Optim Lett (2015) 9:513–521
DOI 10.1007/s11590-014-0767-1

ORIGINAL PAPER

Boundary of subdifferentials and calmness moduli in linear semi-infinite optimization

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Received: 23 December 2013 / Accepted: 26 June 2014 / Published online: 23 July 2014

Abstract This paper was originally motivated by the problem of providing a point-based formula (only involving the nominal data, and not data in a neighborhood) for estimating the calmness modulus of the optimal set mapping in linear semi-infinite optimization under perturbations of all coefficients. With this aim in mind, the paper establishes as a key tool a basic result on finite-valued convex functions in the n-dimensional Euclidean space. Specifically, this result provides an upper limit characterization of the boundary of the subdifferential of such a convex function. When applied to the supremum function associated with our constraint system, this characterization allows us to derive an upper estimate for the aimed calmness modulus in linear semi-infinite optimization under the uniqueness of nominal optimal solution.

Keywords Variational analysis · Calmness · Semi-infinite programming · Linear programming

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1 Introduction

We consider the parameterized family of linear optimization problems

$$\begin{aligned} P(c,a,b): \text{minimize} \;\; & c'x \\ \text{subject to} \;\; & a_t'x \leq b_t, \;\; t \in T, \end{aligned} \tag{1}$$

where $c, x \in \mathbb{R}^n$ are regarded as column-vectors, y' denotes the transpose of $y \in \mathbb{R}^n$, T is a compact Hausdorff space and the functions $t \mapsto a_t \in \mathbb{R}^n$ and $t \mapsto b_t \in \mathbb{R}$ are continuous on T. In our model, the parameter to be perturbed is $(c, a, b) \in \mathbb{R}^n \times C\left(T, \mathbb{R}^{n+1}\right)$, where $(a, b) \equiv (a_t, b_t)_{t \in T}$. In other words, we are working in the context of linear semi-infinite optimization problems under perturbations of *all data*.

The parameter space $\mathbb{R}^n \times C\left(T, \mathbb{R}^{n+1}\right)$ is considered to be endowed with the uniform convergence topology through the norm

$$\|(c,a,b)\| := \max\left\{\|c\|_*, \|(a,b)\|_\infty\right\}, \tag{2}$$

where $\mathbb{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, with *dual norm*, $\|\cdot\|_*$, given by $\|u\|_* = \max_{\|x\| \leq 1} \left|u'x\right|$, and $\|(a,b)\|_\infty := \max_{t \in T} \|(a_t, b_t)\|$, where

$$\|(a_t, b_t)\| := \max\left\{\|a_t\|_*, |b_t|\right\}. \tag{3}$$

Associated with the parameterized problem (1) we consider the *feasible set mapping*, $\mathcal{F} : C\left(T, \mathbb{R}^{n+1}\right) \rightrightarrows \mathbb{R}^n$, the *optimal value function*, $\vartheta : \mathbb{R}^n \times C\left(T, \mathbb{R}^{n+1}\right) \to [-\infty, +\infty]$, and the *optimal* (*solution*) *set mapping* (also called *argmin mapping*), $\mathcal{S} : \mathbb{R}^n \times C\left(T, \mathbb{R}^{n+1}\right) \rightrightarrows \mathbb{R}^n$, which are given by:

$$\begin{aligned} \mathcal{F}(a,b) &:= \left\{x \in \mathbb{R}^n \mid a_t'x \leq b_t, \;\; t \in T\right\}, \\ \vartheta(c,a,b) &:= \inf\left\{c'x \mid x \in \mathcal{F}(a,b)\right\} \;\; (\text{with } \inf \emptyset := +\infty), \\ \mathcal{S}(c,a,b) &:= \left\{x \in \mathcal{F}(a,b) \mid c'x = \vartheta(c,a,b)\right\}. \end{aligned}$$

At this moment, we declare the main contributions of the present paper, which has [3] as an immediate antecedent in the different context of *canonical perturbations* (i.e., only perturbing c and b, and keeping a fixed at its nominal value, denoted by $\overline{a}$). We underline Theorem 4.1 (see also Corollary 4.1), which provides an upper bound on the calmness modulus of $\mathcal{S}$ at a given $\left(\overline{c}, \overline{a}, \overline{b}\right)$, assuming that $\mathcal{S}\left(\overline{c}, \overline{a}, \overline{b}\right)$ is a singleton $\{\overline{x}\}$. The main tool in order to get this result is given in Theorem 3.1, which is of interest by itself. This result characterizes the boundary of the subdifferential, at a point $\overline{x}$, of a convex function defined on $\mathbb{R}^n$, in terms of the upper limit of subdifferentials around $\overline{x}$, and, as far as we know and in spite of being so basic, it seems to be new in the literature of convex analysis.

