Representations of Generalized Almost-Jordan Algebras

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Abstract. This paper deals with the variety of commutative algebras satisfying the identity

$$\beta \{(yx^2)x - ((yx)x)x\} + \gamma \{yx^3 - ((yx)x)x\} = 0$$

where $\alpha, \beta$ are scalars. These algebras appeared as one of the four families of degree four identities in Carini, Hentzel and Piacentini-Cattaneo [6]. We give a characterization of representations and irreducible modules on these algebras. Our results require that the characteristic of the ground field was different from 2, 3.

1. Introduction

Let $A$ be a commutative not necessarily associative algebra over an infinite field $F$. Let $x$ be an element in $A$. We define the principal powers of $x$ by

$$x^1 = x, \quad x^{n+1} = x^n x \text{ for all } n \geq 1.$$ 

A Jordan algebra is a commutative algebra satisfying the identity $x^2(yx) - (x^2y)x = 0$. It is a well known variety of algebra, that is power-associative, i.e., the subalgebra generated by any element of the algebra, is associative. See [13], [24] for properties of these varieties of algebras. It is known (see [18]) that a Jordan algebra satisfies the identity $3((yx^2)x) = 2((yx)x)x + yx^3$. These algebras, called almost-Jordan algebras have been studied by Osborn [18], [19], Petersson [22], Sidorov [26], and Hentzel and Peresi [11]. In this last paper, the authors proved that every semi-prime almost-Jordan algebra is a Jordan algebra and this fact justified the name of these algebras.

A generalized almost-Jordan algebra is a commutative algebra satisfying the identity

$$\beta \{(yx^2)x - ((yx)x)x\} + \gamma \{yx^3 - ((yx)x)x\} = 0$$

for every $x, y \in A$ where $\alpha, \beta$ are scalars, and $(\beta, \gamma) \neq (0, 0)$. For $\beta = 3$ and $\gamma = -1$, we have an almost-Jordan algebra.

In the study of degree four identities not implied by commutativity, Osborn [19] classified those that were implied by the fact of possessing a unit element. Carini, Hentzel and Piacentini-Cattaneo [6] extended this work by dropping the restriction on the existence of the unit element. This result require that characteristic $F \neq 2, 3$. The identity defining a generalized almost-Jordan algebra with $\beta, \gamma \in F$ appears as one of these identities.

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We observe that there are generalized almost-Jordan algebras that are not Jordan algebras.

**Example 1.1.** Let \( A \) be a commutative algebra over \( F \) with base \( \{ e, a \} \), and multiplication table given by
\[
e^2 = e, \ a^2 = e, \ \text{all other products being zero}
\]
Then \( A \) satisfies identity (1) for \( \beta = 1 \) and \( \gamma = -1 \). Moreover \( A \) is not a Jordan algebra, since it is not power-associative since \( a^2(aa) \neq (a^2a)a \).

**Example 1.2.** Let \( A \) be a commutative algebra over \( F \) with base \( \{ e, a \} \) and multiplication table given by
\[
e^2 = e, \ ea = ae = -e - a, \ a^2 = e + a.
\]
Then \( A \) satisfies identity (1) with \( \beta = 0 \) and \( \gamma \neq 0 \). That is \( A \) satisfies \( x^3y = ((xy)x)x \) for every \( x, y \in A \). Moreover \( A \) is not a Jordan algebra, since \( a^2(aa) = 2a \neq (a^2a)a = 0 \).

**Example 1.3.** Let \( A \) be a commutative \( F \)-algebra with base \( \{ e, a, b \} \) and multiplication table given by
\[
e^2 = e, \ ab = ba = b, \ \text{all other products being zero}
\]
Then \( A \) satisfies identity (1) with \( \beta = 1 \) and \( \gamma = 1 \), for every \( \alpha \in F \). Moreover \( A \) is not power-associative since \( (a+b)^4 = 2b \) and \( (a+b)^2(a+b)^2 = 0 \) and \( A \) is not a Jordan algebra.

Generalized almost-Jordan algebras \( A \) have been studied in [1], where the authors proved that these algebras always have a trace form in terms of the trace of right multiplication operators. They also prove that if \( A \) is finite-dimensional and solvable, then it is nilpotent and found three conditions, any of which implies that a finite-dimensional right-nilalgebra \( A \) is nilpotent. In [2] the author found the Wedderburn decomposition of \( A \) assuming that for every ideal \( I \) of \( A \) either \( I \) has a non zero idempotent or \( I \subset R, R \) the solvable radical of \( A \) and the quotient \( A/R \) is separable and in [10] where, assuming that \( A \) also satisfies \( ((xx)x)x = 0 \) the authors proved the existence of an ideal \( I \) of \( A \) such that \( AI = IA = 0 \) and the quotient algebra \( A/I \) is power-associative.

