Bubbling solutions for supercritical problems on manifolds

Juan Dávila a, Angela Pistoia b,∗, Giusi Vaira c

a Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile
b Dipartimento SBAI, Università di Roma “La Sapienza”, via Antonio Scarpa 16, 00161 Roma, Italy
c Dipartimento di Matematica “G. Castelnuovo”, Università di Roma “La Sapienza”, Piazzale A. Moro 1, 00161 Roma, Italy

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ABSTRACT

Let (M, g) be an n-dimensional compact Riemannian manifold without boundary and Γ be a non-degenerate closed geodesic of (M, g). We prove that the supercritical problem

\[-\Delta_g u + hu = u^{\frac{3n+3}{n-1}}\pm\epsilon, \quad u > 0, \text{ in } (M, g)\]

has a solution that concentrates along Γ as ε goes to zero, provided the function h and the sectional curvatures along Γ satisfy a suitable condition. A connection with the solution of a class of periodic Ordinary Differential Equations with singularity of attractive or repulsive type is established.

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RéSUMé

Soit (M, g) une variété riemannienne compacte sans bord, de dimension n, et Γ une géodésique fermée, non dégénérée de (M, g). On démontre que le problème elliptique supercritique

\[-\Delta_g u + hu = u^{\frac{3n+3}{n-1}}\pm\epsilon, \quad u > 0, \text{ dans } (M, g)\]

admet une solution qui se concentre le long de Γ lorsque le paramètre ε tend vers zéro, à condition que la fonction h et les courbures sectionnelles de M le long de Γ satisfont une certaine condition appropriée. On établit également un lien avec

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E-mail addresses: jdavila@dim.uchile.cl (J. Dávila), pistoia@dmmm.uniroma1.it (A. Pistoia), vaira@mat.uniroma1.it (G. Vaira).

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1. Introduction and statement of main results

We deal with the semilinear elliptic equation

$$-\Delta_g u + hu = u^{p-1}, \quad u > 0, \text{ in } (\mathcal{M}, g)$$ (1.1)

where $(\mathcal{M}, g)$ is an $n$-dimensional compact Riemannian manifold without boundary, $h$ is a $C^1$-real function on $\mathcal{M}$ such that $-\Delta_g + h$ is coercive and $p > 2$.

For any $p \in (2, 2^*_n)$, where $2^*_n := \frac{2n}{n-2}$ if $n \geq 3$ and $2^*_n := +\infty$ if $n = 2$, problem (1.1) has a solution, which can be found by minimization of

$$I_p(u) = \frac{\int_{\mathcal{M}} (|\nabla_g u|^2 + hu^2) d\sigma_g}{(\int_{\mathcal{M}} |u|^p d\sigma_g)^{2/p}}$$

over $H^1_g(\mathcal{M}) \setminus \{0\}$, using the compactness of the embedding $H^1_g(\mathcal{M}) \hookrightarrow L^p_g(\mathcal{M})$.

In the critical case, i.e. $p = 2^*_n$, the situation turns out to be more delicate. In particular, the existence of solutions is related to the position of the potential $h$ with respect to the geometric potential $h_g := \frac{m-2}{4(m-1)} R_g$, where $R_g$ is the scalar curvature of the manifold.

If $h \equiv h_g$, then problem (1.1) is referred to as the Yamabe problem [22] and it has always a solution. After Trudinger [20] discovered a gap in the argument in [22] and gave a proof under some conditions on $(\mathcal{M}, g)$, Aubin [2,3] showed that whenever $Q(\mathcal{M}, g) < Q(S^n, g_0)$, where $(S^n, g_0)$ is the standard sphere and

$$Q(\mathcal{M}, g) := \inf_{u \in H^1_g(\mathcal{M}) \setminus \{0\}} \frac{I_{2^*_n}(u)}{I_{2^*_n}(u)},$$

there is a solution to the problem, and proved that this holds if $n \geq 6$ and $(\mathcal{M}, g)$ is not locally conformally flat. Finally, Schoen [18] gave a proof in full generality using the Positive Mass Theorem [19].

When $h < h_g$ somewhere in $\mathcal{M}$, existence of a solution is guaranteed by a minimization argument, arguing as in Aubin [2,3]. The situation is extremely delicate when $h \geq h_g$ everywhere in $\mathcal{M}$, because blow-up phenomena can occur as pointed out by Druet in [9,10].

The supercritical case $p > 2^*_n$ is even more difficult to deal with. A first result in this direction is a perturbative result due to Micheletti, Pistoia and Vétois [15]. They consider the almost critical problem (1.1) when $p = 2^*_n \pm \epsilon$ with $\epsilon > 0$. If $p = 2^*_n - \epsilon$ the problem (1.1) is slightly subcritical and if $p = 2^*_n + \epsilon$ the problem (1.1) is slightly supercritical. They prove the following results:

**Theorem 1.1.** Assume $n \geq 6$ and $\xi_0 \in M$ is a non-degenerate critical point of $h - \frac{n-2}{4(n-1)} R_g$. Then

(i) if $h(\xi_0) > \frac{n-2}{4(n-1)} R_g(\xi_0)$ then the slightly subcritical problem (1.1) with $p = 2^*_n - 1 - \epsilon$, has a solutions $u_\epsilon$ which concentrates at $\xi_0$ as $\epsilon \to 0$,

(ii) if $h(\xi_0) < \frac{n-2}{4(n-1)} R_g(\xi_0)$ then the slightly supercritical problem (1.1) with $p = 2^*_n - 1 - \epsilon$, has a solutions $u_\epsilon$ which concentrates at $\xi_0$ as $\epsilon \to 0$.

Now, for any integer $0 \leq k \leq n - 3$ let $2^*_{n,k} = \frac{2(n-k)}{n-k-2}$ be the $(k+1)$-st critical exponent. We remark that $2^*_{n,k} = 2^*_{n-k,0}$ is nothing but the critical exponent for the Sobolev embedding $H^1_k(\mathcal{N}) \hookrightarrow L^p_k(\mathcal{N})$ in a
compact \((n-k)\)-dimensional Riemannian manifold \((\mathcal{N}, h)\). In particular, \(2_{n,0}^* = \frac{2n}{n-2}\) is the usual Sobolev critical exponent.

We can summarize the results proved by Micheletti, Pistoia and Vétois just saying that problem (1.1) when \(p \to 2_{n,0}^*\) (i.e. \(k = 0\)) has positive solutions blowing-up at points. Note that a point is a 0-dimensional manifold.

A natural question arises:

does problem (1.1) have solutions blowing-up at \(k\)-dimensional submanifolds when \(p \to 2_{n,k}^*\)?

In the present paper, we give a positive answer when \(k = 1\). More precisely, we prove that if \(p \to 2_{n,1}^*\) problem (1.1) has a solution which concentrates along a geodesic \(\Gamma\) of the manifold provided \(h\) satisfies a suitable condition. Let us state our main result.

We consider the problem (1.1) with \(p = 2_{n,1}^* \pm \epsilon\) and \(\epsilon > 0\), i.e.

\[-\Delta_g u + h u = u^{\frac{n+4}{n+2} \pm \epsilon}, \quad u > 0 \text{ in } (\mathcal{M}, g).\]

(1.2)

We will say that problem (1.2) is slightly 2\textit{nd-supercritical} if \(p = 2_{n,1}^* + \epsilon\) and it is slightly 2\textit{nd-subcritical} if \(p = 2_{n,1}^* - \epsilon\).

In order to state our main result, we need to introduce some geometric notation. Let \(\Gamma\) be a closed nontrivial simple geodesic on \(\mathcal{M}\). Given \(\xi \in \Gamma\) there is a natural splitting \(T_\xi \mathcal{M} = T_\xi \Gamma \oplus N_\xi \Gamma\) into the tangent and normal bundle over \(\Gamma\). It is useful to introduce a local system of coordinates near \(\Gamma\). Let \(\gamma : [0, 2\ell] \to \mathcal{M}\) be an arclength parametrization of \(\Gamma\), where \(2\ell\) is the length of \(\Gamma\). We denote by \(E_0\) a unit tangent vector to \(\Gamma\). In a neighborhood of a point \(\xi\) of \(\Gamma\) we give an orthonormal basis \(E_1, \ldots, E_N\) of \(N_\xi \Gamma\). We can assume that the \(E_i\)'s are parallel along \(\Gamma\), i.e. \(\nabla_{E_i} E_j = 0\) for any \(i = 1, \ldots, N\). The geodesic condition for \(\Gamma\) translates into the condition \(\nabla_{E_i} E_0 = 0\). Here \(\nabla\) is the connection associated with the metric \(g\). Moreover, the non-degeneracy of \(\Gamma\) is equivalent to say that the linear equation

\[J \phi := \nabla^2_{E_0} \phi + R(\phi, E_0) E_0 = 0\]

has only the trivial solution on all of \(\Gamma\).

(1.3)

Here \(J\) is the Jacobi operator on \(\Gamma\) corresponding to the second variation of the length functional on curves. For a generic metric \(g\) on \(\mathcal{M}\) it is well known that all closed geodesics are non-degenerate (see Anosov [1]). To parametrize a neighborhood of a point of \(\Gamma\) in \(\mathcal{M}\) we define the \textit{Fermi coordinates}

\[F(x_0, x_1, \ldots, x_N) = \exp_{\gamma(x_0)} \left( \sum_{i=1}^{N} x_i E_i(x_0) \right),\]

(1.4)

where \(\exp_{\gamma(x_0)}\) is the exponential map in \(\mathcal{M}\) through the point \(\gamma(x_0)\).

Let us introduce the function (see also (4.20))

\[\sigma(x_0) = h(x_0) - \frac{(n-3)}{4(n-2)} [R_g(x_0) - (n-1)Ric(\gamma(x_0), \gamma(x_0))],\]

(1.5)

where \(R_g\) is the scalar curvature and \(Ric\) denotes the Ricci tensor.

Let \(a_n := \frac{2(n-2)}{(n-3)(n+1)}\) and \(b_n := \frac{(n-3)^2(n-5)}{4(n+1)}\). We introduce the periodic ODE problem

\[\begin{aligned}
-\ddot{\mu} + a_n \sigma \mu - \frac{b_n}{\mu} &= 0 \quad \text{in } [0, 2\ell], \\
\mu &> 0 \quad \text{in } [0, 2\ell], \\
\mu(0) &= \mu(2\ell), \quad \dot{\mu}(0) = \ddot{\mu}(2\ell)
\end{aligned}\]

(1.6)
which has a *singularity of attractive type* at the origin and the periodic ODE problem

\[
\begin{cases}
-\ddot{\mu} + a_n \sigma \mu + \frac{b_n}{\mu} = 0 & \text{in } [0, 2\ell], \\
\mu > 0 & \text{in } [0, 2\ell], \\
\mu(0) = \mu(2\ell), & \dot{\mu}(0) = \dot{\mu}(2\ell)
\end{cases}
\]

(1.7)

which has a *singularity of repulsive type* at the origin.

Solvability of the slightly 2nd-subcritical problem is strictly related with solvability of (1.6) with *attractive singularity*, while solvability of the slightly 2nd-supercritical problem is strictly related with solvability of (1.7) with *repulsive singularity*. We remark that in the subcritical side the assumption \(\sigma(s) > 0\) for any \(s \in [0, \ell]\) is enough to find a solution to problem (1.6). In this case, using standard arguments, the solution is just a minimizer of the energy. The supercritical side turns out to be more difficult and the only existence result for problem (1.7) was obtained by del Pino, Manásevich and Montero in [5] when \(\sigma(s) < 0\) for any \(s \in [0, \ell]\) provided some extra non-resonance conditions are satisfied (see also Proposition 2.1).

As usual in this kind of problem, we also need to assume a gap condition of the form

\[
|\epsilon k^2 - \kappa^2| > \nu \sqrt{\epsilon}, \quad k = 1, 2, \ldots
\]

(1.8)

where \(\kappa > 0\) is given explicitly in Lemma 6.2 and \(\nu\) is positive.

Now we can state our main result.

**Theorem 1.2.** Let \(n \geq 8\). Let \(\Gamma\) be a simple closed, non-degenerate geodesic of \(M\) (see (1.3)).

(i) Assume the problem (1.6) has a non-degenerate positive solution \(\mu_0\). Then, for any \(\nu > 0\) there exists \(\epsilon_0 > 0\) such that for any \(\epsilon \in (0, \epsilon_0)\) which satisfies condition (1.8), the slightly 2nd-subcritical problem (1.2) with \(p = 2^*_n - 1 - \epsilon\) has a solution \(u_\epsilon\) that concentrates along \(\Gamma\) as \(\epsilon \to 0\).

(ii) Assume the problem (1.7) has a non-degenerate positive solution \(\mu_0\). Then, for any \(\nu > 0\) there exists \(\epsilon_0 > 0\) such that for any \(\epsilon \in (0, \epsilon_0)\) which satisfies condition (1.8), the slightly 2nd-supercritical problem (1.2) with \(p = 2^*_n - 1 + \epsilon\) has a solution \(u_\epsilon\) that concentrates along \(\Gamma\) as \(\epsilon \to 0\).