Let us recall that a mapping $\mathcal{M} : Y \rightrightarrows X$ between metric spaces (with both distances denoted by d) is said to be *calm* at $(\overline{y}, \overline{x}) \in \text{gph}\mathcal{M}$ (the graph of $\mathcal{M}$) if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d(y, \overline{y}) \tag{4}$$

whenever $x \in \mathcal{M}(y) \cap U$ and $y \in V$; where, as usual, $d(x, \Omega)$ is defined as $\inf\{d(x,z) \mid z \in \Omega\}$ for $\Omega \subset \mathbb{R}^n$, with $d(x, \emptyset) := +\infty$. Equivalently, the calmness property can be written in terms of the existence of a (possibly smaller) neighborhood U of $\overline{x}$ and $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d\left(\overline{y}, \mathcal{M}^{-1}(x)\right), \quad \text{for all} \quad x \in U, \tag{5}$$

where $\mathcal{M}^{-1}(x) := \{y \in Y \mid x \in \mathcal{M}(y)\}$. In these terms, the calmness of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ turns out to be equivalent to the so-called *metric subregularity* of $\mathcal{M}^{-1}$ at $(\overline{x}, \overline{y})$.

The infimum of those $\kappa \geq 0$ for which (4)—or (5)—holds (for some associated U and V) is called the *calmness modulus* of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ and denoted by $\operatorname{clm}\mathcal{M}(\overline{y}, \overline{x})$. The case when $\mathcal{M}$ is not calm at $(\overline{y}, \overline{x})$ corresponds to $\operatorname{clm}\mathcal{M}(\overline{y}, \overline{x}) = +\infty$.

Calmness property is weaker than the Aubin property (also called pseudo-Lipschitz or Lipschitz-like), which holds at $(\overline{y}, \overline{x})$ when (4)—or (5)—are valid when replacing $\overline{y}$ with an arbitrary $\widetilde{y}$ in some neighborhood V of $\overline{y}$. The reader is addressed to the monographs [6,12,16,18] for details and references about calmness and Aubin properties. The existing relationship between the calmness property and *local error bounds* is well known (see, e.g., [1,15]). As far as calmness plays an important role in relation to issues from optimization (theory and algorithms), one can find in the literature deep contributions to the analysis of this property in different linear and nonlinear frameworks, mainly devoted to the calmness of feasible set mappings under right-hand-side perturbations; see, e.g., [7,10,13,14]. Subdifferential approaches to calmness/local error bounds can be found in [1,9,11,15]. See also [2] in relation to the so-called *isolated calmness* (also called calmness on selections) of the argmin mapping $\mathcal{S}$ in a convex semi-infinite framework under canonical perturbations.

The structure of the paper is as follows: Sect. 2 gathers the necessary notation and preliminary results. Sect. 3 provides the announced characterization on the boundary of the subdifferential of convex functions. Sect. 4 establishes a point-based expression for the calmness modulus of the feasible set mapping when the nominal feasible set is a singleton. This result is particularly relevant when applied to the (sub)level set mapping $\mathcal{L}$ associated with an optimization problem (1) with a unique solution $\overline{x}$. An upper bound on the calmness modulus of the argmin mapping $\mathcal{S}$, assuming again the uniqueness of nominal optimal solution, is also provided.

2 Preliminaries

In this section we introduce some additional notation and preliminary results which are needed later on. Given $X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $\operatorname{conv}X$ and $\operatorname{cone}X$ the *convex hull* and the *conical convex hull* of X, respectively. It is assumed that the $\operatorname{cone}X$ always contains the zero-vector 0_k, in particular $\operatorname{cone}(\emptyset) = \{0_k\}$. If X is a subset of any topological space, $\operatorname{int}X$, $\operatorname{cl}X$ and $\operatorname{bd}X$ stand, respectively, for the interior, the closure and the boundary of X.

