Spinor norm for skew-hermitian forms over quaternion algebras

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ABSTRACT

We complete all local spinor norm computations for quaternionic skew-hermitian forms over the field Q of rational numbers. This can be used to compute the number of classes in a genus of skew-hermitian lattices of rank 2 or larger over a maximal order in a quaternion algebra D over Q in many cases, e.g., when D ramifies at infinity. Examples are provided.

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1. Introduction

Let K be a number field and let D be a quaternion algebra over K with canonical involution $q \mapsto \bar{q}$. Let V be a rank-n free D-module. Let $h : V \times V \to D$ be a skew-hermitian form, i.e., $h$ is D-linear in the first variable and it satisfies $h(x, y) = -\bar{h}(y, x)$. A D-linear map $\phi : V \to V$ preserving $h$ is called an isometry. We denote by $U_K$ (resp. $U_K^+$) the unitary group of $h$ (resp. the special unitary group of $h$), i.e., the group of isometries (resp. isometries with trivial reduced norm) of $h$. Skew-hermitian forms share many properties of quadratic forms. In fact, if $D \cong M_2(K)$, skew-hermitian forms in a rank-n free

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$D$-module are naturally in correspondence with quadratic forms in the $2n$-dimensional $K$-vector space $PV$, for any idempotent matrix $P$ of rank 1 in $D$ [3, §3]. In this case, the unitary group of $h$ is isomorphic to the orthogonal group of the corresponding quadratic form. On the other hand, $U_K = U^+_K$ when $D$ is a division algebra [8, §2.6].

As in the quadratic case, the problem of determining if two skew-hermitian lattices in the same space are isometric or not can be approached by the theory of genera and spinor genera [1]. This theory depends on the knowledge of the image, under the spinor norm, of the stabilizer of a given lattice in each local group $U^+_K$. Full computations of this image exist for symmetric integral bilinear forms. Non-dyadic cases can be found in [7] and the dyadic case in [5]. For this reason we assume, from now on, that $D$ is a quaternion division algebra. For skew-hermitian forms, non-dyadic places have been completely studied by Böge in [6]. The dyadic case was studied by Arenas-Carmona in [2] and [4], not covering all the cases, which we complete here when $K_p = \mathbb{Q}_2$. From now on $k = K_p$ denotes a dyadic local field of characteristic 0.

We denote by $| \cdot | : D \to \mathbb{R}_{\geq 0}$ and $| \cdot |_k : k \to \mathbb{R}_{\geq 0}$ the absolute values on $D$ and $k$ respectively, and we assume $|q| = |Nq|_k$, where $N$ is the reduced norm, for any $q \in D$. We use $\nu$ for the surjective valuation $\nu : D^* \to \mathbb{Z}$. Let $\mathcal{O}_D = \nu^{-1}(\mathbb{Z}_{\geq 0}) \cup \{0\}$ be the unique maximal order in $D$ [12, §2]. A skew-hermitian lattice or $\mathcal{O}_D$-lattice in $V$, is a lattice $\Lambda$ in $V$ such that $\mathcal{O}_D \Lambda = \Lambda$. Any skew-hermitian lattice $\Lambda$ has a decomposition of the type

$$\Lambda = \Lambda_1 \perp \cdots \perp \Lambda_t,$$

where each indecomposable lattice $\Lambda_i$ has rank 1 or 2, and the scales satisfy $\mathfrak{s}(\Lambda_{r+1}) \subset \mathfrak{s}(\Lambda_r)$ [2, §5]. If some $\Lambda_m$ in the decomposition of $\Lambda$ has rank 1, then $\Lambda_m = \mathcal{O}_{D_1} a_m$ and $h(s_m, s_m) = a_m$. We usually write $\Lambda_m = \langle a_m \rangle = \mathcal{O}_{D_1} a_m$ in this case. A statement like $\Lambda = \langle a_1 \rangle \perp \cdots \perp a_i = \mathcal{O}_{D_1} a_1 \perp \cdots \perp \mathcal{O}_{D_1} s_i$ must be interpreted similarly. Define $A \subset k^+/k^{*2}$ by $A = \{ N(a_m)k^{*2} | \Lambda_m = \langle a_m \rangle, 1 \leq m \leq t \}$. Following [6], we define the spinor image $H(\Lambda) \subseteq k^*$ by the relation $H(\Lambda)/k^{*2} = \theta(U^+_k(\Lambda))$, where $U^+_k(\Lambda)$ is the stabilizer of $\Lambda$ in $U^+_k$, and $\theta : U^+_k \to k^*/k^{*2}$ denotes the spinor norm. If $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, we let $\mu(\Lambda) = \min\{\nu(a_{i+1}) - \nu(a_i) | 1 \leq i < n\} \in \mathbb{Z}_{\geq 0}$. The lattices $\Lambda$ for which the set $H(\Lambda)$ remains unknown to date are:

**Case I:** $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, where $A = \{-uk^{*2}\}$, for a unit $u \in \mathcal{O}^*_k$ of non-minimal quadratic defect [9, §63], and $0 < \mu(\Lambda) \leq \nu(16)$.

**Case II:** $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, where $A = \{\pi k^{*2}\}$, for a prime $\pi$ in $k$, and $\nu(4) \leq \mu(\Lambda) \leq \nu(16)$.

**Theorem 1.** Table 1 contains all local spinor images when the base field is $\mathbb{Q}_2$. In the table, $s$ denotes the number of indecomposable components of rank 2 in the decomposition (1) of $\Lambda$, and $\Delta \in \mathcal{O}^*_k$ is a unit of minimal quadratic defect [9, §63]. Furthermore, $\pi$, $u$, $A$ and $\mu = \mu(\Lambda)$ are as above. A dash means “irrelevant.”
Our (computer assisted) proof of Theorem 1 goes as follows: We use Theorem 2 below to reduce the computation of $H(\Lambda)$ to low rank $\Lambda$. In our case, this means rank 2 or 3. Then we use Theorems 3 and 4 for constructing an algorithm for binary lattices over unramified local dyadic fields and we apply it to $k = \mathbb{Q}_2$. Then we patch the proof in the remaining “rank 3” case.

Before we state the critical theorems, we recall a few facts about simple rotations in skew-hermitian spaces,\(^1\) see [2, §6] for details. Let $(V, h)$ be a skew-hermitian $D$-space. If $s \in V$ and $\sigma \in D^*$ satisfy $\sigma - \bar{\sigma} = h(s, s)$, the map $(s; \sigma)(x) = x - h(x, s)\sigma^{-1}s$ is called a simple rotation with axis $s$. Its spinor norm [2] is $\theta[(s; \sigma)] = N(\sigma)k^{s^2}$, where $N : D^* \to k^*$ is the reduced norm. The set of simple rotations span the group $\mathcal{U}_k^+$. One way to produce simple rotations, that we use heavily in the sequel, is the next lemma:

**Simple Rotation Generating Lemma (SRGL).** (See [2, Lemma 6.3].) Let $t, u \in V$ be such that $h(u, u) = h(t, t) = a$. Let $s = t - u$ and $\sigma = h(t, s)$. Then $(s; \sigma)$ is a well-defined simple rotation satisfying $(s; \sigma)(t) = u$. Furthermore, if $u = rt + t_0$, where $t_0 \in t^\perp$, we have the identities $\sigma = a(1 - \bar{r})$, $h(t_0, t_0) = a - r\bar{a}$, and $\sigma - \bar{\sigma} = h(s, s)$.