Moreover, the solution \(u_\epsilon\) can be described in Fermi coordinates as follows:

\[
u(x_0, x) = \mu_\epsilon^{-\frac{N-2}{2}} w(\mu_\epsilon^{-1}(x - d_\epsilon)) + o(1),
\]

where

\[
\mu_\epsilon(x_0) \sim \sqrt{\epsilon} \mu_0(x_0) \quad \text{and} \quad d_\epsilon(x_0) \sim \epsilon d_k(x_0), \quad k = 1, \ldots, N,
\]

and \(\mu_0\) solves either problem (1.6) in the slightly 2nd-subcritical case or problem (1.7) in the slightly 2nd-supercritical case, the \(d_j\)'s are smooth functions of \(x_0\) and \(w\) is the standard bubble

\[
w(y) = c_N \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}}}, \quad y \in \mathbb{R}^N, \quad c_N = C(N(N - 2))^{\frac{N-2}{2}}
\]

(1.9)

which is the radial solution of the critical problem \(\Delta w + w^p = 0\) in \(\mathbb{R}^N\), with \(N = n - 1\).

Since the existence of solutions to singular problems (1.6) or (1.7) plays a crucial role in the construction of the solution, in particular in the choice of the concentration parameter \(\mu_\epsilon\), it is important to point out that existence of solutions to problems (1.6) or (1.7) is strictly linked with the sign of the function \(\sigma\) defined in (1.5), as it is showed in the following theorem, whose proof is given in Section 2.
Theorem 1.3. If

\[ \min_{x_0 \in \mathbb{R}} \sigma(x_0) > 0, \]

then problem (1.6) has a non-degenerate solution.

If \( h^* \in C^2(M) \) is such that

\[ - \left( \frac{(k + 1)\pi}{2\ell} \right)^2 < \min_{x_0 \in \mathbb{R}} \sigma_{h^*}(x_0) \leq \max_{t \in \mathbb{R}} \sigma_{h^*}(x_0) < - \left( \frac{k\pi}{2\ell} \right)^2 < 0, \]

then for most functions \( h \in C^2(M) \) with \( \| h - h^* \|_{C^0(M)} \leq r \), provided \( r \) is small enough, the problem (1.7) has a non-degenerate solution.

As far as we know, Theorem 1.2 is the first result about existence of solutions to (1.1) which concentrate along geodesic of the manifold \( M \) when the exponent \( p \) approaches the 2nd-critical exponent from above. Indeed, in the Euclidean setting, del Pino, Musso and Pacard in [7] built bubbling solutions for a Dirichlet problem when the exponent is close to but less than the second critical exponent. Solutions concentrating in higher dimensional sets and the gap condition have been found in elliptic problems in the Euclidean setting. We mention among, among many results, [12,13,11,14] for a Neumann singular perturbation problem and [4] for a Schrödinger equation in the plane.

It would be interesting to find a geometric interpretation to problem (1.2). We only observe that the geometric potential

\[ \Omega_{\Gamma}(x_0) := \frac{(n-3)}{4(n-2)} \left[ R_g(x_0) - (n-1)\text{Ric}(\dot{\gamma}(x_0), \dot{\gamma}(x_0)) \right] \]

introduced in (1.5) when \( \Gamma \) reduces to a point \( x_0 \) is nothing but the usual geometric potential \( \frac{(n-2)}{4(n-1)} R_g(x_0) \) which appears in the Yamabe problem.

We conjecture that our result can be extended to higher \( k \)-dimensional minimal submanifolds \( \Gamma \) of \( M \). Indeed, arguments developed by Del Pino, Mahmoudi and Musso in [6] in the Euclidean setting for a Neumann problem could also be applied to Eq. (1.1). More precisely, we could consider a supercritical problem

\[ -\Delta_g u + hu = u^{\frac{m+2}{m-2} \pm \varepsilon}, \quad u > 0, \quad \text{in } (M, g), \]

and we could find conditions on \( h \) such that it possesses solutions which concentrate along \( \Gamma \) as \( \varepsilon \) goes to zero. It would interesting to determine the function \( \sigma_{\Gamma} \) (the analogue of the function \( \sigma \) introduced in (1.5)) whose sign determines the existence of solutions either to the supercritical case or to the subcritical case.

The proof of our result relies on the infinite-dimensional reduction developed by del Pino, Kowalczyk and Wei in [4] and successively adapted by del Pino, Musso and Pacard in [7] to study a problem quite similar to our problem

\[ -\Delta u = u^{\frac{m+1}{m-1} - \varepsilon} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^m \). We omit many details in several steps of the proof, because they can be carried out, up to some minor modifications, as in [7]. However there is an important difference with respect to [7] concerning the scaling parameter \( \mu_\varepsilon \), whose choice is crucial for building the solution. The difference is that the extra term \( \frac{1}{\mu} \) here is the main order term, see (4.11), and leads to the ODEs (1.6) and (1.7), while in [7] it appears at a higher order.
The paper is organized as follows. In Section 2 we study the singular problems (1.6) and (1.7). In Section 3 we build the approximate solution close to the geodesic and in Section 4 we estimate the error. Then, in Section 5 we reduce the problem to a suitable infinite dimensional set of parameters and in Section 6 we study the reduced problem. Section 7 is devoted to the study of a linear problem.

Notation.

- For sums we use the standard convention of summing terms, where repeated indices appear.
- We will denote by $L^\infty_{2\ell}(\mathbb{R})$, $C^0_{2\ell}(\mathbb{R})$, and $C^2_{2\ell}(\mathbb{R})$ the Banach space of $2\ell$-periodic $L^\infty$, $C^0$, and $C^2$ functions, respectively. We will set $\|u\|_{\infty} := \sup_{\mathbb{R}} |u|$, for any $2\ell$-periodic bounded function $u$.

2. A periodic ODE with repulsive or attractive singularity

Let us consider the periodic boundary value problem

\[
\begin{aligned}
-\ddot{\mu} + \sigma \mu - \frac{c}{\mu} &= 0 \quad \text{in } [0, 2\ell], \\
\mu &> 0 \quad \text{in } [0, 2\ell], \\
\mu(0) &= \mu(2\ell), \\
\dot{\mu}(0) &= \dot{\mu}(2\ell),
\end{aligned}
\]

(2.1)

where $c \in \mathbb{R}$ and $\sigma \in C^0_{2\ell}(\mathbb{R})$. The following existence result holds true.

**Proposition 2.1.** Assume either

\[
\min_{t \in \mathbb{R}} \sigma(t) > 0 \quad \text{and} \quad c > 0
\]

(2.2)

or

\[
-\left( \frac{(k+1)\pi}{2\ell} \right)^2 < \min_{t \in \mathbb{R}} \sigma(t) \leq \max_{t \in \mathbb{R}} \sigma(t) < -\left( \frac{k\pi}{2\ell} \right)^2 < 0 \quad \text{and} \quad c < 0
\]

(2.3)

for some integer $k$. Then problem (2.1) has a periodic solution $\mu_0 \in C^2_{2\ell}(\mathbb{R})$.

**Proof.** If (2.2) holds, the claim follows by standard arguments and if (2.3) holds the claim follows by Theorem 1.1 of [5]. □

Let us consider the linearization of problem (2.1) around $\mu_0$, namely the linear periodic boundary value problem

\[
\begin{aligned}
-\ddot{\mu} + \left( \sigma + \frac{c}{\mu^2_0} \right) \mu &= 0 \quad \text{in } [0, 2\ell], \\
\mu(0) &= \mu(2\ell), \\
\dot{\mu}(0) &= \dot{\mu}(2\ell).
\end{aligned}
\]

(2.4)

The solution $\mu_0$ is non-degenerate if and only if the problem (2.4) has only the trivial solution.

**Proposition 2.2.**

(i) If (2.2) holds, then the solution $\mu_0$ is non-degenerate.
Let \( \sigma^* \in C^0_{2\ell}(\mathbb{R}) \) and \( c \in \mathbb{R} \) as in (2.3). The set

\[
\{ \sigma \in B(\sigma^*, r) : \text{all the positive solutions of (2.1) are non-degenerate} \}
\]

is a dense subset of the ball \( B(\sigma^*, r) := \{ \sigma \in C^0_{2\ell}(\mathbb{R}) : \| \sigma - \sigma^* \|_\infty \leq r \} \) provided the radius \( r \) is small enough.

**Proof.** (i) follows immediately by the maximum principle.

Let us prove (ii). We shall use the following abstract transversality theorem previously used by Quinn [16], Saut and Temam [17] and Uhlenbeck [21].

**Theorem 2.3.** Let \( X, Y, Z \) be three Banach spaces and \( U \subset X, V \subset Y \) open subsets. Let \( F : U \times V \to Z \) be a \( C^\alpha \)-map with \( \alpha \geq 1 \). Assume that

(i) for any \( y \in V, F(\cdot, y) : U \to Z \) is a Fredholm map of index \( l \) with \( l \leq \alpha \);

(ii) \( 0 \) is a regular value of \( F \), i.e. the operator \( F^\ell(x_0, y_0) : X \times Y \to Z \) is onto at any point \( (x_0, y_0) \) such that \( F(x_0, y_0) = 0 \);

(iii) the map \( \pi \circ i : F^{-1}(0) \to Y \) is \( \sigma \)-proper, i.e. \( F^{-1}(0) = \bigcup_{\eta=1}^{+\infty} C_\eta \) where \( C_\eta \) is a closed set and the restriction \( \pi \circ i|_{C_\eta} \) is proper for any \( \eta \); here \( i : F^{-1}(0) \to Y \) is the canonical embedding and \( \pi : X \times Y \to Y \) is the projection.

Then the set \( \Theta := \{ y \in V : 0 \text{ is a regular value of } F(\cdot, y) \} \) is a residual subset of \( V \), i.e. \( V \setminus \Theta \) is a countable union of closet subsets without interior points.

In our case the \( C^2 \)-function \( F \) is defined by

\[
F : C^2_{2\ell}(\mathbb{R}) \times C^0_{2\ell}(\mathbb{R}) \to C^0_{2\ell}(\mathbb{R}), \quad F(\mu, \sigma) := -\ddot{\mu} + \sigma \mu - \frac{c}{\mu},
\]

\( X = C^2_{2\ell}(\mathbb{R}) \) and \( U = \{ \mu \in C^2_{2\ell}(\mathbb{R}) : \min_{\mathbb{R}} \mu > 0 \} \), \( Y = Z = C^0_{2\ell}(\mathbb{R}) \) and \( V = B(\sigma^*, r) \), where \( r \) is small enough so that condition (2.3) holds for any \( \sigma \in V \).

It is not difficult to check that for any \( \sigma \in V \) the map \( \mu \to F(\mu, \sigma) \) is a Fredholm map of index 0 and then assumption (i) holds. Let us prove assumption (ii). We fix \( (\mu_0, \sigma_0) \in U \times V \) such that \( F(\mu_0, \sigma_0) = 0 \). The derivative \( D_\sigma F(\mu_0, \sigma_0) : C^0_{2\ell}(\mathbb{R}) \to C^0_{2\ell}(\mathbb{R}) \) is the linear map defined by \( D_\sigma F(\mu_0, \sigma_0)[\sigma] = \sigma \mu_0 \) and it is surjective, because \( \mu_0 > 0 \).

As far as it concerns assumption (iii), we have that

\[
F^{-1}(0) = \bigcup_{m=1}^{+\infty} \left\{ (C_m \times B_m) \cap F^{-1}(0) \right\}
\]

where

\[
C_m = \left\{ \mu \in C^2_{2\ell}(\mathbb{R}) : \frac{1}{m} \leq \min_{\mathbb{R}} \mu \leq \max_{\mathbb{R}} \mu \leq m \right\} \quad \text{and} \quad B_m = B\left(\sigma^*, r - \frac{1}{m}\right).
\]

We can show that the restriction \( \pi \circ i|_{C_m} \) is proper, namely if the sequence \( (\sigma_n) \subset B_m \) converges to \( \sigma \) and the sequence \( (\mu_n) \subset C_m \) is such that \( F(\mu_n, \sigma_n) = 0 \) then there exists a subsequence of \( (\mu_n) \) which converges to \( \mu \in C_m \) and \( F(\mu, \sigma) = 0 \).

That concludes the proof. \( \square \)
Proof of Theorem 1.3. It follows immediately by Proposition 2.1 and Proposition 2.2. □

3. Construction of the approximate solution close to the geodesic

This section is devoted to the construction of an approximation for a solution to the problem (1.2) in a neighborhood of the geodesic.

3.1. The problem near to the geodesic

Let us consider the system of Fermi coordinates \((x_0, x)\) introduced in (1.4). In this language the geodesic \(\Gamma\) is represented by the \(x_0\)-axis. We recall that \(x_0\) denotes the arclength of the curve, \(2\ell\) represent the total length of the geodesic and \(x = (x_1, \ldots, x_N) \in \mathbb{R}^N\). Let us introduce a neighborhood of the geodesic \(\Gamma\) in this system of coordinates

\[
D := \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^N : x_0 \in [-\epsilon, \epsilon], \ |x| < \delta \},
\]

where \(\delta > 0\) is a fixed small number. Then for a function defined in \(D\) we write

\[
\tilde{u}(x_0, x) = u(F(x_0, x))
\]

and we extend \(\tilde{u}\) in a satisfying the following periodicity condition:

\[
\tilde{u}(2\ell, x) = \tilde{u}(0, Ax),
\]

where \(A = (a_{ij})\) is the invertible matrix defined by the requirement

\[
E_i(2\ell) = \sum_{j=1}^N a_{ji}E_j(0).
\]

Therefore, if \(u\) solves Eq. (1.2) in the neighborhood \(D\) of the geodesic, then \(\tilde{u}\) solves

\[
\begin{aligned}
\begin{cases}
\partial_{00}\tilde{u} + \Delta_{x}\tilde{u} + B(\tilde{u}) - h\tilde{u} + f_\epsilon(\tilde{u}) = 0 & \text{in } D \\
\tilde{u}(x_0 + 2\ell, x) = \tilde{u}(x_0, Ax) & \text{for any } (x_0, x) \in D
\end{cases}
\end{aligned}
\]

where \(f_\epsilon(s) := (s^+)^{p+\epsilon}\). For the sake of simplicity, we will refer to \(f_\epsilon(s) := (s^+)^{p+\epsilon}\) as the supercritical case and to \(f_\epsilon(s) := (s^+)^{p-\epsilon}\) as the subcritical case.