Throughout the paper, we appeal to the *set of active indices* at $x \in \mathcal{F}(a,b)$, $T_{a,b}(x)$, defined as

$$T_{a,b}(x) := \left\{t \in T \mid a_t' x = b_t\right\}.$$

Recall that the *Slater constraint qualification* (SCQ) holds at parameter $(a, b) \in C\left(T, \mathbb{R}^{n+1}\right)$ if there exists $\widehat{x} \in \mathbb{R}^n$ (called a Slater point) such that $a_t'\widehat{x} < b_t$ for all $t \in T$.

In the paper we consider the supremum function associated with a fixed (nominal) element $\left(\overline{a}, \overline{b}\right) \in C\left(T, \mathbb{R}^{n+1}\right)$, $\overline{s} : \mathbb{R}^n \rightrightarrows \mathbb{R}$, given by

$$\overline{s}\,(x) := \max\left\{\overline{a}_t'x - \overline{b}_t,\ t \in T\right\}. \tag{6}$$

If ∂ denotes the usual subdifferential in convex analysis, we have, for $x \in \mathbb{R}^n$,

$$\partial\overline{s}\,(x) = \operatorname{conv}\left\{\overline{a}_t \mid \overline{a}_t'x - \overline{b}_t = \overline{s}\,(x)\,,\ t \in T\right\}.$$

(This result follows immediately from the Ioffe–Tikhomirov theorem; see, for instance, [19, Theorem 2.4.18] and the classical Mazur's theorem). In particular, if $\overline{x} \in \mathcal{F}\left(\overline{a}, \overline{b}\right)$ is not a Slater point (i.e., if $\overline{s}\,(\overline{x}) = 0$),

$$\partial\overline{s}\,(\overline{x}) = \operatorname{conv}\left\{\overline{a}_t \mid t \in T_{\overline{a},\overline{b}}\,(\overline{x})\right\}. \tag{7}$$

The following proposition comes straightforwardly from [5, Theorem 5], taking [15, Theorem 1] into account.

Proposition 2.1 *Let* $(\left(\overline{a}, \overline{b}\right), \overline{x}) \in \operatorname{gph}\mathcal{F}$. *Then*

$$\operatorname{clm}\mathcal{F}(\left(\overline{a}, \overline{b}\right), \overline{x}) = \frac{\|\overline{x}\| + 1}{\liminf\limits_{x \to \overline{x},\ \overline{s}(x) > 0} d_*\left(0_n, \partial\overline{s}\,(x)\right)}$$

(*with the convention* $1/0 = +\infty$), *where* d_* *stands for the distance in* $\mathbb{R}^n$ *associated with* $\|\cdot\|_*$.

3 On the boundary of the subdifferential of convex functions

The next two results, Proposition 3.1 and Corollary 3.1, are devoted to prove Theorem 3.1. As usual, f^* stands for the Fenchel–Legendre conjugate of f, and 'dom' means effective domain.

Proposition 3.1 *Let* $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a convex function such that* $g\,(0_n) = 0$ *and* $g\,(x) > 0$ *for all* $x \in \mathbb{R}^n \backslash \{0_n\}$. *For each* $v \in \mathbb{R}^n$, *let* $\mathcal{M}\,(v) := \arg\min\limits_{x \in \mathbb{R}^n}\left(g\,(x) - v'x\right)$. *Then there exists* $\delta > 0$ *such that* $\mathcal{M}\,(v) \neq \emptyset$ *whenever* $\|v\|_* < \delta$.

Proof As a consequence of [17, Theorem 27.1 (d)], we have that $0_n \in \operatorname{int}\operatorname{dom} g^*$, and then, $0_n \in \operatorname{int}\operatorname{dom}\partial g^*$. Then, just observe that $\mathcal{M}\,(v) = \partial g^*\,(v)$ for all $v \in \mathbb{R}^n$, according to [17, Theorem 23.5 (b)⇔(a*)]. □

Corollary 3.1 *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a convex function,* $\overline{x} \in \mathbb{R}^n$, *and* $\overline{u} \in \partial f\,(\overline{x})$. *For each* $u \in \mathbb{R}^n$ *set* $\mathcal{A}\,(u) := \arg\min_{x \in \mathbb{R}^n}\left(f\,(x) - u'x\right)$ *and assume* $\mathcal{A}\,(\overline{u}) = \{\overline{x}\}$. *Then there exists* $\delta > 0$ *such that* $\mathcal{A}\,(u) \neq \emptyset$ *whenever* $\|u - \overline{u}\|_* < \delta$.

