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Spinor norm for skew-hermitian forms over quaternion algebras

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A B S T R A C T

We complete all local spinor norm computations for quaternionic skew-hermitian forms over the field $\mathbb{Q}$ of rational numbers. This can be used to compute the number of classes in a genus of skew-hermitian lattices of rank 2 or larger over a maximal order in a quaternion algebra D over $\mathbb{Q}$ in many cases, e.g., when D ramifies at infinity. Examples are provided.

1. Introduction

Let K be a number field and let D be a quaternion algebra over K with canonical involution $q \mapsto \bar{q}$. Let V be a rank-n free D-module. Let $h : V \times V \to D$ be a *skew-hermitian form*, i.e., h is D-linear in the first variable and it satisfies $h(x, y) = -\overline{h(y, x)}$. A D-linear map $\phi : V \to V$ preserving h is called an isometry. We denote by $\mathcal{U}_K$ (resp. $\mathcal{U}_K^+$) the unitary group of h (resp. the special unitary group of h), i.e., the group of isometries (resp. isometries with trivial reduced norm) of h. Skew-hermitian forms share many properties of quadratic forms. In fact, if $D \cong \mathbb{M}_2(K)$, skew-hermitian forms in a rank-n free

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D-module are naturally in correspondence with quadratic forms in the $2n$-dimensional K-vector space PV, for any idempotent matrix P of rank 1 in D [3, §3]. In this case, the unitary group of h is isomorphic to the orthogonal group of the corresponding quadratic form. On the other hand, $\mathcal{U}_K = \mathcal{U}_K^+$ when D is a division algebra [8, §2.6].

As in the quadratic case, the problem of determining if two skew-hermitian lattices in the same space are isometric or not can be approached by the theory of genera and spinor genera [1]. This theory depends on the knowledge of the image, under the spinor norm, of the stabilizer of a given lattice in each local group $\mathcal{U}_{K_\wp}^+$. Full computations of this image exist for symmetric integral bilinear forms. Non-dyadic cases can be found in [7] and the dyadic case in [5]. For this reason we assume, from now on, that D is a quaternion division algebra. For skew-hermitian forms, non-dyadic places have been completely studied by Böge in [6]. The dyadic case was studied by Arenas-Carmona in [2] and [4], not covering all the cases, which we complete here when $K_\mathfrak{p} = \mathbb{Q}_2$. From now on $k = K_\mathfrak{p}$ denotes a dyadic local field of characteristic 0.

We denote by $|\cdot| : D \to \mathbb{R}_{\geq 0}$ and $|\cdot|_k : k \to \mathbb{R}_{\geq 0}$ the absolute values on D and k respectively, and we assume $|q| = |Nq|_k$, where N is the reduced norm, for any $q \in D$. We use ν for the surjective valuation $\nu : D^* \to \mathbb{Z}$. Let $\mathcal{O}_D = \nu^{-1}(\mathbb{Z}_{\geq 0}) \cup \{0\}$ be the unique maximal order in D [12, §2]. A skew-hermitian lattice or $\mathcal{O}_D$-lattice in V, is a lattice Λ in V such that $\mathcal{O}_D\Lambda = \Lambda$. Any skew-hermitian lattice Λ has a decomposition of the type

$$\Lambda = \Lambda_1 \perp \cdots \perp \Lambda_t, \tag{1}$$

where each indecomposable lattice Λ_r has rank 1 or 2, and the scales satisfy $\mathbf{s}(\Lambda_{r+1}) \subset \mathbf{s}(\Lambda_r)$ [2, §5]. If some Λ_m in the decomposition of Λ has rank 1, then $\Lambda_m = \mathcal{O}_D s_m$ and $h(s_m, s_m) = a_m$. We usually write $\Lambda_m = \langle a_m \rangle = \mathcal{O}_D s_m$ in this case. A statement like $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_t \rangle = \mathcal{O}_D s_1 \perp \cdots \perp \mathcal{O}_D s_t$ must be interpreted similarly. Define $A \subset k^*/k^{*2}$ by $A = \{N(a_m)k^{*2} \mid \Lambda_m = \langle a_m \rangle,\ 1 \leq m \leq t\}$. Following [6], we define the spinor image $H(\Lambda) \subseteq k^*$ by the relation $H(\Lambda)/k^{*2} = \theta(\mathcal{U}_k^+(\Lambda))$, where $\mathcal{U}_k^+(\Lambda)$ is the stabilizer of Λ in $\mathcal{U}_k^+$, and $\theta : \mathcal{U}_k^+ \to k^*/k^{*2}$ denotes the spinor norm. If $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, we let $\mu(\Lambda) = \min\{\nu(a_{i+1}) - \nu(a_i) \mid 1 \leq i < n\} \in \mathbb{Z}_{\geq 0}$. The lattices Λ for which the set $H(\Lambda)$ remains unknown to date are:

Case I: $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, where $A = \{-uk^{*2}\}$, for a unit $u \in \mathcal{O}_k^*$ of non-minimal quadratic defect [9, §63], and $0 < \mu(\Lambda) \leq \nu(16)$.

Case II: $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, where $A = \{\pi k^{*2}\}$, for a prime π in k, and $\nu(4) \leq \mu(\Lambda) \leq \nu(16)$.

Theorem 1. *Table 1 contains all local spinor images when the base field is $\mathbb{Q}_2$. In the table, s denotes the number of indecomposable components of rank 2 in the decomposition (1) of Λ, and $\Delta \in \mathcal{O}_k^*$ is a unit of minimal quadratic defect [9, §63]. Furthermore, π, u, A and $\mu = \mu(\Lambda)$ are as above. A dash means "irrelevant".*

Table 1
Spinor images for arbitrary lattices over $\mathbb{Q}_2$.

s	$\|A\|$	A	μ	$H(\Lambda)$	Reference
–	>1	–	–	$\mathbb{Q}_2^*$	Proposition 2.1 or [2, Table 2]
0	1	$-\Delta\mathbb{Q}_2^{*2}$	–	$\mathbb{Z}_2^*\mathbb{Q}_2^{*2}$	[2, Table 2]
0	1	$-u\mathbb{Q}_2^{*2}$	$0 \leq \mu \leq \nu(4)$	$\mathbb{Q}_2^*$	Proposition 4.1 and [2, Table 1]
0	1	$-u\mathbb{Q}_2^{*2}$	$\mu \geq \nu(8)$	$N(\mathbb{Q}_2(a_m)^*)$	Proposition 4.3 and [2, Table 2]
0	1	$\pi\mathbb{Q}_2^{*2}$	$0 \leq \mu \leq \nu(16)$	$\mathbb{Q}_2^*$	Proposition 4.4 and [2, Tables 1–2]
0	1	$\pi\mathbb{Q}_2^{*2}$	$\mu \geq \nu(32)$	$N(\mathbb{Q}_2(a_m)^*)$	[2, Table 2]
$\neq 0$	–	–	–	$\mathbb{Q}_2^*$	[4, Theorem 2]

Our (computer assisted) proof of Theorem 1 goes as follows: We use Theorem 2 below to reduce the computation of $H(\Lambda)$ to low rank Λ. In our case, this means rank 2 or 3. Then we use Theorems 3 and 4 for constructing an algorithm for binary lattices over unramified local dyadic fields and we apply it to $k = \mathbb{Q}_2$. Then we patch the proof in the remaining "rank 3" case.

