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On the treatment of non-solvable implicit constitutive relations in solid mechanics

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Abstract. We report results concerning the treatment of elastic bodies described by implicit constitutive relations which are not solvable, in the sense of expressing the stresses as functions of the strains or vice versa. Motivated by the theory of generalized hyperelastic materials, which applies also to polyatomic crystals, the field equations to be solved for a body described by a non-solvable implicit constitutive relation in the above sense are lied down. In addition to the momentum equation, auxiliary variables come into play. These variables are accompanied by an equation governing them. We specialize to the case of an isotropic constitutive relation and give some conditions in order for a broad subclass of an isotropic body to be a generalized material. Then, we lay down the field equations for such a body.

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Keywords. Implicit constitutive relations · Generalized hyperelastic materials · Nonlinear elasticity.

1. Introduction

In the recent years, Rajagopal et al. [10,14–18] proposed constitutive relations for elastic bodies, which cannot be classified as either Cauchy or Green elastic bodies [21]. If $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor and $\mathbf{C}$ is the right Cauchy–Green deformation tensor, one of such new relations correspond to:

$$\boldsymbol{\mathcal{G}}(\mathbf{S}, \mathbf{C}) = \mathbf{0}. \tag{1.1}$$

One application of such general theories is in the study of some subclasses of constitutive relations, where the strains are small, stresses are arbitrarily large, and the relation between these two variables is in general nonlinear (see, for example, [7,8,19]). Other applications can be found in biomechanics, see, for example, [22].

Conditions for the solvability of an implicit constitutive relation in the sense of expressing stresses as functions of strains or vice versa are reported on [20]. In a recent work, [6] attention is focused on the field equations written in terms of the stresses solely. The present contribution is connected with the latter work in the sense that we lay down the field equations for a body described by a non-solvable implicit constitutive relation. The motivation stems from the work of Pitteri and Zanzotto [3, p. 339]. These authors state that when an implicit constitutive relation is not solvable, in the sense of expressing the stresses as functions of the strains or vice versa, auxiliary variables naturally come into play. This observation comes from earlier works of Cardin and Cardin–Spera [1,2], where the idea of generalized hyperelastic materials is introduced. Even though the latter framework is introduced for the geometrical desingularization in ideal holonomic constraints, it can also be applied to bodies described by an implicit constitutive relation. Interestingly, to the class of generalized bodies belong materials described by an implicit constitutive relation [13–15]. It is worth stressing that another important application of such a theory is for polyatomic crystals [4,5].

In this note, we utilize the framework of Cardin [1] for generalized hyperelastic materials for the case of non-solvable implicit constitutive relations. By a non-solvable constitutive relation, it is meant that an equation, for example, of the form (1.1), which in components (in Cartesian coordinates) is written[1]

$$\mathcal{G}_{AB}(\mathbf{S},\mathbf{C})=\mathbf{0},\quad A,B=1,2,3, \tag{1.2}$$

cannot be solved for $\mathbf{C}$ as a function of $\mathbf{S}$ or vice versa. In the theory of Cardin in addition to the momentum equation, auxiliary variables come into play, together with an equation that has the role of a field equation, which when solved renders the required values of the auxiliary parameters.

The article is structured as follows. Section 2 presents some basic kinematic relations, while Sect. 3 gives a short summary of the main elements of the theory of Cardin [1], where the idea of a generalized hyperelastic material plays a vital role. In Sect. 4, we apply this approach to the case of implicit constitutive relations. We give necessary and sufficient conditions for a body described by an implicit constitutive relation to be seen as a generalized hyperelastic material in the sense of Cardin. Then, we apply the Maslov–Hormander theorem according to which there exist a real function (in a local sense), which depends not only on the deformation gradient, but also on a set of auxiliary variables as well. This function fulfills two equations, since it is a Morse family [1]: the momentum equation and an additional equation governing the auxiliary parameters. Section 5 treats the case of an isotropic body. Necessary and sufficient conditions for such a body to be a generalized hyperelastic material in the sense of Cardin [1] are given. Essentially, these are constraints on the scalar functions of the implicit constitutive relation. For a broad class of isotropic bodies, we give some sufficient conditions for their fulfillment. In addition to the momentum equation, there is a new equation ruling the auxiliary variables. The article ends up in Sect. 6 with some concluding remarks.

2. Basic equations: kinematics and field equations

Let $\mathbf{x}$ denotes the position of a particle X of a body $\mathcal{B}$ in the current configuration $\mathcal{B}_t$. Let $\mathbf{X}$ the position of the same particle in the reference configuration $\mathcal{B}_r$. It is assumed that for any instant t there exist a one to one function $\boldsymbol{\chi}$ such that $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X},t)$. The deformation gradient $\mathbf{F}$ and the right Cauchy–Green deformation tensors are defined as:

$$\mathbf{F}=\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}},\quad \mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}, \tag{2.1}$$

where we assume that $J=\det\mathbf{F}>0$.

If $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor, then in the quasi-static case, this tensor has to satisfy the equilibrium equation:

$$\mathrm{div}\boldsymbol{\sigma}+\rho\mathbf{b}=\mathbf{0}, \tag{2.2}$$

where $\mathbf{b}$ is the body force, ρ the mass density, and div is the divergence operator defined in terms of $\mathbf{x}$.

The first and second Piola–Kirchhoff stress tensors, denoted by $\mathbf{T}$ and $\mathbf{S}$, respectively, are defined as:

$$\mathbf{T}=J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}},\quad \mathbf{S}=\mathbf{F}^{-1}\mathbf{T}. \tag{2.3}$$

The equilibrium Eq. (2.2) can be expressed in an alternative form in the reference configuration as:

$$\mathrm{Div}\mathbf{T}+\rho_r\mathbf{b}_r=\mathbf{0}, \tag{2.4}$$

where $\mathbf{b}_r$ is the pull-back version of the body forces $\mathbf{b}$ and $\rho_r=J\rho$ is the mass density in the reference configuration. The divergence operator Div is now defined in terms of $\mathbf{X}$. More details about the different definitions and equations presented here can be found, for example, in [24].