Proof Just apply the previous proposition to $g(x) := f(\overline{x}+x) - f(\overline{x}) - \langle \overline{u}, x\rangle$ and note that the associated $\mathcal{M}$ is given by $\mathcal{M}(v) = \mathcal{A}(\overline{u}+v)$. □

Theorem 3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex function and let $\overline{x} \in \mathbb{R}^n$. Then*

$$\mathrm{bd}\partial f(\overline{x}) = \limsup_{x\to\overline{x},\ x\neq\overline{x}} \partial f(x).$$

Proof First, let us see the inclusion '⊂'. Set, as in the previous corollary, $\mathcal{A}(u) := \arg\min_{x\in\mathbb{R}^n}\left(f(x) - u'x\right)$ for each $u \in \mathbb{R}^n$, and pick an arbitrary $\overline{u} \in \mathrm{bd}\partial f(\overline{x})$ (closed), which entails $\overline{x} \in \mathcal{A}(\overline{u})$. The case $\mathcal{A}(\overline{u}) \neq \{\overline{x}\}$ is trivial, since in such a case $\overline{u} \in \partial f(x)$ for all $x \in \mathcal{A}(\overline{u})\setminus\{\overline{x}\}$, and we may approach $\overline{x}$ within the segment linking any $x \in \mathcal{A}(\overline{u})\setminus\{\overline{x}\}$ and $\overline{x}$. Let us assume the nontrivial case $\mathcal{A}(\overline{u}) = \{\overline{x}\}$. As a consequence of [17, Theorem 24.5], taking the equality $\mathcal{A} = \partial f^*$ into account, $\mathcal{A}$ turns out to be (Berge-) upper semicontinuous at $\overline{u}$; in other words, for each $\varepsilon > 0$ there exists $\delta > 0$ sucht that $\|u - \overline{u}\|_* < \delta$ entails $\mathcal{A}(u) \subset B(\overline{x}, \varepsilon)$. According to Corollary 3.1 we may assume δ small enough to ensure $\mathcal{A}(u) \neq \emptyset$, which entails that $\mathcal{A}$ is also (Berge-) lower semicontinuous at $\overline{u}$. Since $\overline{u} \in \mathrm{bd}\partial f(\overline{x})$, we may approach $\overline{u}$ by means of a sequence $\{u^r\}_{r\in\mathbb{N}}$ with $u^r \notin \partial f(\overline{x})$ for all r. Assuming, without loss of generality, $\mathcal{A}(u^r) \neq \emptyset$ for all $r \in \mathbb{N}$, and picking $x^r \in \mathcal{A}(u^r)$ for each r, we obtain $x^r \to \overline{x}$. Finally observe that $x^r \in \mathcal{A}(u^r) \Leftrightarrow u^r \in \partial f(x^r)$, and, since $u^r \notin \partial f(\overline{x})$, we have $\overline{x} \notin \mathcal{A}(u^r)$, hence $x^r \neq \overline{x}$. Accordingly, we have proved $\overline{u} \in \limsup_{x\to\overline{x},\ x\neq\overline{x}} \partial f(x)$.

Now let us see the converse inclusion: '⊃'. Let $x^r \to \overline{x}$ with $x^r \neq \overline{x}$ for all r, and suposse that $u^r \in \partial f(x^r)$ for all r and $\{u^r\}_{r\in\mathbb{N}}$ converges to some $\overline{u} \in \mathbb{R}^n$. Appealing again to [17, Theorem 24.5], mapping ∂f is upper semicontinuous at $\overline{x}$ (see, e.g., [17, Corollary 24.5.1]), so that $\overline{u} \in \partial f(\overline{x})$. Let us see, reasoning by contradiction, that $\overline{u} \notin \mathrm{int}\partial f(\overline{x})$. On the contrary, suppose that the closed Euclidean ball $\overline{B}_2(u^r, \varepsilon)$ is contained in $\partial f(\overline{x})$ for some $\varepsilon > 0$ and some $r \in \mathbb{N}$ (indeed for all r large enough), which in particular entails $u^r + \varepsilon\frac{x^r - \overline{x}}{\|x^r - \overline{x}\|_2} \in \partial f(\overline{x})$, and hence

$$\begin{pmatrix} u^r + \varepsilon\frac{x^r - \overline{x}}{\|x^r - \overline{x}\|_2} \\ -1 \end{pmatrix}' \begin{pmatrix} x^r - \overline{x} \\ f(x^r) - f(\overline{x}) \end{pmatrix} \leq 0. \tag{8}$$