In (3.3) \(B\) is a second order linear operator defined in the following lemma:

Lemma 3.1. Let \(u\) be a smooth function. Then for any \((x_0, x) \in D\) we have

\[
\Delta_gu = \partial_{00}\tilde{u} + \Delta_{x}\tilde{u} + B(\tilde{u}),
\]

where \(B\) is a second order linear operator defined by

\[
B(\tilde{u}) := A^{00}\partial_{00}\tilde{u} + \sum_j A^{0j}\partial_0\partial_j\tilde{u} + \sum_{i,j} \left( -\frac{1}{3} \sum_{k,l} R_{ikjl}x_kx_l + A^{ij} \right) \partial_i\partial_j\tilde{u}
\]

\[
+ B^0\partial_0\tilde{u} + \sum_j \left( \sum_k \left( \frac{2}{3} R_{ijk} + R_{0ijk} \right) x_k + B^j \right) \partial_j\tilde{u},
\]
where the Riemann tensor $R_{ijkl}$ and the metric $g$ are computed along $\Gamma$, depending only on $x_0$, while the functions $A^{\alpha\beta}$ and $B^\alpha$ do depend on $(x_0,x)$ and enjoy the following decompositions:

$$A^{00} = \sum_{k,l} A_{kl}^{00} x_k x_l; \quad A^{ij} = \sum_{k,l,m} A_{kl}^{ij} x_k x_l x_m; \quad A^{0j} = \sum_{k,l} A_{kl}^{0j} x_k x_l;$$

$$B^0 = \sum_k B_k^0 x_k; \quad B^j = \sum_{k,l} B_{kl}^j x_k x_l,$$

where $A_{kl}^{00}, A_{kl}^{ij}, A_{kl}^{0j}, B_k^0$ and $B_{kl}^j$ are smooth functions depending on $(x_0,x)$.

**Proof.** We argue exactly as in Section 4 of [7] taking into account the following expansion of the metric $g$ in a neighborhood of the geodesic

$$\begin{aligned}
g_{00}(x) &= 1 + \sum_{k,l=1}^{N} R_{0k0l} x_k x_l + O(|x|^3), \\
g_{0j}(x) &= O(|x|^2), \quad j = 1, \ldots, N, \\
g_{ij}(x) &= \delta_{ij} + \frac{1}{2} \sum_{k,l} R_{ikjl} x_k x_l x_l + O(|x|^3), \quad i, j = 1, \ldots, N,
\end{aligned}$$

(3.4)

whose proof is postponed in Appendix A. \(\Box\)

### 3.2. The scaled problem

We write an approximated solution of problem (3.3). Let

$$\tilde{u}_\epsilon(x_0, x) = \mu_\epsilon(x_0)^{-\frac{N-2}{2}} \bigg( \frac{x - d_\epsilon(x_0)}{\mu_\epsilon(x_0)} \bigg),$$

(3.5)

where the bubble $w$ is defined in (1.9), and $d_\epsilon$ satisfies

$$d_\epsilon(0) = Ad_\epsilon(2\ell), \quad \text{with } d_\epsilon(x_0) = (d_{\epsilon1}(x_0), \ldots, d_{\epsilon N}(x_0))$$

(3.6)

and $A = (a_{ij})$ is the matrix defined by (3.2). In the sequel, $C^2_{2\ell}(\mathbb{R}, \mathbb{R}^N)$ is the space of functions $d : [0, 2\ell] \to \mathbb{R}^N$ which satisfy (3.6).

We will take $d_\epsilon(x_0)$ of the form

$$d_{\epsilon j}(x_0) = \epsilon d_j(x_0) \quad \text{with } d_j \in C^2_{2\ell}(\mathbb{R}), \quad j = 1, \ldots, N$$

(3.7)

and the concentration parameter $\mu_\epsilon(x_0)$ is given by

$$\mu_\epsilon(x_0) = \sqrt{\epsilon} \tilde{\mu}_\epsilon(x_0), \quad \tilde{\mu}_\epsilon(x_0) = \mu_0(x_0) + (\epsilon \ln \epsilon) \mu_1(x_0) + \epsilon \mu(x_0),$$

(3.8)

with $\mu_0, \mu_1, \mu \in C^2_{2\ell}(\mathbb{R})$. We point out that in (3.8) and (3.7) the $\mu_0, \mu_1, \mu$ and $d_j, j = 1, \ldots, N$ are unknown functions which will be found in the final step of the infinite-dimensional reduction. In particular, it will turn out that $\mu_0$ is a non-degenerate solution to problem (1.6) in the subcritical case or to problem (1.7) in the supercritical case.

Therefore, it is natural to consider the change of variables

$$\tilde{u}_\epsilon(x_0, x) = \mu_\epsilon^{-\frac{N-2}{2}} v \left( \frac{x_0}{\rho}, \frac{x - d_\epsilon}{\mu_\epsilon} \right), \quad \rho := \sqrt{\epsilon}.$$

(3.9)
Here $v_\epsilon = v_\epsilon(y_0, y)$ is defined in a region of the form

$$\mathcal{D} = \left\{(y_0, y): \; y_0 \in \left[ -\frac{\ell}{\rho}, \frac{\ell}{\rho} \right], \; |y| < \frac{\eta}{\sqrt{\rho}} \right\}. \quad (3.10)$$

It is clear that if $\bar{u}_\epsilon(x_0, x)$ solves Eq. (3.3), then $v_\epsilon = v_\epsilon(y_0, y)$ solves problem

$$\begin{cases}
A(v) - \mu_\epsilon^2 h v + \mu_\epsilon^{\frac{N-2}{2}} f_\epsilon(v) = 0 & \text{in } \mathcal{D} \\
v(y_0 + \frac{2\ell}{\rho}, y) = v(y_0, Ay) & \text{for any } (y_0, y) \in \mathcal{D}.
\end{cases} \quad (3.11)$$

We agree that we take $\mu_\epsilon^{\frac{N-2}{2}}$ in the supercritical case, i.e. $f_\epsilon(s) = (s^+)^{p+\epsilon}$ and $\mu_\epsilon^{-\frac{N-2}{2}}$ in the subcritical case, i.e. $f_\epsilon(s) = (s^+)^{p-\epsilon}$.

In (3.11) $A$ is a second order operator of the form defined in the following lemma, whose proof can be obtained arguing exactly as in Lemma 5.1 of [7].

**Lemma 3.2.** After the change of variable (3.9), the following holds true:

$$A(v) := a_0 \partial_{y_0} v + \Delta_y v + \tilde{A}(v),$$

with

$$a_0(\rho y_0) = \rho^{-2} \mu_\epsilon (\rho y_0)^2 = (\mu_0 + \rho \mu)^2 \quad (3.12)$$

and $\tilde{A}(v) := \sum_{n=0}^{2} A_n(v) + B(v)$ where

$$A_0(v) = \tilde{\mu}_\epsilon \left[ D_{yy} v[y]^2 + ND_y v[y] + \frac{N(N-2)}{4} v \right] + \mu_\epsilon \left[ D_{yy} v[y] + \frac{N-2}{2} D_y v \right] d_\epsilon \right]$$

$$+ D_{yy} v[d_\epsilon]^2 - 2 \mu_\epsilon \left[ \rho^{-1} D_y (\partial_0 v)[\mu_\epsilon y + d_\epsilon] + \frac{N-2}{2} \mu_\epsilon \rho^{-1} \partial_0 v \right] - \mu_\epsilon D_y v[d_\epsilon]$$

$$- \mu_\epsilon \tilde{\mu}_\epsilon \left( \frac{N-2}{2} v + D_y v[y] \right)$$

$$A_1(v) := -\frac{1}{3} \sum R_{ijk}(\mu_\epsilon y_k + d_\epsilon)(\mu_\epsilon y_i + d_\epsilon) \partial_{ij} v$$

$$A_2(v) := \sum \left( \frac{2}{3} R_{ijk} + R_{ijk} \right) (\mu_\epsilon y_k + d_\epsilon) \mu_\epsilon \partial_{ij} v$$

and the operator $B(v)$ satisfies

$$B(v) = O\left( |\mu_\epsilon y + d_\epsilon|^2 \right) A_0(v) + O\left( |\mu_\epsilon y + d_\epsilon|^3 \right) \partial_{ij} v$$

$$+ O\left( |\mu_\epsilon y + d_\epsilon|^2 \right) \left[ \mu_\epsilon \rho^{-1} \partial_{0j} v + \mu_\epsilon \rho^{-1} \partial_0 v - D_y (\partial_j v)[d_\epsilon] \right]$$

$$- \left( \frac{N-2}{2} \partial_j v + D_y (\partial_j v)[y] \right) \tilde{\mu}_\epsilon$$

$$- \mu_\epsilon \left( \frac{N-2}{2} v + D_y v[y] \right) + \mu_\epsilon \partial_j v.$$
Our approximation close to the geodesic is

$$\tilde{\omega} = \omega + \omega_1.$$  \hfill (3.13)

The first order approximation $\omega$ is given in (3.15), while the second order approximation $\omega_1$ is given in (3.25). We also set

$$S_\epsilon(v) := A(v) - \mu_\epsilon^2h_\epsilon + \mu_\epsilon^{\frac{N-2}{2}}f_\epsilon(v).$$  \hfill (3.14)

3.3. The ansatz: the first order approximation

We define $\omega$ to be

$$\omega := (1 + \alpha_\epsilon)w + \epsilon_\epsilon(\rho_0)\chi_\epsilon(y)Z_0(y).$$  \hfill (3.15)

In the first term of (3.15), $w$ is the bubble defined in (1.9) and $\alpha_\epsilon := \mu_\epsilon^{\frac{(N-2)^2}{8}} - 1$ in the subcritical case or $\alpha_\epsilon := \mu_\epsilon^{\frac{(N-2)^2}{8}} - 1$ in the subcritical case. In the second term of (3.15), $\chi_\epsilon(y) := \chi(\epsilon y^2)$ where $\chi$ is a cut-off function such that $\chi(s) = 1$ if $s \leq \delta$ and $\chi(s) = 0$ if $s \geq 2\delta$ with $\delta > 0$ small but fixed. Moreover, $Z_0$ denotes the first eigenfunction in $L^2(\mathbb{R}^N)$ of the problem (see Section 7)

$$\Delta Z_0 + pw^{p-1}Z_0 = \lambda_1Z_0 \quad \text{in} \quad \mathbb{R}^N, \quad \text{with} \quad \lambda_1 > 0 \quad \text{and} \quad \int_{\mathbb{R}^N} Z_0^2 \, dy = 1.$$  \hfill (3.16)

Finally, the function $e_\epsilon(x_0)$ is given by

$$e_\epsilon = e \tilde{e}_\epsilon, \quad \tilde{e}_\epsilon = e_0 + (\epsilon \ln \epsilon)e_1 + \epsilon e,$$  \hfill (3.17)

with $e_0, e_1, e \in C^2_{\tilde{Z}}(\mathbb{R})$. We point out that $e_0, e_1$ and $e$ are unknown functions which will be chosen in the final step of the infinite-dimensional reduction, together with the functions $\mu_0$, $\mu$ and $d_j$ introduced in (3.7) and (3.8).

Let us estimate the error $S_\epsilon(\omega)$ one commits by considering $\omega$ a real solution to (3.11), which is itself a function of the parameter functions $\mu, d, e$.

Assume that the functions $\mu, d, e$ defined respectively in (3.8), (3.7) and (3.17), satisfy the assumption

$$\|(\mu, d, e)\| := \|\mu\| + \|d\| + \|e\|_\epsilon \leq C$$  \hfill (3.18)

for some constant $C > 0$, independent of $\epsilon$, where

$$\|\mu\| := \|\tilde{\mu}\|_\infty + \|\hat{\mu}\|_\infty + \|\tilde{\mu}\|_\infty, \quad \|d\| := \sum_{j=1}^N \|d_j\|_\infty,$$  \hfill (3.19)

$$\|e\|_\epsilon := \|\tilde{e}\|_\infty + \|\hat{e}\|_\infty + \|\tilde{e}\|_\infty.$$  \hfill (3.20)

Here and in the rest of the paper, the dot denotes the derivative with respect to $x_0$.