In this paper we deal with representations of algebras. Let \( A \) be an algebra which belongs to a class \( \mathcal{C} \) of commutative algebras over a field \( K \) and let \( M \) be a vector space over \( F \). As in Eilenberg [9], we say that a linear map \( \rho : A \to \text{End}(M) \) is a representation of \( A \) in the class \( \mathcal{C} \) if the split null extension \( S = A \oplus M \) of \( M \), with multiplication given by
\[
(a + m)(b + n) = ab + \rho(a)(n) + \rho(b)(m) \quad \forall \ a, b \in A, m, n \in M
\]
belongs to the class \( \mathcal{C} \).

Representations have been studied for different algebras, for example, in [15], and [21] for Jordan algebras, in [14], [23] and [25] for alternative
Lemma 2.1. Let $A$ be a generalized almost-Jordan algebra and $\rho : A \to End(M)$ a linear map. Then $\rho$ is a representation of $A$, if and only if for every $a, b \in A$ the following identities hold

\begin{align*}
(2) \quad & (\beta + \gamma)\rho_a^3 - \beta \rho_a \rho_{a^2} - \gamma \rho_{a^3} = 0 \\
(3) \quad & (\beta + \gamma)(\rho_a \rho_b + \rho_a^2 \rho_b + \rho_{(a^2)\rho_b}) - \beta(2\rho_a \rho_b \rho_a + \rho_a z_b) - \gamma(2\rho_b \rho_a^2 + \rho_b \rho_{a^2}) = 0
\end{align*}

where $\rho_a := \rho(a) \in End(M)$, and for every $a, b \in A$, $\rho_a \circ \rho_b$ will be denoted by $\rho_{a^\rho_b}$.

Proof. $\rho$ is a representation of $A$ if and only if for every $a + m, b + n \in A \oplus M$ satisfy the identity (1). Straightforward calculations give

\begin{align*}
[(a + m)^2(b + n)(a + m)](a + m) &= (a^2b)a + 2\rho_a(\rho_b(\rho_a(m))) + \rho_a(\rho_a(\rho_a(m))) + \rho_a z_b(m) \\
[(a + m)^3](b + n) &= a^3b + 2\rho_b(\rho_a(\rho_a(m))) + \rho_b(\rho_a^2(m)) + \rho_{a^3}(n)
\end{align*}

Replacing $x = a + m, y = b + n$ in identity (1) we get

\begin{align*}
\beta\{2\rho_a(\rho_b(\rho_a(m))) + \rho_a(\rho_{a^2}(n)) + \rho_a z_b(m) - \rho_a^2((\rho_b(m)) - \rho_a(\rho_{ab}(m)) - \rho_a(\rho_{a^2}(n))
\end{align*}
\[
\rho_{(ab)a}(m) + \gamma(2\rho_b(\rho_a^2(m)) + \rho_b(\rho_a^2(m)) + \rho_a(\rho_a^2(m)) - \rho_a^3(m) - \rho_{(ab)a}(m)) = 0
\]

Now it is easy to see that this relation holds if and only if identities (2) and (3) hold in \(A\).

In the following suppose that \(A\) has an idempotent element \(e \neq 0\). Taking \(a = e\) in identity (2), we obtain

\[
(\beta + \gamma)\rho_e^3 - \beta\rho_e^2 - \gamma\rho_e = 0
\]

**Proposition 2.2.** Let \(A\) be a generalized almost-Jordan algebra and \(\beta, \gamma\) satisfying \(0 \notin \{\gamma, \beta + \gamma, \beta + 2\gamma\}\). Suppose that \(A\) has an idempotent element \(e \neq 0\). Let \(\rho : A \rightarrow \text{End}(M)\) be a representation of \(A\). Then

\[
M = M_0 \oplus M_1 \oplus M_\lambda
\]

where \(M_i = \{m \in M|\rho_e(m) = im\}\), and \(i \in \{0, 1, \lambda\}\).

**Proof.** Using identity (4) we see that \(\rho_e\) satisfies the polynomial \(p(x) = (\beta + \gamma)x^3 - \beta x^2 - \gamma x = 0\). Since \(\beta + \gamma \neq 0\) we define \(\lambda = \frac{2\gamma}{\beta + \gamma} \in \mathbb{F}\) and \(p(x) = (\beta + \gamma)(x - 1)(x - \lambda)\). Moreover, when \(\gamma \neq 0, \beta + 2\gamma \neq 0\), we have that \(\lambda \neq 0\) and \(\lambda \neq 1\), therefore \(p(x)\) has only simple roots. On the other hand, since the minimal polynomial of the operator \(\rho_e\), is a divisor of \(p(x)\) then \(\rho_e\) is diagonalizable and

\[
M = M_0 \oplus M_1 \oplus M_\lambda
\]