On the other hand, $u^r \in \partial f(x^r)$ entails

$$\begin{pmatrix} u^r \\ -1 \end{pmatrix}' \begin{pmatrix} \overline{x} - x^r \\ f(\overline{x}) - f(x^r) \end{pmatrix} \leq 0,$$

and adding up this last inequality with (8), and recalling $x^r \neq \overline{x}$, we obtain the contradiction $\varepsilon\|x^r - \overline{x}\|_2 \leq 0$. □

Remark 3.1 An alternative proof of the direct inclusion in the previous theorem, given $\overline{u} \in \mathrm{bd}\partial f(\overline{x})$ and assuming the nontrivial case $\mathcal{A}(\overline{u}) = \{\overline{x}\}$, is the following: Since $\partial f^*(\overline{u}) = \mathcal{A}(\overline{u}) = \{\overline{x}\}$ and f^* is continuous at $\overline{u}$, f^* is Gâteaux-differentiable at $\overline{u}$, and so, a fortiori, it is Fréchet differentiable (due to the finite-dimensional setting) at $\overline{u}$.Take a sequence $\{u^r\}_{r\in\mathbb{N}}$ with $u^r \notin \partial f(\overline{x})$ for all r, converging to $\overline{u}$. Because $\overline{u} \in \operatorname{int}(\operatorname{dom} f^*)$, there exists $x^r \in \partial f^*(u^r)$ different of $\overline{x}$ which, thanks to the differentiability property of f^*, converges to $\overline{x}$. Accordingly, we have proved $\overline{u} \in \limsup_{x\to\overline{x},\ x\neq\overline{x}} \partial f(x)$.

The next result is a direct consequence of the previous theorem.

Corollary 3.2 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex function and let $\overline{x} \in \mathbb{R}^n$. Then*

$$d\left(0_n, \operatorname{bd}\partial f(\overline{x})\right) = \liminf_{x\to\overline{x},\ x\neq\overline{x}} d\left(0_n, \partial f(x)\right).$$

Remark 3.2 From Theorem 3.1 we can immediately derive the classical result on continuous differentiability of convex functions given in [17, Corollary 25.5.1].

4 Calmness moduli of sub-level sets and argmin mappings

To start with, this section provides a point-based expression for the calmness modulus of $\mathcal{F}$ when the nominal feasible set is a singleton. This seems very restrictive, but turns out to be relevant when applied to the *lower level set mapping associated with* (1), $\mathcal{L} : \mathbb{R}^{n+1} \times C\left(T, \mathbb{R}^{n+1}\right) \rightrightarrows \mathbb{R}^n$, which is given by

$$\mathcal{L}(c, \alpha, a, b) := \left\{x \in \mathbb{R}^n \mid c'x \leq \alpha;\ a_t'x \leq b_t,\ \ t \in T\right\}, \tag{9}$$

for $(c, \alpha, a, b) \in \mathbb{R}^{n+1} \times C\left(T, \mathbb{R}^{n+1}\right)$. The last space is endowed with the norm

$$\|(c, \alpha, a, b)\| := \max\left\{\|(c, \alpha)\|, \|(a, b)\|_\infty\right\}. \tag{10}$$

Proposition 4.1 *Let $\left(\overline{a}, \overline{b}\right) \in C\left(T, \mathbb{R}^{n+1}\right)$ and assume $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$. Then*

$$\operatorname{clm}\mathcal{F}\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right) = \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t, t \in T_{\overline{a},\overline{b}}(\overline{x})\right\}\right)}.$$

Proof The assumption $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$ entails that $\overline{s}(x) > 0$ for all $x \neq \overline{x}$. Consequently, by applying Proposition 2.1, Corollary 3.2, and taking (7) into account, we obtain

$$\operatorname{clm}\mathcal{F}\left(\left(\overline{a},\overline{b}\right),\overline{x}\right)=\frac{\|\overline{x}\|+1}{\liminf\limits_{x\to\overline{x},\ x\neq\overline{x}} d_*\left(0_n,\partial\overline{s}\left(x\right)\right)}=\frac{\|\overline{x}\|+1}{d_*\left(0_n,\operatorname{bd}\partial\overline{s}\left(\overline{x}\right)\right)}$$
$$=\frac{\|\overline{x}\|+1}{d_*\left(0_n,\operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\, t\in T_{\overline{a},\overline{b}}\left(\overline{x}\right)\right\}\right)}.$$

□

The following Corollary comes directly from the previous proposition taking into account the fact that $\mathcal{L}\left(\overline{c},\overline{c}'\overline{x},\overline{a},\overline{b}\right)=\mathcal{S}\left(\overline{c},\overline{a},\overline{b}\right)$.