Before we state the critical theorems, we recall a few facts about simple rotations in skew-hermitian spaces,[1] see [2, §6] for details. Let (V, h) be a skew-hermitian D-space. If $s \in V$ and $\sigma \in D^*$ satisfy $\sigma - \bar{\sigma} = h(s, s)$, the map $(s; \sigma) \in \mathcal{U}_K$ defined by $(s; \sigma)(x) = x - h(x, s)\sigma^{-1}s$ is called a simple rotation with axis s. Its spinor norm [2] is $\theta[(s; \sigma)] = N(\sigma)k^{*2}$, where $N : D^* \to k^*$ is the reduced norm. The set of simple rotations span the group $\mathcal{U}_k^+$. One way to produce simple rotations, that we use heavily in the sequel, is the next lemma:

Simple Rotation Generating Lemma (SRGL). *(See [2, Lemma 6.3].) Let $t, u \in V$ be such that $h(u, u) = h(t, t) = a$. Let $s = t - u$ and $\sigma = h(t, s)$. Then $(s; \sigma)$ is a well-defined simple rotation satisfying $(s; \sigma)(t) = u$. Furthermore, if $u = rt + t_0$, where $t_0 \in t^\perp$, we have the identities $\sigma = a(1 - \bar{r})$, $h(t_0, t_0) = a - ra\bar{r}$, and $\sigma - \bar{\sigma} = h(s, s)$.*

Let $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, with $\langle a_i \rangle = \mathcal{O}_D s_i$ as in (1). Assume also $|2a_m| \geq |a_l|$ for $m < l$. Then the first author proved in [2, Lemma 6.7] that the unitary group $\mathcal{U}_k^+(\Lambda)$ of the lattice is generated by $\mathcal{A}(\Lambda) \cup \mathcal{B}(\Lambda)$, where

1. $\mathcal{A}(\Lambda)$ is the set of simple rotations with axis s_m, for some $m = 1, \ldots, n$.
2. $\mathcal{B}(\Lambda)$ is the set of simple rotations of the form $(s; \sigma)$, where $s = s_m - t$ for some $t = rs_m + s_0$ with $s_0 \in \mathcal{O}_D s_{m+1} \perp \cdots \perp \mathcal{O}_D s_n$, and $1 - r \notin (2i)$.

In particular, the elements of $\mathcal{B}(\Lambda)$ satisfy all relations in **SRGL**. Note that $\mathcal{A}(\Lambda)$ and $\mathcal{B}(\Lambda)$ depend on the splitting (1).

[1] Some authors call these elements reflections. We prefer the name simple rotation since $(s; \sigma)$ acts on the 2-dimensional subspace $k[\sigma]s$ by $v \mapsto uv$, where $u = \bar{\sigma}\sigma^{-1}$ is an element of norm 1.

Theorem 2. *Let* $\Lambda = \langle a_1\rangle\perp\cdots\perp\langle a_n\rangle$, *with* $\langle a_i\rangle = \mathcal{O}_D s_i$, *be a skew-hermitian lattice and let* $\mu = \mu(\Lambda)$ *be as above. Assume* $\mu > \nu(4)$ *and* $N(a_2),\ldots,N(a_n) \in N(a_1)k^{*2}$. *Let* $(s;\sigma) \in \mathcal{B}(\Lambda)$, *i.e.,* $s = (1-r)s_m - s_0$, *where* $s_0 = \lambda_{m+1}s_{m+1}+\cdots+\lambda_n s_n$, $\sigma = a_m(1-\bar{r})$ *and* $|1-r| \geq |2|$. *If* $|\lambda_{m+t}| \geq |2^{l-1}\lambda_{m+t+l}|$, *for some* $t \in \{1,\ldots,n-m\}$ *and for all* $l \in \{1,\ldots,n-m-t\}$, *then there exists* $\Lambda' = \langle b_1\rangle\perp\cdots\perp\langle b_{t+1}\rangle \subset \Lambda$ *satisfying the following conditions:*

1. $(s;\sigma) \in \mathcal{U}_k^+(\Lambda')$.
2. $\mu(\Lambda') \geq \mu(\Lambda)$.
3. $N(b_i) \in N(a_1)k^{*2}$, *for all* $i = 1,\ldots,t+1$.

We say that an element $r \in \mathcal{O}_D$ satisfies the k-star conditions for a lattice $\Lambda = \langle a_1\rangle\perp\cdots\perp\langle a_n\rangle$ if $z = a_1 - ra_1\bar{r}$ satisfies $NzNa_1 \in k^{*2}$ and $NzN(\pi^t a_1)^{-1} \in \mathcal{O}_k$, where $\nu(\pi^t) = \mu(\Lambda)$, while the Hilbert symbol $(\frac{N(1-r),-Na_1}{\mathfrak{p}})$ equals -1.

Theorem 3. *Let* $\Lambda = \langle a_1\rangle\perp\langle a_2\rangle$ *be a skew-hermitian lattice such that* $|2a_1| \geq |a_2|$ *and* $N(a_2) \in N(a_1)k^{*2}$. *The following statements are equivalent:*

1. $H(\Lambda) = k^*$.
2. *There exists* $(s;\sigma) \in \mathcal{B}(\Lambda)$ *such that* $N\sigma \notin N(k(a_1)^*)$.
3. *There exists* $r \in \mathcal{O}_D$ *satisfying the* k*-star conditions for* Λ.

It is known that the (unique) quaternion division k-algebra D has a basis $\{1, i, j, ij\}$, where $i^2 = \pi$, $j^2 = \Delta$, $ij = -ji$. Moreover, if $\omega = \frac{j+1}{2}$, then $\{1, \omega, i, i\omega\}$ is an $\mathcal{O}_k$-basis for $\mathcal{O}_D$. Let $e = \nu(2)/2$ be the ramification index of $k/\mathbb{Q}_2$, and assume $\nu(\pi^t) = \mu(\Lambda)$.

Theorem 4. *Let* Λ *be as in Theorem 3. There exists* $r \in \mathcal{O}_D$ *satisfying the* k*-star conditions for* Λ *if and only if there exists* $\alpha \in \mathcal{S} \oplus \mathcal{S}\omega \oplus \mathcal{S}i \oplus \mathcal{S}i\omega \subset \mathcal{O}_D$ *satisfying them, for one (any) set of representatives* $\mathcal{S}$ *of* $\mathcal{O}_k/\pi^u\mathcal{O}_k$, *with* $u = t + 6e$ *as above.*

2. Generators of $\mathcal{U}_k^+(\Lambda)$ and their spinor norm

If $\Lambda = \langle a_1\rangle\perp\cdots\perp\langle a_n\rangle$ is a skew-hermitian lattice, then $[k^* : N(k(a_1)^*)] = 2$ [9, §63] and $N(k(a_1)^*) \subset H(\Lambda)$ [2, §6]. As a direct consequence of these facts, we have:

Proposition 2.1. *Let* $\Lambda = \langle a_1\rangle\perp\cdots\perp\langle a_n\rangle$ *be a skew-hermitian lattice. Then* $H(\Lambda) = N(k(a_1)^*)$ *or* $H(\Lambda) = k^*$. *In particular:*

1. $H(\Lambda) = k^*$ *if and only if, there exists* $\phi \in \mathcal{C}(\Lambda)$ *such that* $\theta(\phi) \notin N(k(a_1)^*)/k^{*2}$, *for one (any) set of generators* $\mathcal{C}(\Lambda)$ *for* $\mathcal{U}_k^+(\Lambda)$.
2. *If there exists* $b \in \mathcal{O}_D$ *with* $N(b) \notin N(a_1)k^{*2}$ *such that* $\Lambda = \langle b\rangle\perp\Lambda'$, *for some lattice* Λ', *we have* $H(\Lambda) = k^*$.