In [2] Theorem 1] M. Arenas proves that the Peirce decomposition of a generalized almost-Jordan algebra \(A\) is given by \(A = A_0 \oplus A_1 \oplus A_\lambda, A_i = \{a \in A|\lambda a = ia\}\), where \(\lambda = \frac{2\gamma}{\beta + \gamma}\). Moreover, when \(0 \notin \{\gamma, \beta + \gamma, \beta + 2\gamma\}\) we have the following relations among these subspaces

\[
(A_0)^2 \subseteq A_0, \quad (A_1)^2 \subseteq A_1, \quad A_0A_1 = \{0\}
\]

\[
A_\lambda A_0 \subseteq A_\lambda, \quad A_\lambda A_1 \subseteq A_\lambda, \quad (A_\lambda)^2 \subseteq A_0 \oplus A_1
\]

Next we obtain relations among \(A_i\) and \(M_i\), where \(i \in \{0, 1, \lambda\}\). Linearising identity (2) we have

\[
(\beta + \gamma)(\rho_a\rho_b\rho_a + \rho_b\rho_a^2 + \rho_a^2\rho_b) - 2\beta\rho_a\rho_ab - \beta\rho_b\rho_a^2 - 2\gamma\rho_{(ab)a} - \gamma\rho_{a^2b} = 0
\]

Moreover subtracting identities (3) and (5), we obtain

\[
(\gamma - \beta)(\rho_a\rho_b\rho_a + \rho_a\rho_{ab} - \rho_b\rho_a^2 - \rho_{ab}a) + (\beta + \gamma)(2\rho_a^2\rho_b - \rho_b\rho_a^2 - \rho_a^2b) = 0
\]

**Theorem 2.3.** Let \(A\) be a generalized almost-Jordan algebra and \(\beta, \gamma\) satisfying \(0 \notin \{\gamma, \beta + \gamma, \beta + 2\gamma\}\). Suppose that \(A\) has an idempotent element \(e \neq 0\). Let \(\rho : A \rightarrow \text{End}(M)\) be a representation of \(A\). Then the action of \(A\) on \(M\) satisfies the following relations

\[
A_0 \cdot M_0 \subseteq M_0, \quad A_0 \cdot M_1 = \{0\}, \quad A_0 \cdot M_\lambda \subseteq M_\lambda
\]
\[ A_1 \cdot M_0 = \{0\}, \ A_1 \cdot M_1 \subseteq M_1, \ A_1 \cdot M_\lambda \subseteq M_\lambda, \]
\[ A_\lambda \cdot M_0 \subseteq M_\lambda, \ A_\lambda \cdot M_1 \subseteq M_\lambda, \ A_\lambda \cdot M_\lambda = M_0 \oplus M_1. \]
Moreover, if we assume that \( \beta \neq 0 \) and \( \beta + 3\gamma \neq 0 \), then \( A_0 \cdot M_\lambda = A_\lambda \cdot M_0 = \{0\} \) and \( A_\lambda \cdot M_\lambda = \{0\} \), where \( \lambda = -\gamma/\beta + 3\gamma \).

**Proof.** Since \( \rho \) is a representation of \( A \), we have that \( A \) and \( S = A \oplus M \), are generalized almost-Jordan algebra for the same scalars \( (\beta, \gamma) \). If \( e \) is an idempotent element in \( S \), since \( 0 \not\in \{\gamma, \beta + \gamma, \beta + 2\gamma\} \) then the Peirce decomposition of \( S \) relative to \( e \) is \( S = S_0 \oplus S_1 \oplus S_\lambda \), where \( S_i = \{a + m \in S \mid e(a + m) = i(a + m)\} \) for \( i = 0, 1, \lambda \). Moreover, we have the following relations
\[ (S_0)^2 \subseteq S_0, \ (S_1)^2 \subseteq S_1, \ (S_\lambda)^2 \subseteq S_0 \oplus S_1, \]
\[ S_\lambda S_0 \subseteq S_\lambda, \ S_\lambda S_1 \subseteq S_1, \ S_0 S_1 = \{0\}. \]

On the other hand we have that \( A_i = S_i \cap A \) and \( M_i = S_i \cap M \), in fact \( S_i \cap A = \{a + m \in S \mid e(a + m) = i(a + m)\} \cap A = \{a \in A \mid ea = ia\} = A_i \), similarly we have that \( M_i = S_i \cap M \).
\[ A_0 \cdot M_0 = (S_0 \cap A)(S_0 \cap M) \subseteq (S_0 \cap M) = M_0, \]
\[ A_1 \cdot M_1 = (S_1 \cap A)(S_1 \cap M) \subseteq (S_1 \cap M) = M_1, \]
\[ A_\lambda \cdot M_0 = (S_\lambda \cap A)(S_\lambda \cap M) \subseteq (S_\lambda \cap M) = M_0 \oplus M_1, \]
\[ (S_0 \cap A)(S_1 \cap M) = (S_0 \cap M)(S_1 \cap A) = \{0\}, \] that is \( A_0 \cdot M_1 = A_1 \cdot M_0 = \{0\}. \]
\[ A_\lambda \cdot M_1 = (S_\lambda \cap A)(S_1 \cap M) \subseteq (S_\lambda \cap M) = M_\lambda, \] similarly we have that \( A_1 \cdot M_\lambda \subseteq M_\lambda, \)
\[ A_\lambda \cdot M_0 = (S_\lambda \cap A)(S_1 \cap M) \subseteq (S_\lambda \cap M) = M_\lambda. \] In a similar way we prove that \( A_0 \cdot M_\lambda \subseteq M_\lambda. \)