Corollary 4.1 *Let* $\left(\overline{c},\overline{a},\overline{b}\right)\in\mathbb{R}^n\times C\left(T,\mathbb{R}^{n+1}\right)$ *and assume* $\mathcal{S}\left(\overline{c},\overline{a},\overline{b}\right)=\{\overline{x}\}$. *Then*

$$\operatorname{clm}\mathcal{L}\left(\left(\overline{c},\overline{c}'\overline{x},\overline{a},\overline{b}\right),\overline{x}\right)=\frac{\|\overline{x}\|+1}{d_*\left(0_n,\operatorname{bd}\operatorname{conv}\{\overline{c};\ \overline{a}_t,\, t\in T_{\overline{a},\overline{b}}\left(\overline{x}\right)\}\right)}.$$

Theorem 4.1 provides the aforementioned upper bound for the calmness modulus of $\mathcal{S}$ in terms of the calmness modulus of ϑ. Since ϑ is a function (i.e., single-valued), we denote its calmness modulus at $\left(\overline{c},\overline{a},\overline{b}\right)$ by just $\operatorname{clm}\vartheta\left(\overline{c},\overline{a},\overline{b}\right)$. The next remark gathers some preliminary results about ϑ.

Remark 4.1 Under *SCQ* at $\left(\overline{a},\overline{b}\right)$ and the boundedness (and non-emptiness) of $\mathcal{S}\left(\overline{c},\overline{a},\overline{b}\right)$, ϑ turns out to be Lipschitz continuous at $\left(\overline{c},\overline{a},\overline{b}\right)$ (see [8, Theorem 10.1]). Under these hypotheses, [4, Theorem 4.3] provides an explicit expression of a Lipschitz constant L_0 for ϑ at $\left(\overline{c},\overline{a},\overline{b}\right)$, in terms of the distance to infeasibility of the nominal constraint system. Such an L_0 can be computed exclusively in terms of the nominal data $\left(\left(\overline{c},\overline{a},\overline{b}\right),\overline{x}\right)$. Obviously L_0 is an upper bound on $\operatorname{clm}\vartheta\left(\overline{c},\overline{a},\overline{b}\right)$.

Theorem 4.1 *Let* $\left(\overline{c},\overline{a},\overline{b}\right)\in\mathbb{R}^n\times C\left(T,\mathbb{R}^{n+1}\right)$ *and assume* $\mathcal{S}\left(\overline{c},\overline{a},\overline{b}\right)=\{\overline{x}\}$ *and that SCQ holds at* $\left(\overline{a},\overline{b}\right)$. *Then*

$$\operatorname{clm}\mathcal{S}\left(\left(\overline{c},\overline{a},\overline{b}\right),\overline{x}\right)\leq\max\left\{1,\operatorname{clm}\vartheta\left(\overline{c},\overline{a},\overline{b}\right)\right\}\operatorname{clm}\mathcal{L}\left(\left(\overline{c},\overline{c}'\overline{x},\overline{a},\overline{b}\right),\overline{x}\right)$$
$$=\frac{(\|\overline{x}\|+1)\max\left\{1,\operatorname{clm}\vartheta\left(\overline{c},\overline{a},\overline{b}\right)\right\}}{d_*(0_n,\operatorname{bd}\operatorname{conv}\{\overline{c};\ \overline{a}_t,\, t\in T_{\overline{a},\overline{b}}\left(\overline{x}\right)\})}. \tag{11}$$

Proof According to the previous corollary, take any $\kappa>\operatorname{clm}\mathcal{L}\left(\left(\overline{c},\overline{c}'\overline{x},\overline{a},\overline{b}\right),\overline{x}\right)$ and consider neighborhoods W of $\left(\overline{c},\overline{c}'\overline{x},\overline{a},\overline{b}\right)$ in $\mathbb{R}^{n+1}\times C\left(T,\mathbb{R}^{n+1}\right)$ and U of $\overline{x}$ in $\mathbb{R}^n$ such that

$$\|x-\overline{x}\|\leq\kappa\left\|(c,\alpha,a,b)-\left(\overline{c},\overline{c}'\overline{x},\overline{a},\overline{b}\right)\right\|$$

whenever $(c,\alpha,a,b)\in W$ and $x\in\mathcal{L}\left(c,\alpha,a,b\right)\cap U$.