[1] We use capital characters as index for the reference configuration and lower case characters for the current configuration.

3. Generalized hyperelastic materials

We present a short description of Cardin's approach [1], where the idea of a generalized hyperelastic material is introduced. Let $\mathbf{T}$ denote the first Piola–Kirchhoff stress tensor while $\mathbf{F} : \mathrm{Lin}^+ \to \mathcal{R}$ is the deformation gradient, $\mathrm{Lin}^+ = \{\mathbf{F} \in \mathrm{Lin} : det\mathbf{F}>0\}$. One can endow the cotangent space $T^*(\mathrm{Lin}^+)$ with the normal symplectic structure induced by the 2-form $\mathrm{d}\Omega$, Ω being the Liouville 1-form on $T^*(\mathrm{Lin}^+)$

$$\Omega = T_{iL}\mathrm{d}F_{iL}, \quad \mathrm{d}\Omega = \mathrm{d}T_{iL} \wedge \mathrm{d}F_{iL}. \tag{3.1}$$

The Lagrangian submanifolds of the manifold $(T^*(\mathrm{Lin}^+), \mathrm{d}\Omega)$ are those submanifolds satisfying certain conditions. For our purposes, it suffices to say that a Lagrangian submanifold Λ for which the composed map

$$\Lambda \to_{\mathcal{J}} T^*(\mathrm{Lin}^+) \to_{\pi_{\mathrm{Lin}^+}} \mathrm{Lin}^+ \tag{3.2}$$

is of maximum rank

$$\mathrm{rank}D(\pi_{\mathrm{Lin}^+} \circ \mathcal{J}) = 9(= \max), \tag{3.3}$$

corresponds to a standard hyperelastic body; namely the following relation holds:

$$\mathbf{T} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}. \tag{3.4}$$

We note that in Eq. (3.2) $\mathcal{J}$ stands for the inclusion map $\mathcal{J} : \Lambda \to T^*(\mathrm{Lin}^+)$ while $\pi_{\mathrm{Lin}^+} : T^*(\mathrm{Lin}^+) \to \mathrm{Lin}^+$.

When the transversality condition of Eq. (3.3) fails, we can define a generalized hyperelastic body as a Lagrangian submanifold Λ of $T^*(\mathrm{Lin}^+)$. In such a case, the Maslov–Hormander's theorem [9,12] for the local parametrization of Lagrangian submanifolds shows that locally there exists a function $W(\mathbf{F}, \mathbf{p})$, such that Λ is described by the pair $(\mathbf{F}, \mathbf{S}) \in T^*(\mathrm{Lin}^+)$ satisfying

$$\mathbf{T} = \frac{\partial W(\mathbf{F}, \mathbf{p})}{\partial \mathbf{F}}, \quad \mathbf{0} = \frac{\partial W(\mathbf{F}, \mathbf{p})}{\partial \mathbf{p}}. \tag{3.5}$$

The field equations are then

$$\mathrm{Div}\frac{\partial W}{\partial \mathbf{F}} + \rho_r \mathbf{b}_r = \mathbf{0}, \quad \frac{\partial W}{\partial \mathbf{p}} = \mathbf{0}. \tag{3.6}$$

Thus, in addition to the momentum Eq. $(3.6)_1$, one more Eq. $(3.6)_2$ is added that governs the auxiliary parameters $\mathbf{p}$. Failure of the transversality condition means

$$\mathrm{rank}D(\pi_{\mathrm{Lin}^+} \circ \mathcal{J}) < 9(= \max). \tag{3.7}$$

4. Application to implicit constitutive relations

Cardin [1] highlighted the application of the above framework to bodies described by an implicit constitutive relation (see also [14,15]) in Cartesian coordinates:[2]

$$\mathcal{G}_{AB}(\mathbf{S}, \mathbf{C}) = \mathbf{0}. \tag{4.1}$$

According to Cardin, a body described by Eq. (4.1) is a Lagrangian submanifold when

$$\{G_{AB}, G_{CD}\} = 0, \tag{4.2}$$

where

$$\{G_{AB}, G_{CD}\} = \frac{\partial \mathcal{G}_{AB}}{\partial S_{EF}}\frac{\partial \mathcal{G}_{CD}}{\partial C_{EF}} - \frac{\partial \mathcal{G}_{AB}}{\partial C_{EF}}\frac{\partial \mathcal{G}_{CD}}{\partial S_{EF}} \tag{4.3}$$

[2] We have modified the original equations given by Cardin, by replacing $\mathbf{T}$ and $\mathbf{F}$ by the conjugate pair $\mathbf{S}$, $\mathbf{C}$.

is the Poisson bracket canonically associated to the symplectic 2-form Ω. The tranversality condition here is

$$\operatorname{rank}(D_{(\mathbf{S},\mathbf{C})}\boldsymbol{\mathcal{G}}|_{\boldsymbol{\mathcal{G}}=\mathbf{0}}) = 9. \tag{4.4}$$

When this condition fails, the Maslov–Hormander theorem renders the local existence of a Morse family $W(\mathbf{F}, \mathbf{p})$ such that

$$\mathbf{S} = \frac{\partial W(\mathbf{C}, \mathbf{p})}{\partial \mathbf{C}}, \quad \mathbf{0} = \frac{\partial W(\mathbf{C}, \mathbf{p})}{\partial \mathbf{p}}.$$

Failure of the transversality condition means that the implicit constitutive relation cannot be solved to give stresses as functions of the strains or vice versa. The field equations derived from the above framework are in the absence of body forces

$$\operatorname{Div}\left(\frac{\partial W}{\partial \mathbf{C}}\right) = \mathbf{0}, \quad \frac{\partial W}{\partial \mathbf{p}} = \mathbf{0}.$$

Remark. *We note that when the equilibrium equation is written with respect to the second Piola–Kirchhoff stress tensor,* $\mathbf{S}$*, the divergence is taken with respect to the right Cauchy–Green deformation tensor,* $\mathbf{C}$*. The same equation written using the first Piola–Kirchhoff stress tensor utilizes the referential metric for calculating the divergence* [11].