If we add the conditions \( \beta \neq 0 \) and \( \beta + 3\gamma \neq 0 \), we have that \( (S_\lambda)^2 = S_0 S_\lambda = \{0\} \), and the relations \( A_0 \cdot M_\lambda = A_\lambda \cdot M_0 = \{0\} \) and \( A_\lambda \cdot M_\lambda = \{0\} \) follow. \( \square \)

### 3. Exceptional cases

We now look at the three cases which arose as exception in Theorem 2.3.

**3.1. Case \( \gamma = 0 \).** Let \( A \) be a generalized almost-Jordan algebra and \( \gamma = 0 \). Then \( \beta \neq 0 \), and \( A \) satisfies the identity \((yx)x - ((yx)x)x = 0\), for every \( x, y \in A \). Let \( \rho : A \to End(M) \) be a representation of \( A \). For these algebras the minimal polynomial of the operator \( R_e : A \to A \), is the same of the operator \( \rho_e \) and it is given by \( p(t) = t^2(t - 1) \), (see identity 2).

Then the Peirce decomposition of the algebra \( A \) is \( A = A_0 \oplus A_1 \), where \( A_0 = \{x \in A \mid (ex)e = 0\} \) and \( A_1 = \{x \in A \mid (ex) = x\} \). Similarly we have that \( M = M_0 \oplus M_1 \), where \( M_0 = \{m \in M \mid p_e^2(m) = 0\} \) and \( M_1 = \{m \in M \mid \rho_e(m) = m\} \).

Moreover, we have the relations (see 6).
\[ A_0 A_1 \subseteq A_0, \ (A_0)^2 \subseteq A_0. \]
Lemma 3.2. Let $A$ be a generalized almost-Jordan algebra and $\gamma = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be a representation of $A$. Then

$$A_1 \cdot M_0 \subseteq M_0, A_0 \cdot M_1 \subseteq M_0, A_0 \cdot M_0 \subseteq M_0.$$ 

Proof. As in the proof of Theorem 2.3 if we consider the algebra $S = A \oplus M$, then $S$ satisfies identity (1) for $\beta \neq 0$ and $\gamma = 0$. Moreover $e$ is an idempotent element in $S$, so in this case the Peirce decomposition of $S$ relative to $e$ is $S = S_0 \oplus S_1$, where $S_0 = \{a + m \in S \mid e(e(a + m)) = 0\}$ and $S_1 = \{a + m \in S \mid e(a + m) = a + m\}$.

On the other hand we have that $A_i = S_i \cap A$ and $M_i = S_i \cap M$. In fact $S_1 \cap A = \{a + m \in S \mid e(a + m) = (a + m)\} \cap A = \{a \in A \mid ea = a\} = A_1$ and $S_0 \cap A = \{a \in A \mid e(ea) = 0\} = A_0$. Similarly we have that $M_i = S_i \cap M$. Moreover, the subspaces $S_i$ satisfy the following relations

$$S_0S_1 \subseteq S_0, (S_0)^2 \subseteq S_0,$$

and we obtain that

$$A_0 \cdot M_0 = (S_0 \cap A)(S_0 \cap M) \subseteq (S_0 \cap M) = M_0,$$

$$A_1 \cdot M_0 = (S_1 \cap A)(S_0 \cap M) \subseteq (S_0 \cap M) = M_0.$$

Similarly we obtain that $A_0 \cdot M_1 \subseteq M_0$ and Lemma 3.2 follows. 

3.3. Case $\beta + \gamma = 0$. Let $M$ be a vector space and $\rho : A \to \text{End}(M)$ a representation of $A$. We know by identity (1) that the minimal polynomial of $\rho_e$ and $R_e$ is $p(x) = x^2 - x$. Then the Peirce decomposition of the algebra $A$ is $A = A_0 \oplus A_1$, where $A_i = \{a \in A \mid ea = ia\}$ for $i = 0, 1$. Similarly we have that $M = M_0 \oplus M_1$, where $M_i = \{m \in M \mid \rho_e(m) = im\}$ for $i = 0, 1$.

We know that $A_0A_1 = \{0\}$ and $(A_1)^2 \subseteq A_1$, (see [3]).