Due to the continuity of ϑ at $\left(\overline{c},\overline{a},\overline{b}\right)$ under the current hypotheses, let us consider a neighborhood V of $\left(\overline{c},\overline{a},\overline{b}\right)$ in $\mathbb{R}^n\times C\left(T,\mathbb{R}^{n+1}\right)$ such that

$$\left(c,\vartheta\left(c,a,b\right),a,b\right)\in W$$

whenever $(c, a, b) \in V$. Now fix an arbitrarily small $\varepsilon > 0$ and assume that V is small enough to ensure

$$\left|\vartheta\left(c, a, b\right) - \overline{c}'\overline{x}\right| \leq \left(\text{clm}\vartheta\left(\overline{c}, \overline{a}, \overline{b}\right) + \varepsilon\right)\left\|(c, a, b) - \left(\overline{c}, \overline{a}, \overline{b}\right)\right\|$$

provided that $(c, a, b) \in V$. Then, for any $(c, a, b) \in V$ and any $x \in \mathcal{S}\left(c, a, b\right) \cap U$, taking into account the fact that

$$\mathcal{S}\left(c, a, b\right) = \mathcal{L}\left(c, \vartheta\left(c, a, b\right), a, b\right),$$

one has, recalling (2), (3), and (10),

$$\begin{aligned} \|x - \overline{x}\| &\leq \kappa \left\|\left(c, \vartheta\left(c, a, b\right), a, b\right) - \left(\overline{c}, \overline{c}'\overline{x}, \overline{a}, \overline{b}\right)\right\| \\ &= \kappa \max\left\{\left\|\left(c, a, b\right) - \left(\overline{c}, \overline{a}, \overline{b}\right)\right\|, \left|\vartheta\left(c, a, b\right) - \overline{c}'\overline{x}\right|\right\} \\ &\leq \kappa \max\{1, \text{clm}\vartheta\left(\overline{c}, \overline{a}, \overline{b}\right) + \varepsilon\}\left\|\left(c, a, b\right) - \left(\overline{c}, \overline{a}, \overline{b}\right)\right\|. \end{aligned}$$

Consequently, $\kappa \max\{1, \text{clm}\vartheta\left(\overline{c}, \overline{a}, \overline{b}\right) + \varepsilon\}$ turns out to be a calmness constant for $\mathcal{S}$ at $\left(\left(\overline{c}, \overline{a}, \overline{b}\right), \overline{x}\right)$. Due to the fact that κ may be chosen arbitrarily closed to $\text{clm}\mathcal{L}\left(\left(\overline{c}, \overline{c}'\overline{x}, \overline{a}, \overline{b}\right), \overline{x}\right)$ and $\varepsilon > 0$ was arbitrarily chosen, we conclude

$$\text{clm}\mathcal{S}\left(\left(\overline{c}, \overline{a}, \overline{b}\right), \overline{x}\right) \leq \max\left\{1, \text{clm}\vartheta\left(\overline{c}, \overline{a}, \overline{b}\right)\right\}\text{clm}\mathcal{L}\left(\left(\overline{c}, \overline{c}'\overline{x}, \overline{a}, \overline{b}\right), \overline{x}\right),$$

which entails the aimed result according to Corollary 4.1. □

Acknowledgments The authors would like to acknowledge Professor Marco A. López for his support and valuable hints and comments, specially in relation to Sect. 3. We would also like to thank the anonymous referee for his/her comments and suggestions about further connections between the calmness moduli of ϑ and $\mathcal{S}$, which will be helpful for future research. This research has been partially supported by Grant MTM2011-29064-C03-03 from MINECO, Spain. The research of the second author is also partially supported by Fondecyt Project No 1110019, ECOS-Conicyt project No C10E08, and Math-Amsud No. 13MATH-01 2013

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