5. Application to isotropic bodies

In this section, we consider an implicit constitutive relation and use the results presented in Sect. 3 for an isotropic body, where (1.1) becomes [14,15,23,25]:

$$\begin{aligned}\boldsymbol{\mathcal{G}}(\mathbf{S}, \mathbf{C}) = {} & \alpha_0 \mathbf{I} + \alpha_1 \mathbf{S} + \alpha_2 \mathbf{C} + \alpha_3 \mathbf{S}^2 + \alpha_4 \mathbf{C}^2 + \alpha_5 (\mathbf{SC} + \mathbf{CS}) \\ & + \alpha_6 (\mathbf{S}^2\mathbf{C} + \mathbf{CS}^2) + \alpha_7 (\mathbf{C}^2\mathbf{S} + \mathbf{SC}^2) + \alpha_8 (\mathbf{S}^2\mathbf{C}^2 + \mathbf{C}^2\mathbf{S}^2).\end{aligned} \tag{5.1}$$

The functions $\alpha_i, i = 0, 1, 2, \ldots, 8$ are scalar functions that depend on the invariants

$$\begin{aligned}&\operatorname{tr}\mathbf{S}, \quad \operatorname{tr}\mathbf{C}, \quad \operatorname{tr}(\mathbf{S}^2), \quad \operatorname{tr}(\mathbf{C}^2), \quad \operatorname{tr}(\mathbf{S}^3), \quad \operatorname{tr}(\mathbf{C}^3), \quad \operatorname{tr}(\mathbf{CS}), \quad \operatorname{tr}(\mathbf{S}^2\mathbf{C}), \\ &\operatorname{tr}(\mathbf{SC}^2), \quad \operatorname{tr}(\mathbf{S}^2\mathbf{C}^2).\end{aligned} \tag{5.2}$$

In order for a body described by Eq. (5.1) to be a Lagrangian submanifold in the sense of Cardin [1], the Poisson bracket (4.3) should equal to zero.

Let us consider as an example the particular case when in Eq. (5.1) $\alpha_6, \alpha_7, \alpha_8$ are equal to zero; namely when we have the simplified constitutive law of the form

$$\boldsymbol{\mathcal{G}}(\mathbf{S}, \mathbf{C}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{S} + \alpha_2 \mathbf{C} + \alpha_3 \mathbf{S}^2 + \alpha_4 \mathbf{C}^2 + \alpha_5 (\mathbf{SC} + \mathbf{CS}). \tag{5.3}$$

We can see that this particular constitutive relation includes, as a special case, the definition of a Cauchy elastic body, for which we have (if $\alpha_i = 0, i = 1, 3, 5$ and if $\alpha_j = \alpha_j(\mathbf{C}), j = 0, 2, 4$):

$$\mathbf{S} = \alpha_0 \mathbf{I} + \alpha_2 \mathbf{C} + \alpha_4 \mathbf{C}^2,$$

while if $\alpha_i = \alpha_i(\mathbf{S}), i = 0, 1, 3$ and if $\alpha_j = 0, j = 2, 4, 5$ we obtain the new class of elastic bodies:

$$\mathbf{C} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{S} + \alpha_3 \mathbf{S}^2.$$

For Eq. (5.1), the constraint of the vanishing of the Poisson bracket renders, after a lengthy calculation, Eq. (A.1) of the "Appendix". Essentially, this condition is a necessary and sufficient condition for the body to be a Lagrangian submanifold. Conditions sufficient for its fulfillment are the vanishing of each bracket separately, namely Eq. (A.2) of the "Appendix". Application of the Maslov–Hormander theorem then renders the existence of an isotropic function $W = W(C_{AB}, p_A)$, such that for the field equations it holds (if there are no body force)

$$\mathrm{Div}\left(\frac{\partial W}{\partial \mathbf{C}}\right) = \mathbf{0}, \quad \frac{\partial W}{\partial \mathbf{p}} = \mathbf{0}. \tag{5.4}$$

The unknown variables in this case are the pair $\mathbf{C}$ and $\mathbf{p}$. Essentially, in addition to the momentum equation, we have an equation to find the auxiliary variables; this is Eq. $(5.4)_2$ which has to be solved in parallel with Eq. $(5.4)_1$.

Since the function W is isotropic in its arguments, we have for a complete and irreducible representation (see [23])

$$W(\mathbf{C}, \mathbf{p}) = W(\mathrm{tr}\mathbf{C}, \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3), \mathbf{p}\cdot\mathbf{p}, \mathbf{p}\cdot(\mathbf{C}\mathbf{p}), \mathbf{p}\cdot(\mathbf{C}^2\mathbf{p})). \tag{5.5}$$

Therefore, for the quantities that appear in Eq. (5.4) we take

$$\frac{\partial W}{\partial \mathbf{p}} = \phi_0\mathbf{p} + \phi_1\mathbf{C}\mathbf{p} + \phi_2\mathbf{C}^2\mathbf{p}, \tag{5.6}$$

and

$$\frac{\partial W}{\partial \mathbf{C}} = \phi_3\mathbf{I} + \phi_4\mathbf{C} + \phi_5\mathbf{C}^2 + \phi_6\mathbf{p}\otimes\mathbf{p} + \phi_7\mathbf{p}\otimes\mathbf{C}\mathbf{p}, \tag{5.7}$$

where $\phi_i = \phi_i(\mathrm{tr}\mathbf{C}, \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3), \mathbf{p}\cdot\mathbf{p}, \mathbf{p}\cdot(\mathbf{C}\mathbf{p}), \mathbf{p}\cdot(\mathbf{C}^2\mathbf{p}))$. So, collectively for an isotropic body described by Eq. (5.3) one has to solve the following system of equations

$$\begin{aligned} &\mathrm{Div}(\phi_3\mathbf{I} + \phi_4\mathbf{C} + \phi_5\mathbf{C}^2 + \phi_6\mathbf{p}\otimes\mathbf{p} + \phi_7\mathbf{p}\otimes\mathbf{C}\mathbf{p}) = \mathbf{0}, \\ &\phi_0\mathbf{p} + \phi_1\mathbf{C}\mathbf{p} + \phi_2\mathbf{C}^2\mathbf{p} = \mathbf{0}. \end{aligned} \tag{5.8}$$