Let $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, with $\langle a_i \rangle = \mathcal{O}_D s_i$ as in (1). Assume also $|2a_m| \geq |a_i|$ for $m < l$. Then the first author proved in [2, Lemma 6.7] that the unitary group $\mathcal{U}_k^+(\Lambda)$ of the lattice is generated by $\mathcal{A}(\Lambda) \cup \mathcal{B}(\Lambda)$, where

1. $\mathcal{A}(\Lambda)$ is the set of simple rotations with axis $s_m$, for some $m = 1, \ldots, n$.
2. $\mathcal{B}(\Lambda)$ is the set of simple rotations of the form $(s; \sigma)$, where $s = s_m - t$ for some $t = rs_m + s_0$ with $s_0 \in \mathcal{O}_D s_{m+1} \perp \cdots \perp \mathcal{O}_D s_n$, and $1 - r \notin (2i)$.

In particular, the elements of $\mathcal{B}(\Lambda)$ satisfy all relations in SRGL. Note that $\mathcal{A}(\Lambda)$ and $\mathcal{B}(\Lambda)$ depend on the splitting (1).

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\(^1\) Some authors call these elements reflections. We prefer the name simple rotation since $(s; \sigma)$ acts on the 2-dimensional subspace $k[\sigma]s$ by $v \mapsto uv$, where $u = \bar{\sigma}^{-1}$ is an element of norm 1.
Theorem 2. Let $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, with $\langle a_i \rangle = \mathcal{O}_{D} s_i$, be a skew-hermitian lattice and let $\mu = \mu(\Lambda)$ be as above. Assume $\mu > \nu(4)$ and $N(a_2), \ldots, N(a_n) \in N(a_1)k^{*2}$. Let $(s; \sigma) \in \mathcal{B}(\Lambda)$, i.e., $s = (1 - r)s_m - s_0$, where $s_0 = \lambda_{m+1}s_{m+1} + \cdots + \lambda_n s_n$, $\sigma = a_m(1 - r)$ and $|1 - r| \geq |2|$. If $|\lambda_{m+1}| \geq |2^{l} - 1\lambda_{m+l+1}|$, for some $t \in \{1, \ldots, n - m\}$ and for all $l \in \{1, \ldots, n - m - t\}$, then there exists $\Lambda' = \langle b_1 \rangle \perp \cdots \perp \langle b_{t+1} \rangle \subset \Lambda$ satisfying the following conditions:

1. $(s; \sigma) \in \mathcal{U}^{+}_{k}(\Lambda')$.
2. $\mu(\Lambda') \geq \mu(\Lambda)$.
3. $N(b_i) \in N(a_1)k^{*2}$, for all $i = 1, \ldots, t + 1$.

We say that an element $r \in \mathcal{O}_{D}$ satisfies the k-star conditions for a lattice $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ if $z = a_1 - ra_1r$ satisfies $NzNa_1 \in k^{*2}$ and $NzN(\pi^{*}a_1)^{-1} \in \mathcal{O}_{k}$, where $\nu(\pi^{*}) = \mu(\Lambda)$, while the Hilbert symbol $\left( \frac{N(1 - r), -Na_1}{p} \right)$ equals $-1$.

Theorem 3. Let $\Lambda = \langle a_1 \rangle \perp \langle a_2 \rangle$ be a skew-hermitian lattice such that $|2a_1| \geq |a_2|$ and $N(a_2) \in N(a_1)k^{*2}$. The following statements are equivalent:

1. $H(\Lambda) = k^{*}$.
2. There exists $(s; \sigma) \in \mathcal{B}(\Lambda)$ such that $N\sigma \notin N(k(a_1)^{*})$.
3. There exists $r \in \mathcal{O}_{D}$ satisfying the k-star conditions for $\Lambda$.

It is known that the (unique) quaternion division $k$-algebra $D$ has a basis $\{1, i, j, ij\}$, where $i^2 = \pi$, $j^2 = \Delta$, $ij = -ji$. Moreover, if $\omega = \frac{j + 1}{2}$, then $\{1, \omega, i, i\omega\}$ is an $\mathcal{O}_{k}$-basis for $\mathcal{O}_{D}$. Let $e = \nu(2)/2$ be the ramification index of $k/\mathbb{Q}_{2}$, and assume $\nu(\pi^{*}) = \mu(\Lambda)$.

Theorem 4. Let $\Lambda$ be as in Theorem 3. There exists $r \in \mathcal{O}_{D}$ satisfying the k-star conditions for $\Lambda$ if and only if there exists $\alpha \in S \oplus S\omega \oplus Si \oplus Si\omega \subset \mathcal{O}_{D}$ satisfying them, for one (any) set of representatives $S$ of $\mathcal{O}_{k}/\pi^{u}\mathcal{O}_{k}$, with $u = t + 6e$, as above.

2. Generators of $\mathcal{U}^{+}_{k}(\Lambda)$ and their spinor norm

If $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ is a skew-hermitian lattice, then $[k^{*} : N(k(a_1)^{*})] = 2$ [9, §63] and $N(k(a_1)^{*}) \subset H(\Lambda)$ [2, §6]. As a direct consequence of these facts, we have:

Proposition 2.1. Let $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ be a skew-hermitian lattice. Then $H(\Lambda) = N(k(a_1)^{*})$ or $H(\Lambda) = k^{*}$. In particular:

1. $H(\Lambda) = k^{*}$ if and only if, there exists $\phi \in \mathcal{C}(\Lambda)$ such that $\theta(\phi) \notin N(k(a_1)^{*})/k^{*2}$, for one (any) set of generators $\mathcal{C}(\Lambda)$ for $\mathcal{U}^{+}_{k}(\Lambda)$.
2. If there exists $b \in \mathcal{O}_{D}$ with $N(b) \notin N(a_1)k^{*2}$ such that $\Lambda = \langle b \rangle \perp \Lambda'$, for some lattice $\Lambda'$, we have $H(\Lambda) = k^{*}$.
Remark 2.1. In particular, if \(|2a_m| \geq |a_l|\) for \(m < l\), and \(\mathcal{C}(\Lambda) = \mathcal{A}(\Lambda) \cup \mathcal{B}(\Lambda)\), we just need to check the property for the elements in \(\mathcal{B}(\Lambda)\), since simple rotations \((s_m; \sigma) \in \mathcal{A}(\Lambda)\) have spinor norm \(N(\sigma)k^{*2} \in N(k(a_m)^*)/k^{*2}\). Our strategy includes to replace \(\mathcal{B}(\Lambda)\) by a smaller set that still generates but it is easier to control (cf. Lemma 4.7).