It is possible to compute the expansion of the error $S_\epsilon(\omega)$ as showed in the following lemma whose proof is postponed in Section 4.1.
Lemma 3.3. If $\epsilon > 0$ small enough, then for any $(y_0, y) \in \mathcal{D}$ the following expansion holds

$$S_\epsilon(\omega) = \pm \epsilon w^p \ln w + \epsilon \lambda_1 c_0 Z_0 - \epsilon \mu_0^2 hw$$

$$+ \epsilon \left[ \mu_0^2 \left( D_{yy}[y]^2 + N D_{y}[y] + \frac{N(N - 2)}{4} w \right) - \mu_0 \bar{\mu}_0 Z_{N+1} \right]$$

$$+ \epsilon^2 \left[ -\mu_0 \partial_j w \bar{d}_j - \frac{1}{3} \mu_0 R_{ikjl} y_k y_l \partial_{ij} w + \mu_0 \left( \frac{2}{3} R_{ijik} + R_{0j0k} \right) d_k \partial_j w - 2 \mu_0 \partial_j Z_{N+1} d_j \right]$$

$$+ \epsilon^2 \left[ (\rho^2 a_0 \hat{\epsilon} + \lambda_1 \epsilon) Z_0 + \left( \sum_{i,j} \hat{d}_i \hat{d}_j - \frac{1}{3} R_{ijkl} d_k d_l \right) \partial_{ij} w + \Upsilon_0 \right]$$

$$- 2 \mu_0 \mu_0 hw + b(p \mu_0, \mu, d, \epsilon) w^p + 2 \mu_0 \mu \left( D_{yy}[y]^2 + N D_{y}[y] + \frac{N(N - 2)}{4} w \right)$$

$$- \mu_0 \bar{\mu}_0 Z_{N+1} + 2 \mu_0 \left( -\frac{1}{3} R_{ikjl} y_k y_l \partial_{ij} w + \left( \frac{2}{3} R_{ijik} + R_{0j0k} \right) y_k \partial_j w \right)$$

$$- e_0 \mu_0 \mu_0 Z_{N+1} + \mu_0^2 e_0 \left( -\frac{1}{3} R_{ikjl} y_k y_l \partial_{ij} Z_0 + \left( \frac{2}{3} R_{ijik} + R_{0j0k} \right) y_k \partial_j Z_0 \right)$$

$$+ \mu_0^2 \left( D_{yy}Z_0[y]^2 + N D_{y}Z_0[y] + \frac{N(N - 2)}{4} Z_0 \right) - \mu_0^2 h Z_0$$

$$+ \epsilon^2 \left[ -\mu_0 \partial_j \bar{d}_j - \frac{1}{3} \mu_0 R_{ikjl} y_k y_l d_i \partial_{ij} w - \mu \left( \frac{2}{3} R_{ijik} + R_{0j0k} \right) d_k \mu_0 \partial_j w - 2 \mu_0 \partial_j Z_{N+1} \bar{d}_j \right]$$

$$- \mu_0 e_0 \partial_j \bar{Z}_0 \bar{d}_j - \frac{1}{3} \mu_0 e_0 R_{ikjl} y_k y_l d_i \partial_{ij} Z_0 + \mu_0 e_0 \left( \frac{2}{3} R_{ijik} + R_{0j0k} \right) d_k \partial_j Z_0$$

$$- 2 \mu_0 e_0 \left( \frac{N - 2}{2} D_y Z_0 + D_{yy}Z_0[y] \right) \bar{d}_j \right] + \epsilon^3 \Theta,$$  \hfill (3.21)

where

- $Z_0$ is defined in (3.16) and $Z_{N+1}$ is defined in (3.23)
- the first term is “$-\epsilon w^p \ln w$” in the subcritical case or “$+\epsilon w^p \ln w$” in the supercritical case.

$$\Upsilon_0 = p(p - 1)c_0^2 w^{p - 2} Z_0^2 + p c_0 w^{p - 1} \ln w Z_0$$  \hfill (3.22)

- $\Theta = \Theta(y_0, y)$ is a sum of functions of the form

$$h_0(p \lambda_0) \left[ f_1(\mu, d, \bar{\mu}, \bar{d}) + o(1) f_2(\mu, d, \epsilon, \bar{\mu}, \bar{d}, \bar{\epsilon}) \right] f_3(y)$$

with

- $h_0$ a smooth function uniformly bounded in $\epsilon$
- $f_1$ and $f_2$ smooth functions of their arguments, uniformly bounded in $\epsilon$ when $\mu, d$ and $\epsilon$ satisfy (3.18)
- $f_2$ depending linearly on the argument $(\bar{\mu}, \bar{d}, \bar{\epsilon})$
- $o(1) \rightarrow 0$ as $\epsilon \rightarrow 0$ uniformly when $\mu, d$ and $\epsilon$ satisfy (3.18)
- $\sup_{y \in R} (1 + |y|^{N - 2}) |f_3(y)| < +\infty$
Now, we use formula (3.21) to compute, for each \( y_0 \in [-\ell/\rho, +\ell/\rho] \), the \( L^2(D_{y_0}) \) the projection of the error \( \mathcal{S}_\epsilon(\omega) \) along the elements of the kernel of the linear operator \( \mathcal{L}_0 := \Delta_{R^N} + p\omega^{p-1}I \) (see Section 7), i.e. the functions

\[
Z_k(y) := \partial_k w(y), \quad k = 1, \ldots, N \quad \text{and} \quad Z_{N+1}(y) := y \cdot \nabla w(y) + \frac{N - 2}{2} w(y). \tag{3.23}
\]

Lemma 3.4. If \( \epsilon > 0 \) small enough, then for any \( x_0 = \rho y_0 \) with \( y_0 \in [-\ell/\rho, +\ell/\rho] \) the following expansion holds:

\[
\int_{D_{y_0}} \mathcal{S}_\epsilon(\omega) Z_k \, dy = \epsilon^2 c_1 \mu_0 \left( -\ddot{a}_k + \sum R_{0k0d} \right) + \epsilon^2 \theta, \quad \text{for any } k = 1, \ldots, N;
\]

moreover, if \( \mu_0 \) solves either (1.6) or (1.7) there exist \( \mu_1, \epsilon_0, c_1 \in C^2_0(\mathbb{R}) \) such that

\[
\int_{D_{y_0}} \mathcal{S}_\epsilon(\omega) Z_{N+1} \, dy = \epsilon^2 c_2 \mu_0 \left[ \alpha_{N+1}(x_0) + c_3 Q(x_0, d) - \mu + \left( a_n \sigma + \frac{b_n}{\rho_0} \right) \mu \right] + \epsilon^3 \ln \epsilon \theta
\]

and

\[
\int_{D_{y_0}} \mathcal{S}_\epsilon(\omega) Z_0 \, dy = \epsilon^2 \left[ \epsilon_0 \dot{\rho} + \lambda_1 \epsilon + \alpha_0(x_0) + c_4 Q(\rho y_0, d) + \beta(x_0) \mu \right] + \epsilon^3 \ln \epsilon \theta.
\]

Here

- \( \sigma \) is defined in (1.5) and \( a_n, b_n \) are positive constants depending only on \( n \) defined in (4.16)
- \( Q(x_0, d) := \sum (d_j^2 - \frac{1}{2} R_{kij} d_k d_l) \)
- \( c_i \)’s are constants which depend only on \( n \)
- \( \alpha_i \)’s and \( \beta \) are explicit smooth functions, uniformly bounded in \( \epsilon \) when \( \mu, d \) and \( \epsilon \) satisfy (3.18)
- \( \theta = \theta(x_0) \) denotes a sum of functions of the form

\[
h_0(x_0) \left[ h_1(\mu, d, e, \dot{\mu}, \dot{e}, \dot{d}) + o(1) h_2(\mu, d, e, \dot{\mu}, \dot{e}, \dot{d}, \dot{e}) \right],
\]

where

- \( h_0 \) is a smooth function uniformly bounded in \( \epsilon \)
- \( h_1 \) and \( h_2 \) are smooth functions of their arguments, uniformly bounded in \( \epsilon \) when \( \mu, d \) and \( \epsilon \) satisfy (3.18)
- \( h_2 \) depends linearly on the argument \((\mu, \dot{d}, \dot{e})\)
- \( o(1) \to 0 \) as \( \epsilon \to 0 \) uniformly when \( \mu, d \) and \( \epsilon \) satisfy (3.18)

The proof is postponed in Section 4.2.

In the sequel we will use the following norms, which are motivated by the linear theory presented in Section 7. For functions \( \phi, g \) defined on a set \( D \) as in (3.10), and for a fixed \( 2 \leq \nu < N \), let

\[
\|
\phi
\|_*: = \sup_{D \setminus \\overline{\Omega}} \left( 1 + |y|^{-2} \right) \left| \phi(y_0, y) \right| + \sup_{D \setminus \\overline{\Omega}} \left( 1 + |x|^{-\nu} \right) |D \phi(x_0, x)|;
\]

\[
\|
\phi
\|_{**} := \sup_{D \setminus \\overline{\Omega}} \left( 1 + |y|^{\nu} \right) |g(y_0, y)|.
\]

Therefore, from the expansion given in (3.21) we conclude that the error \( \mathcal{S}_\epsilon(\omega) \), computed in (3.21), has the properties listed in the following lemma:
Lemma 3.5. Let $\mu_0$ and $c_0$ as in Lemma 3.4 If $\epsilon$ is small enough
\[ S_{\epsilon}(\omega) = \epsilon S_0 + \epsilon [\rho^2 a_0 \bar{e} + \lambda_1 e] \chi e Z_0 + N_0, \] (3.24)
where
- $S_0$ is a smooth function of $\rho y_0$ uniformly bounded in $\epsilon$
- $S_0$ does not depend on $\mu, d$ and $\epsilon$
- $\int_{D_{\rho_0}} S_0 Z_j dy = 0$ for any $y_0 \in (-\rho^{-1} \ell, \rho^{-1} \ell)$ and for any $j = 0, \ldots, N + 1$
- $\|N_0\|_* \leq c \epsilon^2$.

Here $c$ is a positive constant independent of $\epsilon$. All the estimates are uniform with respect to $\mu, d$ and $\epsilon$ which satisfy (3.18).

3.4. The ansatz: the second order approximation

Now we introduce a further correction $\omega_1$ to $\omega$, to get the final approximation $\tilde{\omega} := \omega + \omega_1$. The correction $\omega_1$ is chosen to reduce the size of the error (3.24), killing the term $\epsilon S_0$ and it is found in the following lemma, whose proof can be carried out arguing exactly as in Section 5 of [7].

Lemma 3.6. If $\epsilon$ is small enough there exists a unique solution $\omega_1$ of the problem
\[
\begin{cases}
A(\omega_1) - \mu_0^2 h \omega_1 + p w^{p-1} \omega_1 = -\epsilon S_0 + \sum_{j=0}^{N} \sigma_j Z_j & \text{in } D, \\
\int_{D_{\rho_0}} \omega_1(y_0, y) Z_j dy = 0 & \text{for any } y_0 \in \left[ -\frac{\ell}{\rho}, \frac{\ell}{\rho} \right], \ j = 0, \ldots, N + 1.
\end{cases}
\] (3.25)

Moreover, the function $\omega_1$ satisfies
- $\|\omega_1\|_* \leq c\epsilon$ and $\|\partial_0 \omega_1\|_* \leq c \epsilon^2$
- $\omega_1$ depends smoothly on $\mu$ and $d$ and it is independent of $\epsilon$
- $\|\omega_1(\mu_1, d_1) - \omega_1(\mu_2, d_2)\|_* \leq c\| (\mu_1 - \mu_2, d_1 - d_2)\|

and each function $\sigma_j$ satisfies
- $\|\sigma_j\|_\infty \leq o(1) \epsilon^3$
- $\sigma_j$ depends smoothly on $\mu$ and $d$ and it is independent of $\epsilon$
- $\|\sigma_j(\mu_1, d_1) - \sigma_j(\mu_2, d_2)\|_\infty \leq c \epsilon^2 \| (\mu_1 - \mu_2, d_1 - d_2)\|.$

Moreover, it holds true
\[ S_{\epsilon}(\tilde{\omega}) = \epsilon^2 S_1 + \epsilon [\rho^2 a_0 \bar{e} + \lambda_1 e] \chi e Z_0 + N_1 + \sum_{j=0}^{N} \sigma_j Z_j, \] (3.26)
where
- $S_1$ is a smooth function of $\rho y_0$ uniformly bounded in $\epsilon$.
- $S_1$ depends smoothly on $\mu, d$ and $e$
- $\|S_1(\mu_1, d_1, e_1) - S_1(\mu_2, d_2, e_2)\|_{**} \leq c(\|\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2\|)$
- $\|N_1\|_{**} \leq c\epsilon^2$.

Here $c$ is a positive constant independent of $\epsilon$. All the estimates are uniform with respect to $\mu, d$ and $e$ which satisfy (3.18). Moreover, the components of $S_\epsilon(\tilde{\omega})$ along the $Z_j$'s satisfy the estimate in Lemma 3.4.

4. The error $S_\epsilon(\omega)$

4.1. The pointwise estimate of the error

We recall that

$$S_\epsilon(\omega) = A(\omega) - \mu_\epsilon^2 h\omega + \mu_\epsilon^{\frac{N-2}{2}} f_\epsilon(\omega)$$

where by Lemma 3.2

$$A(\omega) = a_0\partial_{y0}\omega + \Delta_y\omega + \sum_{k=0}^{2} A_k(\omega) + B(\omega)$$

and

$$\omega(y) = (1 + \alpha_\epsilon)w(y) + e_\epsilon(\rho y_0)\chi_\epsilon(y)Z_0(y).$$

Here we recall that

$$\alpha_\epsilon = \mu_\epsilon^{\frac{(N-2)^2}{4}} - 1$$

and

$$\Delta((1 + \alpha_\epsilon)w) + \mu_\epsilon^{\frac{N-2}{2}} f_0((1 + \alpha_\epsilon)w) = 0 \text{ in } \mathbb{R}^N.$$

Proof of Lemma 3.3. We use Lemma 3.2.