6. Conclusions

We studied a general class of implicit constitutive relations for elastic bodies presenting some conditions which when satisfied there exists a scalar function W depending on the deformation $\mathbf{C}$ and an auxiliary vector field $\mathbf{p}$. The stress tensor $\mathbf{S}$ can then be expressed locally in the usual manner as the derivative of W with respect to $\mathbf{C}$ and participates in the standard form of the momentum equation. Additionally to the momentum equation, an equation ruling the auxiliary variables should be solved in these cases. These additional degrees of freedom are the price one pays when working with a real implicit constitutive relation.

Appendix A

The Poisson bracket of Eq. (4.3) for the constitutive relation of Eq. (5.3) renders

$$\begin{aligned} &\frac{\partial\alpha_1}{\partial S_{EF}}\frac{\partial\alpha_0}{\partial C_{EF}}\left[S_{AB}I_{CD} - I_{AB}S_{CD}\right] + \frac{\partial\alpha_0}{\partial S_{EF}}\frac{\partial\alpha_2}{\partial C_{EF}}\left[I_{AB}C_{CD} - C_{AB}I_{CD}\right] \\ &\quad + \alpha_2\left[\frac{\partial\alpha_0}{\partial S_{CD}}I_{AB} - \frac{\partial\alpha_0}{\partial S_{AB}}I_{CD}\right] + \frac{\partial\alpha_0}{\partial S_{EF}}\frac{\partial\alpha_4}{\partial C_{EF}}\left[I_{AB}C_{CP}C_{PD} - C_{AN}C_{NB}I_{CD}\right] \\ &\quad + \alpha_4\left[\frac{\partial\alpha_0}{\partial S_{CD}}I_{AB}C_{PD} + \frac{\partial\alpha_0}{\partial S_{PD}}I_{AB}C_{CP} - C_{NB}\frac{\partial\alpha_0}{\partial S_{AN}}I_{CD} - C_{AN}\frac{\partial\alpha_0}{\partial S_{NB}}I_{CD}\right] \\ &\quad + \frac{\partial\alpha_0}{\partial S_{EF}}\frac{\partial\alpha_5}{\partial C_{EF}}\left[I_{AB}S_{CP}C_{PD} - C_{AN}S_{NB}I_{CD} + I_{AB}C_{CP}S_{PD} - S_{AN}C_{NB}I_{CD}\right] \\ &\quad + \alpha_5\left[\frac{\partial\alpha_0}{\partial S_{PD}}I_{AB}S_{CP} + \frac{\partial\alpha_0}{\partial S_{CP}} - S_{AN}\frac{\partial\alpha_0}{\partial S_{NB}}I_{CD} - S_{NB}\frac{\partial\alpha_0}{\partial S_{AN}}I_{CD}\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial\alpha_1}{\partial S_{EF}}\frac{\partial\alpha_2}{\partial C_{EF}}\left[S_{AB}C_{CD}-C_{AB}S_{CD}\right]+\alpha_2\left[\frac{\partial\alpha_2}{\partial S_{AB}}-\frac{\partial\alpha_1}{\partial S_{AB}}S_{CD}\right]\\
&+\frac{\partial\alpha_1}{\partial S_{EF}}\frac{\partial\alpha_4}{\partial C_{EF}}\left[S_{AB}C_{CP}C_{PD}-C_{AN}C_{NB}S_{CD}\right]\\
&+\alpha_4\left[\frac{\partial\alpha_1}{\partial S_{CP}}S_{AB}C_{PD}+\frac{\partial\alpha_1}{\partial S_{PD}}-C_{NB}\frac{\partial\alpha_1}{\partial S_{AN}}S_{CD}-C_{AN}\frac{\partial\alpha_1}{\partial S_{NB}}S_{CD}\right]\\
&+\frac{\partial\alpha_1}{\partial S_{EF}}\frac{\partial\alpha_5}{\partial C_{EF}}\left[S_{AB}S_{CP}C_{PD}+S_{AB}C_{CP}S_{PD}-S_{AN}C_{NB}S_{CD}-C_{AN}S_{NB}S_{CD}\right]\\
&+\alpha_5\left[\frac{\partial\alpha_1}{\partial S_{PD}}S_{AB}S_{CP}+\frac{\partial\alpha_1}{\partial S_{CP}}S_{AB}S_{PD}-S_{AN}\frac{\partial\alpha_1}{\partial S_{NB}}S_{CD}-S_{NB}\frac{\partial\alpha_1}{\partial S_{AN}}S_{CD}\right]\\
&+\alpha_1\left[\frac{\partial\alpha_0}{\partial C_{AB}}I_{CD}-\frac{\partial\alpha_0}{\partial C_{CD}}I_{AB}\right]+\alpha_1\left[\frac{\partial\alpha_0}{\partial C_{AB}}C_{CD}-\frac{\partial\alpha_0}{\partial C_{CD}}C_{AB}\right]\\