Lemma 2.1. Let \(\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle\), with \(\langle a_i \rangle = \mathcal{O}_D s_i\), be a skew-hermitian lattice such that \(|2a_m| \geq |a_l|\) for \(m < l\). Take \((s; \sigma) \in \mathcal{B}(\Lambda)\), i.e., \(s = (1 - r)s_m - s_0\), where \(s_0 = \lambda_{m+1}s_{m+1} + \cdots + \lambda_n s_n\), \(\sigma = a_m(1 - \bar{r})\), and \(|1 - r| \geq |2|\). If any of the following conditions is satisfied:

1. \(|1 - r| > |2|\) and \(|\lambda_{m+1}| < 1\), while \(\mu(\Lambda) \geq \nu(8)\), and \(k/\mathbb{Q}_2\) is unramified,
2. \(|1 - r| = |2|\) and \(|\lambda_{m+1}| \leq |2|\), while \(\mu(\Lambda) \geq \nu(4\pi)\),
3. \(|1 - r| > |2|\) or \(|\lambda_{m+1}| < 1\), while \(\mu(\Lambda) \geq \nu(16)\),
4. \(|\lambda_{m+1}| \leq |4|, |\lambda_{m+2}| \leq |2|\) and \(|\lambda_{m+3}| < 1\).

then \(\theta[(s; \sigma)] \in N(k(a_m)^*)/k^{*2}\).

Proof. It suffices to prove that if \(a = h(s, s)\), then \(N(a) \in N(a_m)k^{*2}\), since \(\sigma \in k(a)\). In fact, we have \(s = (1 - r)s_m - s_0\), so that \(a = (1 - r)a_m(1 - \bar{r}) + a_0\), where \(a_0 = h(s_0, s_0)\). It follows that

\[
N(a) = N(a_m)N(1 - r)^2N(1 + (1 - r)^{-1}a_0(1 - \bar{r})^{-1}a_m^{-1}).
\]

(2)

Now, \(a_0 = \lambda_{m+1}a_{m+1}\bar{\lambda}_{m+1} + \cdots + \lambda_n a_n\bar{\lambda}_n\) and \(|(1 - r)^{-1}a_0(1 - \bar{r})^{-1}a_m^{-1}| = |a_0|a_m^{-1}|/|1 - r|^2 < |4|\) if any of the conditions above is satisfied. This implies the last norm in (2) is a square. \(\Box\)

The following result, together with \text{SRGL}, gives us an easy method to construct simple rotations in \(\mathcal{B}(\Lambda)\) as in the introduction.

Lemma 2.2. Let \(r \in \mathcal{O}_D\) be a non-zero quaternion and let \(a_1, a_2 \in \mathcal{O}_D\) be non-zero pure quaternions. There exists \(\lambda \in \mathcal{O}_D\) different from zero such that \(a_1 = ra_1\bar{r} + \lambda a_2\bar{\lambda}\) if and only if \(NzNa_2 \in k^{*2}\) and \(NzNa_2^{-1} \in \mathcal{O}_k\), where \(z = a_1 - ra_1\bar{r}\).

Proof. The equation \(a_1 = ra_1\bar{r} + \lambda a_2\bar{\lambda}\) has a solution \(\lambda \in D^*\) if and only if the binary skew-hermitian form \(h'\) whose Gramm matrix is \(
\begin{pmatrix}
\frac{z}{0}
\frac{0}{-a_2}
\end{pmatrix}
\)

is isotropic. Now, \(h'\) is isotropic if and only if \(NzNa_2 = \text{disc}(h') \in k^{*2}\) [10, Chapter 10, §3, Theorem 3.6]. We conclude that there exists \(\lambda \in D^*\) satisfying \(a_1 = ra_1\bar{r} + \lambda a_2\bar{\lambda}\) if and only if \(NzNa_2 \in k^{*2}\). Finally, we have \(Nz = Na_2N\lambda^2\), whence \(\lambda \in \mathcal{O}_D\) if and only if \(NzNa_2^{-1} \in \mathcal{O}_k\). \(\Box\)
3. Proof of Theorems 2, 3 and 4

Proof of Theorem 2. Set $\Lambda' = \langle b_1 \rangle \perp \cdots \perp \langle b_{t+1} \rangle = \mathcal{O}_D s_m \perp \cdots \perp \mathcal{O}_D s_{m+t-1} \perp \mathcal{O}_D s_0'$, where $s_0' = \sum_{i=m+1}^{n} \lambda_i s_i$. It is clear that $\Lambda' \subset \Lambda$. To prove condition (1) in the theorem, we note that $s_0 = s_0' - \sum_{i=m+1}^{m+t-1} \lambda_i s_i \in \Lambda'$. We compute

- $(s; \sigma)(s_i) = s_i - h(s_i,s)\sigma^{-1}s = s_i + h(s_i,s_0)\sigma^{-1}s$, for $m + 1 \leq i \leq m + t - 1$,
- $(s; \sigma)(s_0') = s_0' - h(s_0', s_0)\sigma^{-1}s = s_0' + h(s_0', s_0)\sigma^{-1}s$, and
- $(s; \sigma)(s_m) = r s_m + s_0 \in \Lambda'$.

Hence, $(s; \sigma)(\Lambda') \subset \Lambda'$ if $h(s_i, s_0)\sigma^{-1}, h(s_0', s_0)\sigma^{-1} \in \mathcal{O}_D$. The latter holds since $|\sigma| = |a_m(1 - \bar{r})| \geq 2|a_m|$ is larger than the height of $s_0$. We conclude that $(s; \sigma) \in \mathcal{U}_{k}^{+}(\Lambda')$.

On the other hand, as

$$b_{t+1} = h(s_0', s_0') = \sum_{u=m+t}^{n} \lambda_u a_u \lambda_u,$$

we have $|b_{t+1}| = |a_{m+t}|\lambda_{m+t}|^2$, since $|\lambda_{m+t}| \geq |2^{l-1}\lambda_{m+t+l}|$ when $1 \leq l \leq n - m - t$ and $\mu(\Lambda) > \nu(4)$. From here $\mu(\Lambda') \geq \mu(\Lambda)$, proving condition (2). Finally, to prove the last condition, we consider

$$N(b_{t+1}) = N(\lambda_{m+t})^2 N(a_{m+t}) N\left(1 + (\lambda_{m+t} a_{m+t} \bar{\lambda}_{m+t})^{-1} \sum_{u=m+t+1}^{n} \lambda_u a_u \lambda_u\right),$$

where $|(\lambda_{m+t} a_{m+t} \bar{\lambda}_{m+t})^{-1}| = |a_{m+t}|^{-1}|\lambda_{m+t}|^{-2}$. Since $|a_{m+t+l}| < |4'| a_{m+t}$ and $|\lambda_{m+t}| \geq |2^{l-1}\lambda_{m+t+l}|$ for all $l \in \{1, \ldots, n - m - t\}$, the last term in (3) is a square, whence $N(b_{t+1}) \in N(a_{m+t})k^{2}$ and the proof of the condition (3) is completed. □