A straightforward computation shows that

$$S_\epsilon(\omega) = \sum_{\kappa=0}^{2} A_{\kappa}(w) - \mu_\epsilon^2 h\omega \pm \epsilon w^p \ln w + [\rho^2 a_0 e_\epsilon(\rho y_0) + \lambda_1 e_\epsilon(\rho y_0)] \chi_\epsilon Z_0$$

$$+ B(\omega) + a_0 w\partial_{y0}\alpha_\epsilon + \bar{A}(\alpha_\epsilon w) - \mu_\epsilon^2 \alpha_\epsilon h\omega$$

$$+ \mu_\epsilon^{\frac{N-2}{2}} [f_\epsilon((1 + \alpha_\epsilon)w) - f_0((1 + \alpha_\epsilon)w)] \pm \epsilon w^p \ln w$$

$$+ \sum_{\kappa=0}^{2} A_{\kappa}(\epsilon_\epsilon\chi_\epsilon Z_0) - \mu_\epsilon^2 e_\epsilon\chi_\epsilon Z_0 h$$
\[+ B(e_\epsilon \chi, Z_0) + e_\epsilon Z_0 \Delta \chi \epsilon + 2e_\epsilon \nabla \chi \epsilon \nabla Z_0 \]
\[+ \sum \nabla_{\epsilon}^2 \left[ f_\epsilon(w) - f_\epsilon((1 + \alpha_\epsilon)w) \right] - f'_\epsilon(w)e_\epsilon \chi, Z_0. \]  
(4.1)

By Lemma 3.2, we get the first term of \( J_0 \)
\[2 \sum_{\kappa = 0} A_\kappa(w) = \mu^2 \left[ D_{y_\epsilon} w[y]^2 + ND_{y_\epsilon} w[y] + \frac{N(N - 2)}{4} w \right] + \mu \epsilon \left[ D_{y_\epsilon} w[y] + \frac{N - 2}{2} D_{y_\epsilon} w \right][d_\epsilon] + D_{y_\epsilon} w[d_\epsilon]^2 \]
\[- \mu D_{y_\epsilon} w[d_\epsilon] - \mu \epsilon \mu_\epsilon \left( \frac{N - 2}{2} w + D_{y_\epsilon} w[y] \right) \]
\[+ \frac{1}{3} \sum R_{j_\epsilon k_\epsilon} (\mu_\epsilon y_\epsilon + d_\epsilon)(\mu_\epsilon y_\epsilon + d_\epsilon) \partial_j w \]
\[+ \frac{1}{3} \sum \left( \frac{2}{3} R_{i_\epsilon j_\epsilon} + R_{0_\epsilon j_\epsilon} \right) (\mu_\epsilon y_\epsilon + d_\epsilon) \mu_\epsilon \partial_j w + \epsilon^3 \Theta \]
\[= \epsilon^2 \left[ \sum \left( d_\epsilon d_\epsilon - \frac{1}{3} R_{j_\epsilon k_\epsilon} d_\epsilon d_\epsilon \right) \partial_j w \right] + \epsilon^3 \Theta, \]  
(4.2)

where \( \Theta = \Theta(\rho y_0, y) \) has the required properties.

By Lemma 3.2, we deduce that \( B(w) \) is of lower order with respect to \( \sum A_\kappa(w) \). Moreover, by definition of \( \alpha_\epsilon \) we get that \( \alpha_\epsilon = O(\epsilon |\ln \epsilon|) \) as \( \epsilon \to 0 \). Hence \( \alpha_\epsilon \hat{A}(w) \) and \( \mu_\epsilon \alpha_\epsilon \hat{h}_\epsilon w \) are terms of lower order with respect to the others. Furthermore \( \partial_{\epsilon_\alpha_\epsilon} = \epsilon^2 O(\alpha_\epsilon) \), so also \( \alpha_\epsilon \partial_{\epsilon_\alpha_\epsilon}[\alpha_\epsilon w] = O(\epsilon^2 |\ln \epsilon|)w \). Therefore,
\[J_1 = \epsilon^3 \Theta \]

where \( \Theta = \Theta(\rho y_0, y) \) is a sum of functions of the form \( h_0(\rho y_0) f_1(\mu, d, \mu, d) f_2(y) \), with \( h_0 \) a smooth function uniformly bounded in \( \epsilon \), \( f_1 \) a smooth function of its arguments, homogeneous of degree 3, uniformly bounded in \( \epsilon \) and \( \sup y \in \mathbb{R} (1 + |y|^{-2}) f_2(y) |< +\infty \).

By mean value theorem we deduce that
\[J_2 = \pm \left( \frac{n - 2}{8} \right)^2 (\epsilon^2 \ln \epsilon) \omega^p (\ln \omega - 1) \pm \epsilon^2 \omega^p \left( \frac{(n - 2)^2}{8} (\ln \omega - 1) \ln \mu + \frac{1}{2} \ln \omega \right) \]
\[+ O(\epsilon^3 |\ln \epsilon|). \]  
(4.3)

By Lemma 3.2 we also get that
where \( \Theta = \Theta(\rho y_0, y) \) has the required properties.

Finally, standard estimates yield to

\[
J_5 = \epsilon^2 \left[ p(p-1)\epsilon_0^2 w^{p-2} Z_0^2 + p\epsilon_0 w^{p-1} \ln w Z_0 \right] + \epsilon^3 |\ln \epsilon| \Theta,
\]

where \( \Theta = \Theta(\rho y_0, y) \) is a sum of functions of the form \( h_0(\rho y_0) h_1(\mu, d, \epsilon) h_2(y) \) with \( h_0 \) a smooth function, uniformly bounded in \( \epsilon, h_1 \) a smooth function of its arguments and \( \sup_{y \in \mathbb{R}} (1 + |y|^N) |h_2(y)| < +\infty \).

Collecting all the previous estimates we get the proof. \( \square \)

4.2. The components of the error along the \( Z_j \)'s

**Proof of Lemma 3.4.** The proof consists of two steps. In the first part we compute the expansion in \( \epsilon \) of the projection assuming that

\[
\mu_\epsilon = \rho \mu, \quad d_{\epsilon j} = \epsilon d_j, \quad e_\epsilon = \epsilon \bar{e}.
\]

In the second part we will choose the \( \epsilon \)-order terms \( \mu_0 \) and \( e_0 \) and the \( \epsilon \ln \epsilon \)-order terms \( \mu_1 \) and \( e_1 \) in the expansion of \( \mu, \bar{e} \).

Arguing as in the proof of Lemma 3.3, we have

\[
S_\epsilon(\omega) = \pm \epsilon w^p \ln w - \rho^2 \bar{\mu}^2 \epsilon w^2 + \sum_{k=0}^{2} A_k(w) + \epsilon \left[ \rho^2 \bar{a}_0 \bar{e} + \lambda_1 \bar{e} \right] \chi_\epsilon Z_0 + J_1 + \cdots + J_5.
\]

We stress the fact that the first term in \( I_1 \) is \( +\epsilon w^p \ln w \) in the super-critical case and \( -\epsilon w^p \ln w \) in the subcritical case.

- **The projection of \( I_1 \).**

\[
\int_{\mathcal{D}_{y_0}} I_1 Z_{N+1} dy = \pm \epsilon \int_{\mathcal{D}_{y_0}} w^p \ln w Z_{N+1} dy - \rho^2 \bar{\mu}^2 \int_{\mathcal{D}_{y_0}} hw Z_{N+1} dy
\]
\[= -\epsilon A_1 + O(\rho^N) - \rho^2 \bar{\mu}^2 h(\rho y_0) \int_{\mathbb{R}^N} w Z_{N+1} \, dy + O(\rho^N)\]

where

\[A_1 = \int_{\mathbb{R}^N} w^p \ln w Z_{N+1} \, dy = \frac{N}{(p+1)^2} \int_{\mathbb{R}^N} w^{p+1} \, dy > 0 \quad \text{(see Remark 4.1)} \quad (4.4)\]

and

\[A_2 = \int_{\mathbb{R}^N} w Z_{N+1} \, dy < 0 \quad \text{(see Remark 4.1).} \quad (4.5)\]

\[\int_{\mathcal{D}_{y_0}} I_1 Z_k \, dy = \epsilon \int_{\mathcal{D}_{y_0}} w^p \ln w Z_j \, dy + \rho^2 \bar{\mu}^2 \int_{\mathcal{D}_{y_0}} h w Z_j \, dy\]

\[= \epsilon \int_{\mathbb{R}^N} w^p \ln w Z_j \, dy + \rho^2 \bar{\mu}^2 h(\rho y_0) \int_{\mathbb{R}^N} w Z_j \, dy + O(\rho^{N+1})\]

\[= O(\rho^{N+1}) \quad \text{for } k = 1, \ldots, N. \]

\[\int_{\mathcal{D}_{y_0}} I_1 Z_0 \, dy = -\epsilon \int_{\mathcal{D}_{y_0}} w^p \ln w Z_0 \, dy - \rho^2 \bar{\mu}^2 \int_{\mathcal{D}_{y_0}} h w Z_0 \, dy\]

\[= \epsilon [-A_3 - \bar{\mu}^2 h(\rho y_0) A_4] + O(\rho^N), \]

where

\[A_3 := \int_{\mathbb{R}^N} w^p \ln w Z_0 \, dy, \quad A_4 := \int_{\mathbb{R}^N} w Z_0 \, dy. \quad (4.6)\]

- **The projection of \( I_2. \)**
  - We use estimate (4.2).

\[\int_{\mathcal{D}_{y_0}} I_2 Z_{N+1} \, dy = \epsilon^2 \sum \left( \dddot{d}_i d_j - \frac{1}{3} R_{ikjl} d_k d_l \right) \int_{\mathcal{D}_{y_0}} \partial_{ij} w Z_{N+1} \, dy\]

\[- \rho \bar{\mu} \sum \dddot{d}_j \int_{\mathcal{D}_{y_0}} \partial_j w Z_{N+1} \, dy\]

\[- \frac{1}{3} \bar{\mu} \rho \epsilon \sum R_{ikjl} d_l \int_{\mathcal{D}_{y_0}} y_k \partial_{ij} w Z_{N+1}\]

\[+ \rho \bar{\mu} \epsilon \left( \frac{2}{3} R_{ijik} + R_{0ijk} \right) d_k \int_{\mathcal{D}_{y_0}} \partial_j w Z_{N+1} \, dy\]

\[- 2 \bar{\mu} \rho \epsilon \sum \dddot{d}_j \int_{\mathcal{D}_{y_0}} \partial_j Z_{N+1} Z_{N+1} \, dy\]
\[
+ \dot{\bar{\mu}}^2 \rho^2 \int_{\mathcal{D}_{y_0}} \left[ D_{yy}w[y]^2 + ND_y[w] + \frac{N(N-2)}{4}w \right] Z_{N+1} dy
\]

\[- \ddot{\bar{\mu}} \rho^2 \int_{\mathcal{D}_{y_0}} Z_{N+1}^2 dy \]

\[- \rho^2 \ddot{\bar{\mu}}^2 \frac{1}{3} \sum_{R_{ijkl}} y_k y_l \partial_{ij} w Z_{N+1} dy \]

\[+ \dot{\bar{\mu}}^2 \rho^2 \sum_{\left( \frac{2}{3} R_{ijij} + R_{0000} \right)} \int_{\mathcal{D}_{y_0}} y_j \partial_j w Z_{N+1} dy \]

\[= \epsilon^2 \sum_{\mathbb{R}^N} \left[ \ddot{d}_i^2 - \frac{1}{3} R_{ikkl} d_k d_l \right] \int_{\mathbb{R}^N} \partial_i w Z_{N+1} dy \]

\[+ \dot{\bar{\mu}}^2 \rho^2 \sum_{\left( \frac{2}{3} R_{ijij} + R_{0000} \right)} \int_{\mathbb{R}^N} y_j \partial_j w Z_{N+1} dy \]

\[- \frac{1}{3} \rho^2 \ddot{\bar{\mu}}^2 \sum_{R_{ijkl}} y_k y_l \partial_{ij} w Z_{N+1} dy + \epsilon^3 \theta \]

\[= \epsilon^2 B_1 \sum_{\mathbb{R}^N} \left[ \ddot{d}_i^2 - \frac{1}{3} R_{ikkl} d_k d_l \right] \int_{\mathbb{R}^N} \partial_i w Z_{N+1} dy \]

\[+ \epsilon \left[ \ddot{\bar{\mu}}^2 \sum_{\left( \frac{1}{3} R_{ijij} + R_{0000} \right)} B_2 - \ddot{\bar{\mu}} \ddot{B}_3 \right] + \epsilon^3 \theta, \]

where the function \( \theta = \theta(\rho y_0) \) has the required properties and

\[
B_1 := \int_{\mathbb{R}^N} \partial_i w Z_{N+1} dy, \quad B_2 := \int_{\mathbb{R}^N} y_j \partial_j w Z_{N+1} dy < 0, \quad B_3 := \int_{\mathbb{R}^N} Z_{N+1}^2 dy. \quad (4.7)
\]

