigma_1+r_1)x^{-\alpha}\pi_2+(\sigma_2+r_2)q^{-\alpha}(\eta_2-\pi_2)\right)-\lambda_h\,p_2=0$$

where $x=\left(\pi_2 y_{h,2}/\pi_1 y_{h,1}\right)$ and $q=\left((\eta_2-\pi_2)y_{h,2}/(\eta_1-\pi_1)y_{h,1}\right)$. Now, let consider an interior solution $\gamma_\ell=\delta_\ell=0$, for all $\ell\in L$. From the two first equations we conclude that $x=q$. Hence, the optimal distribution of use should follow the rule $\left(\pi_2/\pi_1\right)=\left(\eta_2/\eta_1\right)$. Considering this last and using the other two equations we conclude that, $\overline{y}_{h,1}=\left(\alpha\,p\cdot w_h/p_1\right)$ and $\overline{y}_{h,2}=\left((1-\alpha)\,p\cdot w_h/p_2\right)$.

$$\mathcal{W}_h=\sum_{i=1}^{2}\sigma_i\,U_i=[\sigma_1\pi_2+\sigma_2(\eta_2-\pi_2)]\left(\frac{\eta_1}{\eta_2}\right)^{\alpha}(p\cdot w_h)\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{(1-\alpha)}{p_2}\right)^{1-\alpha}.$$

Also, if $\sigma_1 = \sigma_2$ we obtain a continuum of solutions, where $\pi_1 \in (\eta_1 - 1, 1)$ and $\pi_2 \in (\eta_2 - 1, 1)$, such that $\big(\pi_2/\pi_1\big) = \big(\eta_2/\eta_1\big)$ and the constraints associated to the utility level of every member are met,[4] Finally, notice that the welfare function takes the value,

$$\mathcal{W}_h = \sum_{i=1}^{2} \sigma_i \, U_i = \sigma_i \, \eta_2^{1-\alpha} \, \eta_1^{\alpha} \, (p \cdot w_h) \, \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)}{p_2}\right)^{1-\alpha} .$$

[4]That is,

$$\pi_1 \in \left[\frac{V_1}{C_1}, \left(\eta_1 - \frac{V_2}{C_1}\right)\right]$$
$$\pi_2 \in \left[\frac{V_1}{C_2}, \left(\eta_2 - \frac{V_2}{C_2}\right)\right]$$

where,

$$C_1 = \left(\frac{\eta_2}{\eta_1}\right)^{1-\alpha} \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)}{p_2}\right)^{1-\alpha} p \cdot w_h \, ;$$
$$C_2 = \left(\frac{\eta_1}{\eta_2}\right)^{\alpha} \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)}{p_2}\right)^{1-\alpha} p \cdot w_h \, ;$$
$$V_1 = v^1(p) + \mu \, \nu^1(p) \, ;$$
$$V_2 = v^2(p) + \mu \, \nu^2(p) \, ;$$