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Numerical solution of some boundary value problems in nonlinear magneto-elasticity

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Abstract

In the context of the theory of nonlinear magneto-elastic deformations, the problem of the extension (shortening) of a cylinder of finite length under the influence of a magnetic field applied far away in free space is studied. The boundary value problem is solved using the finite element method. There exist exact solutions for the problem, which are based on the assumption of working with infinitely long cylinders. In this communication, results are obtained for different relations between the radius of the cylinder and its length, comparing the results for the magnetic field between *short* and *long* cylinders. As well as this, the influence of applying such external traction through the direct contact with an *external machine* has been studied.

Keywords

Magneto-elasticity, finite deformations, boundary-value problems, numerical solutions

Introduction

In the recent years there has been a growing interest in the study of the behaviour of a relatively new class of elastomers that react to magnetic fields. These materials consist of a matrix made with a rubber-like material filled with magneto-active particles; see, for example, Albanese and Cunefare (2003), Bednarek (1999), Boczkowska and Awietjan (2009), Bossis et al. (2001), Farshad and Le Roux (2004), Ginder et al. (1999, 2000, 2001, 2002), Lokander and Stenberg (2003), Li and Zhang (2008), Varga et al. (2005, 2006) and Yalcintas and Dai (2004). These magneto-sensitive (MS) elastomers are capable of undergoing large elastic deformations under the application of magnetic fields (see Bednarek, 1999; Bossis et al., 2001; Ginder et al., 2002; Jolly et al., 1996).

The application of these materials requires a deep understanding of their mechanical and magnetic properties. The development of theories to understand the interaction of electromagnetic fields with continua has attracted the attention of the scientific community for a long time; see, for example, the works by Eringen and Maugin (1990), Hutter et al. (2006) and Maugin (1988). In the case of solids interacting with magnetic fields, we mention in particular the monograph by Brown (1966). A revision of these theories has been carried out in the recent years by different researchers, such as Brigadnov and Dorfmann (2003), Dorfmann and Ogden (2003), Kankanala and Triantafyllidis (2004), Ogden and Steigmann (2010) and Steigmann (2004). We mention especially the series of works by Dorfmann and Ogden (2004a,b, 2005a,b), Vu and Steinmann (2007b, 2010a), Barham et al. (2009, 2010) and Maugin (2009) among others. Experimental data is currently scarce and can be found, for example, in the works by Bellan and Bossis (2002), Bossis et al. (2001), Ginder et al. (1999), Jolly et al. (1996) and the recent paper by Danas et al. (2012).

In the present work we consider as a starting point the theory developed by Dorfmann and Ogden (2004a,b, 2005a,b), which is based on the assumption that there exists an amended total energy function and on the use of a total stress tensor, which incorporates in its definition the magnetic body forces. In the case we want to solve some boundary value problems considering bodies made of MS materials, which are surrounded by free space, it is necessary to solve two highly nonlinear coupled partial differential equations for the body, plus the simplified form of the Maxwell

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equations for the surrounding free space; as well as this, it is required the satisfaction of some continuity conditions for the stresses and the magnetic variables across the boundary of the body (see, for example, Kovetz, 2000). To obtain exact solutions for the problem described previously is very difficult, and so far all exact solutions that can be found in the literature (see, for example, Pucci and Saccomandi, 1993; Dorfmann and Ogden, 2004a,b, 2005a,b) have been obtained under the assumption of considering infinite geometries, working, for example, with cylinders and tubes of finite radius but infinite length, or slabs of finite thickness but infinite length and width. The motivation of such assumption lies in the need to deal with the continuity conditions for the magnetic variables (see, for example, Kovetz, 2000).

Owing to the difficulties in solving boundary value problems exactly even for simple finite geometries, there is a need for numerical methods of solutions. Some works presenting numerical results for MS bodies undergoing large deformations have been published recently, we mention, for example, the papers by Barham et al. (2009, 2010), Bustamante et al. (2007, 2011) and Vu et al. (2007) (in the similar nonlinear electro-elastic problem).

In Bustamante et al. (2011) we see some results for a slab (plane strain problem) surrounded completely by vacuum, deforming under the effect of a magnetic field applied far away in the surrounding free space. The boundary value problem was solved using the finite element method with the commercial code (Comsol, 2007).

In the present work we use the same methodology presented in Bustamante et al. (2011) in order to solve an additional boundary value problem; the objective is three-fold:

- For a MS cylinder surrounded by free space, under the effect of an axial uniform magnetic field applied far away, we study the effect of considering different sizes for the exterior free space.[1] Theoretically such free space must be infinite, but when working with a standard finite element formulation, it is necessary to consider a large but finite surrounding free space. The interaction of the body with vacuum implies (in the case of finite geometries) that the surrounding magnetic field is in general non-uniform (see, for example, Bustamante et al. (2007, 2011) and for the similar electro-elastic problem see Vu and Steinmann (2010b)).
  It is important to assess the effect of the size of the surrounding free space on the distribution of magnetic field inside the body, in order to obtain (approximately) the minimum size of such exterior free space, such that the body would behave as if it would be surrounded by infinite free space.
- A second main problem considered in this work is to study the mechanical and especially the magnetic behaviour of a cylinder, for different relations between the radius and its length. The exact solutions found, for example, in Pucci and Saccomandi (1993) for the case of MS cylinders or tubes, have been obtained under the assumption of infinitely long bodies; under such assumption, if the external (far away) axial component of the magnetic field is uniform, then inside the cylinder the field is also uniform and only the axial component is not zero. When we model the behaviour of cylinders of finite length, near its corners the field is highly non-uniform (see, for example, Bustamante et al., 2007), and an interesting issue is to study the distribution of magnetic field for cylinders of different lengths (for a fixed value for the initial radius), in order to see the difference in behaviour between the cases of *short* and *long* MS cylinders.
- In some studies on the behaviour of magneto- and electro-active bodies, only the MS and electro-sensitive materials have been considered in the analysis. As a first approximation that seems to be a good starting point, however, as pointed out, for example, by Bustamante et al. (2007), the surrounding free space may have an important effect on the behaviour of a MS body, in particular when the geometries show corners or other irregularities in their surfaces. Although the model of a body surrounded by free space is an improvement in comparison with the case of considering only a MS body, from the point of view of future applications, an infinite surrounding free space may not be the best approximation for many practical situations, where we may have the MS body interacting with other bodies in a given device (see, for example, Figures 3, 4 and 5 in Bustamante, 2009). Having this in mind, we study the behaviour of a body made of a MS elastomer in contact with two large magneto-active (approximately rigid) bodies, which are used in order to simulate the interaction of the MS body with an *external machine* (see Batra (1972) and Bustamante (2009) for details about that concept).

The structure of the paper is described now. In Section 2 we present briefly the main equations of the kinematics of deformation, the equations of magnetostatics and the main elements of the theory for MS elastomers of Dorfmann and Ogden (2004a,b, 2005b). In Section 3 we discuss the boundary value problem in nonlinear magneto-elasticity, and details are given on the problem treated here, namely the extension (shortening) of a cylinder under a uniform axial magnetic field applied far away. In Section 4 some numerical results are presented for that problem. Finally, in Section 5 some final remarks are given.

This work is based on the results presented in the thesis by Salas (2012).

Basic equations

Kinematics and the equations of magneto-statics

We consider a body that in the reference configuration $\mathcal{B}_0$ is stress free. The vector $\mathbf{X}$ denotes a point of the body in the reference configuration. Under the effect of external magnetic fields and mechanical loads, the particles of the body occupy the new position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, where we are assuming there exists a one-to-one mapping $\boldsymbol{\chi}$, and we only consider quasi-static deformations. The current configuration is denoted by $\mathcal{B}$. The displacement field $\mathbf{u}$ is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X}. \tag{1}$$

The deformation gradient $\mathbf{F}$ and the right and left Cauchy–Green tensors[2] $\mathbf{c}$, $\mathbf{b}$ are defined as

$$\mathbf{F} = \mathrm{Grad}\boldsymbol{\chi}, \quad \mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \tag{2}$$

where $(\ )^{\mathrm{T}}$ is the transpose of a second-order tensor. The differential operators gradient, divergence and curl in the current configuration are denoted as

$$\mathrm{grad}, \quad \mathrm{div}, \quad \mathrm{curl},$$

respectively. In the reference configuration, where the derivatives are taken with respect to $\mathbf{X}$, the operators are denoted as

$$\mathrm{Grad}, \quad \mathrm{Div}, \quad \mathrm{Curl}.$$

We use the convention that the divergence operator acts on the left leg of a second order tensor. We also use the notation

$$J = \det \mathbf{F}, \tag{3}$$

and it is assumed that $\boldsymbol{\chi}$ is such that $J>0$. For more details, see, for example, Ogden (1997).

The results presented in this paper are based on the assumption of time-independent deformation and magnetic fields, considering also there is no free electric current within the bodies.