Proof of Theorem 3. The equivalence between (1) and (2) is a direct consequence of Proposition 2.1 and the subsequent remark to it. To prove that (2) implies (3), let $(s; \sigma)$ be a simple rotation such that $\theta[(s; \sigma)] = N(\sigma) k^{2} \notin N(k(a_1)^*)/k^{2}$. As isometry $(s; \sigma) \in \mathcal{B}(\Lambda)$ satisfies $a_1 = h(s_1, s_1) = ra \bar{r} + \lambda a \lambda$, where $(s; \sigma)(s_1) = rs + \lambda s$. Such an $r \in \mathcal{O}_D$ satisfies $\sigma = a_1(1 - \bar{r})$ by SRGL. Hence, $\theta[(s; \sigma)] \notin N(k(a_1)^*)/k^{2}$ if and only if $N(1 - r) \notin N(k(a_1)^*)$, or equivalently $(N(1 - r) - Na_1)^{p} = -1$. On the other hand, Lemma 2.2 tells us that $NzNa_2 \in k^{2}$ and $NzNa_2^{-1} \in \mathcal{O}_k$, where $z = a_1 - ra \bar{r}$. The result follows since $Na_2 \in N(a_1)k^{2}$ and $\mu = \nu(a_2) - \nu(a_1) = \nu(\pi^t)$. Conversely, if $r \in \mathcal{O}_D$ satisfies the k-star conditions, then Lemma 2.2 and SRGL imply the existence of $\phi \in \mathcal{B}(\Lambda)$ such that $\theta(\phi) = N(a_1)N(1 - r)k^{2}$ and the result follows as before. □

Corollary 3.1. Let $\Lambda$ be as in Theorem 3. Let $t$ be such that $\mu = \nu(\pi^t)$. If $H(\Lambda) = k^{*}$, then $H(\Lambda') = k^{*}$ for every lattice $\Lambda' = \langle a_1 \rangle \perp \langle b \rangle$ with $N(b) \in N(a_1)k^{2}$ and $\mu(\Lambda') = \nu(\pi^t)$, for $e \leq s < t$. 
Remark 3.1. Due to Lemma 2.1, in the condition (2) of Theorem 3, it is enough to consider simple rotations \( (s; \sigma) \in \mathcal{B}(\Lambda) \) with \(|\lambda| > |4|\), where \( s = (1 - r)s_1 - \lambda s_2 \). Remember that \(|1 - r| \geq 2| \) for \((s; \sigma) \in \mathcal{B}(\Lambda)\).

Proof of Theorem 4. Assume \( r \in \mathcal{O}_D \) satisfies the \( k \)-star conditions. Let \( \alpha \in \mathcal{O}_D \) be a representative of the class of \( r \) modulo \( \pi^u \) as in the statement. Then, \( r = \alpha + \pi^u \beta \), with \( \beta \in \mathcal{O}_D \) and \( \alpha \in \mathcal{S} \oplus \mathcal{S} \omega \oplus \mathcal{S} \iota \oplus \mathcal{S} \iota \omega \subset \mathcal{O}_D \). As \( 1 - r = 1 - \alpha - \pi^u \beta \) we have \( N(1 - r) = N(1 - \alpha)N(1 - (1 - \alpha)^{-1}\pi^u \beta) \). Now, \(|1 - r| \geq 2| \) implies \(|1 - \alpha| \geq 2| \).

Therefore, \( N(1 - (1 - \alpha)^{-1}\pi^u \beta) \) is a square. Hence, \( (\frac{N(1-r)}{\pi^u})_p - Na_1 = (\frac{N(1-\alpha)}{\pi^u})_p - Na_1 \).

On the other hand, if \( z = a_1 - ra_1 \bar{r} = \pi^t \lambda a_1 \bar{\lambda} \) and \( z' = a_1 - \alpha a_1 \bar{\alpha} \), then \( z = z' - \pi^u \gamma \), with \( \gamma = \alpha a_1 \bar{\beta} + \beta a_1 \bar{\alpha} + \pi^u \beta a_1 \bar{\beta} \in \mathcal{O}_D \). Note that \( a_1^{-1} \gamma \in \mathcal{O}_D \). We have \(|z'| = |z| |\), since \(|z| = |\pi^t \lambda a_1 \bar{\lambda}| > |16\pi^t a_1| = |\pi^{4e+4} a_1| > |\pi^u \gamma| \), where we are assuming \(|\lambda| > |4| \) (see Remark 3.1). Furthermore, we have that \( Nz/Nz' = Nz(1 - z'^{-1}\pi^u \gamma) \) with \(|z'^{-1}\pi^u \gamma| < |\pi^{-(4e+4)} a_1^{-1}\pi^u a_1 \bar{\gamma}| \leq |\pi^{2e} a_1^{-1}\gamma| \leq 4| \). Hence, \( NzNa_1 \) is a square if and only if \( NzNa_1 \) is a square. Finally, from \(|z'| = |z| \) we obtain \(|Nz|/\pi^{2t} (a_1)|_k \leq 1 \) if and only if \(|Nz'|/\pi^{2t} (a_1)|_k \leq 1 \).

Remark 3.2. The optimal choice for the number \( u \) in Theorem 4 depends on \(|\lambda| \). For example, since \( z = a_1 - ra_1 \bar{r} = \pi^t \lambda a_1 \bar{\lambda} \), if \( \lambda \) satisfies \(|\lambda| = 1 \), then we would have \(|z'| = |z| \) and so \(|z'^{-1}\pi^u \gamma| = |\pi^{-t} a_1^{-1}\pi^u \gamma| \leq |\pi^{u-t}| < |4| \) if \( u = t + 2e + 1 \). This holds in some cases when \( k = \mathbb{Q}_2 \).

4. Proof of Theorem 1

The following result is a direct consequence of [2, Lemma 4.3]. Note that for either of the remaining cases I or II described in the introduction, the extension \( k(a_1)/k \) is ramified.

Lemma 4.1. Let \( \langle a_1 \rangle \perp \langle a_2 \rangle \) be a skew-hermitian lattice such that \( N(a_2) \in N(a_1)k^{*2} \) and the extension \( k(a_1)/k \) is ramified. Then, there exists a skew-hermitian lattice \( L = \langle q \rangle \perp \langle \epsilon q \rangle \), where \( q \in D^* \) and \( \epsilon \in k^* \), such that \( H(L) = H(\Lambda) \). Moreover, we can assume that \( q = q' \), for any quaternion \( q' \in D^* \) with \( N(q') \in N(a_1)k^{*2} \).