&+\alpha_1\left[\frac{\partial\alpha_4}{\partial C_{AB}}C_{CP}C_{PD}-\frac{\partial\alpha_4}{\partial C_{CD}}C_{AN}C_{NB}\right]\\
&+\alpha_1\left[\frac{\partial\alpha_5}{\partial C_{AB}}S_{CP}C_{PD}+\frac{\partial\alpha_5}{\partial C_{AB}}C_{CP}C_{PD}-\frac{\partial\alpha_5}{\partial C_{CD}}S_{AN}C_{NB}-\frac{\partial\alpha_5}{\partial C_{CD}}C_{AN}S_{NB}\right]\\
&+\frac{\partial\alpha_3}{\partial S_{EF}}\frac{\partial\alpha_0}{\partial C_{EF}}\left[S_{AN}S_{NB}I_{CD}-I_{AB}S_{CP}S_{PD}\right]+\frac{\partial\alpha_3}{\partial S_{EF}}\frac{\partial\alpha_2}{\partial C_{EF}}\left[S_{AN}S_{NB}C_{CD}-C_{AB}S_{CP}S_{PD}\right]\\
&+\alpha_2\left[\frac{\partial\alpha_3}{\partial S_{CD}}S_{AN}S_{NB}-\frac{\partial\alpha_3}{\partial S_{AB}}S_{CP}S_{PD}\right]\\
&+\frac{\partial\alpha_3}{\partial S_{EF}}\frac{\partial\alpha_4}{\partial C_{EF}}\left[S_{AN}S_{NB}C_{CP}C_{PD}-C_{AN}C_{NB}S_{CP}S_{PD}\right]\\
&+\alpha_4\left[\frac{\partial\alpha_3}{\partial S_{CP}}S_{AN}S_{NB}C_{PD}+\frac{\partial\alpha_3}{\partial S_{PD}}S_{AN}S_{NB}C_{CP}-C_{NB}\frac{\partial\alpha_3}{\partial S_{AN}}S_{CP}S_{PD}-C_{AN}\frac{\partial\alpha_3}{\partial S_{NB}}S_{CP}S_{PD}\right]\\
&+\frac{\partial\alpha_3}{\partial S_{EF}}\frac{\partial\alpha_5}{\partial C_{EF}}\left[S_{AN}S_{NB}S_{CP}C_{PD}+S_{AN}S_{NB}C_{CP}S_{PD}-S_{AN}C_{NB}S_{CP}S_{PD}-C_{AN}S_{NB}S_{CP}S_{PD}\right]\\
&+\alpha_5\left[\frac{\partial\alpha_3}{\partial S_{PD}}S_{AN}S_{NB}S_{CP}+\frac{\partial\alpha_3}{\partial S_{CP}}S_{AN}S_{NB}S_{PD}-S_{AN}\frac{\partial\alpha_3}{\partial S_{NB}}S_{CP}S_{PD}-S_{NB}\frac{\partial\alpha_3}{\partial S_{AN}}S_{CP}S_{PD}\right]\\
&+\alpha_3\left[S_{NB}\frac{\partial\alpha_0}{\partial C_{AN}}I_{CD}+S_{AN}\frac{\partial\alpha_0}{\partial C_{NB}}I_{CD}-\frac{\partial\alpha_0}{\partial C_{CP}}I_{AB}S_{PD}-\frac{\partial\alpha_0}{\partial C_{PD}}I_{AB}S_{CP}\right]\\
&+\alpha_3\left[S_{NB}\frac{\partial\alpha_0}{\partial C_{AN}}C_{CD}+S_{AN}\frac{\partial\alpha_0}{\partial C_{NB}}C_{CD}-\frac{\partial\alpha_0}{\partial C_{CP}}C_{AB}S_{PD}-\frac{\partial\alpha_0}{\partial C_{PD}}C_{AB}S_{CP}\right]\\
&+\alpha_3\left[S_{NB}\frac{\partial\alpha_4}{\partial C_{AN}}C_{CP}C_{PD}+S_{AN}\frac{\partial\alpha_4}{\partial C_{NB}}C_{CP}C_{PD}-\frac{\partial\alpha_4}{\partial C_{CP}}C_{AN}C_{NB}S_{PD}-\frac{\partial\alpha_4}{\partial C_{PD}}C_{AN}C_{NB}S_{CP}\right]\\
&+\alpha_3\alpha_4\left[I_{AC}S_{PB}C_{PD}-I_{AC}C_{PB}S_{PD}\right]+\alpha_3\alpha_4\left[S_{AP}I_{BD}C_{CP}-C_{AP}S_{CP}I_{BD}\right]\\
&+\alpha_3\left[S_{NB}\frac{\partial\alpha_5}{\partial C_{AN}}S_{CP}C_{PD}+S_{NB}\frac{\partial\alpha_5}{\partial C_{AN}}C_{CP}S_{PD}+S_{AN}\frac{\partial\alpha_5}{\partial C_{NB}}S_{CP}C_{PD}+S_{AN}\frac{\partial\alpha_5}{\partial C_{NB}}C_{CP}S_{PD}\right.\\
&\left.-\frac{\partial\alpha_5}{\partial C_{CP}}S_{AN}C_{NB}S_{PD}-\frac{\partial\alpha_5}{\partial C_{PD}}S_{AN}C_{NB}S_{CP}-\frac{\partial\alpha_5}{\partial C_{CP}}C_{AN}S_{NB}S_{PD}-\frac{\partial\alpha_5}{\partial C_{PD}}C_{AN}S_{NB}S_{CP}\right]\\