The magnetic field and the magnetic induction in the current configuration are denoted by $\mathbf{H}$ and $\mathbf{B}$, respectively. In free space they are related by (see Kovetz, 2000)

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{4}$$

where μ_0 is the permeability of vacuum.

For magnetizable materials an additional vector field, the magnetization $\mathbf{M}$ is defined as

$$\mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H}. \tag{5}$$

For the problems considered in this work, under the assumptions stated previously, the simplified form of the Maxwell equations are (see Kovetz, 2000):

$$\mathrm{curl}\,\mathbf{H} = \mathbf{0}, \quad \mathrm{div}\,\mathbf{B} = 0. \tag{6}$$

The magnetic field and the magnetic induction have to satisfy the continuity conditions across the boundary $\partial\mathcal{B}$ (in absence of free surface electric currents):

$$\mathbf{n} \cdot [\![\mathbf{B}]\!] = 0, \quad \mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0}, \tag{7}$$

where $\mathbf{n}$ is the outward unit normal vector to $\partial\mathcal{B}$, and where $[\![\ \ ]\!]$ is defined, for example, as $[\![\mathbf{B}]\!] = \mathbf{B}^{\mathrm{o}} - \mathbf{B}^{\mathrm{i}}$, where $^{\mathrm{o}}$ and $^{\mathrm{i}}$ signify *outside* and *inside* the material, respectively.

Equations (6) and the continuity conditions (7) are in Eulerian form. We can define Lagrangian counterparts of $\mathbf{B}$ and $\mathbf{H}$ as (see, for example, Dorfmann and Ogden, 2004a,b, 2005b):

$$\mathbf{B}_{\mathrm{l}} = J\mathbf{F}^{-1}\mathbf{B}, \quad \mathbf{H}_{\mathrm{l}} = \mathbf{F}^{\mathrm{T}}\mathbf{H}. \tag{8}$$

Considering the identities $J\,\mathrm{div}\,\mathbf{B} = \mathrm{Div}(J\mathbf{F}^{-1}\mathbf{B})$ and $J\mathbf{F}^{-1}\mathrm{curl}\,\mathbf{H} = \mathrm{Curl}\,(\mathbf{F}^{\mathrm{T}}\mathbf{H})$, Equations (6) can be recast in Lagrangian form as

$$\mathrm{Curl}\,\mathbf{H}_{\mathrm{l}} = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{B}_{\mathrm{l}} = 0, \tag{9}$$

while (7) become

$$\mathbf{N} \cdot [\![\mathbf{B}_{\mathrm{l}}]\!] = 0, \quad \mathbf{N} \times [\![\mathbf{H}_{\mathrm{l}}]\!] = \mathbf{0}, \tag{10}$$

where $\mathbf{N}$ is the unit outward normal vector to the reference boundary $\partial\mathcal{B}_0$, which is associated with $\mathbf{n}$ through Nanson's formula.

Magneto-mechanical equilibrium and constitutive equations

We assume there are no mechanical body forces. Following, for example, the treatment by Dorfmann and Ogden (2004a,b, 2005b), the magnetic body forces are calculated as the divergence of a second-order tensor, which is added to the Cauchy stress tensor, obtaining a total stress tensor $\boldsymbol{\tau}$ (in the current configuration). The equilibrium equation can be expressed as

$$\mathrm{div}\,\boldsymbol{\tau} = \mathbf{0}, \tag{11}$$

and the total stress tensor has to satisfy the boundary conditions

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \boldsymbol{\tau}_{\mathrm{m}}\mathbf{n}, \quad \text{on} \quad \partial\mathcal{B}, \tag{12}$$

where $\mathbf{t}_a$ is the mechanical traction and $\boldsymbol{\tau}_m$ is the Maxwell stress tensor, which is evaluated on the free space near the boundary of the body, defined by (see, for example, Jackson, 1999; Kovetz, 2000):

$$\boldsymbol{\tau}_m = \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{H})\mathbf{I}. \tag{13}$$

Remark. As mentioned in the Introduction, one objective of this work is to study the behaviour of some bodies made with MS elastomers, when considering interactions with free space and also with other external bodies. The boundary condition (12) means we are assuming that when a MS body is under the effect of a mechanical traction, we need to add the Maxwell stresses as an external load as well (see Bustamante et al., 2008). When the MS body is in contact only with free space, we still need to consider the term $\boldsymbol{\tau}_m\mathbf{n}$ as external traction. As discussed by Bustamante (2009) an open question is: What would be a better model for the situation where the body is under the effect of a mechanical surface traction? The mechanical surface traction is a simplified model of the actual contact of our MS body with an external body, and if we agree about this, the question is whether (12), where we also incorporated the Maxwell stresses, is actually the best model for the external traction. This is why in the present work we explore the behaviour of a MS body, assuming that in some parts of its boundary is perfectly attached to an external body, which we call a 'machine'.

In order to solve boundary value problems, we require constitutive equations that relate the different variables of the problem. We assume that the total stress tensor can be expressed as a function of the deformation gradient and the magnetic field. We use the constitutive theory developed by Dorfmann and Ogden (2004a,b, 2005b) due to its simplicity; in such a case for compressible bodies we have

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{B} = -J^{-1}\mathbf{F}\frac{\partial\Omega^*}{\partial\mathbf{H}_l}, \tag{14}$$

where $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{H}_l)$ is called the amended energy function per unit of volume.

For an isotropic magneto-active body we have that $\Omega^* = \Omega^*(I_1, I_2, I_3, I_4, I_5, I_6)$, where the invariants I_i, $i = 1, 2, \ldots, 6$ are given as (see Spencer, 1971):

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,\mathbf{c}^2], \quad I_3 = \det\mathbf{c}, \tag{15}$$

$$I_4 = \mathbf{H}_l \cdot \mathbf{H}_l, \quad I_5 = (\mathbf{c}\mathbf{H}_l) \cdot \mathbf{H}_l, \quad I_6 = (\mathbf{c}^2\mathbf{H}_l) \cdot \mathbf{H}_l. \tag{16}$$

From (14) the explicit expressions for $\boldsymbol{\tau}$ and $\mathbf{B}$ for isotropic bodies are:

$$\boldsymbol{\tau} = J^{-1}[2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) + 2I_3\Omega_3^*\mathbf{I} + 2\Omega_5^*\mathbf{bH} \otimes \mathbf{bH} + 2\Omega_6^*(\mathbf{bH} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{bH})], \tag{17}$$

$$\mathbf{B} = -2J^{-1}(\Omega_4^*\mathbf{bH} + \Omega_5^*\mathbf{b}^2\mathbf{H} + \Omega_6^*\mathbf{b}^3\mathbf{H}), \tag{18}$$

respectively, where $\Omega_i^* = \frac{\partial\Omega^*}{\partial I_i}$, $i = 1, 2, \ldots, 6$.

A prototype energy function

For the numerical calculations, the same energy function Ω^* used in Bustamante et al. (2011) is considered here, which is a modified version of the function that was proposed by Bustamante (2010) from experimental data found in the papers by Ginder et al. (1999) and Bellan and Bossis (2002). The energy function has the form

$$\Omega^* = \Omega^*_{\mathrm{iso}} + \Omega^*_{\mathrm{vol}} + \Omega^*_0, \tag{19}$$

where Ω^*_0 is a constant value and

$$\Omega^*_{\mathrm{iso}} = \frac{1}{2}(g_0 + g_1 I_4)(\bar{I}_1 - 3) - m_0 m_1 \log\left[\cosh\left(\frac{\sqrt{I_4}}{m_1}\right)\right] - c_0 I_4 + c_1\mu_0\bar{I}_5, \tag{20}$$

$$\Omega^*_{\mathrm{vol}} = \frac{1}{2}\kappa(J-1)^2, \tag{21}$$

where g_0, g_1, m_0, m_1, $c_0 = \mu_0(c_1 - 1)/2$ and κ are constants. Regarding $\bar{I}_1$ and $\bar{I}_5$, these modified invariants are given as (see, for example, Flory, 1961; Ogden, 1976)

$$\bar{I}_1 = J^{-2/3}I_1, \quad \bar{I}_5 = J^{-2/3}I_5. \tag{22}$$

The values of the different constants in (20), (21) are shown in Table 1.

Boundary value problem

In this section we speak about the boundary value problem in nonlinear magneto-elasticity, and thereafter details of the specific problems to be solved are given.