Remark 2.1. In particular, if $|2a_m| \geq |a_l|$ for $m < l$, and $\mathcal{C}(\Lambda) = \mathcal{A}(\Lambda) \cup \mathcal{B}(\Lambda)$, we just need to check the property for the elements in $\mathcal{B}(\Lambda)$, since simple rotations $(s_m; \sigma) \in \mathcal{A}(\Lambda)$ have spinor norm $N(\sigma)k^{*2} \in N(k(a_m)^*)/k^{*2}$. Our strategy includes to replace $\mathcal{B}(\Lambda)$ by a smaller set that still generates but it is easier to control (cf. Lemma 4.7).

Lemma 2.1. *Let $\Lambda = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, with $\langle a_i \rangle = \mathcal{O}_D s_i$, be a skew-hermitian lattice such that $|2a_m| \geq |a_l|$ for $m < l$. Take $(s; \sigma) \in \mathcal{B}(\Lambda)$, i.e., $s = (1 - r)s_m - s_0$, where $s_0 = \lambda_{m+1}s_{m+1} + \cdots + \lambda_n s_n$, $\sigma = a_m(1 - \bar{r})$, and $|1 - r| \geq |2|$. If any of the following conditions is satisfied:*

1. $|1 - r| > |2|$ *and* $|\lambda_{m+1}| < 1$, *while* $\mu(\Lambda) \geq \nu(8)$, *and* $k/\mathbb{Q}_2$ *is unramified,*
2. $|1 - r| = |2|$ *and* $|\lambda_{m+1}| \leq |2|$, *while* $\mu(\Lambda) \geq \nu(4\pi)$,
3. $|1 - r| > |2|$ *or* $|\lambda_{m+1}| < 1$, *while* $\mu(\Lambda) \geq \nu(16)$,
4. $|\lambda_{m+1}| \leq |4|$, $|\lambda_{m+2}| \leq |2|$ *and* $|\lambda_{m+3}| < 1$.

then $\theta[(s; \sigma)] \in N(k(a_m)^*)/k^{*2}$.

Proof. It suffices to prove that if $a = h(s, s)$, then $N(a) \in N(a_m)k^{*2}$, since $\sigma \in k(a)$. In fact, we have $s = (1 - r)s_m - s_0$, so that $a = (1 - r)a_m(1 - \bar{r}) + a_0$, where $a_0 = h(s_0, s_0)$. It follows that

$$N(a) = N(a_m)N(1 - r)^2 N\big(1 + (1 - r)^{-1}a_0(1 - \bar{r})^{-1}a_m^{-1}\big). \tag{2}$$

Now, $a_0 = \lambda_{m+1}a_{m+1}\overline{\lambda_{m+1}} + \cdots + \lambda_n a_n \overline{\lambda_n}$ and $|(1 - r)^{-1}a_0(1 - \bar{r})^{-1}a_m^{-1}| = |a_0||a_m^{-1}|/|1 - r|^2 < |4|$ if any of the conditions above is satisfied. This implies the last norm in (2) is a square. □

The following result, together with **SRGL**, gives us an easy method to construct simple rotations in $\mathcal{B}(\Lambda)$ as in the introduction.

Lemma 2.2. *Let $r \in \mathcal{O}_D$ be a non-zero quaternion and let $a_1, a_2 \in \mathcal{O}_D$ be non-zero pure quaternions. There exists $\lambda \in \mathcal{O}_D$ different from zero such that $a_1 = ra_1\bar{r} + \lambda a_2 \bar{\lambda}$ if and only if $NzNa_2 \in k^{*2}$ and $NzNa_2^{-1} \in \mathcal{O}_k$, where $z = a_1 - ra_1\bar{r}$.*

Proof. The equation $a_1 = ra_1\bar{r} + \lambda a_2 \bar{\lambda}$ has a solution $\lambda \in D^*$ if and only if the binary skew-hermitian form h' whose Gramm matrix is $\left(\begin{smallmatrix} z & 0 \\ 0 & -a_2 \end{smallmatrix}\right)$ is isotropic. Now, h' is isotropic if and only if $NzNa_2 = \mathrm{disc}(h') \in k^{*2}$ [10, Chapter 10, §3, Theorem 3.6]. We conclude that there exists $\lambda \in D^*$ satisfying $a_1 = ra_1\bar{r} + \lambda a_2 \bar{\lambda}$ if and only if $NzNa_2 \in k^{*2}$. Finally, we have $Nz = Na_2 N\lambda^2$, whence $\lambda \in \mathcal{O}_D$ if and only if $NzNa_2^{-1} \in \mathcal{O}_k$. □

3. Proof of Theorems 2, 3 and 4

Proof of Theorem 2. Set $\Lambda' = \langle b_1\rangle\perp\cdots\perp\langle b_{t+1}\rangle = \mathcal{O}_D s_m\perp\cdots\perp\mathcal{O}_D s_{m+t-1}\perp\mathcal{O}_D s_0'$, where $s_0' = \sum_{i=m+t}^{n}\lambda_i s_i$. It is clear that $\Lambda'\subset\Lambda$. To prove condition (1) in the theorem, we note that $s_0 = s_0' - \sum_{i=m+1}^{m+t-1}\lambda_i s_i \in \Lambda'$. We compute

- $(s;\sigma)(s_i) = s_i - h(s_i,s)\sigma^{-1}s = s_i + h(s_i,s_0)\sigma^{-1}s$, for $m+1\le i\le m+t-1$,
- $(s;\sigma)(s_0') = s_0' - h(s_0',s)\sigma^{-1}s = s_0' + h(s_0',s_0)\sigma^{-1}s$, and
- $(s;\sigma)(s_m) = rs_m + s_0 \in \Lambda'$.