Here we used the fact that

\[
\sum_{R_{ijkl}} y_k y_l \partial_{ij} w Z_{N+1} dy = \sum_{R_{jiij}} y_j \partial_j w Z_{N+1} dy,
\]

because \( R_{ijkl} \) is antisymmetric (i.e. \( R_{ijkl} = -R_{kijl} \)),

\[
\int_{\mathbb{R}^N} y_k y_l \partial_{ij} w Z_{N+1} dy \]

\[= \int_{\mathbb{R}^N} y_k y_l \left( -c_N(N-2) \frac{\delta_{ij}}{(1+|y|^2)^{N+2}} + c_N N(N-2) \frac{y_i y_j}{(1+|y|^2)^{N+2}} \right) Z_{N+1} dy \quad (4.8)
\]

and \( \int_{\mathbb{R}^N} \frac{y_k y_l y_i y_j}{(1+|y|^2)^{N+2}} Z_{N+1} dy \) is symmetric,
\[
\int_{D_0} I_2 Z_k dy = \rho \tilde{\mu} \left[ -\ddot{a}_k \int_{\mathbb{R}^N} Z_j^2 dy - \frac{2}{3} R_{ijkl} d_l \int_{\mathbb{R}^N} y_m \partial_{ij} w Z_k dy \right]
+ \left( \frac{2}{3} R_{ijkl} + R_{ij0l} \right) d_l \int_{\mathbb{R}^N} Z_j^2 dy + \rho^2 \epsilon \theta
\]
= \epsilon^2 \tilde{\mu} B_4 [\ddot{a}_k + R_{ij0l} d_l] + \rho^2 \epsilon \theta,
\]
where
\[
B_4 := \int_{\mathbb{R}^N} Z_j^2 dy, \quad j = 1, \ldots, N. \tag{4.9}
\]

Here we used the fact that
\[
-\frac{2}{3} R_{ijkl} \int_{\mathbb{R}^N} y_m \partial_{ij} w Z_k dy
= -\frac{2}{3} \left[ R_{disk} \int y_k \partial_i w Z_k dy + R_{iktj} \int y_l \partial_{ik} w Z_k dy + R_{kjij} \int y_j \partial_{kj} w Z_k dy \right]
= -\frac{1}{3} B_4 [R_{disk} - R_{iktj}] = -\frac{2}{3} B_4 R_{disk}.
\]

\[
\int_{D_0} I_2 Z_0 dy = \epsilon^2 \left[ \sum \left( d_l^2 - \frac{1}{3} R_{ijkl} d_k d_l \right) \int_{\mathbb{R}^N} \partial_{ij} w Z_0 dy \right]
+ \tilde{\mu}^2 \rho \sum \left( \frac{2}{3} R_{ijkl} + R_{ij0j} \right) \int_{\mathbb{R}^N} y_j \partial_j w Z_0 dy
- \rho^2 \left[ \mu \frac{1}{3} \sum R_{ijkl} \int_{\mathbb{R}^N} y_k y_l \partial_{ij} w Z_0 dy + \epsilon^3 r \right]
= \epsilon^2 B_5 \sum \left[ d_l^2 - \frac{1}{3} R_{ijkl} d_k d_l \right]_{Q(\epsilon, \rho)} + \epsilon \tilde{\mu}^2 B_6 \sum \left( \frac{1}{3} R_{ijkl} + R_{ij0j} \right) + \epsilon^3 \theta,
\]
where
\[
B_5 := \int_{\mathbb{R}^N} \partial_{ij} w Z_0 dy, \quad B_6 := \int_{\mathbb{R}^N} y_j \partial_j w Z_0 dy. \tag{4.10}
\]

Here we used (4.8) and argued as before.

• The projection of $I_3$.

\[
\int_{D_0} I_3 Z_{N+1} dy = o(1) \epsilon^3 \quad \text{and} \quad \int_{D_0} I_3 Z_k dy = o(1) \epsilon^3 \quad \text{for any} \quad k = 1, \ldots, N,
\]
because of the symmetry and of the orthogonality of $Z_0$ with $Z_{N+1}$ and $Z_j$.

\[
\int_{D_0} I_3 Z_0 dy = \epsilon [\rho^2 a_0 \tilde{e} + \lambda_1 \tilde{e}] + o(1) \epsilon^3
\]
because $\int_{\mathbb{R}^N} Z_0^2 dy = 1$. 

• The projection of $I_4$.

\[
\int_{D_{y_0}} I_4 Z_{N+1} \, dy = \epsilon^2 \ln \epsilon D_1 + \epsilon^2 b_1(\rho y_0) + \epsilon^3 |\ln \epsilon| \theta
\]

\[
\int_{D_{y_0}} I_4 Z_k \, dy = \epsilon^2 \theta \quad \text{for any } k = 1, \ldots, N.
\]

\[
\int_{D_{y_0}} I_4 Z_0 \, dy = \epsilon^2 \ln \epsilon D_2 + \epsilon^2 b_2(\rho y_0) + \epsilon^3 |\ln \epsilon| \theta,
\]

where

\[
D_1 := \pm \frac{(N-2)^2}{16} A_1, \quad D_2 := \pm \frac{(N-2)^2}{16} A_3 \quad \text{(see (4.4) and (4.6))},
\]

$b_1, b_2$ are explicit functions and the function $\theta = \theta(\rho y_0)$ has the required properties.

Hence, summing up the previous calculations we conclude that

\[
\int_{D_{y_0}} S_\epsilon(\omega) Z_{N+1} \, dy = \epsilon \left( \pm A_1 - \mu_0 \bar{\mu}_0 B_3 + \mu_0^2 g_1 \right)
\]

\[
\quad + \epsilon^2 \ln \epsilon \left( -\bar{\mu}_1 \mu_0 B_3 + \mu_1 (\bar{\mu}_0 B_3 + 2 \mu_0 g_1) + D_1 \right)
\]

\[
\quad + \epsilon^2 \left( -\bar{\mu}_1 \mu_0 B_3 + \mu (\bar{\mu}_0 B_3 + 2 \mu_0 g_1) + B_1 Q(d, x_0) + b_1(x_0) \right)
\]

\[
\quad + O(\epsilon^3 |\ln \epsilon|),
\]

(4.11)

where (see Remark 4.1)

\[
g_1(x_0) := -A_2 h(x_0) + \sum \left( \frac{1}{3} R_{ijij} + R_{0j0j} \right) B_2 = -A_2 \sigma(x_0)
\]

(4.12)

and

\[
\int_{D_{y_0}} S_\epsilon(\omega) Z_0 \, dy = \epsilon \left( \lambda_1 e_0 - A_3 + \mu_0^2 g_2 \right)
\]

\[
\quad + \epsilon^2 \ln \epsilon \left( \lambda_1 e_1 + 2 \mu_0 \mu_1 + D_2 \right)
\]

\[
\quad + \epsilon^2 \left( \epsilon a_0 \bar{e} + \lambda_1 e + a_0 \bar{e} + b_2(x_0) + 2 \mu_0 g_2 + B_5 Q(d, x_0) \right)
\]

\[
\quad + O(\epsilon^3 |\ln \epsilon|),
\]

(4.13)

where

\[
g_2(x_0) := -A_4 h(x_0) + \sum \left( \frac{1}{3} R_{ijij} + R_{0j0j} \right) B_6.
\]

(4.14)

More precisely, $\mu_0$ solves the periodic ODE
\[-\ddot{\mu}_0 B_3 + g_1 \mu_0 \pm \frac{A_1}{\mu_0} = 0, \quad \mu_0 > 0 \text{ in } [0, 2\ell]. \quad (4.15)\]

which is nothing but problem (1.6) or (1.7) where (see Remark 4.1)

\[a_n := -\frac{A_2}{B_3} > 0 \quad \text{and} \quad b_n := \frac{A_1}{B_3} > 0 \quad (\text{see } (4.4), (4.5) \text{ and } (4.7)). \quad (4.16)\]

Moreover,

\[e_0 = \frac{A_3 - \mu_0^2 g_2}{\lambda_1}. \quad (4.17)\]

Finally, \( \mu_1 \) solves the periodic ODE

\[-\ddot{\mu}_1 \mu_0 B_3 + \mu_1 (\ddot{\mu}_0 B_3 + 2 \mu_0 g_1) + D_1 = 0 \quad \text{in } [0, 2\ell]. \quad (4.18)\]

We point out that \( \mu_1 \) does exist, because \( \mu_0 \) is a non-degenerate solution of (4.15) (see also Lemma 6.1). Moreover,

\[e_1 = \frac{-2 \mu_0 \mu_1 - D_2}{\lambda_1}. \quad (4.19)\]

That concludes the proof. \( \Box \)

**Remark 4.1.** It holds

- \( g_1(x_0) = -A_2 \sigma(x_0) \) with \( A_2 < 0 \) (see (4.5)),
- \( A_1 > 0 \) (see (4.4)),
- \( a_n = -\frac{A_2}{B_3} = \frac{2(N-1)}{(N-2)(N+2)} = \frac{2(n-2)}{(n-3)(n+1)} \) (see (4.5) and (4.7)),
- \( b_n = \frac{A_1}{B_3} = \frac{(N-2)^2(N-4)}{4(N+2)} = \frac{(n-3)^2(n-5)}{4(n+1)} \) (see (4.4) and (4.7)).

**Proof.** It is useful to point out that

\[\frac{B_2}{A_2} = \frac{3(N-2)}{4(N-1)}.\]

Indeed, if we denote

\[I^q_p := \int_0^{+\infty} \frac{r^q}{(1+r)^p} \, dr \quad \text{if } p - q > 1\]

and we use the properties

\[I^q_{p+1} = \frac{p-(q+1)}{p} I^q_p \quad \text{and} \quad I^{q+1}_{p+1} = \frac{q+1}{p-(q+1)} I^q_{p+1}\]

a straightforward computation shows that
\[ A_1 = \frac{N}{(p+1)^2} \int_{\mathbb{R}^N} w^{p+1} dy = c_N^2 \frac{(N-2)^4}{8N} \omega_N I_N^{N/2} > 0, \]
\[ A_2 = \int_{\mathbb{R}^N} wZ_{N+1}^1 dy = -c_N^2 \frac{2(N-1)(N-2)}{N(N-4)} \omega_N I_N^{N/2} < 0, \]
\[ B_2 = \int_{\mathbb{R}^N} y_j \partial_j wZ_{N+1} dy = -c_N^2 \frac{3(N-2)^2}{2N(N-4)} \omega_N I_N^{N/2} < 0 \]

and
\[ B_3 = \int_{\mathbb{R}^N} Z_{N+1}^2 dy = c_N^2 \frac{(N-2)^2(N+2)}{2N(N-4)} \omega_N I_N^{N/2} > 0, \]

where \( \omega_N \) is the measure of the sphere \( S^{N-1} \). Therefore, we immediately deduce the quantities \( a_n \) and \( b_n \), taking into account that \( N = n - 1 \).

Moreover, it is easy to check that
\[
\frac{1}{3} \sum_{i,j=1}^{N} R_{iijj}(x_0) + \sum_{j=1}^{N} R_{0j0j}(x_0) = \frac{1}{3} \sum_{i,j=0}^{N} R_{iijj}(x_0) - \frac{1}{3} \sum_{j=1}^{N} R_{0j0j}(x_0) = \frac{1}{3} R_g(x_0) - \frac{N}{3} \text{Ric}(\gamma(x_0), \gamma(x_0)). \tag{4.20}
\]

Therefore, the claim follows. \( \Box \)

5. The infinite dimensional reduction

5.1. The gluing procedure

Here we perform a gluing procedure that reduces the full problem (1.2) to the scaled problem (3.11) in the neighborhood of the scaled geodesic.

Since the procedure is very similar to that of [7] we briefly sketch it.

We denote by \( M_\rho \) the scaled manifold \( \tfrac{1}{\rho} M \), by \( z \) the original variable in \( M_\rho \) and by \( \xi := \rho z \) the corresponding point in \( M \). It is clear that the function \( u(x) \) is a solution to (1.2) if and only if the function \( v(z) := \rho^{\frac{N-2}{2}} u(\rho z) \) solves the problem
\[
\Delta_g v - \rho^2 h v + \rho^{-\frac{N-2}{2}} v^{p-\epsilon} = 0 \quad \text{in } M_\rho \tag{5.1}
\]

The function \( \tilde{\omega}(y_0, y) \) constructed in (3.13) defines an approximation to a solution of (1.2) near the geodesic through the natural change of variables (3.9).

It is useful to introduce the following notation. Let \( f(z) \) be a function defined in a small neighborhood of the scaled geodesic \( \Gamma_\rho := \tfrac{1}{\rho} \Gamma \). Through the change of variables (3.9) we denote
\[
\tilde{f}(y_0, y) = \mu e^{-\frac{N-2}{2}}(\rho y_0) f\left( \frac{1}{\rho} F(\rho y_0, \mu(\rho y_0) + d(\rho y_0)) \right), \tag{5.2}
\]

where the point \( \rho z = F(\rho y_0, \mu(\rho y_0) + d(\rho y_0)) \in M \) and \( \mu e, \mu_0 \) and \( d_0 \) are defined in (3.8) and (3.7). According this notation, we set \( \omega = \omega(z) \) the function corresponding to \( \tilde{\omega} = \tilde{\omega}(y_0, y) \).
Let $\delta > 0$ be a fixed number with $4\delta < \delta$, where $\delta$ is given in (3.1). We consider a smooth cut-off function $\zeta_{\delta}(s)$ such that $\zeta_{\delta}(s) = 1$ if $0 < s < \delta$ and $\zeta_{\delta}(s) = 0$ if $s > 2\delta$. Let us consider the cut-off function $\eta_{\delta}$ defined on the manifold $M_\rho$ by

$$
\eta_{\delta}(z) = \zeta_{\delta}\left(\frac{\text{dist}_g(\xi, \Gamma)}{\rho}\right) \quad \text{for } \rho z = \xi \in M.
$$

We remark that with this definition $\eta_{\delta}(z)$ does not depend on the parameter functions.