Two general types of boundary value problems are studied, namely a body completely surrounded by free space, as depicted in Figure 1(A), and a body in contact

Table 1. Values for the constants used in (20) and (21).

g_0 (Pa)	g_1 ($\mathrm{PaA^{-2}m^2}$)	m_0 (T)	m_1 ($\mathrm{Am^{-1}}$)	c_1	μ_0 ($\mathrm{NA^{-2}}$)	κ (Pa)
10^5	-10^{-6}	0.4998	309339.5	1250	1.2566×10^{-6}	10^5

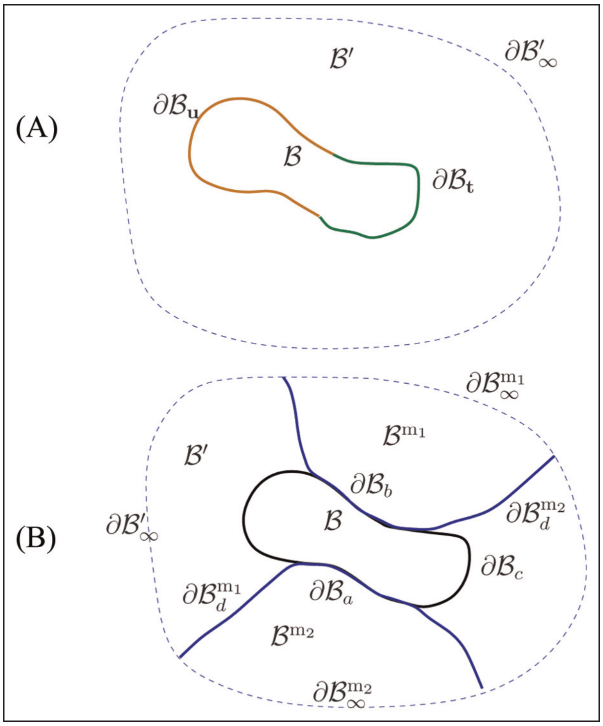

Figure 1. (A) MS body surrounded completely by free space. (B) MS body surrounded by free space and in contact with two rigid bodies $\mathcal{B}^{m_1}$, $\mathcal{B}^{m_2}$.

with two rigid magnetizable bodies, as depicted in Figure 1(B), which is used in order to model the interaction with a *machine*.

Boundary value problem for a MS body surrounded completely by free space

For this boundary value problem we need to solve Equations (6) for the body $\mathcal{B}$ and the surrounding free space $\mathcal{B}'$. Let's assume there exists a scalar potential φ such that

$$\mathbf{H} = -\operatorname{grad}\varphi, \tag{23}$$

then $(6)_1$ is satisfied automatically.[3] Therefore, using (23) and (4) for free space $\mathcal{B}'$ we need to solve $(6)_2$

$$\operatorname{div}\operatorname{grad}\varphi = 0 \quad \text{in} \quad \mathcal{B}'. \tag{24}$$

From (18) for $\mathcal{B}$ we have $\mathbf{B} = \mathcal{C}\operatorname{grad}\varphi$, where $\mathcal{C} = 2J^{-1}(\Omega_4^*\mathbf{b} + \Omega_5^*\mathbf{b}^2 + \Omega_6^*\mathbf{b}^3)$ is a second-order tensor that depends on $\boldsymbol{\chi}$ and φ. For the body $\mathcal{B}$ we need to solve the nonlinear partial differential equation

$$\operatorname{div}(\mathcal{C}\operatorname{grad}\varphi) = 0 \quad \text{in} \quad \mathcal{B}. \tag{25}$$

For the body we also need to solve (11)

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}. \tag{26}$$

We have two coupled nonlinear partial differential equations for the body $\mathcal{B}$, which have to be solved in order to find $\boldsymbol{\chi}$ and φ, while we have one linear partial differential equation in $\mathcal{B}'$ for φ in vacuum.

Regarding the boundary or continuity conditions, from the point of view of the magnetic variables, using (23) the continuity conditions (7) becomes

$$[\![\varphi]\!] = 0, \quad \mathbf{n}\cdot[\![\mathbf{B}]\!] = 0, \quad \text{on} \quad \partial\mathcal{B}. \tag{27}$$

If $\partial\mathcal{B}' = \partial\mathcal{B}\cup\partial\mathcal{B}'_\infty$ (see Figure 1(A)), for the far away surface $\partial\mathcal{B}'_\infty$ some boundary conditions are also needed. In this work we assume that

$$\varphi = \tilde{\varphi} \quad \text{on} \quad \partial\mathcal{B}'^{\varphi}_\infty, \qquad \mathbf{B}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}'^{\mathbf{B}}_\infty, \tag{28}$$

where $\tilde{\varphi}$ is a prescribed value for the potential and $\partial\mathcal{B}'^{\varphi}_\infty \cap \partial\mathcal{B}'^{\mathbf{B}}_\infty = \emptyset$, $\partial\mathcal{B}'^{\varphi}_\infty \cup \partial\mathcal{B}'^{\mathbf{B}}_\infty = \partial\mathcal{B}'_\infty$.

Regarding the boundary conditions for the mechanical part of the problem we have (see Figure 1(A)):

$$\mathbf{u} = \tilde{\mathbf{u}} \quad \text{on} \quad \partial\mathcal{B}_\mathbf{u}, \qquad \boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{a} \quad \text{on} \quad \partial\mathcal{B}_\mathbf{t}, \tag{29}$$

where $\tilde{\mathbf{u}}$ is a given value for the displacement on a part of $\partial\mathcal{B}$.

Remark. Regarding the boundary condition $(29)_2$, the original continuity condition for the stress (12) requires the addition of the term $\boldsymbol{\tau}_\mathrm{m}\mathbf{n}$ as external traction. This external traction, due to the Maxwell stresses calculated with the magnetic field outside the body, has to be applied on the surfaces free of mechanical traction (see Bustamante et al., 2008); however, the application of that condition is not clear for the surface $\partial\mathcal{B}_\mathbf{u}$ and $(29)_1$, where a Dirichlet boundary condition is assumed to exist.

In this work as a first approximation, we do not consider the effect of $\tau_\mathrm{m}\mathbf{n}$ as external traction for the parts of the surface of the MS body in contact with free space. Regarding the parts of the surface of the body under the effect of mechanical traction, following the arguments given by McMeeking and Landis (2005, paragraph after Equation (11) in that paper) for the similar problem considering electro-elastic bodies, the term $\tau_\mathrm{m}\mathbf{n}$ can be considered as a part of the definition of $\mathbf{t}_\mathrm{a}$.

In Figure 2(A) there is a depiction of a cylinder of finite length, which is under the effect of a uniform axial magnetic field applied far away.

It is assumed that $\boldsymbol{\chi}$ and φ do not depend on the azimuthal position θ (in a system of cylindrical coordinates), and in Figure 2(B) is shown the axisymmetric model of the cylinder and the free space surrounding it. The MS cylinder has a radius r_b and a length l_b. The free space is assumed to have a cylindrical shape far away, it has a radius R_f and a length L_f. The boundaries of the cylinder and the free space are denoted as (a), (b), …, (h), respectively.

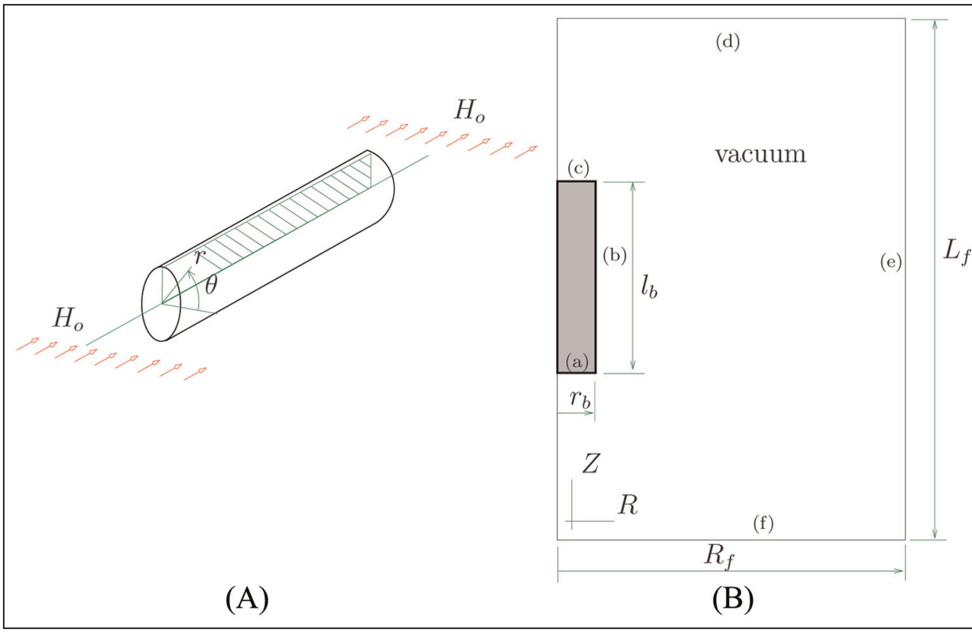

Figure 2. Model of a cylinder under the influence of an external axial magnetic field, which is uniform far away.