Note that, due to Corollary 3.1, if \( \mu(\Lambda) = \nu(\pi^t) \) for \( \Lambda \) as in Theorem 3, we can take \( \epsilon \) in last lemma equals to \( \pi^t \), for any prime \( \pi \) of \( k \). If \( k = \mathbb{Q}_2 \), we have \( \mathcal{O}_k = \mathbb{Z}_2 \), \( \mathcal{O}_D = \mathbb{Z}_2[\omega] \oplus i\mathbb{Z}_2[\omega] \) and \( \mathcal{O}_k/\pi^u \mathcal{O}_k \cong \mathbb{Z}/2^u \mathbb{Z} \). By considering Theorems 3, 4 and the lemma above, we are able to construct an algorithm for computing \( H(\Lambda) \), for all binary \( \mathcal{O}_D \)-lattices \( \Lambda \), as follows:

1. By Lemma 4.1, we are reduced to compute \( H(L) \) for \( L = \langle q \rangle \perp \langle 2^t q \rangle \), for \( q^2 \) running over representative of all suitable square classes, and a few values of \( t \) for each \( q \).
2. Fix a set of representatives \( S \) of the finite ring \( \mathbb{Z}_2/2^u \mathbb{Z}_2 \): We can choose \( S = \{0, 1, \ldots, 2^u - 1\} \) for \( u \) large enough (see Remark 3.2).
3. For \( r = a + b\omega + ci + di\omega \in S \oplus S\omega \oplus Si \oplus Si\omega \subseteq \mathcal{O}_D \), check if the \( k \)-star conditions are satisfied. This verification can be done by using the computer algebra system Sage [11].

4. Conclude that \( H(\Lambda) = \mathbb{Q}_2^* \) if some \( r \) in the last step satisfies the \( k \)-star conditions. Otherwise, \( H(\Lambda) = N(\mathbb{Q}_2(a_1)^*) \) in virtue of Theorems 3, 4 and Proposition 2.1.

**Remark 4.1.** The algorithm can be extended to any unramified finite extension \( k \) of \( \mathbb{Q}_2 \). The condition \( |2a_1| \geq |a_2| \) in Theorems 3 and 2 is essential. Hence, the algorithm does not work, for \( \mu < \nu(2) \), if the extension \( k/\mathbb{Q}_2 \) ramifies, unless the algorithm returns the value \( k^* \) for \( \mu < \nu(2) \).

**4.1. Computations using Sage**

In all that follows we assume \( i^2 = 2 \), \( j^2 = 5 \), and \( ij = -ji \). Whenever a different uniformizing parameter \( \pi \) makes computations easier we use \( i_\pi = u_\pi i \), for some unit \( u_\pi \in \mathbb{Q}_2(j) \), such that \( i_\pi^2 = \pi \), or equivalently \( N(u_\pi) = \pi/2 \). The following results are obtained by computer search. When the algorithm does find solutions, we actually list them. Otherwise it is just stated that no solutions were found.

**Lemma 4.2.** (See Table 2.) For any \( q \in \{j+i, i+j\} \) and \( t \in \{3, 4\} \), there exist \( r_1, r_2 \in \mathcal{O}_D \) such that:

1. \(|1 - r_1| = |2|, \ NzNq \in \mathbb{Q}_2^{*2} \) and \( NzN(2^tq)^{-1} \in \mathbb{Z}_2^{*} \), where \( z = q - r_1q\overline{r_1} \).
2. \(|1 - r_2| = |i|, \ NzNq \in \mathbb{Q}_2^{*2} \) and \( NzN(2^tq)^{-1} \in \mathbb{Z}_2^{*} \), where \( z = q - r_2q\overline{r_2} \).

**Lemma 4.3.** (See Table 3.) Let \( L = \langle q \rangle \perp \langle 4q \rangle \) be a skew-hermitian lattice satisfying the conditions in Theorem 3, for \( q \in \{j+i, i+j\} \). Then there exists \( r \in \mathcal{O}_D \) satisfying the \( k \)-star conditions for \( L \).

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Proof of Lemma 4.2.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q )</td>
<td>( t )</td>
</tr>
<tr>
<td>( j + ij )</td>
<td>3</td>
</tr>
<tr>
<td>( j + ij )</td>
<td>4</td>
</tr>
<tr>
<td>( i + j )</td>
<td>3</td>
</tr>
<tr>
<td>( i + j )</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Proof of Lemma 4.3.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q )</td>
<td>( r )</td>
</tr>
<tr>
<td>( j + ij )</td>
<td>(+1 + 2i\omega )</td>
</tr>
<tr>
<td>( i + j )</td>
<td>(+1 + 2i + 2i\omega )</td>
</tr>
</tbody>
</table>
Lemma 4.4. (See Table 4.) Let \( L = \langle i_\pi \rangle \perp \langle 16i_\pi \rangle \) be a skew-hermitian lattice satisfying the hypothesis in Theorem 3, for \( \pi \in \{\pm 2, \pm 10\} \) as above. Then, there exists \( r \in \mathcal{O}_D \) satisfying the \( k \)-star conditions for \( L \).

Lemma 4.5. There is no \( r = a + b\omega + ci + di\omega \in \mathbb{Z} \oplus \mathbb{Z}\omega \oplus \mathbb{Z}i \oplus \mathbb{Z}i\omega = \mathcal{O}_D \), with \( 0 \leq a, b, d, c < 2^{1+3} \) satisfying the \( k \)-star conditions for \( L = \langle q \rangle \perp \langle 2^t q \rangle \), if \( t \in \{3, 4\} \) and \( q \in \{j + ij, j + i\} \).

4.2. Proof of Theorem 1 in Case I

Assume \( \Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle \), where \( N(a_m) \in -u\mathbb{Q}_2^2 \), for each \( m = 1, \ldots, n \) and \( u \in \mathbb{Z}_2^* \) is a unit of non-minimal quadratic defect independent of \( m \). As \( \mathbb{Z}_2^*/\mathbb{Z}_2^2 = \{\pm 1, \pm 3\} \) and a pure quaternion cannot have reduced norm \(-1\), we have two options for \( u \): \( u = -5 \) or \( u = -1 \).

In virtue of Lemma 4.1, we consider binary lattices \( \Lambda = \langle q \rangle \perp \langle 2^t q \rangle \), with \( 1 \leq t \leq 4 \), where we can choose any pure quaternion \( q \in \mathcal{O}_D^0 \) satisfying \( N(q) \in -u\mathbb{Q}_2^2 \). Here, \( q = q_u \) satisfy \( N(q) \in -u\mathbb{Q}_2^2 \), for \( u \) running over the set \( \{-5, -1\} \) of units of non-minimal quadratic defect. We choose \( q_{-5} = j + ij \) and \( q_{-1} = i + j \).

Proposition 4.1. Let \( \Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle \) be a skew-hermitian lattice such that \( N(a_1), \ldots, N(a_n) \in -u\mathbb{Q}_2^2 \) and \( 0 < \mu(\Lambda) \leq \nu(4) \). Then \( H(\Lambda) = \mathbb{Q}_2^* \).

Proof. We can assume \( n = 2 \) and \( \Lambda = \langle q_u \rangle \perp \langle 2^t q_u \rangle \), with \( q_u \in \{q_{-5}, q_{-1}\} = \{j + ij, i + j\} \) and \( t \in \{1, 2\} \). In virtue of Corollary 3.1 it suffices to prove the result for \( t = 2 \). Lemma 4.3 tells us that there exists \( r \in \mathcal{O}_D \) satisfying the \( k \)-star conditions. This is equivalent to \( H(\Lambda) = \mathbb{Q}_2^* \) by Theorem 3. □

To handle the cases where \( \mu = \nu(8) \) or \( \mu = \nu(16) \) we use the following result, which is used to improve the set of generators \( \mathcal{B}(\Lambda) \). The proof is a routine computation.