Hence, $(s;\sigma)(\Lambda')\subseteq\Lambda'$ if $h(s_i,s_0)\sigma^{-1}, h(s_0',s_0)\sigma^{-1}\in\mathcal{O}_D$. The latter holds since $|\sigma| = |a_m(1-\bar{r})| \ge |2a_m|$ is larger than the height of s_0. We conclude that $(s;\sigma)\in\mathcal{U}_k^+(\Lambda')$. On the other hand, as

$$b_{t+1} = h\big(s_0',s_0'\big) = \sum_{u=m+t}^{n}\lambda_u a_u\overline{\lambda_u},$$

we have $|b_{t+1}| = |a_{m+t}||\lambda_{m+t}|^2$, since $|\lambda_{m+t}| \ge |2^{l-1}\lambda_{m+t+l}|$ when $1\le l\le n-m-t$ and $\mu(\Lambda) > \nu(4)$. From here $\mu(\Lambda')\ge\mu(\Lambda)$, proving condition (2). Finally, to prove the last condition, we consider

$$N(b_{t+1}) = N(\lambda_{m+t})^2 N(a_{m+t}) N\left(1 + (\lambda_{m+t}a_{m+t}\overline{\lambda_{m+t}})^{-1}\sum_{u=m+t+1}^{n}\lambda_u a_u\overline{\lambda_u}\right), \tag{3}$$

where $|(\lambda_{m+t}a_{m+t}\overline{\lambda_{m+t}})^{-1}| = |a_{m+t}|^{-1}|\lambda_{m+t}|^{-2}$. Since $|a_{m+t+l}| < |4^l a_{m+t}|$ and $|\lambda_{m+t}| \ge |2^{l-1}\lambda_{m+t+l}|$ for all $l\in\{1,\dots,n-m-t\}$, the last term in (3) is a square, whence $N(b_{t+1})\in N(a_{m+t})k^{*2}$ and the proof of the condition (3) is completed. □

Proof of Theorem 3. The equivalence between (1) and (2) is a direct consequence of Proposition 2.1 and the subsequent remark to it. To prove that (2) implies (3), let $(s;\sigma)$ be a simple rotation such that $\theta[(s;\sigma)] = N(\sigma)k^{*2}\notin N(k(a_1)^*)/k^{*2}$. As isometry $(s;\sigma)\in\mathcal{B}(\Lambda)$ satisfies $a_1 = h(s_1,s_1) = ra_1\bar{r} + \lambda a_2\bar{\lambda}$, where $(s;\sigma)(s_1) = rs_1 + \lambda s_2$. Such an $r\in\mathcal{O}_D$ satisfies $\sigma = a_1(1-\bar{r})$ by **SRGL**. Hence, $\theta[(s;\sigma)]\notin N(k(a_1)^*)/k^{*2}$ if and only if $N(1-r)\notin N(k(a_1)^*)$, or equivalently $\left(\frac{N(1-r),-Na_1}{\mathfrak{p}}\right) = -1$. On the other hand, Lemma 2.2 tells us that $NzNa_2\in k^{*2}$ and $NzNa_2^{-1}\in\mathcal{O}_k$, where $z = a_1 - ra_1\bar{r}$. The result follows since $Na_2\in N(a_1)k^{*2}$ and $\mu = \nu(a_2) - \nu(a_1) = \nu(\pi^t)$. Conversely, if $r\in\mathcal{O}_D$ satisfies the k-star conditions, then Lemma 2.2 and **SRGL** imply the existence of $\phi\in\mathcal{B}(\Lambda)$ such that $\theta(\phi) = N(a_1)N(1-\bar{r})k^{*2}$ and the result follows as before. □

Corollary 3.1. *Let Λ be as in Theorem 3. Let t be such that $\mu = \nu(\pi^t)$. If $H(\Lambda) = k^*$, then $H(\Lambda') = k^*$ for every lattice $\Lambda' = \langle a_1\rangle\perp\langle b\rangle$ with $N(b)\in N(a_1)k^{*2}$ and $\mu(\Lambda') = \nu(\pi^s)$, for $e\le s<t$.*

Remark 3.1. Due to Lemma 2.1, in the condition (2) of Theorem 3, it is enough to consider simple rotations $(s;\sigma) \in \mathcal{B}(\Lambda)$ with $|\lambda| > |4|$, where $s = (1-r)s_1 - \lambda s_2$. Remember that $|1-r| \geq |2|$ for $(s;\sigma) \in \mathcal{B}(\Lambda)$.

Proof of Theorem 4. Assume $r \in \mathcal{O}_D$ satisfies the k-star conditions. Let $\alpha \in \mathcal{O}_D$ be a representative of the class of r modulo π^u as in the statement. Then, $r = \alpha + \pi^u\beta$, with $\beta \in \mathcal{O}_D$ and $\alpha \in \mathcal{S} \oplus \mathcal{S}\omega \oplus \mathcal{S}i \oplus \mathcal{S}i\omega \subset \mathcal{O}_D$. As $1-r = 1-\alpha-\pi^u\beta$ we have $N(1-r) = N(1-\alpha)N(1-(1-\alpha)^{-1}\pi^u\beta)$. Now, $|1-r| \geq |2|$ implies $|1-\alpha| \geq |2|$. Therefore, $N(1-(1-\alpha)^{-1}\pi^u\beta)$ is a square. Hence, $\left(\frac{N(1-r),-Na_1}{\mathfrak{p}}\right) = \left(\frac{N(1-\alpha),-Na_1}{\mathfrak{p}}\right)$. On the other hand, if $z = a_1 - ra_1\bar{r} = \pi^t\lambda a_1\bar{\lambda}$ and $z' = a_1 - \alpha a_1\bar{\alpha}$, then $z = z' - \pi^u\gamma$, with $\gamma = \alpha a_1\bar{\beta} + \beta a_1\bar{\alpha} + \pi^u\beta a_1\bar{\beta} \in \mathcal{O}_D$. Note that $a_1^{-1}\gamma \in \mathcal{O}_D$. We have $|z'| = |z|$, since $|z| = |\pi^t\lambda a_1\bar{\lambda}| > |16\pi^t a_1| = |\pi^{4e+t}a_1| > |\pi^u\gamma|$, where we are assuming $|\lambda| > |4|$ (see Remark 3.1). Furthermore, we have that $Nz = Nz'N(1-z'^{-1}\pi^u\gamma)$ with $|z'^{-1}\pi^u\gamma| < |\pi^{-(4e+t)}a_1^{-1}\pi^{t+6e}\gamma| = |\pi^{2e}a_1^{-1}\gamma| \leq |4|$. Hence, $NzNa_1$ is a square if and only if $Nz'Na_1$ is a square. Finally, from $|z'| = |z|$ we obtain $|Nz/\pi^{2t}N(a_1)|_k \leq 1$ if and only if $|Nz'/\pi^{2t}N(a_1)|_k \leq 1$. $\square$

Remark 3.2. The optimal choice for the number u in Theorem 4 depends on $|\lambda|$. For example, since $z = a_1 - ra_1\bar{r} = \pi^t\lambda a_1\bar{\lambda}$, if λ satisfies $|\lambda| = 1$, then we would have $|z'| = |\pi^t a_1|$ and so $|z'^{-1}\pi^u\gamma| = |\pi^{-t}a_1^{-1}\pi^u\gamma| \leq |\pi^{u-t}| < |4|$ if $u = t + 2e + 1$. This holds in some cases when $k = \mathbb{Q}_2$.

4. Proof of Theorem 1

The following result is a direct consequence of [2, Lemma 4.3]. Note that for either of the remaining cases I or II described in the introduction, the extension $k(a_1)/k$ is ramified.