For this problem the boundary and the continuity conditions (discussed previously in Section 2.1) are (see Figure 2(B)) as follows:

- Boundaries (d), (f): A given constant value for $\varphi = \pm\tilde{\varphi}$, respectively.[4]
- Boundaries (a), (c): If $^{(i)}$ and $^{(o)}$ denote inside and outside the body, when the body is still not deformed, the condition $(7)_1$ is equivalent to $B_z^{(i)} = B_z^{(o)}$, where $B_z^{(o)} = \mu_0 H_z^{(o)}$. The scalar potential φ is continuous. On (c) an external (uniform in the radial direction r) mechanical traction $\mathbf{t}_{\mathrm{a}} = t_{\mathrm{a}_z}\mathbf{e}_z$ can be applied. On (a) we assume that $u_z = 0$.
- Boundary (b): When the body is still not deformed, the condition $(7)_1$ is equivalent to $B_r^{(i)} = B_r^{(o)}$, where $B_r^{(o)} = \mu_0 H_r^{(o)}$ and φ is continuous. The surface is free of mechanical traction.
- Boundary (e): We assume that $B_r^{(o)} = \mu_0 H_r^{(o)} = 0$.

The corners formed by the intersections of (a) with (b) and (c) with (b) have a small radius, the presence of that small radius is explained later on.

Boundary value problem for a MS elastomer in contact with two semi-infinite almost rigid bodies

The interaction of a MS elastomer with rigid magnetizable bodies was proposed by Bustamante (2009), as a better model to consider external mechanical loads acting on a MS body. In Bustamante (2009) it was assumed that some parts of the MS body were perfectly attached to rigid magnetizable semi-infinite external bodies, which are used to model the interaction with other bodies in a given device. In Figure 1(B) we have a MS body $\mathcal{B}$ interacting with two almost rigid bodies $\mathcal{B}^{m_1}$, $\mathcal{B}^{m_2}$, which are called the *machine*. The body $\mathcal{B}$ is perfectly attached to $\mathcal{B}^{m_1}$ and $\mathcal{B}^{m_2}$ on $\partial\mathcal{B}_a$ and $\partial\mathcal{B}_b$, respectively. The bodies $\mathcal{B}$, $\mathcal{B}^{m_1}$, $\mathcal{B}^{m_2}$ are surrounded by free space $\mathcal{B}'$. The interface of $\mathcal{B}$ with $\mathcal{B}'$ is denoted by $\partial\mathcal{B}_c$, whereas the interfaces of $\mathcal{B}^{m_1}$ and $\mathcal{B}^{m_2}$ with $\mathcal{B}'$ are denoted by $\partial\mathcal{B}_d^{m_1}$, $\partial\mathcal{B}_d^{m_2}$, respectively. The surface of $\mathcal{B}'$ far away is denoted by $\partial\mathcal{B}'_\infty$, while the surfaces of $\mathcal{B}^{m_1}$ and $\mathcal{B}^{m_2}$ far away are denoted by $\partial\mathcal{B}_\infty^{m_1}$ and $\partial\mathcal{B}_\infty^{m_2}$, respectively.

In the present work, we assume that $\mathcal{B}^{m_1}$, $\mathcal{B}^{m_2}$ are magnetizable bodies, which can be modelled as linear isotropic magneto-elastic materials, i.e. we use the decoupled constitutive equations:

$$\mathbf{B} = \mu_r\mu_0\mathbf{H}, \quad \boldsymbol{\tau} = \mathcal{C}\varepsilon, \tag{30}$$

where $\mathcal{C}$ is a fourth-order constant tensor and ε is the linearized strain tensor. The following values for the material parameters for the *machine* are used:

$$\begin{aligned} \mu_r &= 4000, \quad E = 2 * 10^{11}\,\mathrm{Pa}, \\ \nu &= 0.29, \quad \rho = 7870\,\mathrm{kgm}^{-3}, \end{aligned} \tag{31}$$

where E, ν and ρ are the Young modulus, Poisson ratio and density of the material, respectively.

Assuming that (23) is valid here, the boundary value problem to solve is

$$\operatorname{divgrad}\varphi = 0, \quad \text{in} \quad \mathcal{B}', \tag{32}$$

$$\text{divgrad}\,\varphi = 0, \quad \text{div}\boldsymbol{\tau} = \mathbf{0}, \quad \text{in} \quad \mathcal{B}^{\text{m}} = \mathcal{B}^{\text{m}_1} \cup \mathcal{B}^{\text{m}_2}, \tag{33}$$

where $\boldsymbol{\tau}$ for $\mathcal{B}^{\text{m}}$ is given by $(30)_2$.

For $\mathcal{B}$ we need to solve

$$\text{div}(\mathcal{C}\text{grad}\varphi) = 0, \quad \text{div}\boldsymbol{\tau} = \mathbf{0}, \quad \text{in} \quad \mathcal{B}, \tag{34}$$

where $\boldsymbol{\tau}$ is given by (17) for $\mathcal{B}$.

As for the magnetic boundary and continuity conditions we assume (see Figure 1(B))

$$[\![\varphi]\!] = 0, \quad \mathbf{n}\cdot[\![\mathbf{B}]\!] = 0, \quad \text{on} \quad \partial\mathcal{B} = \partial\mathcal{B}_a \cup \partial\mathcal{B}_b \cup \partial\mathcal{B}_c. \tag{35}$$

The same conditions hold for $\partial\mathcal{B}_d^{\text{m}_1}$ and $\partial\mathcal{B}_d^{\text{m}_2}$.

As for $\partial\mathcal{B}'_\infty$ and $\partial\mathcal{B}_\infty^{\text{m}_1}$, $\partial\mathcal{B}_\infty^{\text{m}_2}$ we have

$$\varphi = \tilde{\varphi} \quad \text{on} \quad \partial\mathcal{B}_\infty^{'\varphi}, \quad \mathbf{B}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}_\infty^{'\mathbf{B}}, \tag{36}$$

$$\varphi = \tilde{\varphi} \quad \text{on} \quad \partial\mathcal{B}_\infty^{\text{m}_1\varphi}, \quad \mathbf{B}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}_\infty^{\text{m}_1\mathbf{B}}, \tag{37}$$

$$\varphi = \tilde{\varphi} \quad \text{on} \quad \partial\mathcal{B}_\infty^{\text{m}_2\varphi}, \quad \mathbf{B}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}_\infty^{\text{m}_2\mathbf{B}}, \tag{38}$$

where $\tilde{\varphi}$ is a prescribed value for φ and $\partial\mathcal{B}_\infty^{'\varphi} \cup \partial\mathcal{B}_\infty^{'\mathbf{B}} = \partial\mathcal{B}'_\infty$, $\partial\mathcal{B}_\infty^{'\varphi} \cap \partial\mathcal{B}_\infty^{'\mathbf{B}} = \varnothing$, $\partial\mathcal{B}_\infty^{\text{m}_1\varphi} \cup \partial\mathcal{B}_\infty^{\text{m}_1\mathbf{B}} = \partial\mathcal{B}_\infty^{\text{m}_1}$, $\partial\mathcal{B}_\infty^{\text{m}_1\varphi} \cap \partial\mathcal{B}_\infty^{\text{m}_1\mathbf{B}} = \varnothing$, $\partial\mathcal{B}_\infty^{\text{m}_2\varphi} \cup \partial\mathcal{B}_\infty^{\text{m}_2\mathbf{B}} = \partial\mathcal{B}_\infty^{\text{m}_2}$ and $\partial\mathcal{B}_\infty^{\text{m}_2\varphi} \cap \partial\mathcal{B}_\infty^{\text{m}_2\mathbf{B}} = \varnothing$.

Regarding the mechanical boundary conditions, on $\partial\mathcal{B}_a \cup \partial\mathcal{B}_b$ we assume $[\![\mathbf{u}]\!] = \mathbf{0}$, i.e. the bodies $\mathcal{B}$ and $\mathcal{B}^{\text{m}_1}$, $\mathcal{B}^{\text{m}_2}$ are perfectly attached. On $\partial\mathcal{B}_c$, if we assume as a first approximation that we do not consider the Maxwell stresses, we have $\boldsymbol{\tau}\mathbf{n} = \mathbf{0}$. On $\partial\mathcal{B}_\infty^{\text{m}_1}$ we have $\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\text{a}$, where $\mathbf{t}_\text{a}$ is a given mechanical load; on the other hand on $\partial\mathcal{B}_\infty^{\text{m}_2}$ we assume that $\mathbf{u} = \tilde{\mathbf{u}}$, with $\tilde{\mathbf{u}}$ is a given displacement field (which is usually zero).

In this problem the mechanical forces are applied indirectly on the surface of the body $\mathcal{B}$ through the contact with the bodies $\mathcal{B}^{\text{m}_1}$, $\mathcal{B}^{\text{m}_2}$.

In Figure 3 there is a depiction of the MS cylinder and a *machine*, which is used to apply traction on it.