Lemma 3.4. Let $A$ be a generalized almost-Jordan algebra and $\beta + \gamma = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be a representation of $A$. Then

$$A_0 \cdot M_1 = A_1 \cdot M_0 = \{0\}, A_1 \cdot M_1 \subseteq M_1.$$ 

Proof. The algebra $S = A \oplus M$, is a generalized almost-Jordan algebra so $S$ satisfies identity (1) for $\beta$ and $\gamma$ satisfying $\beta + \gamma = 0$. Moreover the Peirce decomposition of $S$ relative to the idempotent is $S = S_0 \oplus S_1$, where $S_i = \{a + m \in S \mid e(a + m) = i(a + m)\}$ for $i = 0, 1$.

As the above case we have that $A_i = S_i \cap A$ and $M_i = S_i \cap M$, and in this case we have the following relations

$$S_0S_1 = \{0\}, (S_1)^2 \subseteq S_1.$$ 

Therefore, we have that

$$A_1 \cdot M_1 = (S_1 \cap A)(S_1 \cap M) \subseteq (S_1 \cap M) = M_1,$$

$$A_1 \cdot M_0 = (S_1 \cap A)(S_0 \cap M) = \{0\},$$

Similarly we have that $A_0 \cdot M_1 = \{0\}$. 

\[\square\]
3.5. **Case** $\beta + 2\gamma = 0$. Let $\rho : A \to \text{End}(M)$ be a representation of a generalized almost-Jordan algebra and $\beta + 2\gamma = 0$. For these algebras the minimal polynomial of the operator $R_e : A \to A$, is the same of the operator $\rho_e$ and it is given by $p(t) = t(t - 1)^2$, (see identity (4)). Then the Peirce decomposition of the algebra $A$ is $A = A_0 \oplus A_1$, where $A_0 = \{x \in A \mid (ex) = 0\}$ and $A_1 = \{x \in A \mid (ex)e - 2(ex) + x = 0\}$. Similarly we have that $M = M_0 \oplus M_1$, where $M_0 = \{m \in M \mid \rho_e(m) = 0\}$ and $M_1 = \{m \in M \mid (\rho_e - \text{id})^2(m) = 0\}$.

Moreover we have the following relations (see [6]).

$$A_0 A_1 = \{0\}, \quad (A_0)^2 \subseteq A_0.$$

**Lemma 3.6.** Let $A$ be a generalized almost-Jordan algebra and $\beta + 2\gamma = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be a representation of $A$. Then

$$A_0 \cdot M_1 = A_1 \cdot M_0 = \{0\}, \quad A_0 \cdot M_0 \subseteq M_0.$$

**Proof.** The algebra $S = A \oplus M$, is a generalized almost-Jordan algebra so $S$ satisfies identity (1) for $\beta$ and $\gamma$ satisfying $\beta + 2\gamma = 0$. Moreover the Peirce decomposition of $S$ relative to the idempotent is $S = S_0 \oplus S_1$, where $S_0 = \{a + m \in S \mid e(a + m) = 0\}$ and $S_1 = \{a + m \in S \mid e(a + m) - 2e(a + m) + (a + m) = 0\}$.

As in the above Lemmas we have that $A_i = S_i \cap A$ and $M_i = S_i \cap M$. Moreover we have the following relations

$$S_0 S_1 = \{0\}, \quad (S_0)^2 \subseteq S_0.$$

Therefore we have

$$A_0 \cdot M_0 = (S_0 \cap A)(S_0 \cap M) \subseteq (S_0 \cap M) = M_0,$$

$$A_1 \cdot M_0 = (S_1 \cap A)(S_0 \cap M) = \{0\}.$$

Similarly we obtain that $A_0 \cdot M_1 = \{0\}$ and Lemma 3.6 follows. $\square$

4. **Irreducible Modules**

Let $A$ be an algebra over $K$ and $\rho : A \to \text{End}(M)$ a representation of $A$.

**Definition 4.1.** Let $N$ be a subspace of $M$, we will say that $N$ is a submodule of $M$ if and only if $A \cdot N \subseteq N$.

**Definition 4.2.** We will say that $M$ is an irreducible module or that $\rho$ is an irreducible representation, if $M \neq 0$, and $M$ has no proper submodules or equivalently there is no proper subspace of $M$ which are invariant under all the transformations $\rho(a), a \in A$.

**Proposition 4.3.** Let $A$ be a generalized almost-Jordan algebra and $\beta, \gamma$ satisfying $0 \notin \{\gamma, \beta + \gamma, \beta + 2\gamma\}$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be an irreducible representation of $A$. Then one of the following conditions hold.

(i) $M = M_\lambda$ or (ii) $M = M_0$ or (iii) $M = M_1$
Proof. Using Theorem 2.3 we obtain that $M_0$ and $M_\lambda$ are submodules of $M$. Since $M$ is irreducible, then $M = M_0$ or $M_0 = \{0\}$. On the other hand, $M = M_\lambda$ or $M_\lambda = \{0\}$ and the Proposition follows.