&+\frac{\partial\alpha_5}{\partial S_{EF}}\frac{\partial\alpha_0}{\partial C_{EF}}\left[S_{AN}C_{NB}I_{CD}+C_{AN}S_{NB}I_{CD}-I_{AB}S_{CP}C_{PD}-I_{AB}C_{CP}S_{PD}\right]\\
&+\frac{\partial\alpha_5}{\partial S_{EF}}\frac{\partial\alpha_2}{\partial C_{EF}}\left[S_{AN}C_{NB}C_{CD}+C_{AN}S_{NB}C_{CD}-C_{AB}S_{CP}C_{PD}-C_{AB}C_{CP}S_{PD}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_2\left[\frac{\alpha_5}{\partial S_{CD}}S_{AN}C_{NB}+\frac{\partial\alpha_5}{\partial S_{CD}}C_{AN}S_{NB}-\frac{\partial\alpha_5}{\partial S_{AB}}S_{CP}C_{PD}-\frac{\partial\alpha_5}{\partial S_{AB}}C_{CP}S_{PD}\right]\\
&+\frac{\partial\alpha_5}{\partial S_{EF}}\frac{\partial\alpha_4}{\partial C_{EF}}\left[S_{AN}C_{NB}C_{CP}C_{PD}+C_{AN}S_{NB}C_{CP}C_{PD}-C_{AN}C_{NB}S_{CP}C_{PD}-C_{AN}C_{NB}C_{CP}S_{PD}\right]\\
&+\alpha_4\left[\frac{\partial\alpha_5}{\partial S_{CP}}S_{AN}C_{NB}C_{PD}+\frac{\partial\alpha_5}{\partial S_{PD}}S_{AN}C_{NB}C_{CP}+\frac{\partial\alpha_5}{\partial S_{CP}}C_{AN}S_{NB}C_{PD}+\frac{\partial\alpha_5}{\partial S_{PD}}C_{AN}S_{NB}C_{CD}\right.\\
&\left.+-C_{NB}\frac{\partial\alpha_5}{\partial S_{AN}}S_{CP}C_{PD}-C_{NB}\frac{\partial\alpha_5}{\partial S_{AN}}C_{CP}S_{PD}-C_{AN}\frac{\partial\alpha_5}{\partial S_{NB}}S_{CP}C_{PD}-C_{AN}\frac{\partial\alpha_5}{\partial S_{NB}}C_{CP}S_{PD}\right]\\
&+\alpha_5\left[\frac{\partial\alpha_5}{\partial S_{PD}}S_{AN}C_{NB}S_{CP}+\frac{\partial\alpha_5}{\partial S_{CP}}S_{AN}C_{NB}S_{PD}+\frac{\partial\alpha_5}{\partial S_{PD}}C_{AN}S_{NB}S_{CP}+\frac{\partial\alpha_5}{\partial S_{CP}}C_{AN}S_{NB}S_{PD}\right.\\
&\left.-S_{AN}\frac{\partial\alpha_5}{\partial S_{NB}}S_{CP}C_{PD}-S_{AN}\frac{\partial\alpha_5}{\partial S_{NB}}C_{CP}S_{PD}-S_{NB}\frac{\partial\alpha_5}{\partial S_{AN}}S_{CP}C_{PD}-S_{NB}\frac{\partial\alpha_5}{\partial S_{AN}}C_{CP}S_{PD}\right]\\
&+\alpha_5\left[C_{NB}\frac{\partial\alpha_0}{\partial C_{AN}}I_{CD}+C_{AN}\frac{\partial\alpha_0}{\partial C_{NB}}I_{CD}-\frac{\partial\alpha_0}{\partial C_{CP}}I_{AB}C_{PD}-\frac{\partial\alpha_0}{\partial C_{PD}}I_{AB}C_{CP}\right]\\
&+\alpha_5\left[C_{NB}\frac{\partial\alpha_2}{\partial C_{AN}}C_{CD}+C_{AN}\frac{\partial\alpha_2}{\partial C_{NB}}C_{CD}-\frac{\partial\alpha_2}{\partial C_{CP}}C_{AB}C_{PD}-\frac{\partial\alpha_2}{\partial C_{PD}}C_{AB}C_{CP}\right]\\
&+\alpha_5\left[C_{NB}\frac{\partial\alpha_4}{\partial C_{AN}}C_{CP}C_{PD}+C_{AN}\frac{\partial\alpha_4}{\partial C_{NB}}C_{CP}C_{PD}-\frac{\partial\alpha_4}{\partial C_{CP}}C_{AN}C_{NB}C_{PD}-\frac{\partial\alpha_4}{\partial C_{CP}}C_{AN}C_{NB}C_{CP}\right]\\
&+\alpha_5\left[C_{NB}\frac{\partial\alpha_5}{\partial C_{AN}}S_{CP}C_{PD}+C_{NB}\frac{\partial\alpha_5}{\partial C_{AN}}C_{CP}S_{PD}+C_{AN}\frac{\partial\alpha_5}{\partial C_{NB}}S_{CP}C_{PD}+C_{AN}\frac{\partial\alpha_5}{\partial C_{NB}}C_{CP}S_{PD}\right.\\
&\left.-\frac{\partial\alpha_5}{\partial C_{CP}}S_{AN}C_{NB}C_{PD}-\frac{\partial\alpha_5}{\partial C_{PD}}S_{AN}C_{NB}C_{CP}-\frac{\partial\alpha_5}{\partial C_{CP}}C_{AN}C_{NB}C_{CP}-\frac{\partial\alpha_5}{\partial C_{PD}}C_{AN}S_{NB}C_{CP}\right]\\
&+\alpha_5^2\left[I_{AC}C_{PB}S_{PD}+C_{AP}I_{BD}S_{CP}-S_{AP}I_{BD}C_{CP}-I_{AC}S_{PB}C_{PD}\right]=0. \qquad \text{(A.1)}
\end{aligned}
$$