Assuming axial symmetry, φ and $\boldsymbol{\chi}$ only depend on the radial and axial positions r, z, respectively. The free space and the *machine* are very large. The different boundaries and interfaces are denoted as (a), (b), …, (j). For these boundaries we have the following:

- Boundaries (d), (f): We assume we give constant values for $\varphi = \pm\tilde{\varphi}$, respectively.
- Boundary (e): We have $B_r = \mu_0 H_r = 0$.
- Boundaries (a), (g) and (c), (h): For $R < r_\text{b}$, the scalar field φ is continuous; regarding the continuity condition $\mathbf{n}\cdot[\![\mathbf{B}]\!] = 0$, when the body is still undeformed that conditions is equivalent to[5] $B_z^\text{i} = B_z^\text{m}$, where we have used the notation $\mathbf{B}^\text{m}$ to denote the magnetic induction in the machine. Regarding the mechanical conditions, we have $[\![\mathbf{u}]\!] = 0$, which is equal to $\mathbf{u}^\text{i} = \mathbf{u}^\text{m}$ across the interfaces (a), (c).

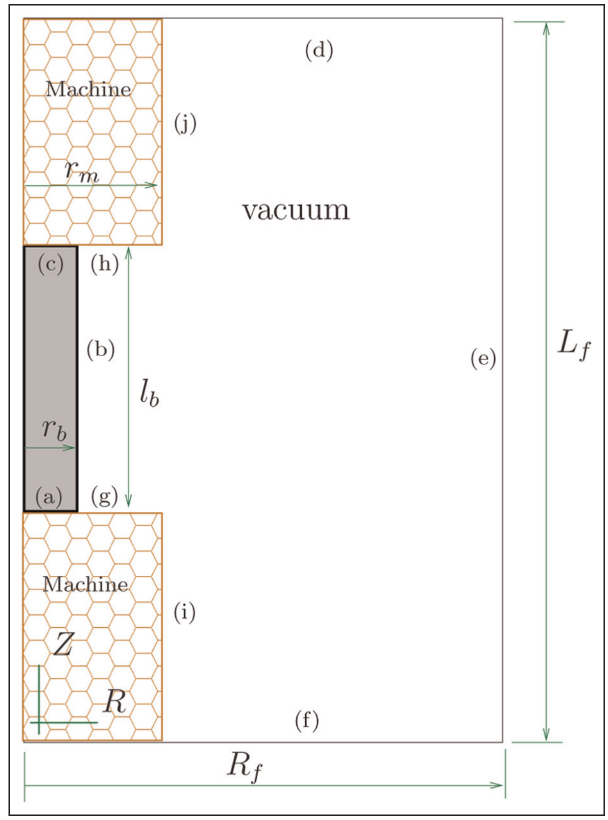

Figure 3. Model of a cylinder interacting with two external bodies (machine).

- For $r_b < R < r_\text{m}$ (where r_m is radius of the machine) the continuity condition $(27)_2$ is equivalent to $B_z^\text{m} = B_z^\text{o}$ and φ is continuous, where we have used the notation $\mathbf{B}^\text{o}$ to denote the magnetic induction outside the MS body in vacuum.
- Regarding the boundary (b), φ is continuous, and $\mathbf{n}\cdot[\![\mathbf{B}]\!] = 0$ must hold. When the body is undeformed that condition is equivalent to $B_r = B_r^\text{o}$. The condition $(29)_2$ is equivalent to $\boldsymbol{\tau}\mathbf{n} = \mathbf{0}$.
- Finally, across (i) and (j) the continuity condition $[\![\varphi]\!] = 0$ is valid and $(27)_2$ becomes $B_r^\text{m} = B_r^\text{o}$.

Numerical results for a cylinder under the effect of a uniform axial magnetic field applied far away

For the problems described in Sections 3.1 and 3.2 we show some numerical results, which have been obtained using the program Comsol multiphysics (Comsol, 2007). The behaviour of the MS body was modeled modifying the piezoelectric modulus, considering that in the quasi-static case, the fundamental equations of electrostatics have the same form as in magnetostatics. The magnetic field in vacuum was obtained using the electrostatic modulus, and the influence of the deformation of the body on the calculations of the field in

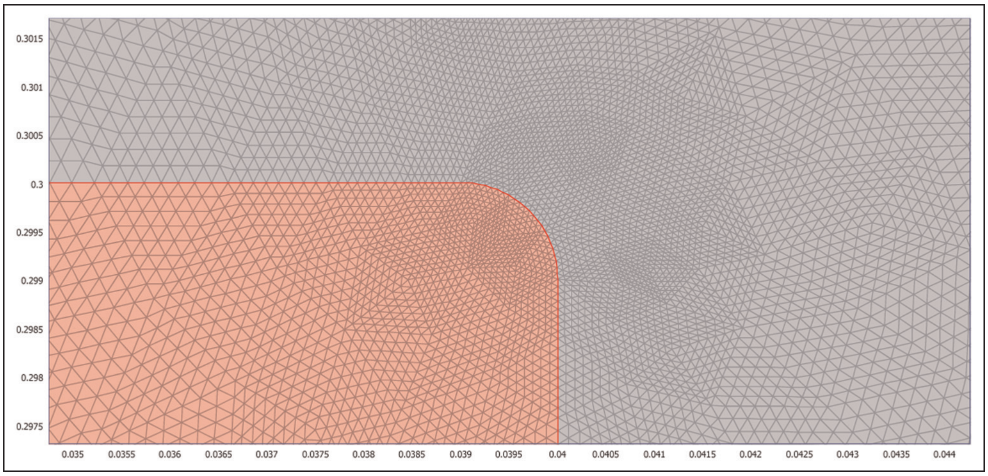

Figure 4. Mesh density near one of the corners of the cylinder.

vacuum, was incorporated using the moving mesh tool ALE. In order to obtain solutions for the nonlinear problem, the damped Newton method was used (Comsol, 2007).

The continuity conditions (27) are particularly critical for our numeric work, as mentioned in detail in the Introduction. If a cylinder would have sharp corners, then it is possible to show that to require (27) to be satisfied, would mean to have a magnetic field with a magnitude that goes to infinite in a small region near the corners; such phenomena may create problems for standard numerical methods, therefore, in order to avoid this, the corners were assumed to have a small radius. For the results shown in this work a radius of 0.001 m was assumed. In Figure 4 there is a detailed view of the mesh density near one of the corners of a cylinder.

In the theory developed by Dorfmann and Ogden (2004a,b, 2005b), the MS bodies are surrounded by an infinite vacuum space. With standard finite element methods, it is not possible to model infinite geometries, therefore, for the surrounding free space depicted in Figure 2(B), we assume the existence of a large enough but finite surrounding space, and the hypothesis is that for large enough surrounding spaces, the behaviour of the body would be the same as in the case of an infinite free space.

In our case for a cylinder of dimensions $r_b = 0.04$ m, $l_b = 0.3$ m and free space of variable size (see Table 2), for a far away field of magnitude, different sizes for the surrounding free space were considered as shown in Table 2.

In Figure 5 the influence of the size of the surrounding free space is considered, in this case for the axial component of the magnetic induction B_z as a function of the axial position Z, for the line $R = 1$ cm from the axis of symmetry.

From $(27)_2$ this component must be continuous across the surface of the cylinder. The limits of the region where the cylinder is located are denoted by vertical dashed lines. The cases considered are from Table 2, and we can notice that for cases 1—3 there is almost no difference in the behaviour of B_z for the different sizes of the surrounding space, there is a slight

Table 2. Geometry of the surrounding free space used to study the influence of the size of this space.

Model	Vacuum length L_f (m)	Vacuum radius R_f (m)	Number elements
1	2.3	1.0	526.952
2	1.7	0.7	131.648
3	0.9	0.4	32.912
4	0.7	0.2	8.228
5	0.5	0.1	2.057

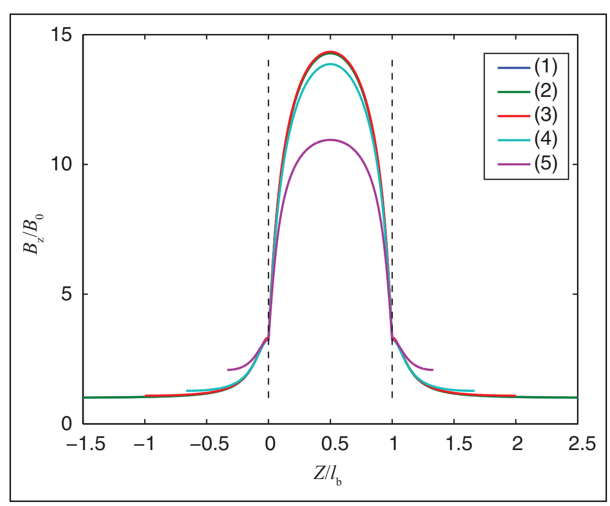

Figure 5. Effect of the size of the free surrounding space for the axial component of the magnetic induction. The cases are taken from Table 2. The line for the plot is $R = 1$ cm.

Table 3. Geometry of the cylinder and free space used to study the effect in the field of considering different ratios between the length and radius of the cylinder.