Lemma 4.1. *Let $\langle a_1\rangle \perp \langle a_2\rangle$ be a skew-hermitian lattice such that $N(a_2) \in N(a_1)k^{*2}$ and the extension $k(a_1)/k$ is ramified. Then, there exists a skew-hermitian lattice $L = \langle q\rangle \perp \langle \epsilon q\rangle$, where $q \in D^*$ and $\epsilon \in k^*$, such that $H(L) = H(\Lambda)$. Moreover, we can assume that $q = q'$, for any quaternion $q' \in D^*$ with $N(q') \in N(a_1)k^{*2}$.*

Note that, due to Corollary 3.1, if $\mu(\Lambda) = \nu(\pi^t)$ for Λ as in Theorem 3, we can take ϵ in last lemma equals to π^t, for any prime π of k. If $k = \mathbb{Q}_2$, we have $\mathcal{O}_k = \mathbb{Z}_2$, $\mathcal{O}_D = \mathbb{Z}_2[\omega] \oplus i\mathbb{Z}_2[\omega]$ and $\mathcal{O}_k/\pi^u\mathcal{O}_k \cong \mathbb{Z}/2^u\mathbb{Z}$. By considering Theorems 3, 4 and the lemma above, we are able to construct an algorithm for computing $H(\Lambda)$, for all binary $\mathcal{O}_D$-lattices Λ, as follows:

1. By Lemma 4.1, we are reduced to compute $H(L)$ for $L = \langle q\rangle \perp \langle 2^t q\rangle$, for q^2 running over representative of all suitable square classes, and a few values of t for each q.
2. Fix a set of representatives $\mathcal{S}$ of the finite ring $\mathbb{Z}_2/2^u\mathbb{Z}_2$: We can choose $\mathcal{S} = \{0, 1, \ldots, 2^u - 1\}$ for u large enough (see Remark 3.2).

Table 2
Proof of Lemma 4.2.

q	t	r_1	r_2
$j+ij$	3	$-1-4i-4i\omega$	$1-14\omega-i-10i\omega$
$j+ij$	4	$-1-8i-8i\omega$	$1-6\omega-13i-6i\omega$
$i+j$	3	$-1-4i\omega$	$1-2\omega-i$
$i+j$	4	$-1-8i\omega$	$-1-6\omega-3i$

Table 3
Proof of Lemma 4.3.

q	r	$N(1-r)$	z	$NzNq$
$j+ij$	$1+2i\omega$	$2\cdot 2^2$	$4(-1+2\omega-4i-7i\omega)$	$2^4\cdot 5^2$
$i+j$	$1+2i+2i\omega$	$-2\cdot 2^2$	$4(1-2\omega+3i+3i\omega)$	$2^4(1+8\cdot 20)$

3. For $r=a+b\omega+ci+di\omega\in\mathcal{S}\oplus\mathcal{S}\omega\oplus\mathcal{S}i\oplus\mathcal{S}i\omega\subset\mathcal{O}_D$, check if the k-star conditions are satisfied. This verification can be done by using the computer algebra system Sage [11].
4. Conclude that $H(\Lambda)=\mathbb{Q}_2^*$ if some r in the last step satisfies the k-star conditions. Otherwise, $H(\Lambda)=N(\mathbb{Q}_2(a_1)^*)$ in virtue of Theorems 3, 4 and Proposition 2.1.

Remark 4.1. The algorithm can be extended to any unramified finite extension k of $\mathbb{Q}_2$. The condition $|2a_1|\geq|a_2|$ in Theorems 3 and 2 is essential. Hence, the algorithm does not work, for $\mu<\nu(2)$, if the extension $k/\mathbb{Q}_2$ ramifies, unless the algorithm returns the value k^* for $\mu<\nu(2)$.

4.1. Computations using Sage

In all that follows we assume $i^2=2$, $j^2=5$, and $ij=-ji$. Whenever a different uniformizing parameter π makes computations easier we use $i_\pi=u_\pi i$, for some unit $u_\pi\in\mathbb{Q}_2(j)$, such that $i_\pi^2=\pi$, or equivalently $N(u_\pi)=\pi/2$. The following results are obtained by computer search. When the algorithm does find solutions, we actually list them. Otherwise it is just stated that no solutions were found.

Lemma 4.2. *(See Table 2.) For any $q\in\{j+ij,i+j\}$ and $t\in\{3,4\}$, there exist $r_1,r_2\in\mathcal{O}_D$ such that:*

1. $|1-r_1|=|2|$, $NzNq\in\mathbb{Q}_2^{*2}$ *and* $NzN(2^tq)^{-1}\in\mathbb{Z}_2^*$, *where* $z=q-r_1q\overline{r_1}$.
2. $|1-r_2|=|i|$, $NzNq\in\mathbb{Q}_2^{*2}$ *and* $NzN(2^tq)^{-1}\in\mathbb{Z}_2^*$, *where* $z=q-r_2q\overline{r_2}$.

Lemma 4.3. *(See Table 3.) Let $L=\langle q\rangle\perp\langle 4q\rangle$ be a skew-hermitian lattice satisfying the conditions in Theorem 3, for $q\in\{j+ij,i+j\}$. Then there exists $r\in\mathcal{O}_D$ satisfying the k-star conditions for L.*

Table 4
Proof of Lemma 4.4.

π	r	$N(1-r)$	z	$NzNi_\pi$
± 2	$15+8\omega$	$5\cdot 2^2(1+8\cdot 5^{-1}\cdot 7)$	$-592i_\pi+304i_\pi\omega$	$2^{10}(1+8\cdot 38)$
± 10	$15+8\omega$	$5\cdot 2^2(1+8\cdot 5^{-1}\cdot 7)$	$-592i_\pi+304i_\pi\omega$	$2^{10}\cdot 5^2(1+8\cdot 38)$

Lemma 4.4. *(See Table 4.) Let $L=\langle i_\pi\rangle\perp\langle 16i_\pi\rangle$ be a skew-hermitian lattice satisfying the hypothesis in Theorem 3, for $\pi\in\{\pm 2,\pm 10\}$ as above. Then, there exists $r\in\mathcal{O}_D$ satisfying the k-star conditions for L.*

Lemma 4.5. *There is no $r=a+b\omega+ci+di\omega\in\mathbb{Z}\oplus\mathbb{Z}\omega\oplus\mathbb{Z}i\oplus\mathbb{Z}i\omega=\mathcal{O}_D$, with $0\le a,b,d,c<2^{t+3}$ satisfying the k-star conditions for $L=\langle q\rangle\perp\langle 2^tq\rangle$, if $t\in\{3,4\}$ and $q\in\{j+ij,j+i\}$.*

4.2. Proof of Theorem 1 in Case I

Assume $\Lambda=\langle a_1\rangle\perp\cdots\perp\langle a_n\rangle$, where $N(a_m)\in -u\mathbb{Q}_2^{*2}$, for each $m=1,\dots,n$ and $u\in\mathbb{Z}_2^*$ is a unit of non-minimal quadratic defect independent of m. As $\mathbb{Z}_2^*/\mathbb{Z}_2^{*2}=\{\overline{\pm 1},\overline{\pm 5}\}$ and a pure quaternion cannot have reduced norm -1, we have two options for u: $u=-5$ or $u=-1$.

In virtue of Lemma 4.1, we consider binary lattices $\Lambda=\langle q\rangle\perp\langle 2^tq\rangle$, with $1\le t\le 4$, where we can choose any pure quaternion $q\in\mathcal{O}_D^0$ satisfying $N(q)\in -u\mathbb{Q}_2^{*2}$. Here, $q=q_u$ satisfy $N(q)\in -u\mathbb{Q}_2^{*2}$, for u running over the set $\{-5,-1\}$ of units of non-minimal quadratic defect. We choose $q_{-5}=j+ij$ and $q_{-1}=i+j$.