Lemma 4.6. If \( r \in \mathcal{O}_D \) satisfies either of the equations

\[
j + ij = r(j + ij)\bar{r} + 2^t\lambda(j + ij)\bar{\lambda}, \quad \text{or} \quad i + j = r(i + j)\bar{r} + 2^t\lambda(i + j)\bar{\lambda},
\]

where \( \lambda \in \mathcal{O}_D, \) and \( t \geq 2, \) then \( 1 - r \in i\mathcal{O}_D. \)

Lemma 4.7. Let \( \Lambda = \langle a_1 \rangle \perp \langle a_2 \rangle = \mathcal{O}_D s_1 \perp \mathcal{O}_D s_2 \) be a skew-hermitian lattice such that \( N(a_1), N(a_2) \in -u\mathbb{Q}_2^2 \) and \( \nu(8) \leq \mu(\Lambda) \leq \nu(16) \). There exists a lattice \( L \) of rank 2

<table>
<thead>
<tr>
<th>( \pi )</th>
<th>( r )</th>
<th>( N(1 - r) )</th>
<th>( z )</th>
<th>( NzN\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm 2 )</td>
<td>( 15 + 8\omega )</td>
<td>( 5 \cdot 2^2(1 + 8 \cdot 5^{-1} \cdot 7) )</td>
<td>( -592i_\pi + 304i_\pi \omega )</td>
<td>( 2^{10}(1 + 8 \cdot 38) )</td>
</tr>
<tr>
<td>( \pm 10 )</td>
<td>( 15 + 8\omega )</td>
<td>( 5 \cdot 2^2(1 + 8 \cdot 5^{-1} \cdot 7) )</td>
<td>( -592i_\pi + 304i_\pi \omega )</td>
<td>( 2^{10} \cdot 5^2(1 + 8 \cdot 38) )</td>
</tr>
</tbody>
</table>
such that $H(L) = H(\Lambda)$, and a suitable splitting of $L$, such that $\mathcal{A}(L) \cup \mathcal{B}_1(L)$ generates $\mathcal{U}^+_{\mathbb{Q}_2}(L)$, for $l = 1, 2$, where $\mathcal{B}_1(L) = \{(s; \sigma) \in \mathcal{B}(L): |1-r| = |i|\}$ and $\mathcal{B}_2(L) = \{(s; \sigma) \in \mathcal{B}(L): |\lambda| = 1\}$, with $r$ as in SRGL and $\lambda$ as in Lemma 2.2.

**Proof.** By Lemma 4.1 there is a lattice $L = O_D s_1 \perp O_D s_2 = \langle q_u \rangle \perp \langle 2^t q_u \rangle$ as above, with $u \in \{-5, -1\}$ and $t \in \{3, 4\}$ satisfying $H(L) = H(\Lambda)$. Let $\phi \in \mathcal{B}(L)$ be such that $\phi(s_1) = rs_1 + \lambda s_2$. We have $|1-r| \in \{|i|, |2|\}$ in virtue of Lemma 4.6. Hence, to prove that $\mathcal{B}_1(L)$ satisfies the required property, it suffices to prove that, if $\phi$ satisfies $|1-r| = |2|$, then there exists $(s; \sigma) \in \mathcal{B}(L)$ such that $|1-r''| = |i|$ and $|1-r'| = |i|$, where $(s; \sigma)(s_1) = r's_1 + \lambda' s_2$ and $(s; \sigma)(s_1) = r'' s_1 + \lambda'' s_2$. In this case, there exists a second element $(s'; \sigma') \in \mathcal{B}_1(L)$ defined by $s' = s_1 - (s; \sigma)(s_1)$, $\sigma' = q(1-r')$ such that $(s'; \sigma')(s; \sigma)(s_1) = s_1$. In fact, by a computation we have

$$1 - r'' = 1 - r + [rq(1-r') + 2^t q \bar{\lambda}] (1-r')^{-1} q^{-1} (1-r'), \quad \lambda'' = \lambda + [rq(1-r') + 2^t q \bar{\lambda}] (1-r')^{-1} q^{-1} \lambda', \quad (5)$$

Lemma 4.2 implies the existence of an element $r' \in O_D$ such that $|1-r'| = |i|$, $NzN(2^t q) \in \mathbb{Q}_2^2$, and $NzN(2^t q)^{-1} \in \mathbb{Z}_2$, where $z = q - r' q \bar{\lambda}$ and $t \in \{3, 4\}$. Hence, by Lemma 2.2, there exists $\lambda' \in O_D$, such that $q = r'q \bar{\lambda} + 2^t \lambda' q \bar{\lambda}$. Then $(s; \sigma)$, where $s = (1-r')s_1 - \lambda' s_2$ and $\sigma = q(1-r')$, belongs to $\mathcal{B}_1(L)$ (cf. SRGL). On the other hand, as

$$- \left[ rq(1-r') + 2^t \lambda q \bar{\lambda} \right] (1-r')^{-1} q^{-1} (1-r') = |rq(1-r') + 2^t \lambda q \bar{\lambda}| = |1-r'| = |i| \quad (7)$$

and $|1-r| = |2|$, it follows that $|1-r''| = |i|$. In particular, $\mathcal{A}(L) \cup \mathcal{B}_1(L)$ generates $\mathcal{U}^+_{\mathbb{Q}_2}(L)$.

Now, to prove that $\mathcal{A}(L) \cup \mathcal{B}_2(L)$ generates $\mathcal{U}^+(L)$, by a similar argument as for $\mathcal{B}_1(L)$, it suffices to prove that, if $\phi \in \mathcal{B}(L)$ satisfies $|\lambda| < 1$, there exists $(s; \sigma) \in \mathcal{B}(L)$ such that $|\lambda'| = 1$ and $|\lambda''| = 1$, where $\lambda, \lambda', \lambda''$ are defined by $\phi$, $(s; \sigma)$ and $(s; \sigma)\phi$ respectively, as before. From Eq. (6) we see that $|\lambda''| = 1$ if $|\lambda| < 1$ and $|\lambda'| = 1$. By Lemma 4.2, there exists $r' \in O_D$ such that

$$|1-r'| = |i| \text{ or } |2|, \quad NzN(2^t q) \in \mathbb{Q}_2^2 \quad \text{and} \quad NzN(2^t q)^{-1} \in \mathbb{Z}_2^2, \quad (8)$$

where $z = q - r' q \bar{\lambda}$ and $t \in \{3, 4\}$. Hence, by Lemma 2.2, there exists $\lambda' \in O_D$ such that $q = r'q \bar{\lambda} + 2^t \lambda' q \bar{\lambda}$. Then if $s = (1-r')s_1 - \lambda' s_2$ and $\sigma = q(1-r')$, then $(s; \sigma) \in \mathcal{B}(L)$ (cf. SRGL), and $|\lambda'| = 1$ since $NzN(2^t q)^{-1} \in \mathbb{Z}_2^2$. Now, we take $|1-r'| = |i|$ if $|1-r| = |2|$, and $|1-r'| = |2|$ if $|1-r| = |i|$, so that $|1-r'|, |1-r''| \geq |2|$ by (5). The result follows. □