Model	Cylinder length l_b (m)	Vacuum length L_f (m)	Vacuum radius R_f (m)
1	0.1	2.3	1.0
2	0.2	2.5	1.0
3	0.3	3.3	1.0
4	0.5	5.5	2.0
5	0.7	7.7	2.4

difference for case 4, and a rather large difference for case 5, which is the situation with the smallest surrounding free space. Therefore, at least regarding the behaviour of B_z for that particular line, the size of the surrounding space mentioned in case 3 would be large enough, such that it would be equivalent to have the cylinder surrounded by an infinite free space.

The exact solutions presented, for example, in Pucci and Saccomandi (1993) are based in the assumption of working with infinitely long cylinders, such that the conditions (7) are not considered for the surfaces $Z = 0$, $Z = l_b$. The hypothesis is that for cylinders where the length is very large in comparison with the radius, such exact solutions are good approximations for the problem. We study the effect in the distribution of the field of considering different relations between the lengths and radii of the cylinders. In Table 3 we have the data for the cylinders studied, the radius $r_b = 0.04$ m is constant and different lengths l_b for the cylinders are considered. The length of the surrounding free space is adapted to the different lengths of the cylinders. The external far away magnetic field is $1.74 * 10^7 \mathrm{Am}^{-1}$ and the same value is used for all cases.

In Figure 6 the axial component of the magnetic induction B_z is plotted for the axial line $R = 1$ cm for the cases mentioned in Table 3.

Some additional results for the case of a cylinder of the dimensions $r_b = 0.04$ m, $l_b = 0.3$ m are shown, considering a free space of length $L_f = 2.3$ m and radius $R_f = 1.0$ m, without any external mechanical load and a far away magnetic field of magnitude $2.17 \,\mathrm{Am}^{-1}$.

In Figure 7 we have depictions of the deformed cylinder, showing the behaviour of the two components of the magnetic field. The original shape of the cylinder (in the reference configuration) is denoted by the black lines superposed on the figure.

We note that H_r is approximately zero in the center of the cylinder, and that near the surfaces $Z = 0$, $Z = l_b$ (in the reference configuration), the field changes rapidly especially near the corners.

In Figure 8(a) we have a depiction of the distribution of J, which is very close to 1 for almost the whole cylinder but two small regions near the corners. In Figure 8(b) we see the behaviour of the axial component of the total stress tensor τ_{zz}.

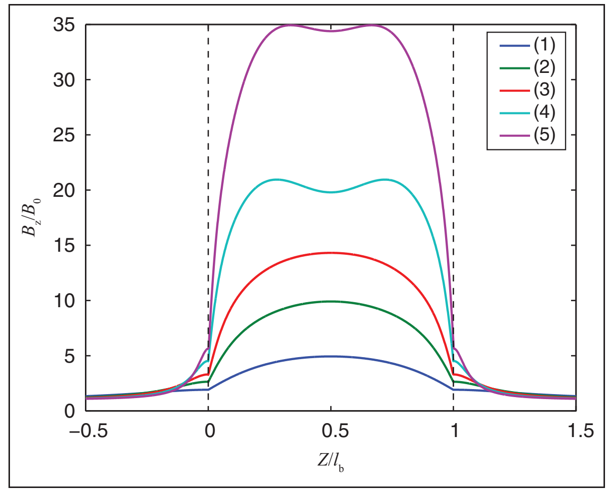

Figure 6. Effect in the axial component of the magnetic induction of considering different relations for the length and radius of the cylinder. The cases correspond to the situations listed in Table 3. The line for the plot is $R = 1$ cm.

The continuity conditions (7) have been already studied in Figures 5 and 6. In Figure 9 we see depictions for the axial component of the magnetic induction and for the radial component of the magnetic field, which accordingly with (7), must be continuous across the surfaces $Z = 0$, $Z = l_b$; we notice that they are indeed continuous.

In Section 3.2 we discussed in detail about how actual mechanical loads can be applied on a MS body. Surface traction can only be applied if a MS body is in contact with an external ‘machine’. We are interested in comparing the behaviour of a MS cylinder interacting with a machine, with the case of a cylinder that is surrounded completely by free space. Far away an external magnetic field of magnitude $2450 \,\mathrm{Am}^{-1}$ is applied on the surfaces (d) and (f) (indirectly by prescribing φ, see Figure 3).

From Figure 10 we can see the difference between the case the traction is applied with a ‘machine’ and the case the external traction $\mathbf{t}_a = 80\mathrm{kPa}\mathbf{e}_z$ is applied directly on the cylinder, assuming that is surrounded completely by free space. A first preliminary conclusion from these results, is that the application of a force through a machine does have an influence on the behaviour of the MS cylinder. This conclusion is reinforced when we note the results shown in Figure 11, where we have depictions of B_z and H_r for some axial lines at $R = 0.01$, 0.02 and 0.03 m, crossing the cylinder and free space.

The continuity conditions (7) are satisfied, and if we compare these results with the results shown in Figure 9, we see important differences between the plots for B_z, although we must remember that the results shown in Figure 9 were obtained for a cylinder free of any external mechanical traction.

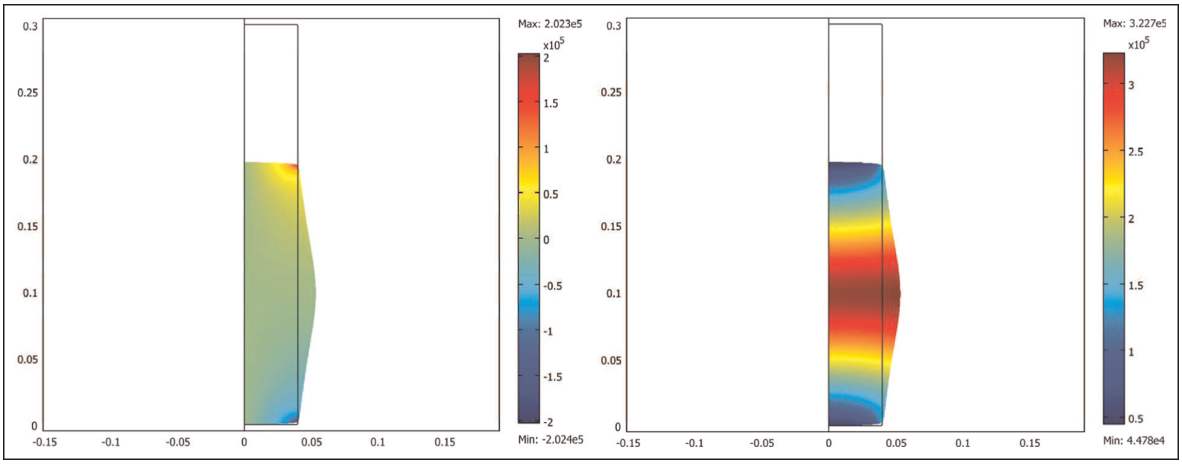

Figure 7. Deformed cylinder. On the left we see the behaviour of the radial component of the magnetic field H_r (current configuration). On the right we see the behaviour of the axial component of the magnetic field H_z.

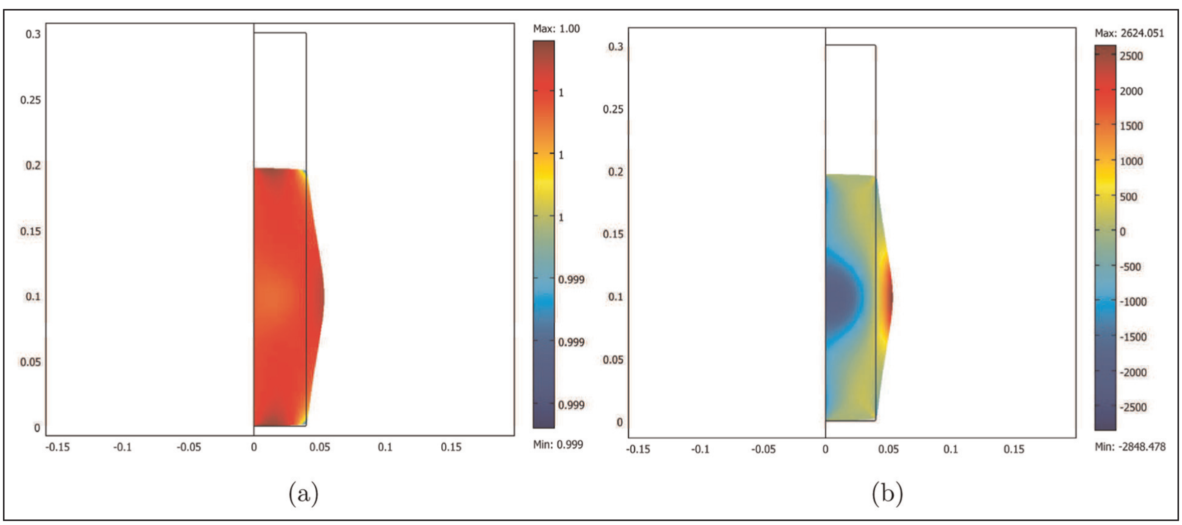

Figure 8. Deformed cylinder. In (a) we see the behaviour of J. On the right we have the distribution of the component τ_{zz} of the total stress tensor.