We define our global first approximation of the problem (1.2) $w(z)$ as

$$
w(z) = \eta_{\delta}(z)\omega(z).
$$

We look for a solution to problem (5.1) of the form $u = w + \phi$, namely

$$
\Delta_g \Phi + p\omega^{p-1}\Phi + N(\Phi) + E = 0 \quad \text{in } M_\rho,
$$

where

$$
N(\Phi) = \rho^{-\frac{N-2}{2}}(w + \phi)^{p-\epsilon} - \omega^{p-\epsilon} - p\omega^{p-1}\Phi - \rho^2 h(w + \phi)
$$

and

$$
E = \Delta_g w + \omega^{p-\epsilon}.
$$

We look for a solution $\Phi$ of (5.4) as $\Phi = \eta_{2\delta}\phi + \psi$ where the function $\phi$ is such that the corresponding function $\tilde{\phi}$ via the change of variables (5.2) is defined only in $\mathcal{D}$. It is immediate to check that $\Phi$ of this form solves (5.4) if the pair $(\psi, \phi)$ solves the following nonlinear coupled system:

$$
\Delta_g \psi + (1 - \eta_{2\delta})p\omega^{p-1}\psi = -2\nabla_g \phi \nabla_g \eta_{2\delta} - \phi \Delta_g \eta_{2\delta} - (1 - \eta_{2\delta})N(\eta_{2\delta}\phi + \psi) \quad \text{in } M_\rho
$$

and

$$
\mathcal{A}(\tilde{\phi}) + p\tilde{\phi}^{p-1}\tilde{\phi} = -N(\eta_{2\delta}\phi + \psi) - \mathcal{S}_c(\omega) - p\tilde{\phi}^{p-1}\tilde{\phi} \quad \text{in } \mathcal{D},
$$

where

$$
N(\tilde{\phi}) = \mu_{2\delta}^{-\frac{N-2}{2}}(\tilde{\omega})^{p-\epsilon} - \omega^{p-\epsilon} - p\tilde{\omega}^{p-1}\tilde{\phi} - \mu_{2\delta}^2 h(\tilde{\phi}), \quad \tilde{\phi} = \eta_{2\delta}\tilde{\phi} + \tilde{\psi}.
$$

Indeed, problem (5.4) in a scaled neighborhood of the geodesic looks like Problem 5.8 and the error $E$ given in (5.6) via the change of variables (5.2) is nothing but the error term $\mathcal{S}_c(\tilde{\omega})$ defined in (3.26).

Given $\phi$ such that $\tilde{\phi}$ is defined in $\mathcal{D}$, we first solve problem (5.7) for $\psi$ (see Section 6 of [7]).

**Lemma 5.1.** For any $R > 0$ there exists $r > 0$ such that for any function $\phi$ such that the corresponding function $\tilde{\phi}$ is defined only in $\mathcal{D}$ with $\|\tilde{\phi}\|_* \leq r$, there exists a unique solution $\psi = \psi(\phi)$ of (5.7) with

$$
\|\psi\|_\infty \leq R e^{\frac{N-2}{2}}\|\tilde{\phi}\|_*.
$$

Moreover, the nonlinear operator $\psi$ satisfies a Lipschitz condition of the form

$$
\|\psi(\phi_1) - \psi(\phi_2)\|_\infty \leq c e^{\frac{N-2}{2}}\|\phi_1 - \phi_2\|_*
$$

for some positive constant $c$ independent on $\epsilon$. 
Finally, we substitute \( \tilde{\psi} = \tilde{\psi}(\phi) \) (via the change of variables (5.2)) in Eq. (5.7) and we reduce the full problem (1.2) to solving the following (nonlocal) problem in \( \mathcal{D} \):

\[
\mathcal{A}(\tilde{\phi}) + p\omega^{p-1} \tilde{\phi} = -\mathcal{N}(\eta_{2\delta}^2 \tilde{\phi} + \tilde{\psi}(\phi)) - S_\epsilon(\tilde{\omega}) - p\omega^{p-1}\tilde{\psi}(\phi) \quad \text{in } \mathcal{D}.
\]  

(5.11)

5.2. The nonlinear projected problem

We can solve the following projected problem associated to (5.11): given \( \mu, d \) and \( \epsilon \) satisfying (3.18), find functions \( \tilde{\phi} \) and \( c_j(y_0) \) for \( j = 0, \ldots, N+1 \) such that

\[
\begin{aligned}
L(\tilde{\phi}) &= -S_\epsilon(\tilde{\omega}) + \mathfrak{N}(\tilde{\phi}) + \sum_{j=0}^{N} c_j Z_j \quad \text{in } \mathcal{D}, \\
\tilde{\phi}(y_0 + \frac{2\ell}{\rho}, y) &= \phi(y_0, Ay) \quad \text{for any } (y_0, y) \in \mathcal{D}, \\
\int_{\mathcal{D}_{y_0}} \tilde{\phi}Z_j dy &= 0 \quad \text{for any } y_0 \in \left[-\frac{\ell}{\rho}, \frac{\ell}{\rho} \right], \quad j = 0, 1, \ldots, N+1.
\end{aligned}
\]  

(5.12)

Here \( S_\epsilon(\tilde{\omega}) \) is given in (3.26) and

\[
L(\tilde{\phi}) := \mathcal{A}(\tilde{\phi}) + p\omega^{p-1} \tilde{\phi} \quad (\mathcal{A} \text{ is in Lemma 3.2 and } \omega \text{ is in (3.5)}),
\]

\[
\mathfrak{N}(\tilde{\phi}) := p(\omega^{p-1} - \tilde{\omega}^{p-1}) \tilde{\phi} - \mathcal{N}(\zeta_{2\delta} \tilde{\phi} + \tilde{\psi}(\phi)) - p\omega^{p-1}\tilde{\psi}(\phi) \quad (\mathcal{N} \text{ is in (5.9)}).
\]

Proposition 5.2. There exists \( c > 0 \) such that for all sufficiently small \( \epsilon \) and all \( \mu, d \) and \( \epsilon \) satisfying (3.18), problem (5.12) has a unique solution \( \tilde{\phi} = \tilde{\phi}(\mu, d, \epsilon) \) and \( c_j = c_j(\mu, d, \epsilon) \) which satisfies

\[
\|\tilde{\phi}\|_\star \leq c\epsilon^2.
\]  

(5.13)

Moreover, \( \tilde{\phi} \) depends Lipschitz continuously on \( \mu, d \) and \( \epsilon \) in the sense

\[
\|\tilde{\phi}(\mu_1, d_1, \epsilon_1) - \tilde{\phi}(\mu_2, d_2, \epsilon_2)\|_\star \leq c\epsilon^2 \|\mu_1 - \mu_2, d_1 - d_2, \epsilon_1 - \epsilon_2\|
\]

for some positive constant \( c \) independent of \( \epsilon \) and uniformly with respect to \( \mu, d \) and \( \epsilon \) which satisfy (3.18).

Proof. We argue exactly as in Section 7 of [7], using a contraction mapping argument and the linear theory developed in Proposition 7.3. \( \square \)

6. The reduced problem

6.1. The reduced system

We find \( N+1 \) equations relating \( \mu, d \) and \( \epsilon \) to get all the coefficients \( c_j \) in (5.12) identically equal to zero. To do this, we multiply Eq. (5.12) by \( Z_j \), for all \( j = 0, \ldots, N+1 \) and we integrate in \( y \). Thus, the system

\[
c_j(\rho y_0) = 0, \quad j = 0, 1, \ldots, N+1
\]

is equivalent to
\[
\int_{D_{y_0}} S_i(\tilde{\omega}) Z_j \, dy + \int_{D_{y_0}} (L(\tilde{\phi}) - \Re(\tilde{\phi})) Z_j \, dy = 0, \quad j = 0, 1, \ldots, N + 1,
\]
for any \( y_0 \in \left[ -\frac{\ell}{2}, \frac{\ell}{2} \right] \).

By Proposition 5.2 it follows that
\[
\int_{D_{y_0}} (L(\tilde{\phi}) - \Re(\tilde{\phi})) Z_j \, dy = \varepsilon^3 \theta,
\]
where \( \theta = \theta(p_{y_0}) \) is as in Lemma 3.4.

Hence the equations \( c_j = 0 \) are equivalent to the following limit system on \( N + 2 \) nonlinear ordinary differential equations:
\[
\begin{align*}
L_{N+1}(\mu) &:= -\ddot{\mu} + \left( a_n \sigma \pm \frac{b_n}{\mu_0^2} \right) \mu = -\alpha_{N+1}(x_0) - c_3 Q(x_0, d) + \varepsilon \ln \varepsilon |M_{N+1}|, \\
L_k(d) &:= -\ddot{d}_k + \sum_{j=1}^{N} R_{0j0k} d_j = \sqrt{\ell} M_k, \quad k = 1, \ldots, N, \\
L_0(\varepsilon) &:= \varepsilon a_0 \ddot{\varepsilon} + A_1 \varepsilon = -\alpha_0(x_0) - c_4 Q(x_0, d) - \beta(x_0) \mu + \varepsilon \ln \varepsilon |M_0|,
\end{align*}
\]
where \( \mu, d_1, \ldots, d_N, \varepsilon \in C^2(\mathbb{R}) \) and

- the functions \( \alpha_i \) and \( \beta \) are explicit functions of \( x_0 \), smooth and uniformly bounded in \( \varepsilon \) given in Lemma 3.4
- the operator \( Q \) is quadratic in \( d \) (see Lemma 3.4) and it is uniformly bounded in \( L^\infty_{2\ell}(\mathbb{R}) \) for \( (\mu, d, \varepsilon) \) satisfying (3.18)
- the operators \( M_i = M_i(\mu, d, \varepsilon) \) can be decomposed as \( M_i(\mu, d, \varepsilon) = A_i(\mu, d, \varepsilon) + K_i(\mu, d, \varepsilon) \), where
  - \( K_i \) is uniformly bounded in \( L^\infty_{2\ell}(\mathbb{R}) \) for \( (\mu, d, \varepsilon) \) satisfying (3.18) and it is compact
  - \( A_i \) depends on \( (\mu, d, \varepsilon) \) and their first and second derivatives and it satisfies
    \[
    \| A_i(\mu_2, d_2, \varepsilon_2) - A_i(\mu_1, d_1, \varepsilon_1) \| \leq o(1)\| (\mu_2 - \mu_1, d_2 - d_1, \varepsilon_2 - \varepsilon_1) \|
    \]
    uniformly for \( (\mu, d, \varepsilon) \) satisfying (3.18)
  - the dependence on \( (\mu, d, \varepsilon) \) is linear.

Our goal is to solve (6.1) in \( \mu, d \) and \( \varepsilon \). To do so, we first analyze the invertibility of the linear operator \( L_{N+1} \).

**Lemma 6.1.** For any \( f \in L^\infty_{2\ell}(\mathbb{R}) \), there exists a unique \( \mu \in C^2_{2\ell}(\mathbb{R}) \) solution of \( L_{N+1}(\mu) = f \). Moreover, there exists \( c \) such that
\[
\| \mu \|_\infty + \| \dot{\mu} \|_\infty \leq c \| f \|_\infty.
\]

**Proof.** The non-degeneracy condition of the solution \( \mu_0 \) translates into the fact that the periodic ODE
\[
-\ddot{\mu} + \left( a_n \sigma \pm \frac{b_n}{\mu_0^2} \right) \mu = 0 \quad \text{in} \ [0, 2\ell]
\]
has only the trivial solutions. Therefore the claim follows. \( \square \)

Next, we analyze the invertibility of the linear operator \( L_0 \).
Lemma 6.2. Assume

\[ |\varepsilon m^2 - \kappa^2| > \nu \sqrt{\varepsilon} \quad \text{for any } m = 1, 2, \ldots \]

for some \( \nu \) positive, where

\[ \kappa := \pi \sqrt{\lambda_1} \int_{-\ell}^{\ell} \frac{1}{\sqrt{a_0(s)}} ds. \]

For any \( f \in C^0_{2\ell}(\mathbb{R}) \cap L^\infty_{2\ell}(\mathbb{R}) \), there exists a unique solution \( e \in C^2_{2\ell}(\mathbb{R}) \) of \( L_0(e) = f \). Moreover, there exists \( c \) such that

\[ \varepsilon \| \dot{e} \|_\infty + \sqrt{\varepsilon} \| \ddot{e} \|_\infty + \| e \|_\infty \leq c \frac{1}{\sqrt{\varepsilon}} \| f \|_\infty. \]

Finally, if \( f \in C^2_{2\ell}(\mathbb{R}) \), then

\[ \varepsilon \| \dot{e} \|_\infty + \sqrt{\varepsilon} \| \ddot{e} \|_\infty + \| e \|_\infty \leq c [ \| \dot{f} \|_\infty + \| \ddot{f} \|_\infty + \| f \|_\infty ] . \]

Proof. We argue as in Lemma 8.2 of [7]. \( \Box \)

Finally, we consider the invertibility of the linear operator \( (L_1, \ldots, L_N) \).