**Remark 4.2.** Notice that, for a lattice $\Lambda$ as in the previous lemma, we can replace $\mathcal{B}(\Lambda)$ by $\mathcal{B}_l(\Lambda)$, for $l = 1, 2$, in Theorem 3. Hence, since $|\lambda| = 1$ for $(s; \sigma) \in \mathcal{B}_2(\Lambda)$, we can improve the number $u$ in Theorem 4 in virtue of Remark 3.2.
Proposition 4.2. Let $\Lambda = \langle a_1 \rangle \perp \langle a_2 \rangle$ be as in Theorem 3. There exists $r \in \mathcal{O}_D$ satisfying the $k$-star conditions for $t \in \{3, 4\}$ and $Na_1 \in -w\mathbb{Q}_2^2$, with $u$ a unit of non-minimal quadratic defect, if and only if there exists $\alpha = a + b\omega + ci + di\omega \in \mathbb{Z} \oplus \mathbb{Z} \omega \oplus \mathbb{Z} i \oplus \mathbb{Z} i\omega = \mathcal{O}_D$, with $0 \leq a, b, c, d < 2^{t+3}$, satisfying them.

Combining this result with Theorem 3, Lemma 4.1 and Lemma 4.5, we obtain

Corollary 4.1. Let $\Lambda = \langle a_1 \rangle \perp \langle a_2 \rangle$ be a skew-hermitian lattice such that $N(a_1), N(a_2) \in -u\mathbb{Q}_2^2$, where $u$ is a unit of non-minimal quadratic defect and $\mu = \nu(a_2) - \nu(a_1)$ satisfies $\nu(8) \leq \mu \leq \nu(16)$. Then $H(\Lambda) = N(Q_2(a_1)^*)$.

We need the following result to handle ternary lattices $\Lambda$ with $\mu(\Lambda) = \nu(8)$. For the sake of generality we state it for an arbitrary dyadic field $k$.

Lemma 4.8. If $|\eta| = |i|$ and $a_1$ is a pure unit, then $T(2(\eta a_1 \bar{\eta})^{-1}a_1) \in \pi\mathcal{O}_k$, where $T$ is the trace map.

Proof. Set $\eta = i\rho$, for $\rho \in \mathcal{O}_D^\times$. Note that $a_1 i \equiv i\bar{a}_1 \mod \pi$, while $\rho$ and $\bar{a}_1$ commute modulo $i$. We conclude that $\eta a_1 \bar{\eta} \equiv -N(\rho)\pi\bar{a}_1 \mod \pi i$. In other words $\frac{1}{\pi}\eta a_1 \bar{\eta} = -N(\rho)a_1 + \varepsilon$, where $\varepsilon \in i\mathcal{O}_D$, whence $\pi(\eta a_1 \bar{\eta})^{-1} = -\frac{\mu}{\pi N(\rho a_1)}$, for some $\delta \in i\mathcal{O}_D$. Hence

$$T(2(\eta a_1 \bar{\eta})^{-1}a_1) \equiv \frac{-4a_1^2}{\pi N(\rho a_1)} + \frac{2}{\pi}T(\delta a_1) \mod \pi$$

and the result follows since $\delta \in i\mathcal{O}_D$ implies $T(\delta a_1) \in \pi\mathcal{O}_k$. □

Note that if $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ is a skew-hermitian lattice and $(s; \sigma) \in \mathcal{B}(\Lambda)$, then $s = (1-r)s_m - s_0$, for some $s_0 \in \mathcal{O}_D s_{m+1} \perp \cdots \perp \mathcal{O}_D s_n$. Hence, if $m > 1$ then $(s; \sigma)$ fixes $\langle a_1 \rangle$, so we can assume $m = 1$ in order to compute spinor norms in the next result.

Proposition 4.3. Let $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ be a skew-hermitian lattice such that $N(a_1), \ldots, N(a_n) \in -u\mathbb{Q}_2^2$, where $u$ is a unit of non-minimal quadratic defect. If $\mu = \mu(\Lambda)$ satisfies $\nu(8) \leq \mu \leq \nu(16)$, then $H(\Lambda) = N(Q_2(a_1)^*)$.

Proof when $\mu = \nu(16)$. In virtue of Lemma 2.1 it suffices to consider rotations $(s; \sigma) \in \mathcal{B}(\Lambda)$ such that $|1 - r| = |2|$ and $|\lambda_2| = 1$. In this case, Theorem 2 tells us we can set $n = 2$ in the statement of the proposition. For $n = 2$, because of Lemma 4.7, we can replace $\Lambda$ by a lattice $L$ such that $H(L) = H(\Lambda)$ and a set of generators of $U_{Q_2}(L)$ is $A(L) \cup B_1(L)$. It follows that $H(\Lambda) = N(Q_2(a_1)^*)$ since rotations in $B_1(L)$ have spinor norm belonging to $N(Q_2(a_1)^*)$ in virtue of Lemma 2.1. □

Proof when $\mu = \nu(8)$. In virtue of Lemma 2.1, any rotation $(s; \sigma) \in \mathcal{B}(\Lambda)$ satisfies $\theta[(s; \sigma)] \in N(Q_2(a_1)^*)/\mathbb{Q}_2^2$ unless one of the following conditions is satisfied:
1. $|1 - r| = |i|$, $|\lambda_2| = 1$, 
2. $|1 - r| = |2|$, $|\lambda_2| \in \{1, |i|\}$.

As in the previous case, by Theorem 2, when $|\lambda_2| = 1$ we are reduced to consider binary lattices and when $|1 - r| = |2|$, $|\lambda_2| = |i|$ to study either binary lattices or rank 3 lattices with $|\lambda_3| = 1$. For rank 2 lattices, Corollary 4.1 tells us that $\mathcal{H}(\Lambda) = N(Q_2(\sigma_1)^*)$. We prove that, for rank 3 lattices $\Lambda$ such that $(s; \sigma) \in B(\Lambda)$ satisfies $|1 - r| = |2|$, $|\lambda_2| = |i|$, $|\lambda_3| = 1$ we also have $\theta/(s; \sigma) \in N(Q_2(\sigma_1)^*/Q_2^2$. In fact, in virtue of $[2, \text{Lemma 4.3}]$ we can assume that $\Lambda = \langle a_1 \rangle \perp (8e_2a_1) \perp (64e_3a_1)$, with $e_2, e_3 \in Z^2$. Hence, SRGL tells us that $r, \lambda_2, \lambda_3 \in \mathcal{O}_D$, with $|1 - r| \geq |2|$, define an element $\phi$ if $\phi \in B(\Lambda)$ as before if and only if they satisfy the relation

$$z = a_1 - ra_1 \bar{r} = 8\lambda_2 e_2a_1 \bar{\lambda}_2 + 64\lambda_3 e_3a_1 \bar{\lambda}_3.$$ 