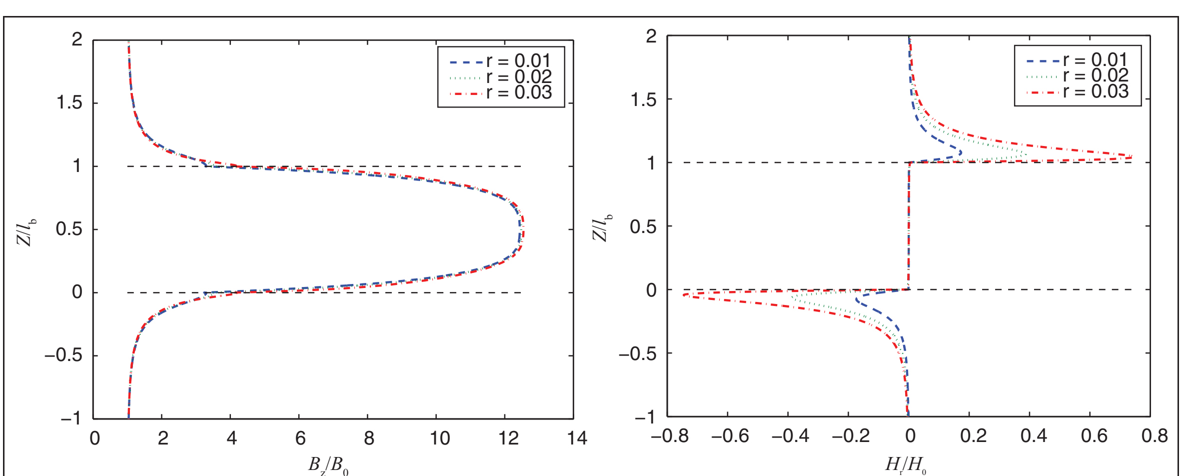

Figure 9. Deformed cylinder. Continuity for B_z and for H_r for different axial lines.

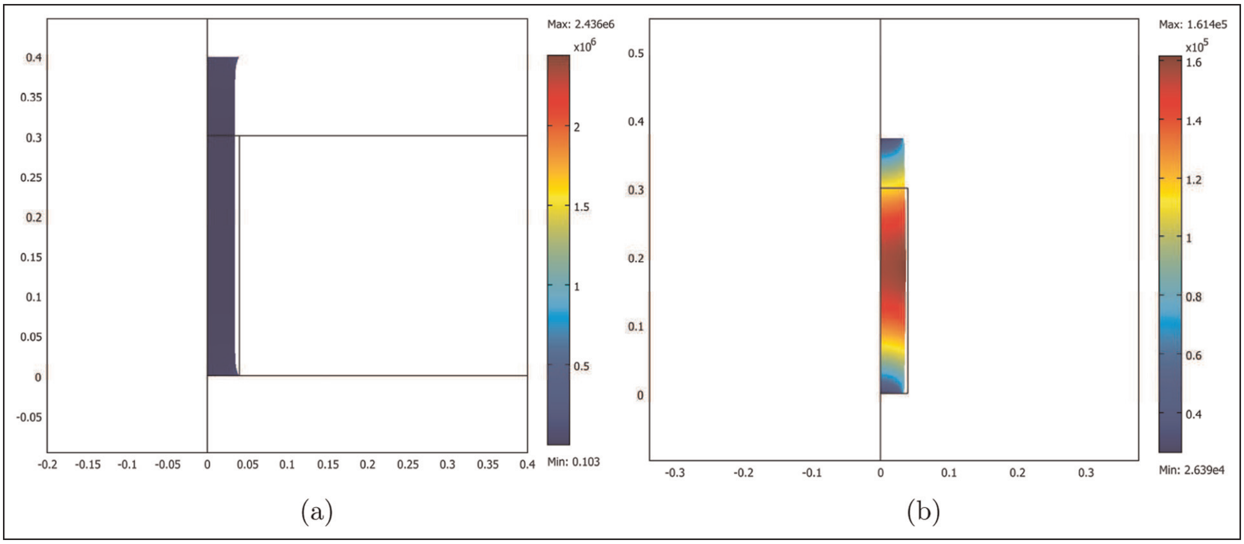

Figure 10. Axial component of the magnetic field H_z for the deformed cylinder. In (a) a MS cylinder is depicted interacting with a *traction machine*, where the reference configuration is presented by the black lines, and the machine is represented by the upper and lower cylinders. In (b) there is a depiction of a deformed MS cylinder assuming the body completely surrounded by free space, and a traction force $\mathbf{t}_a$ is applied on the surface $Z = l_b$.

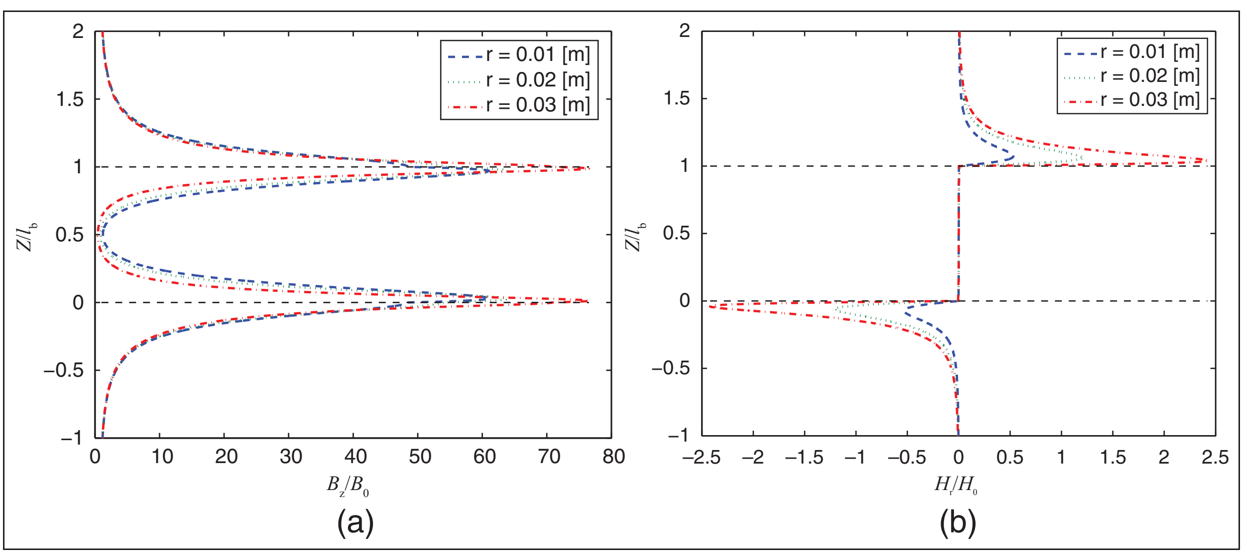

Figure 11. Continuity conditions for a cylinder interacting with a machine: (a) B_z and (b) H_r plotted for different axial lines.

Final remarks

As explained in the Introduction, in some of the recent works on nonlinear magneto-elasticity, we see models where only the MS bodies are taken into account, disregarding the effect that the surrounding free space may have in the distribution of magnetic field and deformation inside the MS body. However, as shown, for example, by Vu and Steinmann (2010b) for a similar problem involving electro-elastic bodies, in some situations it may be important to consider the interaction with the surrounding free space. In the present work we have studied different problems concerning the behaviour of a MS cylinder, such as to find the smallest size of the surrounding free space to have an approximate model of an infinite surrounding space, to study the effect on the distribution of magnetic field of considering larger or shorter cylinders and finally, to analyze the effect of applying external mechanical forces through the direct contact with an external body, which we have called a 'machine'.

The results shown in the present work were obtained under the simplification that Maxwell stresses were not taken into account as external traction (see

Bustamante, 2009). When considering MS elastomers surrounded by free space, there are five main important problems in order to obtain good results from the numerical point of view:

- the MS bodies behave in a nonlinear manner with a strong coupling between strains and magnetic fields;
- the MS bodies can show relatively large deformations;
- the deformation of the MS body has an influence on the way the external magnetic field is distributed;
- high concentration of external field can be observed near points of the surface of the body with rapid changes in the geometry;
- the requirement the continuity conditions for the magnetic field and the stresses are satisfied (see Kovetz, 2000; Bustamante et al., 2007) implies additional problems from the numerical point of view.

In a recent work by Bustamante and Ogden (2012) the second variation was calculated for this problem, and from those results it becomes clear that a solution for the boundary value problem may not be a minimum for the associate functional, further complicating the search for numerical results. An alternative method to look for solutions could be to use a bisection method to look for a first approximation for the solution, and as a second step, to use that first approximation as a starting point for a standard Newton method, in order to find a final improved solution.