Linearizing identity (3) we obtain

\[(\beta + \gamma)(\rho_c \rho_{ab} + \rho_a \rho_{cb} + \rho_a \rho_c \rho_b + \rho_c \rho_a \rho_b + \rho_{(ab)c} + \rho_{(cb)a}) - 2\beta(\rho_a \rho_b \rho_c + \rho_c \rho_b \rho_a + \rho_{(ac)b}) - 2\gamma(\rho_b \rho_a \rho_c + \rho_b \rho_c \rho_a + \rho_b \rho_{ac}) = 0\]

**Theorem 4.4.** Let $A$ be a generalized almost-Jordan algebra with $\beta$ and $\gamma$ satisfying $0 \notin \{\beta, \gamma, \beta + \gamma, \beta + 2\gamma, \beta + 3\gamma\}$. Suppose that $A$ has an idempotent element $e \neq 0$, and $M$ be an irreducible module. If $M = M_1$, then $M$ is an associative module.

Proof. $M$ is associative if and only if

\[(a, b, m) = 0 \quad \forall a, b \in A, \ m \in M \]

\[(a, m, b) = 0 \quad \forall a, b \in A, \ m \in M \]

Since $M = M_1$ we have $\rho_e = id_M$. We must prove relations (8) and (9). Let $a, b \in A$ and $m \in M$, then $a = a_0 + a_1 + a_\lambda$ and $b = b_0 + b_1 + b_\lambda$ with $a_i, b_i \in A_i$, $i = 0, 1, \lambda$. We have that

\[(a, b, m) = (a_0 + a_1 + a_\lambda, b_0 + b_1 + b_\lambda, m) = (a_1, b_1, m)\]

Similarly we have that $(a, m, b) = (a_1, m, b_1)$. Therefore for proving that $M$ is associative, we must verify relations (8) and (9) for all $a, b \in A_1$ and $m \in M = M_1$.

\[(a, b, m) = ab \cdot m - a \cdot (b \cdot m) = \rho_{ab}(m) - \rho_a(\rho_b(m)) = (\rho_{ab} - \rho_a \rho_b)(m)\]

and

\[(a, m, b) = (a \cdot m) \cdot b - a \cdot (m \cdot b) = b \cdot (a \cdot m) - a \cdot (b \cdot m) = (\rho_b \rho_a - \rho_{ab})(m).\]

Therefore we need to prove that $\rho_b \rho_a = \rho_a \rho_b = \rho_{ab}$.

Replacing $a, b \in A_1$ and $c = e$ in identity (7) we have

\[(\beta + \gamma)(3\rho_{ab} + 3\rho_a \rho_b) - 2\beta(\rho_a \rho_b + \rho_b \rho_a + \rho_{ab}) - 6\gamma(\rho_b \rho_a) = 0.\]

Reordering the terms we have

\[(\beta + 3\gamma)\rho_{ab} + (\beta + 3\gamma)\rho_a \rho_b - 2(\beta + 3\gamma)\rho_b \rho_a = 0.\]

Since $\beta + 3\gamma \neq 0$ we obtain

\[(\rho_{ab} + \rho_a \rho_b - 2\rho_b \rho_a = 0)\]

Interchanging $a$ and $b$ in identity (10) we obtain

\[(\rho_{ab} + \rho_b \rho_a - 2\rho_a \rho_b = 0)\]

Finally subtracting identity (10) and identity (11) and using that char$(F) \neq 3$, we obtain $\rho_a \rho_b = \rho_b \rho_a$. Then we have that $\rho_{ab} = \rho_a \rho_b$. That is (8) and (9) are valid for all $a, b \in A_1$, and $M$ is an associative module.}

In the case $M = M_\lambda$ we have the following result
Theorem 4.5. Let $A$ be a generalized almost-Jordan algebra with $\beta$ and $\gamma$ satisfying $0 \notin \{ \beta, \gamma, \beta + \gamma, \beta + 2\gamma, \beta + 3\gamma, \beta - 2\gamma \}$. Suppose that $A$ has an idempotent element $e \neq 0$, and $M$ be an irreducible module. If $M = M_\lambda$, then the following relations hold

(i) $(a, m, b) = 0 \quad \forall \ a, b \in A, \ m \in M$

(ii) $(ab)m = \lambda^{-1}a(bm) \quad \forall \ a, b \in A, \ m \in M$.