Proposition 4.1. *Let $\Lambda=\langle a_1\rangle\perp\cdots\perp\langle a_n\rangle$ be a skew-hermitian lattice such that $N(a_1),\dots,N(a_n)\in -u\mathbb{Q}_2^{*2}$ and $0<\mu(\Lambda)\le\nu(4)$. Then $H(\Lambda)=\mathbb{Q}_2^*$.*

Proof. We can assume $n=2$ and $\Lambda=\langle q_u\rangle\perp\langle 2^tq_u\rangle$, with $q_u\in\{q_{-5},q_{-1}\}=\{j+ij,i+j\}$ and $t\in\{1,2\}$. In virtue of Corollary 3.1 it suffices to prove the result for $t=2$. Lemma 4.3 tells us that there exists $r\in\mathcal{O}_D$ satisfying the k-star conditions. This is equivalent to $H(\Lambda)=\mathbb{Q}_2^*$ by Theorem 3. □

To handle the cases where $\mu=\nu(8)$ or $\mu=\nu(16)$ we use the following result, which is used to improve the set of generators $\mathcal{B}(\Lambda)$. The proof is a routine computation.

Lemma 4.6. *If $r\in\mathcal{O}_D$ satisfies either of the equations*

$$j+ij=r(j+ij)\bar{r}+2^t\lambda(j+ij)\bar{\lambda},\quad \text{or}\quad i+j=r(i+j)\bar{r}+2^t\lambda(i+j)\bar{\lambda},\tag{4}$$

where $\lambda\in\mathcal{O}_D$, and $t\ge 2$, then $1-r\in i\mathcal{O}_D$.

Lemma 4.7. *Let $\Lambda=\langle a_1\rangle\perp\langle a_2\rangle=\mathcal{O}_Ds_1\perp\mathcal{O}_Ds_2$ be a skew-hermitian lattice such that $N(a_1),N(a_2)\in -u\mathbb{Q}_2^{*2}$ and $\nu(8)\le\mu(\Lambda)\le\nu(16)$. There exists a lattice L of rank 2* *such that $H(L) = H(\Lambda)$, and a suitable splitting of L, such that $\mathcal{A}(L) \cup \mathcal{B}_l(L)$ generates $\mathcal{U}^+_{\mathbb{Q}_2}(L)$, for $l = 1, 2$, where $\mathcal{B}_1(L) = \{(s;\sigma) \in \mathcal{B}(L)\colon |1-r| = |i|\}$ and $\mathcal{B}_2(L) = \{(s;\sigma) \in \mathcal{B}(L)\colon |\lambda| = 1\}$, with r as in* ***SRGL*** *and λ as in Lemma 2.2.*

Proof. By Lemma 4.1 there is a lattice $L = \mathcal{O}_D s_1 \perp \mathcal{O}_D s_2 = \langle q_u\rangle \perp \langle 2^t q_u\rangle$ as above, with $u \in \{-5, -1\}$ and $t \in \{3, 4\}$ satisfying $H(L) = H(\Lambda)$. Let $\phi \in \mathcal{B}(L)$ be such that $\phi(s_1) = rs_1 + \lambda s_2$. We have $|1-r| \in \{|i|, |2|\}$ in virtue of Lemma 4.6. Hence, to prove that $\mathcal{B}_1(L)$ satisfies the required property, it suffices to prove that, if ϕ satisfies $|1-r| = |2|$, then there exists $(s;\sigma) \in \mathcal{B}(L)$ such that $|1-r''| = |i|$ and $|1-r'| = |i|$, where $(s;\sigma)(s_1) = r's_1 + \lambda' s_2$ and $(s;\sigma)\phi(s_1) = r''s_1 + \lambda'' s_2$. In this case, there exists a second element $(s';\sigma') \in \mathcal{B}_1(L)$ defined by $s' = s_1 - (s;\sigma)\phi(s_1)$, $\sigma' = q(1-\overline{r''})$ such that $(s';\sigma')(s;\sigma)\phi(s_1) = s_1$. In fact, by a computation we have

$$1 - r'' = 1 - r + \left[rq\left(1-\bar{r'}\right) + 2^t\lambda q\bar{\lambda'}\right]\left(1-\bar{r'}\right)^{-1} q^{-1}\left(1-r'\right), \tag{5}$$

$$\lambda'' = \lambda + \left[rq\left(1-\bar{r'}\right) + 2^t\lambda q\bar{\lambda'}\right]\left(1-\bar{r'}\right)^{-1} q^{-1}\lambda'. \tag{6}$$

Lemma 4.2 implies the existence of an element $r' \in \mathcal{O}_D$ such that $|1-r'| = |i|$, $NzN(2^tq) \in \mathbb{Q}_2^{*2}$, and $NzN(2^tq)^{-1} \in \mathbb{Z}_2$, where $z = q - r'q\overline{r'}$ and $t \in \{3, 4\}$. Hence, by Lemma 2.2, there exists $\lambda' \in \mathcal{O}_D$, such that $q = r'q\overline{r'} + 2^t\lambda' q\overline{\lambda'}$. Then $(s;\sigma)$, where $s = (1-r')s_1 - \lambda' s_2$ and $\sigma = q(1-\bar{r'})$, belongs to $\mathcal{B}_1(L)$ (cf. **SRGL**). On the other hand, as

$$\begin{aligned}\left|\left[rq\left(1-\bar{r'}\right) + 2^t\lambda q\bar{\lambda'}\right]\left(1-\bar{r'}\right)^{-1} q^{-1}\left(1-r'\right)\right| &= \left|rq\left(1-\bar{r'}\right) + 2^t\lambda q\bar{\lambda'}\right| \\ &= \left|1-\bar{r'}\right| = |i|\end{aligned} \tag{7}$$

and $|1-r| = |2|$, it follows that $|1-r''| = |i|$. In particular, $\mathcal{A}(L) \cup \mathcal{B}_1(L)$ generates $\mathcal{U}^+_{\mathbb{Q}_2}(L)$.

Now, to prove that $\mathcal{A}(L) \cup \mathcal{B}_2(L)$ generates $\mathcal{U}^+(L)$, by a similar argument as for $\mathcal{B}_1(L)$, it suffices to prove that, if $\phi \in \mathcal{B}(L)$ satisfies $|\lambda| < 1$, there exists $(s;\sigma) \in \mathcal{B}(L)$ such that $|\lambda'| = 1$ and $|\lambda''| = 1$, where λ, λ', λ'' are defined by ϕ, $(s;\sigma)$ and $(s;\sigma)\phi$ respectively, as before. From Eq. (6) we see that $|\lambda''| = 1$ if $|\lambda| < 1$ and $|\lambda'| = 1$. By Lemma 4.2, there exists $r' \in \mathcal{O}_D$ such that

$$\left|1-r'\right| = |i| \text{ or } |2|, \qquad NzN\left(2^tq\right) \in \mathbb{Q}_2^{*2} \quad \text{and} \quad NzN\left(2^tq\right)^{-1} \in \mathbb{Z}_2^*,$$

where $z = q - r'q\overline{r'}$ and $t \in \{3, 4\}$. Hence, by Lemma 2.2, there exists $\lambda' \in \mathcal{O}_D$ such that $q = r'q\overline{r'} + 2^t\lambda' q\overline{\lambda'}$. Then if $s = (1-r')s_1 - \lambda' s_2$ and $\sigma = q(1-\bar{r'})$, then $(s;\sigma) \in \mathcal{B}(\Lambda)$ (cf. **SRGL**), and $|\lambda'| = 1$ since $NzN(2^tq)^{-1} \in \mathbb{Z}_2^*$. Now, we take $|1-r'| = |i|$ if $|1-r| = |2|$, and $|1-r'| = |2|$ if $|1-r| = |i|$, so that $|1-r'|, |1-r''| \geq |2|$ by (5). The result follows. □

Remark 4.2. Notice that, for a lattice Λ as in the previous lemma, we can replace $\mathcal{B}(\Lambda)$ by $\mathcal{B}_l(\Lambda)$, for $l = 1, 2$, in Theorem 3. Hence, since $|\lambda| = 1$ for $(s;\sigma) \in \mathcal{B}_2(\Lambda)$, we can improve the number u in Theorem 4 in virtue of Remark 3.2.