Regarding the incorporation of the Maxwell stresses, these are calculated with the magnetic field outside the body (see Equation (13)); but $\mathbf{H} = -\operatorname{grad}\varphi$, therefore the error in the calculations of φ due to the discretization used in the finite element method, increases with the application of the gradient operator; moreover, near the surface of the body, especially near the corners, there can be a rapid change in the field outside, which may create some problems with the convergence of the standard Newton method.

Despite the problems described previously, we obtained interesting results, for example, for relatively short cylinders we observed a relatively marked departure of the numerical solutions regarding the deformations of the bodies, as can be seen from Figure 7, where we note non-homogeneous deformations, especially near the middle part of the cylinder, which should be compared with the exact solutions, where cylinders are assumed to deform (in the axial direction) in a uniform way.

Second, we studied in a detailed manner the effect of considering surrounding free spaces of different sizes, finding a 'minimum' size, for which a MS cylinder cannot feel the difference between that finite space and a larger or infinite surrounding space.

Realizing that for more realistic models we need to study the interaction of our MS bodies with other external bodies, we obtained some results, where our cylinders interacted with external 'machines', which are made of stiffer materials. From Figures 10 and 11 we observe that the deformation and distribution of magnetic field (or magnetic induction) can be affected significantly with such interactions, therefore, it is necessary to study such problems in more detail for future works.

It has been shown with some numerical examples that there is a need to study the effect of the surrounding space, when modelling the behaviour of MS bodies of finite size. Such analysis may not only be important as a tool for design when using such materials, but can also be of some use when performing experimental work, in order to determine the properties of, for example, MS elastomers, since if we are able to measure the magnetic field near the surface of a body, then we could determine in a more precise manner how the field is distributed inside the MS bodies by considering the continuity condition (7). The effect of considering the Maxwell stresses will be studied in a future communication, which will require the development of new nonlinear solvers, in order to overcome the numerical problems mentioned previously.

Let us elaborate more about the experimental data available and the status of the results presented in our communication. The numerical results presented in our paper are based on a constitutive equation proposed by Bustamante (2010) (see Section 5.2.1 and Figures 2 and 3 therein). In that work there is a detailed description about the experimental data available and the difficulties encountered in order to propose good and realistic expressions for the total energy function. The main data used in order to propose Equation (130) of Bustamante (2010) was obtained from the papers by Bellan and Bossis (2002) and Ginder et al. (1999). In Bellan and Bossis (2002) one can find experimental information for a MS elastomer composed of 15% of magneto-active particles per volume. The most important information corresponds to the results for the stresses in terms of the strains for traction tests considering different external magnetic fields (see Bellan and Bossis, 2002, Figures 2 and 3). The rest of the information, in particular regarding the magnetic behaviour, was obtained from Ginder et al. (1999, Figure 4), which is necessary to point out was obtained for a MS elastomer composed of a different proportion of particles (27% of particles per volume).

The information provided by Bellan and Bossis (2002) and Ginder et al. (1999) is not enough to propose a definite expression for the total energy function. Not only has the paucity of experimental information been a problem for all researchers working on the theoretical and numerical modelling for these materials, but also the fact that the little experimental information provided and the procedures used to perform such experiments are in general not very well described by the experimentalists.[6]

Another problem is that most of the experimental results presented in the literature would correspond to average values for the stresses, strains and magnetic variables. One of the main points addressed in our work was the effect in the behaviour of a MS body if one considers the continuity condition for the magnetic variables. One must agree that the results presented in this communication suggest that near points, where the geometry of the MS body changes rapidly, we can expect large values for the magnetic field (although presented only in small regions). Moreover, such non-uniform fields may cause non-uniform deformations for the body, which can be observed in Varga et al. (2005, Figure 4) and Böse (2007, Figure 7). In addition, many researchers doing experiments are mostly interested in the dynamic behaviour of these materials, because of the possible applications in vibration control (see, for example, Farshad and Le Roux, 2004), therefore, they have obtained quantities such as the dynamic modulus, which is a concept useful for problems considering small strains, which may not be valid in the case of moderately large elastic deformation in MS solids.[7]

Therefore, one should understand that to propose more realistic expressions for the energy function is a difficult work, which has not been addressed completely yet. Several researchers have considered other methods to find such constitute expressions (see, for example, Borcea and Bruno, 2001; Chatzigeorgiou et al., 2012; Ivaneyko et al., 2011; Ponte-Castañeda and Galipeau, 2011).

In order to be able to propose more realistic expressions for the energy function, the experimentalists should use as a basis for their work some of the exact solutions for MS bodies presented, for example, by Dorfmann and Ogden (2004a,b, 2005a,b) and Pucci and Saccomandi (1993). These solutions are approximations of real problems, since they are valid for infinite or semi-infinite bodies (in order to avoid problems with the continuity conditions for the magnetic variables). These solutions are approximations also because of the way external forces are applied on such infinite bodies. There are several such solutions, for example, the extension of an infinitely long bar, the extension and inflation of a tube, the inflation, extension and torsion of a tube (in these three examples under the effect of an axial magnetic field), plus other problems involving bending and shear of rectangular slabs. The experimentalists could use two or three of these simple problems, such as the extension of a cylindrical bar under traction and the finite shear of a slab, as theoretical basis to obtain meaningful data, in order to propose a first approximation for the energy function. Such expression for the energy function can be used in a finite element code to study the same simple problems numerically; however, such numerical code needs to address problems such as the proper way to consider the interface between the exterior free space and the machine with the MS bodies. To develop such codes is still a work in progress, only some issues have been addressed by the different research groups working on this topic worldwide. Using such a code, one could model the behaviour of the same bodies considered for the experimental work, and in an iterative process to adjust the values of the constants until obtaining a good degree of agreement for the different results. Additional geometries could be studied using such a code and that refined expression for the energy function, and one could compare those results with the experimental information obtained from those additional problems.

Considering the shortcomings mentioned previously, it was difficult to compare the results presented in this paper with the experimental data available in the literature. The main idea of the present work was not necessarily to obtain numerical results that would fit perfectly with the little experimental information available, but to call attention of the interested researchers to the possible influence of considering the external free space on the numerical results, and also of considering different and more realistic ways to apply external mechanical loads on a MS body. To consider the surrounding free space may have an important influence on the results obtained, and the same happens if one considers (from the point of view of mathematical modelling) a load applied through the contact (bonding) with an external machine. The results of that last case are very different from the case the forces are applied in the traditional manner, where the forces are applied in the model without considering the need of interaction with an external body.

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Notes

1. In the original theoretical works by Dorfmann and Ogden (2004a,b, 2005a,b), it is assumed that the free space which surrounds the bodies is of infinite size. The same assumption has been used in Bustamante et al. (2008).
2. We use lowercase characters in order to denote the Cauchy–Green strain tensors, to avoid problems with the notation used to denote the magnetic induction field **B** (commonly used in the literature on electromagnetism).
3. We need to remark that from the physical point of view the scalar magnetic potential does not have a clear

physical meaning; we use it due to the simplicity of working with this potential from the mathematical point of view. From the physical point of view, the vector potential **A**, which is connected to the magnetic induction through the relation $\mathbf{B} = \text{curl}\,\mathbf{A}$, is considered to have a clearer physical meaning (see, for example, Semon and Taylor, 1996).

4. Far away from the boundaries of the cylinder, the application of $\pm\tilde{\varphi}$ on (d) and (f) generates a uniform axial magnetic field **H** in vacuum.
5. The bodies $\mathcal{B}^{m_1}$, $\mathcal{B}^{m_2}$ are assumed to be made of a material that is much stiffer than $\mathcal{B}$, therefore, we can expect that the surfaces (h), (f) do not deform much, as a result in this problem $B_z^i = B_z^m$ can be considered valid for the whole process.
6. In a paper by Danas et al. (2012) one can find more recent experimental data for MS elastomers (see Figures 3 and 4 therein), but it cannot be used in the numerical modelling of the present paper, since their work was based on a different theory, considering the magnetization field instead the magnetic field as the independent magnetic variable (and a MS elastomer composed of 25% of particles per volume). Another recent reference where one can find experimental data is the paper by Boczkowska and Awietjan (2012), see in particular Figure 12 therein, where one find plots for the magnetization versus the magnetic field and Figure 14 for the stresses versus the strains in a compression test, where it is interesting to note that the stress is zero when there is no strain, but the magnetization is different to zero, which is a strange result, considering that in such a situation from the physical point of view we must expect a non-zero stress in order to maintain that zero strain (see, for example, Bellan and Bossis, 2002, Figures 2 and 3 therein).
7. See, for example, Bellan and Bossis (2002, Figures 8 and 9), Ginder et al. (1999, Figures 2 and 3) and the results presented by Deng and Gong (2008), Major et al. (2009), Mitsumata (2009); Mitsumata and Ohori (2011), Miedzinska et al. (2010), Gordaninejad et al. (2012), Kashima et al. (2012), Bica (2012), Zhu et al. (2012), Ying et al. (2013) and Ghafoorianfar et al. (2013).

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