Lemma 6.3. Assume the geodesic is non-degenerate. For any \( f = (f_1, \ldots, f_N) \) with \( f_k \in L^\infty_{2\ell}(\mathbb{R}) \), there exists a \( \tilde{d} = (d_1, \ldots, d_N) \) with \( d_k \in C^2_{2\ell}(\mathbb{R}) \) such that \( L_k(d) = f_k \) for any \( k = 1, \ldots, N \). Moreover, there exists \( c \) such that

\[ \| \tilde{d} \|_\infty + \| \dddot{d} \|_\infty + \| \ddot{d} \|_\infty \leq c \| f \|_\infty. \]

Proof. It is useful to point out that assumption (1.3) about non-degeneracy of \( \Gamma \) in normal coordinates translates exactly into the fact that the linear system of ODE’s

\[- \ddot{d}_k + \sum_{j=1}^{N} R_{0j0k} d_j = 0, \quad \text{in } [0, 2\ell], \quad k = 1, \ldots, N \]

has only the trivial solution \( d \equiv 0 \) satisfying the periodicity condition (3.6). Therefore, the claim follows. \( \Box \)

6.2. The choice of parameters: the proof completed!

Now, we are ready to complete the proof, finding parameters which solve the reduced problem (6.1). First, by Lemma 6.1 we find \( \tilde{\mu}_0 \) solution of

\[ L_{N+1}(\tilde{\mu}_0) = -\alpha_{N+1}(x_0), \quad \text{with } \| \tilde{\mu}_0 \|_\infty + \| \dot{\tilde{\mu}}_0 \|_\infty + \| \ddot{\tilde{\mu}}_0 \|_\infty \leq c. \]

Then, by Lemma 6.2 we find \( \tilde{\epsilon}_0 \) solution of

\[ L_0(\tilde{\epsilon}_0) = -\alpha_0 - \beta \tilde{\mu}_0, \quad \text{with } \varepsilon \| \tilde{\epsilon}_0 \|_\infty + \sqrt{\varepsilon} \| \ddot{\tilde{\epsilon}}_0 \|_\infty + \| \dot{\tilde{\epsilon}}_0 \|_\infty \leq c. \]

Therefore, \( \| (\tilde{\mu}_0, 0, \tilde{\epsilon}_0) \| \leq c \). Let us define
\[
\mu = \hat{\mu}_0 + \hat{\mu}_1, \quad d = \hat{d}_1, \quad e = \hat{e}_0 + \hat{e}_1.
\]

The system (6.1) reduces to

\[
\begin{aligned}
L_{N+1}(\hat{\mu}_1) &= -c_3Q(x_0, \hat{d}_1) + \epsilon|\ln \epsilon|M_{N+1}, \\
L_k(\hat{d}_1) &= \sqrt{\epsilon}M_k, \quad k = 1, \ldots, N, \\
L_0(\hat{e}_1) &= -c_4Q(x_0, \hat{d}_1) - \beta(x_0)\hat{\mu}_1 + \epsilon|\ln \epsilon|M_0.
\end{aligned}
\]

(6.2)

Let us observe now that the linear operator

\[
\mathcal{L}(\hat{\mu}_1, \hat{d}_1, \hat{e}_1) = (L_{N+1}(\hat{\mu}_1), L_N(\hat{d}_1), \ldots, L_1(\hat{d}_1), L_0(\hat{e}_1))
\]

is invertible with bounds for \(\mathcal{L}(\hat{\mu}_1, \hat{d}_1, \hat{e}_1) = (f, g, h)\) given by

\[
\|\langle \hat{\mu}_1, \hat{d}_1, \hat{e}_1 \rangle\| \leq C[\|f\|_\infty + \|g\|_\infty + \epsilon^{-1/2}\|h\|_\infty].
\]

Finally, by the contraction mapping principle it follows that, the problem (6.2) has a unique solution with

\[
\|
\hat{\mu}_1\|_\infty < c\epsilon|\ln \epsilon|, \quad \|
\hat{d}_1\|_\infty < \sqrt{\epsilon}, \quad \|
\hat{e}_1\|_\infty < \sqrt{\epsilon|\ln \epsilon|}.
\]

That concludes the proof.

7. The linear theory

Here we recall a linear theory necessary to solve problem (3.11), which has been developed in Section 3 of [7].

Let us consider the operator \(\mathcal{L}_0 := \Delta_{\mathbb{R}^N} + pw^{p-1}\). It is well-known that the \(L^2\)-null space of the operator \(\mathcal{L}_0\) is \(N + 1\)-dimensional and spanned by the functions

\[
Z_j(y) := \partial_j w(y), \ j = 1, \ldots, N \quad \text{and} \quad Z_{N+1}(y) := y \cdot \nabla w(y) + \frac{N-2}{2} w(y).
\]

Moreover it is known that (see [7]) the operator \(\mathcal{L}_0\) has one negative eigenvalue \(-\lambda_1 < 0\), whose corresponding eigenfunction \(Z_0\) (normalized to have \(L^2\)-norm equal to 1) decays exponentially at infinity with exponential order \(O(e^{-\sqrt{\lambda_1}|x|})\).

The following results (see Lemma 3.1 of [7] and also [8]) are useful in order to obtain a priori estimates and a solvability theory for problem (3.11).

**Lemma 7.1.** Assume that \(\lambda \notin \{0, \pm \sqrt{\lambda_1}\}\). Then for \(g \in L^\infty(\mathbb{R}^N)\), there exists a unique bounded solution of

\[
(\mathcal{L}_0 - |\lambda|^2)\psi = g
\]

in \(\mathbb{R}^N\). Moreover

\[
\|\psi\|_{L^\infty} \leq c_\lambda \|g\|_{L^\infty}
\]

for some constant \(c_\lambda > 0\) only depending on \(\lambda\).
Lemma 7.2. Let $\phi$ a bounded solution of
\[
\partial_{00}\phi + \Delta_y\phi + pw^{p-1}\phi = 0 \quad \text{in } \mathbb{R}^{N+1}.
\]
Then $\phi(y_0, y)$ is a linear combination of the functions $Z_j$, $j = 1, \ldots, N+1$, $Z_0(y) \cos(\sqrt{\lambda_1} y_0)$, $Z_0(y) \sin(\sqrt{\lambda_1} y_0)$.

Now, we study a slightly more general problem than (3.11) that involves the essential features needed. For any constant $M > 0$ we consider the domain $\mathcal{D}$ defined as
\[
\mathcal{D} := \{(y_0, y) \in \mathbb{R} \times \mathbb{R}^N : |y| < M\}
\]
and given a function $\phi$ defined on $\mathcal{D}$, an operator of the form
\[
L(\phi) := b(y_0)\partial_{00}\phi + \Delta_y\phi + pw^{p-1}\phi + \sum_{i,j} b_{ij}(y_0, y) \partial_{ij}\phi + \sum_i b_i(y_0, y) \partial_i\phi + d(y_0, y) \phi.
\]
Then for a given function $g$ we want to solve the following projected problem:
\[
\begin{cases}
L(\phi) = g + \sum_{j=0}^{N+1} c_j(y_0) Z_j(y) & \text{in } \mathcal{D}, \\
\int_{\mathcal{D}_{y_0}} \phi(y_0, y) Z_j(y) dy = 0 & \text{for any } y_0 \in \mathbb{R}, j = 0, \ldots, N,
\end{cases}
\]
where
\[
\mathcal{D}_{y_0} := \{y \in \mathbb{R}^N : (y_0, y) \in \mathcal{D}\}.
\]
We fix a number $2 \leqslant \nu < N$ and consider the $L^\infty$-weighted norms
\[
\|\phi\|_* := \sup_{\mathcal{D}} (1 + |y|^{\nu-2}) |\phi(y_0, y)| + \sup_{\mathcal{D}} (1 + |x|^{\nu-1}) |D\phi(x_0, x)|,
\]
\[
\|g\|_{**} := \sup_{\mathcal{D}} (1 + |y|^{\nu}) |g(y_0, y)|.
\]
We assume that all functions involved are smooth. The following result (see Proposition 3.2 of [7]) establishes existence and uniform a priori estimates for problem (7.2) in the above norms, provided that appropriate bounds for the coefficients hold.

Proposition 7.3. Assume that $N \geqslant 7$ and $N - 2 \leqslant \nu < N$. Assume that there exists $m > 0$ such that
\[
m \leqslant b(y_0) \leqslant m^{-1} \quad \text{for any } y_0 \in \mathbb{R}.
\]
There exist $\delta > 0$ and $C > 0$ such that if
\[
M \|\partial_{00}b\|_\infty + \sum_{i,j} (\|b_{ij}\|_\infty + \|Db_{ij}\|_\infty) + \sum_i (\|b_i\|_\infty + \|D_i\|) + \|1 + |y|\|_\infty < \delta
\]
then for any $g$ with $\|g\|_{**} < \infty$ there exists a unique solution $\phi = T(g)$ of problem (7.2) with $\|\phi\|_* < \infty$ and it holds true that
\[
\|\phi\|_* \leqslant C\|g\|_{**}.
\]
Appendix A

A.1. **Proof of (3.4)**

Let \( E_0, E_1, \ldots, E_N \) be the coordinate vectors as given in the Introduction. By our choice of coordinates it follows that \( \nabla_E E = 0 \) on \( \Gamma \) for any vector field \( E \), that is a linear combination (with coefficients depending only on \( x_0 \)) of the \( E_j \)'s, \( j = 1, \ldots, N \).

In particular, for any \( i, j = 1, \ldots, N \) and for any \( t \in \mathbb{R} \), we have \( \nabla_{E_i + tE_j}(E_i + tE_j) = 0 \) on \( \Gamma \), which implies \( \nabla_{E_i}E_j + \nabla_{E_j}E_i = 0 \) for every \( i, j = 1, \ldots, N \).

Using the fact that \( E_i \)'s are coordinate vectors for \( j = 1, \ldots, N \) and in particular \( \nabla_{E_a}E_b = \nabla_{E_b}E_a \) for all \( a, b = 0, \ldots, N \), we obtain that \( \nabla E_j E_i = 0 \) for every \( i, j = 1, \ldots, N \). The geodesic coordinate for \( \Gamma \) translates precisely into \( \nabla E_0 E_0 = 0 \).

These facts immediately yield

\[
\partial_m g_{ij} = E_m(E_i, E_j) = \langle \nabla_{E_m}E_i, E_j \rangle + \langle E_i, \nabla_{E_m}E_j \rangle = 0
\]  
(A.1)

on \( \Gamma \) with \( i, j, m = 1, \ldots, N \).

Moreover, since \( E_a \)'s are coordinate vectors for \( a = 0, \ldots, N \), we obtain

\[
\partial_m g_{0j} = E_m(E_0, E_j)
= \langle \nabla_{E_m}E_0, E_j \rangle + \langle E_0, \nabla_{E_m}E_j \rangle
= \langle \nabla_{E_0}E_m, E_j \rangle + \langle E_0, \nabla_{E_m}E_j \rangle = 0
\]  
(A.2)

on \( \Gamma \) with \( m, j = 1, \ldots, N \).

Here we used the fact that \( \nabla_{E_0}E_m = 0 \) on \( \Gamma \), namely that \( \nabla_{E_0}E_m \) has zero normal components. Moreover by (A.1) it follows that

\[
\partial_m g_{00} = 0 \quad \text{on} \ \Gamma.
\]  
(A.3)

We can also prove that the components \( R_{0m0j} \) of the curvature tensor are given by

\[
R_{0m0j} = -\frac{1}{2} \partial_{mj}g_{00}.
\]  
(A.4)

Indeed, we have

\[
-R_{0m0j} = \langle R(E_0, E_j)E_0, E_m \rangle
= \langle \nabla_{E_0}E_j E_0, E_m \rangle - \langle \nabla_{E_j}E_0 E_0, E_m \rangle
= \langle \nabla_{E_0} E_j E_0, E_m \rangle - E_j \langle \nabla_{E_0} E_0, E_m \rangle - E_j \langle E_0, \nabla_{E_0} E_j \rangle
= \langle \nabla_{E_0} E_j E_0, E_m \rangle - E_j \langle \nabla_{E_0} E_0, E_m \rangle + E_j \langle E_0, \nabla_{E_0} E_j \rangle
= \langle \nabla_{E_0} E_j E_0, E_m \rangle + E_j \langle E_0, \nabla_{E_m} E_0 \rangle
= \frac{1}{2} E_j E_m \langle E_0, E_0 \rangle + E_0 \langle \nabla_{E_i} E_0, E_m \rangle - \langle \nabla_{E_j} E_0, E_0 \rangle
= \frac{1}{2} \partial_{mj}g_{00},
\]

where here we have used the above properties and the fact that
\[ \nabla E_j E_0 = \nabla E_0 E_j = \frac{1}{2} \partial_{ij} g_{00} E_0 = 0. \]

By (A.2), (A.4), (A.3) and (A.1) the claim follows.

References