Proposition 4.2. *Let $\Lambda = \langle a_1\rangle \perp \langle a_2\rangle$ be as in Theorem 3. There exists $r \in \mathcal{O}_D$ satisfying the k-star conditions for $t \in \{3, 4\}$ and $Na_1 \in -u\mathbb{Q}_2^{*2}$, with u a unit of non-minimal quadratic defect, if and only if there exists $\alpha = a+b\omega+ci+di\omega \in \mathbb{Z}\oplus\mathbb{Z}\omega\oplus\mathbb{Z}i\oplus\mathbb{Z}i\omega = \mathcal{O}_D$, with $0 \le a, b, c, d < 2^{t+3}$, satisfying them.*

Combining this result with Theorem 3, Lemma 4.1 and Lemma 4.5, we obtain

Corollary 4.1. *Let $\Lambda = \langle a_1\rangle \perp \langle a_2\rangle$ be a skew-hermitian lattice such that $N(a_1), N(a_2) \in -u\mathbb{Q}_2^{*2}$, where u is a unit of non-minimal quadratic defect and $\mu = \nu(a_2)-\nu(a_1)$ satisfies $\nu(8) \le \mu \le \nu(16)$. Then $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$.*

We need the following result to handle ternary lattices Λ with $\mu(\Lambda) = \nu(8)$. For the sake of generality we state it for an arbitrary dyadic field k.

Lemma 4.8. *If $|\eta| = |i|$ and a_1 is a pure unit, then $T(2(\eta a_1\bar{\eta})^{-1}a_1) \in \pi\mathcal{O}_k$, where T is the trace map.*

Proof. Set $\eta = i\rho$, for $\rho \in \mathcal{O}_D^*$. Note that $a_1 i \equiv i\bar{a_1} \pmod{\pi}$, while ρ and $\bar{a_1}$ commute modulo i. We conclude that $\eta a_1\bar{\eta} \equiv -N(\rho)\pi\bar{a_1} \pmod{\pi i}$. In other words $\frac{1}{\pi}\eta a_1\bar{\eta} = -N(\rho)\bar{a_1} + \varepsilon$, where $\varepsilon \in i\mathcal{O}_D$, whence $\pi(\eta a_1\bar{\eta})^{-1} = -(N(\rho)\bar{a_1})^{-1} + \delta = \frac{-a_1}{N(\rho a_1)} + \delta$, for some $\delta \in i\mathcal{O}_D$. Hence

$$T\big(2(\eta a_1\bar{\eta})^{-1}a_1\big) \equiv \frac{-4a_1^2}{\pi N(\rho a_1)} + \frac{2}{\pi}T(\delta a_1) \pmod{\pi}$$

and the result follows since $\delta \in i\mathcal{O}_D$ implies $T(\delta a_1) \in \pi\mathcal{O}_k$. □

Note that if $\Lambda = \langle a_1\rangle \perp \cdots \perp \langle a_n\rangle$ is a skew-hermitian lattice and $(s;\sigma) \in \mathcal{B}(\Lambda)$, then $s = (1-r)s_m - s_0$, for some $s_0 \in \mathcal{O}_D s_{m+1} \perp \cdots \perp \mathcal{O}_D s_n$. Hence, if $m > 1$ then $(s;\sigma)$ fixes $\langle a_1\rangle$, so we can assume $m = 1$ in order to compute spinor norms in the next result.

Proposition 4.3. *Let $\Lambda = \langle a_1\rangle \perp \cdots \perp \langle a_n\rangle$ be a skew-hermitian lattice such that $N(a_1), \ldots, N(a_n) \in -u\mathbb{Q}_2^{*2}$, where u is a unit of non-minimal quadratic defect. If $\mu = \mu(\Lambda)$ satisfies $\nu(8) \le \mu \le \nu(16)$, then $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$.*

Proof when $\mu = \nu(16)$. In virtue of Lemma 2.1 it suffices to consider rotations $(s;\sigma) \in \mathcal{B}(\Lambda)$ such that $|1-r| = |2|$ and $|\lambda_2| = 1$. In this case, Theorem 2 tells us we can set $n = 2$ in the statement of the proposition. For $n = 2$, because of Lemma 4.7, we can replace Λ by a lattice L such that $H(L) = H(\Lambda)$ and a set of generators of $\mathcal{U}^+_{\mathbb{Q}_2}(L)$ is $\mathcal{A}(L) \cup \mathcal{B}_1(L)$. It follows that $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$ since rotations in $\mathcal{B}_1(L)$ have spinor norm belonging to $N(\mathbb{Q}_2(a_1)^*)$ in virtue of Lemma 2.1. □

Proof when $\mu = \nu(8)$. In virtue of Lemma 2.1, any rotation $(s;\sigma) \in \mathcal{B}(\Lambda)$ satisfies $\theta[(s;\sigma)] \in N(\mathbb{Q}_2(a_1)^*)/\mathbb{Q}_2^{*2}$ unless one of the following conditions is satisfied:

1. $|1-r| = |i|$, $|\lambda_2| = 1$,
2. $|1-r| = |2|$, $|\lambda_2| \in \{1, |i|\}$.