We can rewrite this equation as $z = 8\lambda_3 w \bar{\lambda}_3$, where $w = e_2 \eta a_1 \bar{\eta} + 8e_3a_1$ and $\eta = \lambda_3^{-1} \lambda_2$. Remember that, in this case, $|\lambda_2| = |i|$ and $|\lambda_3| = 1$. Hence, by Lemma 2.2, the existence of $r, \lambda_2, \lambda_3$ satisfying the equation above is equivalent to the existence of $r, \eta \in \mathcal{O}_D$, with $|\eta| = |i|$ such that $NzN(w) \in Q_2^2$ and $NzN(8w)^{-1} \in Z^2$. We know that $|w| = |2|$, so $NzN(8w)^{-1} \in Z_2$ if and only if $NzN(w)^{-1} \in Z_2$. On the other hand, $N(w) = N(\epsilon(e_2 \eta a_1 \bar{\eta})N(1 + 8\epsilon(e_2 \eta a_1 \bar{\eta})^{-1}a_1)$, where $\epsilon = \epsilon_2^{-1} \epsilon_3 \in Z^2$. Here, as $|8\epsilon(e_2 \eta a_1 \bar{\eta})^{-1}a_1| = |4|$, we can write $8\epsilon(e_2 \eta a_1 \bar{\eta})^{-1}a_1 = 4\epsilon \xi$ with $\xi = 2(e_2 \eta a_1 \bar{\eta})^{-1}a_1 \in \mathcal{O}_D$. As we have the relation $N(1 + 4\epsilon \xi) = 1 + 4\epsilon T(\xi) + 16\epsilon^2 N(\xi)$, it follows that $N(1 + 4\epsilon \xi) \in Q_2^2 \cup 5Q_2^2$, hence, $N(e_2 \eta a_1 \bar{\eta} + 8e_3a_1) \in N(a_1)Q_2^2 \cup 5N(a_1)Q_2^2$, where $N(w) \in 5Na_1Q_2^2$ if and only if $T(\xi) \equiv 1 \pmod{2}$. The last condition is not satisfied in virtue of Lemma 4.8. Therefore, we are reduced to the following result, which is an analogue of Theorem 3: $\mathcal{H}(\Lambda) = Q_2^*$ if and only if there exists $r \in \mathcal{O}_D$, with $|1 - r| = |2|$ satisfying the conditions:

$$\left( \frac{N(1 - r), -Na_1}{p} \right) = -1, \quad NzNa_1 \in Q_2^*, \quad |z| = |16|.$$ 

These are the $k$-star conditions for $\langle a_1 \rangle \perp (16a_1)$, and Lemma 4.5 implies that there is no $r \in \mathcal{O}_D$ satisfying them. Hence, we conclude that $\mathcal{H}(\Lambda) = N(Q_2(\sigma_1)^*)$.

4.3. Proof of Theorem 1 in Case II

By Lemma 4.1, in rank 2 case, we consider lattices $\Lambda$ of the form $\langle i_{\pi} \rangle \perp (2^{i_{\nu}}i_{\pi})$, where $\nu(4) \leq t \leq \nu(16)$, and for every prime $\pi$, we set $i_{\pi} \in O_{k(i)}$ such that $i_{\pi}^2 = \pi$. Remember that, if we prove that $\mathcal{H}(\Lambda) = Q_2^*$, then $\mathcal{H}(\Lambda) = Q_2^*$ for lattices $\Lambda$ of arbitrary rank. By Corollary 3.1, we can assume $t = 4$. Hence, the next result follows from Lemma 4.4 and Theorem 3:

Proposition 4.4. Let $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ be a skew-hermitian lattice such that $N(a_1), \ldots, N(a_n) \in \pi Q_2^*$ and $0 < \mu(\Lambda) \leq \nu(16)$. Then $\mathcal{H}(\Lambda) = Q_2^*$. 

4.4. Examples

Let consider the family of lattices $\Lambda = \langle i \rangle \perp (2^t i)$, for $t > 0$, where $D = (\frac{2.5}{Q})$ and $i^2 = 2$. $D$ ramifies only at 2 and 5. The lattice $\Lambda$ is unimodular for $p \neq 2$. We have that $H(\Lambda_p) = \mathbb{Z}_p^* \mathbb{Q}_p^{*2}$ for $p \neq 2$, in virtue of the computations in [7] (for $p \neq 5$) and [6, Theorem 4] (for $p = 5$). Hence, the spinor class field $\Sigma_{\Lambda}$ can ramify only at 2 and $\infty$, so $\Sigma_{\Lambda} \subset \mathbb{Q}(\sqrt{-1}, \sqrt{2})$. Observe that the algebra $D$ decomposes at infinity and the quadratic form corresponding to $\Lambda$ is indefinite. Hence, class and spinor genus of $\Lambda$ coincide and $\Sigma_{\Lambda} \subset \mathbb{R}$. On the other hand, for $p = 2$, Table 1 tells us that $H(\Lambda_2) = \mathbb{Q}_2^*$ if $t \leq 4$ and $H(\Lambda_2) = N(\mathbb{Q}_2(i^*)^*)$ if $t > 4$, whence $\Sigma_{\Lambda}$ decomposes at 2 for $t \leq 4$ and ramifies at 2 for $t > 4$. We conclude that $\Sigma_{\Lambda} = \mathbb{Q}$ for $t \leq 4$, while $\Sigma_{\Lambda} = \mathbb{Q}(\sqrt{2})$ for $t > 4$. In the first case, Hasse principle holds for $\Lambda$. In the second case, the class number of $\Lambda$ is 2.

Now consider the family of lattices $\Lambda = \langle i \rangle \perp (2^t i^t)$, where $D = (\frac{-1}{Q})$ and $i^2 = -1$. $D$ ramifies only at 2 and $\infty$. As before $H(\Lambda_p) = \mathbb{Z}_p^* \mathbb{Q}_p^{*2}$ for $p \neq 2$, hence $\Sigma_{\Lambda} \subset \mathbb{Q}(\sqrt{-1}, \sqrt{2})$. Since the form has discriminant 1 it is isotropic and therefore $\mathcal{U}_K^+$ is non-compact at infinity. In fact, the same holds for every binary lattice over a quaternion algebra ramifying at $\infty$, since all pure quaternions in the Hamilton Algebra are congruent. Hence, class and spinor genus of $\Lambda$ coincide also in this case. On the other hand, the spinor image is $\mathbb{R}^+$ at infinity and, since $-1$ is a ramified unit at 2, Table 1 tells us that $\Sigma_{\Lambda} = \mathbb{Q}$ for $t \leq 2$, while $\Sigma_{\Lambda} = \mathbb{Q}(\sqrt{-1})$ for $t > 2$.

Acknowledgments

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References