Proof. Since $M = M_\lambda$ we have that $\rho e = \lambda \text{id}$. We must prove (i) and (ii) for all $a, b \in A_1$ and $m \in M$. Replacing $a, b \in A_1$ and $c = e$ in identity (7) we have that

$$(\beta + \gamma)(\lambda \rho_{ab} + \rho_a \rho_b + 2\lambda \rho_a \rho_b + 2\rho_{ab}) - 2\beta(\lambda \rho_a \rho_b + \lambda \rho_b \rho_a + \rho_{ab})$$

$- 2\gamma(2\lambda \rho_b \rho_a + \rho_{ab}) = 0$

Reordering in term of $\rho_{ab}, \rho_a \rho_b$ and $\rho_b \rho_a$ we have

$$(\beta + \gamma)(\lambda + 2) - 2\beta)\rho_{ab} + ((\beta + \gamma)(2\lambda + 1) - 2\beta\lambda)\rho_a \rho_b$$

$- 2(2\gamma \lambda + \gamma + \beta\lambda)\rho_b \rho_a = 0$

Developing each coefficient and in the case of the coefficient of $\rho_{ab}$ we use the value of $\lambda$, to get the identity

$$\gamma \rho_{ab} + (\beta + \gamma + 2\gamma \lambda)\rho_a \rho_b - 2\gamma \lambda \rho_b \rho_a = 0$$

Interchanging $a$ and $b$ in identity (12) we have

$$\gamma \rho_{ab} + (\beta + \gamma + 2\gamma \lambda)\rho_b \rho_a - 2\gamma \lambda \rho_a \rho_b = 0$$

Subtracting identity (12) and identity (13) we obtain

$$(\beta + \gamma + 4\gamma \lambda)(\rho_a \rho_b - \rho_b \rho_a) = 0$$

Replacing the value of $\lambda$ we obtain

$$\frac{(\beta + 3\gamma)(\beta - \gamma)}{\beta + \gamma}(\rho_a \rho_b - \rho_b \rho_a) = 0$$

Since $(\beta + 3\gamma) \neq 0$ and $(\beta - \gamma) \neq 0$, we have $\rho_a \rho_b = \rho_b \rho_a$. Therefore we have (i). Using (12) we have

$$\gamma \rho_{ab} + (\beta + \gamma)\rho_a \rho_b = 0$$

Since $\beta + \gamma \neq 0$, and using the value of $\lambda$ we obtain

$$- \lambda \rho_{ab} + \rho_a \rho_b = 0$$

Therefore $\rho_{ab} = \lambda^{-1} \rho_a \rho_b$, we prove (ii), and the Theorem follows. $\square$

5. Exceptional cases

We now look at five cases that arose as exception in Theorem 4.4 and in Theorem 4.5.
5.1. Case $\beta = 0$. In this case, since $\beta = 0$ using Theorem \ref{thm:1.2} we have that $M_0$ is submodule and we have the following result

**Lemma 5.2.** Let $A$ be a generalized almost-Jordan algebra with $\beta = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be an irreducible representation of $A$. Then $M = M_0$ or $M = M_1 \oplus M_{-1}$, where $M_i = \{m \in M \mid \rho_e(m) = im\}$ para $i = 0, 1, -1$.

5.3. Case $\gamma = 0$. In this case, using Lemma \ref{lem:3.3} we have that $M_0$ is submodule and we have:

**Lemma 5.4.** Let $A$ be a generalized almost-Jordan algebra with $\gamma = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be an irreducible representation of $A$. Then $M = M_0$ or $M = M_1$, where $M_0 = \{m \in M \mid \rho_e^2(m) = 0\}$ and $M_1 = \{m \in M \mid \rho_e(m) = m\}$.

**Proposition 5.5.** Let $A$ be a generalized almost-Jordan algebra with $\gamma = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be an irreducible representation of $A$. If $(A_1)^2 \subseteq A_1$ and $M = M_1$, then $M$ is an associative module.

**Proof.** Suppose $\gamma = 0$, $(A_1)^2 \subseteq A_1$ and $M = M_1$, that is $\rho_e = id_M$. We need to prove that $(a, b, m) = (a, m, b) = 0$ for all $a, b \in A_1$, $m \in M$. But

$$(a, b, m) = ab \cdot m - a \cdot (b \cdot m) = \rho_{ab}(m) - \rho_a(\rho_b(m)) = (\rho_{ab} - \rho_a \rho_b)(m)$$

and

$$(a, m, b) = (a \cdot m) \cdot b - a \cdot (m \cdot b) = b \cdot (a \cdot m) - a \cdot (b \cdot m) = (\rho_b \rho_a - \rho_a \rho_b)(m)$$

Replacing $a, b \in A_1$ and $c = e$ en relation \ref{eq:7} we obtain $\rho_{a \rho_b} + \rho_{ab} - 2 \rho_b \rho_a = 0$. Interchanging $a$ and $b$ in the above identity we obtain $\rho_b \rho_a + \rho_{ab} - 2 \rho_a \rho_b = 0$. Subtracting both identities we have that $\rho_{a \rho_b} = \rho_b \rho_a$. So $\rho_{ab} = \rho_a \rho_b$ and $M$ is an associative module. \hfill $\square$

5.6. Case $\beta + \gamma = 0$. In this case using Lemma \ref{lem:3.4} we have that $M_1$ is submodule of $M$ and we have the following result

**Lemma 5.7.** Let $A$ be a generalized almost-Jordan algebra with $\beta + \gamma = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be an irreducible representation of $A$. Then $M = M_0 \oplus M = M_1$.