Essentially, this condition is a necessary and sufficient condition for the body to be a Lagrangian submanifold. Conditions sufficient for its fulfillment are the vanishing of each bracket separately, namely

$$
\begin{aligned}
&S_{AB}I_{CD}-I_{AB}S_{CD}=0,\quad I_{AB}C_{CD}-C_{AB}I_{CD}=0,\\
&\frac{\partial\alpha_0}{\partial S_{CD}}I_{AB}-\frac{\partial\alpha_0}{\partial S_{AB}}I_{CD}=0,\quad I_{AB}C_{CP}C_{PD}-C_{AN}C_{NB}I_{CD}=0,\\
&\frac{\partial\alpha_0}{\partial S_{CD}}I_{AB}C_{PD}+\frac{\partial\alpha_0}{\partial S_{PD}}I_{AB}C_{CP}-C_{NB}\frac{\partial\alpha_0}{\partial S_{AN}}I_{CD}-C_{AN}\frac{\partial\alpha_0}{\partial S_{NB}}I_{CD}=0,\\
&I_{AB}S_{CP}C_{PD}-C_{AN}S_{NB}I_{CD}+I_{AB}C_{CP}S_{PD}-S_{AN}C_{NB}I_{CD}=0,\\
&\frac{\partial\alpha_0}{\partial S_{PD}}I_{AB}S_{CP}+\frac{\partial\alpha_0}{\partial S_{CP}}-S_{AN}\frac{\partial\alpha_0}{\partial S_{NB}}I_{CD}-S_{NB}\frac{\partial\alpha_0}{\partial S_{AN}}I_{CD}=0,\\
&S_{AB}C_{CD}-C_{AB}S_{CD}=0,\quad \frac{\partial\alpha_2}{\partial S_{AB}}-\frac{\partial\alpha_1}{\partial S_{AB}}S_{CD}=0,\\
&S_{AB}C_{CP}C_{PD}-C_{AN}C_{NB}S_{CD}=0,\\
&\frac{\partial\alpha_1}{\partial S_{CP}}S_{AB}C_{PD}+\frac{\partial\alpha_1}{\partial S_{PD}}-C_{NB}\frac{\partial\alpha_1}{\partial S_{AN}}S_{CD}-C_{AN}\frac{\partial\alpha_1}{\partial S_{NB}}S_{CD}=0,\\
&S_{AB}S_{CP}C_{PD}+S_{AB}C_{CP}S_{PD}-S_{AN}C_{NB}S_{CD}-C_{AN}S_{NB}S_{CD}=0,\\
&\frac{\partial\alpha_1}{\partial S_{PD}}S_{AB}S_{CP}+\frac{\partial\alpha_1}{\partial S_{CP}}S_{AB}S_{PD}-S_{AN}\frac{\partial\alpha_1}{\partial S_{NB}}S_{CD}-S_{NB}\frac{\partial\alpha_1}{\partial S_{AN}}S_{CD}=0,\\
&\frac{\partial\alpha_0}{\partial C_{AB}}I_{CD}-\frac{\partial\alpha_0}{\partial C_{CD}}I_{AB}=0,\quad \frac{\partial\alpha_0}{\partial C_{AB}}C_{CD}-\frac{\partial\alpha_0}{\partial C_{CD}}C_{AB}=0,
\end{aligned}
$$