As in the previous case, by Theorem 2, when $|\lambda_2| = 1$ we are reduced to consider binary lattices and when $|1-r| = |2|$, $|\lambda_2| = |i|$ to study either binary lattices or rank 3 lattices with $|\lambda_3| = 1$. For rank 2 lattices, Corollary 4.1 tells us that $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$. We prove that, for rank 3 lattices Λ such that $(s;\sigma) \in \mathcal{B}(\Lambda)$ satisfies $|1-r| = |2|$, $|\lambda_2| = |i|$, $|\lambda_3| = 1$ we also have $\theta[(s;\sigma)] \in N(\mathbb{Q}_2(a_1)^*)/\mathbb{Q}_2^{*2}$. In fact, in virtue of [2, Lemma 4.3] we can assume that $\Lambda = \langle a_1\rangle \perp \langle 8\epsilon_2 a_1\rangle \perp \langle 64\epsilon_3 a_1\rangle$, with $\epsilon_2, \epsilon_3 \in \mathbb{Z}_2^*$. Hence, **SRGL** tells us that $r, \lambda_2, \lambda_3 \in \mathcal{O}_D$, with $|1-r| \geq |2|$, define an element $\phi \in \mathcal{B}(\Lambda)$ as before if and only if they satisfy the relation

$$z = a_1 - ra_1\bar{r} = 8\lambda_2\epsilon_2 a_1\overline{\lambda_2} + 64\lambda_3\epsilon_3 a_1\overline{\lambda_3}.$$

We can rewrite this equation as $z = 8\lambda_3 w\overline{\lambda_3}$, where $w = \epsilon_2\eta a_1\bar{\eta} + 8\epsilon_3 a_1$ and $\eta = \lambda_3^{-1}\lambda_2$. Remember that, in this case, $|\lambda_2| = |i|$ and $|\lambda_3| = 1$. Hence, by Lemma 2.2, the existence of r, λ_2, λ_3 satisfying the equation above is equivalent to the existence of $r, \eta \in \mathcal{O}_D$, with $|\eta| = |i|$ such that $NzN(w) \in \mathbb{Q}_2^{*2}$ and $NzN(8w)^{-1} \in \mathbb{Z}_2$. We know that $|w| = |2|$, so $NzN(8w)^{-1} \in \mathbb{Z}_2$ if and only if $\frac{Nz}{2^8} \in \mathbb{Z}_2$. On the other hand, $N(w) = N(\epsilon_2\eta a_1\bar{\eta})N(1 + 8\epsilon(\eta a_1\bar{\eta})^{-1}a_1)$, where $\epsilon = \epsilon_2^{-1}\epsilon_3 \in \mathbb{Z}_2^*$. Here, as $|8\epsilon(\eta a_1\bar{\eta})^{-1}a_1| = |4|$, we can write $8\epsilon(\eta a_1\bar{\eta})^{-1}a_1 = 4\epsilon\xi$ with $\xi = 2(\eta a_1\bar{\eta})^{-1}a_1 \in \mathcal{O}_D^*$. As we have the relation $N(1+4\epsilon\xi) = 1 + 4\epsilon T(\xi) + 16\epsilon^2 N(\xi)$, it follows that $N(1+4\epsilon\xi) \in \mathbb{Q}_2^{*2} \cup 5\mathbb{Q}_2^{*2}$, hence, $N(\epsilon_2\eta a_1\bar{\eta} + 8\epsilon_3 a_1) \in N(a_1)\mathbb{Q}_2^{*2} \cup 5N(a_1)\mathbb{Q}_2^{*2}$, where $N(w) \in 5Na_1\mathbb{Q}_2^{*2}$ if and only if $T(\xi) \equiv 1 \pmod 2$. The last condition is not satisfied in virtue of Lemma 4.8. Therefore, we are reduced to the following result, which is an analogue of Theorem 3: $H(\Lambda) = \mathbb{Q}_2^*$ *if and only if there exists* $r \in \mathcal{O}_D$, *with* $|1-r| = |2|$ *satisfying the conditions*:

$$\left(\frac{N(1-r), -Na_1}{\mathfrak{p}}\right) = -1, \quad NzNa_1 \in \mathbb{Q}_2^{*2}, \quad |z| = |16|.$$

These are the k-star conditions for $\langle a_1\rangle \perp \langle 16a_1\rangle$, and Lemma 4.5 implies that there is no $r \in \mathcal{O}_D$ satisfying them. Hence, we conclude that $H(\Lambda) = N(\mathbb{Q}_2(a_1)^*)$. □

4.3. Proof of Theorem 1 in Case II

By Lemma 4.1, in rank 2 case, we consider lattices Λ of the form $\langle i_\pi\rangle \perp \langle 2^t i_\pi\rangle$, where $\nu(4) \leq t \leq \nu(16)$, and for every prime π, we set $i_\pi \in \mathcal{O}_{k(j)}i$ such that $i_\pi^2 = \pi$. Remember that, if we prove that $H(\Lambda) = \mathbb{Q}_2^*$, then $H(\Lambda) = \mathbb{Q}_2^*$ for lattices Λ of arbitrary rank. By Corollary 3.1, we can assume $t = 4$. Hence, the next result follows from Lemma 4.4 and Theorem 3:

Proposition 4.4. *Let* $\Lambda = \langle a_1\rangle \perp \cdots \perp \langle a_n\rangle$ *be a skew-hermitian lattice such that* $N(a_1), \ldots, N(a_n) \in \pi\mathbb{Q}_2^{*2}$ *and* $0 < \mu(\Lambda) \leq \nu(16)$. *Then* $H(\Lambda) = \mathbb{Q}_2^*$.

4.4. Examples

Let consider the family of lattices $\Lambda = \langle i\rangle \perp \langle 2^t i\rangle$, for $t > 0$, where $D = \left(\frac{2,5}{\mathbb{Q}}\right)$ and $i^2 = 2$. D ramifies only at 2 and 5. The lattice Λ is unimodular for $p \neq 2$. We have that $H(\Lambda_p) = \mathbb{Z}_p^*\mathbb{Q}_p^{*2}$ for $p \neq 2$, in virtue of the computations in [7] (for $p \neq 5$) and [6, Theorem 4] (for $p = 5$). Hence, the spinor class field Σ_Λ can ramify only at 2 and ∞, so $\Sigma_\Lambda \subset \mathbb{Q}(\sqrt{-1}, \sqrt{2})$. Observe that the algebra D decomposes at infinity and the quadratic form corresponding to Λ is indefinite. Hence, class and spinor genus of Λ coincide and $\Sigma_\Lambda \subset \mathbb{R}$. On the other hand, for $p = 2$, Table 1 tells us that $H(\Lambda_2) = \mathbb{Q}_2^*$ if $t \leq 4$ and $H(\Lambda_2) = N(\mathbb{Q}_2(i)^*)$ if $t > 4$, whence Σ_Λ decomposes at 2 for $t \leq 4$ and ramifies at 2 for $t > 4$. We conclude that $\Sigma_\Lambda = \mathbb{Q}$ for $t \leq 4$, while $\Sigma_\Lambda = \mathbb{Q}(\sqrt{2})$ for $t > 4$. In the first case, Hasse principle holds for Λ. In the second case, the class number of Λ is 2.

Now consider the family of lattices $\Lambda = \langle i\rangle \perp \langle 2^t i\rangle$, where $D = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ and $i^2 = -1$. D ramifies only at 2 and ∞. As before $H(\Lambda_p) = \mathbb{Z}_p^*\mathbb{Q}_p^{*2}$ for $p \neq 2$, hence $\Sigma_\Lambda \subset \mathbb{Q}(\sqrt{-1}, \sqrt{2})$. Since the form has discriminant 1 it is isotropic and therefore $\mathcal{U}_K^+$ is non-compact at infinity. In fact, the same holds for every binary lattice over a quaternion algebra ramifying at ∞, since all pure quaternions in the Hamilton Algebra are congruent. Hence, class and spinor genus of Λ coincide also in this case. On the other hand, the spinor image is $\mathbb{R}^+$ at infinity and, since -1 is a ramified unit at 2, Table 1 tells us that $\Sigma_\Lambda = \mathbb{Q}$ for $t \leq 2$, while $\Sigma_\Lambda = \mathbb{Q}(\sqrt{-1})$ for $t > 2$.

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