**Proposition 5.8.** Let $A$ be a generalized almost-Jordan algebra with $\beta + \gamma = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \to \text{End}(M)$ be an irreducible representation of $A$. If $(A_0)^2 \subseteq A_0$ and $\rho_e \neq 0$, then $M$ is an associative module.

**Proof.** As the above results we need to prove that $(a, b, m) = (a, m, b) = 0 \forall a, b \in A_1, m \in M$. Since $\rho_e \neq 0$ Lemma \ref{lem:5.5} implies that $\rho_e = id_M$. Moreover we have that $(A_0)^2 \subseteq A_0$, so if $a = a_0 + a_1$ and $b = b_0 + b_1$ with $a_i, b_i \in A_i$, we have that $(a, b, m) = (a_1, b_1, m)$ y $(a, m, b) = (a_1, m, b_1)$ and we only need to take $a, b \in A_1$. With the same argument using in the proof of Teorema \ref{thm:4.4} we prove that $M$ is an associative module. \hfill $\square$
The next example shows an irreducible module of dimension 2, in the case $\beta + \gamma = 0$

**Example 5.9.** Let us consider the algebra $A$ of base $\{e, a\}$ and multiplication table $e^2 = e, ea = ae = 0, a^2 = e$ given in Example 1.1. Let $M$ be a 2-dimensional $\mathbb{R}$-vector space and $\{v, w\}$ a base of $M$. We define a linear map $\rho : A \rightarrow \text{End}(M)$ by $\rho_e = 0$ and $\rho_a (\lambda_1 v + \lambda_2 w) = (2\lambda_2 - \lambda_1)v + (\lambda_2 - \lambda_1)w$. Then $\rho$ satisfies (2) and (3), so $\rho$ is a representation of $A$. Suppose that $M$ is not irreducible, that is, there exists a submodule $N = \mathbb{R}m$ for some $m \in M - \{0\}$. Let $m = \lambda_1 v + \lambda_2 w \neq 0$, since $N$ is a submodule of $M$, we have that $\rho_x (m) = b_x m$ for some $b_x \in \mathbb{R}$, and for all $x \in A$. Taking $x = a$ we have that $\rho_a (m) = b_a m$, and we obtain that

$$(2\lambda_2 - \lambda_1) = b_a \lambda_1, \quad (\lambda_2 - \lambda_1) = b_a \lambda_2$$

From the first identity we have $\lambda_2 = \frac{(b_a + 1)}{2} \lambda_1$, and replacing this value in the second identity we have

$$\frac{(b_a + 1)}{2} \lambda_1 - \lambda_1 = b_a \frac{(b_a + 1)}{2} \lambda_1$$

$$(b_a + 1) \lambda_1 - 2\lambda_1 = b_a (b_a + 1) \lambda_1$$

$$((b_a)^2 + b_a - b_a - 1 + 2) \lambda_1 = 0$$

$$((b_a)^2 + 1) \lambda_1 = 0$$

Since the polynomial $x^2 + 1 = 0$ is irreducible in $\mathbb{R}[x]$, we obtain that $\lambda_1 = 0$, and then $\lambda_2 = 0$. A contradiction since $m \neq 0$. Therefore $M$ is a 2-dimensional irreducible module.

5.10. $\beta + 2\gamma = 0$. Lemma 3.6 implies that $M_0$ is un submodule of $M$, and we have

**Lemma 5.11.** Let $A$ be a generalized almost-Jordan algebra with $\beta + 2\gamma = 0$. Suppose that $A$ has an idempotent element $e \neq 0$. Let $\rho : A \rightarrow \text{End}(M)$ be an irreducible representation of $A$. Then $M = M_0$ or $M = M_1$, where $M_0 = \{m \in M \mid \rho_e (m) = 0\}$ and $M_1 = \{m \in M \mid (\rho_e - \text{id})^2 (m) = 0\}$.

5.12. $\beta + 3\gamma = 0$. These algebras are the almost-Jordan algebras and it is known that for this kind of algebras every irreducible module is a Jordan module (see [26]).

**Open problems:** We do not know which is the situation with an irreducible module $M$,

1. In the case $M = M_0$
2. In the case $\beta - \gamma = 0$, that is $A$ satisfies the identity, $(yx^2)x + yx^3 - 2((yx)x)x = 0$.
3. In the case $\beta = 0$, that is $A$ satisfies the identity, $yx^3 - ((yx)x)x = 0$.
4. In the case $\beta + 2\gamma = 0$, that is $A$ satisfies the identity, $yx^3 - 2(yx^2)x + ((yx)x)x = 0$.

In the last two cases we only know that $M_0$ is a submodule of $M$. 
References


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