$$\frac{\partial \alpha_4}{\partial C_{AB}} C_{CP} C_{PD} - \frac{\partial \alpha_4}{\partial C_{CD}} C_{AN} C_{NB} = 0,$$
$$\frac{\partial \alpha_5}{\partial C_{AB}} S_{CP} C_{PD} + \frac{\partial \alpha_5}{\partial C_{AB}} C_{CP} C_{PD} - \frac{\partial \alpha_5}{\partial C_{CD}} S_{AN} C_{NB} - \frac{\partial \alpha_5}{\partial C_{CD}} C_{AN} S_{NB} = 0,$$
$$S_{AN} S_{NB} I_{CD} - I_{AB} S_{CP} S_{PD} = 0, \quad S_{AN} S_{NB} C_{CD} - C_{AB} S_{CP} S_{PD} = 0,$$
$$\frac{\partial \alpha_3}{\partial S_{CD}} S_{AN} S_{NB} - \frac{\partial \alpha_3}{\partial S_{AB}} S_{CP} S_{PD} = 0,$$
$$S_{AN} S_{NB} C_{CP} C_{PD} - C_{AN} C_{NB} S_{CP} S_{PD} = 0,$$
$$\frac{\partial \alpha_3}{\partial S_{CP}} S_{AN} S_{NB} C_{PD} + \frac{\partial \alpha_3}{\partial S_{PD}} S_{AN} S_{NB} C_{CP} - C_{NB} \frac{\partial \alpha_3}{\partial S_{AN}} S_{CP} S_{PD} - C_{AN} \frac{\partial \alpha_3}{\partial S_{NB}} S_{CP} S_{PD} = 0,$$
$$S_{AN} S_{NB} S_{CP} C_{PD} + S_{AN} S_{NB} C_{CP} S_{PD} - S_{AN} C_{NB} S_{CP} S_{PD} - C_{AN} S_{NB} S_{CP} S_{PD} = 0,$$
$$\frac{\partial \alpha_3}{\partial S_{PD}} S_{AN} S_{NB} S_{CP} + \frac{\partial \alpha_3}{\partial S_{CP}} S_{AN} S_{NB} S_{PD} - S_{AN} \frac{\partial \alpha_3}{\partial S_{NB}} S_{CP} S_{PD} - S_{NB} \frac{\partial \alpha_3}{\partial S_{AN}} S_{CP} S_{PD} = 0,$$
$$S_{NB} \frac{\partial \alpha_0}{\partial C_{AN}} I_{CD} + S_{AN} \frac{\partial \alpha_0}{\partial C_{NB}} I_{CD} - \frac{\partial \alpha_0}{\partial C_{CP}} I_{AB} S_{PD} - \frac{\partial \alpha_0}{\partial C_{PD}} I_{AB} S_{CP} = 0,$$
$$S_{NB} \frac{\partial \alpha_0}{\partial C_{AN}} C_{CD} + S_{AN} \frac{\partial \alpha_0}{\partial C_{NB}} C_{CD} - \frac{\partial \alpha_0}{\partial C_{CP}} C_{AB} S_{PD} - \frac{\partial \alpha_0}{\partial C_{PD}} C_{AB} S_{CP} = 0,$$
$$S_{NB} \frac{\partial \alpha_4}{\partial C_{AN}} C_{CP} C_{PD} + S_{AN} \frac{\partial \alpha_4}{\partial C_{NB}} C_{CP} C_{PD} - \frac{\partial \alpha_4}{\partial C_{CP}} C_{AN} C_{NB} S_{PD} - \frac{\partial \alpha_4}{\partial C_{PD}} C_{AN} C_{NB} S_{CP} = 0,$$
$$I_{AC} S_{PB} C_{PD} - I_{AC} C_{PB} S_{PD} = 0, \; S_{AP} I_{BD} C_{CP} - C_{AP} S_{CP} I_{BD} = 0,$$
$$S_{NB} \frac{\partial \alpha_5}{\partial C_{AN}} S_{CP} C_{PD} + S_{NB} \frac{\partial \alpha_5}{\partial C_{AN}} C_{CP} S_{PD} + S_{AN} \frac{\partial \alpha_5}{\partial C_{NB}} S_{CP} C_{PD} + S_{AN} \frac{\partial \alpha_5}{\partial C_{NB}} C_{CP} S_{PD}$$
$$- \frac{\partial \alpha_5}{\partial C_{CP}} S_{AN} C_{NB} S_{PD} - \frac{\partial \alpha_5}{\partial C_{PD}} S_{AN} C_{NB} S_{CP} - \frac{\partial \alpha_5}{\partial C_{CP}} C_{AN} S_{NB} S_{PD} - \frac{\partial \alpha_5}{\partial C_{PD}} C_{AN} S_{NB} S_{CP} = 0,$$
$$S_{AN} C_{NB} I_{CD} + C_{AN} S_{NB} I_{CD} - I_{AB} S_{CP} C_{PD} - I_{AB} C_{CP} S_{PD} = 0,$$
$$S_{AN} C_{NB} C_{CD} + C_{AN} S_{NB} C_{CD} - C_{AB} S_{CP} C_{PD} - C_{AB} C_{CP} S_{PD} = 0,$$
$$\frac{\alpha_5}{\partial S_{CD}} S_{AN} C_{NB} + \frac{\partial \alpha_5}{\partial S_{CD}} C_{AN} S_{NB} - \frac{\partial \alpha_5}{\partial S_{AB}} S_{CP} C_{PD} - \frac{\partial \alpha_5}{\partial S_{AB}} C_{CP} S_{PD} = 0,$$
$$S_{AN} C_{NB} C_{CP} C_{PD} + C_{AN} S_{NB} C_{CP} C_{PD} - C_{AN} C_{NB} S_{CP} C_{PD} - C_{AN} C_{NB} C_{CP} S_{PD} = 0,$$
$$\frac{\partial \alpha_5}{\partial S_{CP}} S_{AN} C_{NB} C_{PD} + \frac{\partial \alpha_5}{\partial S_{PD}} S_{AN} C_{NB} C_{CP} + \frac{\partial \alpha_5}{\partial S_{CP}} C_{AN} S_{NB} C_{PD} + \frac{\partial \alpha_5}{\partial S_{PD}} C_{AN} S_{NB} C_{CD}$$
$$+ - C_{NB} \frac{\partial \alpha_5}{\partial S_{AN}} S_{CP} C_{PD} - C_{NB} \frac{\partial \alpha_5}{\partial S_{AN}} C_{CP} S_{PD} - C_{AN} \frac{\partial \alpha_5}{\partial S_{NB}} S_{CP} C_{PD} - C_{AN} \frac{\partial \alpha_5}{\partial S_{NB}} C_{CP} S_{PD} = 0,$$
$$\frac{\partial \alpha_5}{\partial S_{PD}} S_{AN} C_{NB} S_{CP} + \frac{\partial \alpha_5}{\partial S_{CP}} S_{AN} C_{NB} S_{PD} + \frac{\partial \alpha_5}{\partial S_{PD}} C_{AN} S_{NB} S_{CP} + \frac{\partial \alpha_5}{\partial S_{CP}} C_{AN} S_{NB} S_{PD}$$
$$- S_{AN} \frac{\partial \alpha_5}{\partial S_{NB}} S_{CP} C_{PD} - S_{AN} \frac{\partial \alpha_5}{\partial S_{NB}} C_{CP} S_{PD} - S_{NB} \frac{\partial \alpha_5}{\partial S_{AN}} S_{CP} C_{PD} - S_{NB} \frac{\partial \alpha_5}{\partial S_{AN}} C_{CP} S_{PD} = 0,$$
$$C_{NB} \frac{\partial \alpha_0}{\partial C_{AN}} I_{CD} + C_{AN} \frac{\partial \alpha_0}{\partial C_{NB}} I_{CD} - \frac{\partial \alpha_0}{\partial C_{CP}} I_{AB} C_{PD} - \frac{\partial \alpha_0}{\partial C_{PD}} I_{AB} C_{CP} = 0,$$
$$C_{NB} \frac{\partial \alpha_2}{\partial C_{AN}} C_{CD} + C_{AN} \frac{\partial \alpha_2}{\partial C_{NB}} C_{CD} - \frac{\partial \alpha_2}{\partial C_{CP}} C_{AB} C_{PD} - \frac{\partial \alpha_2}{\partial C_{PD}} C_{AB} C_{CP} = 0,$$
$$C_{NB} \frac{\partial \alpha_4}{\partial C_{AN}} C_{CP} C_{PD} + C_{AN} \frac{\partial \alpha_4}{\partial C_{NB}} C_{CP} C_{PD} - \frac{\partial \alpha_4}{\partial C_{CP}} C_{AN} C_{NB} C_{PD} - \frac{\partial \alpha_4}{\partial C_{CP}} C_{AN} C_{NB} C_{CP} = 0,$$
$$C_{NB} \frac{\partial \alpha_5}{\partial C_{AN}} S_{CP} C_{PD} + C_{NB} \frac{\partial \alpha_5}{\partial C_{AN}} C_{CP} S_{PD} + C_{AN} \frac{\partial \alpha_5}{\partial C_{NB}} S_{CP} C_{PD} + C_{AN} \frac{\partial \alpha_5}{\partial C_{NB}} C_{CP} S_{PD}$$

$$-\frac{\partial \alpha_5}{\partial C_{CP}} S_{AN} C_{NB} C_{PD} - \frac{\partial \alpha_5}{\partial C_{PD}} S_{AN} C_{NB} C_{CP} - \frac{\partial \alpha_5}{\partial C_{CP}} C_{AN} C_{NB} C_{CP} - \frac{\partial \alpha_5}{\partial C_{PD}} C_{AN} S_{NB} C_{CP} = 0,$$
$$I_{AC} C_{PB} S_{PD} + C_{AP} I_{BD} S_{CP} - S_{AP} I_{BD} C_{CP} - I_{AC} S_{PB} C_{PD} = 0. \quad \text{(A.2)}$$

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