REPRESENTATIONS OF GENERALIZED ALMOST-JORDAN ALGEBRAS

MARCELO FLORES AND ALICIA LABRA*

ABSTRACT. This paper deals with the variety of commutative algebras satisfying the identity $\beta\{(yx^2)x-((yx)x)x\}+\gamma\{yx^3-((yx)x)x\}=0$ where α,β are scalars. These algebras appeared as one of the four families of degree four identities in Carini, Hentzel and Piacentini-Cattaneo [6]. We give a characterization of representations and irreducible modules on these algebras. Our results require that the characteristic of the ground field was different from $2,3$.

1. INTRODUCTION

Let A be a commutative not necessarily associative algebra over an infinite field F. Let x be an element in A. We define the principal powers of x by $x^1=x,\ x^{n+1}=x^nx$ for all $n\geq 1$.

A *Jordan algebra* is a commutative algebra satisfying the identity $x^2(yx)-(x^2y)x=0$. It is a well known variety of algebra, that is *power-associative,* i.e., the subalgebra generated by any element of the algebra, is associative. See [13], [24] for properties of these varieties of algebras. It is known (see [18]) that a Jordan algebra satisfies the identity $3((yx^2)x)=2((yx)x)x+yx^3$. These algebras, called *almost-Jordan algebras* have been studied by Osborn [18], [19], Petersson [22], Sidorov [26], and Hentzel and Peresi [11]. In this last paper, the authors proved that every semi-prime almost-Jordan algebra is a Jordan algebra and this fact justified the name of these algebras.

A *generalized almost-Jordan algebra* is a commutative algebra satisfying the identity

$$\beta\{(yx^2)x-((yx)x)x\}+\gamma\{yx^3-((yx)x)x\}=0 \tag{1}$$

for every x,y $\in A$ where α,β are scalars, and $(\beta,\gamma)\neq(0,0)$. For $\beta=3$ and $\gamma=-1$, we have an almost-Jordan algebra.

In the study of degree four identities not implied by conmutativity, Osborn [19] classified those that were implied by the fact of possessing a unit element. Carini, Hentzel and Piacentini-Cattaneo [6] extended this work by dropping the restriction on the existence of the unit element. This result require that characteristic $F\neq 2,3$. The identity defining a generalized almost-Jordan algebra with $\beta,\gamma\in F$ appears as one of these identities.

*Supported by Fondecyt 1120844.

We observe that there are generalized almost-Jordan algebras that are not Jordan algebras.

Example 1.1. *Let A be a commutative algebra over F with base $\{e, a\}$, and multiplication table given by*

$$e^2 = e,\ a^2 = e,\ \text{ all other products being zero}$$

Then A satisfies identity (1) for $\beta = 1$ y $\gamma = -1$. Moreover A is not a Jordan algebra, since it is not power-associative since $a^2(aa) \neq (a^2a)a$.

Example 1.2. *Let A be a commutative algebra over F with base $\{e, a\}$ and multiplication table given by*

$$e^2 = e,\ ea = ae = -e - a,\ a^2 = e + a.$$

Then A satisfies identity (1) with $\beta = 0$ and $\gamma \neq 0$. That is A satisfies $x^3y = ((xy)x)x$ for every $x, y \in A$. Moreover A is not a Jordan algebra, since $a^2(aa) = 2a \neq (a^2a)a = 0$.

Example 1.3. *Let A be a commutative F-algebra with base $\{e, a, b\}$ and multiplication table given by*

$$e^2 = e,\ ab = ba = b,\ \text{ all other products being zero}$$

Then A satisfies identity (1) with $\beta = 1$ and $\gamma = 1$, for every $\alpha \in F$. Moreover A is not power-associative since $(a+b)^4 = 2b$ and $(a+b)^2(a+b)^2 = 0$ and A is not a Jordan algebra

Generalized almost-Jordan algebras A have been studied in [1], where the authors proved that these algebras always have a trace form in terms of the trace of right multiplication operators. They also prove that if A is finite-dimensional and solvable, then it is nilpotent and found three conditions, any of which implies that a finite-dimensional right-nilalgebra A is nilpotent. In [2] the author found the Wedderburn decomposition of A assuming that for every ideal I of A either I has a non zero idempotent or $I \subset R, R$ the solvable radical of A and the quotient A/R is separable and in [10] where, assuming that A also satisfies $((xx)x)x = 0$ the authors proved the existence of an ideal I of A such that $AI = IA = 0$ and the quotient algebra A/I is power-associative.

In this paper we deal with representations of algebras. Let A be an algebra which belongs to a class $\mathcal{C}$ of commutative algebras over a field K and let M be a vector space over F. As in Eilenberg [9], we say that a linear map $\rho : A \to End(M)$ is a *a representation of A in the class $\mathcal{C}$* if the split null extension $S = A \oplus M$ of M, with multiplication given by

$$(a + m)(b + n) = ab + \rho(a)(n) + \rho(b)(m) \ \ \forall\, a, b \in A, m, n \in M$$

belongs to the class $\mathcal{C}$.

Representations have been studied for different algebras, for example, in [15], and [21] for Jordan algebras, in[14], [23] and [25] for alternative algebras, in [17] for composition algebras, in [12] for Lie algebras, in [26] for Lie triple algebras, in [20] for Novikov algebras, in [5] for Bernstein algebras, in [16] for train algebras of rank 3, in [3] for power-associative train algebras of rank 4, in [4] for algebras of rank 3, in [7] for Malcev algebras and in [8] for Malcev super algebras.

A representation $\rho : A \to End(M)$ is said to be *irreducible* if $M \neq 0$ and there is no proper subspace of M which is invariant under all the transformations $\rho(a)$, $a \in A$, and is said to be *r-dimensional* if $\dim M = r$.

In this paper we study representations and irreducible modules over generalized almost-Jordan algebras A. The paper is organized as follows. In Section 2 we find necessary and sufficient conditions for a linear map ρ to be a representation. We find the action of A over M when $0 \notin \{\gamma, \beta+\gamma, \beta+2\gamma\}$ (see Theorem 2.3). In Section 3 we look at the three cases that arose as exception in the above Theorem. In Section 4 we study irreducible modules over generalized almost-Jordan algebras and we prove two theorems when $0 \notin \{\beta, \gamma, \beta+\gamma, \beta+2\gamma, \beta+2\gamma\}$ (see Theorem 4.4 and Theorem 4.5). The last Theorem need also the condition $\beta - \gamma \neq 0$. In Section 5 we look at five cases that arose as exception in Theorem 4.4 and Theorem 4.5. Moreover, in the case $\beta + \gamma = 0$ we give an example of a 2-dimensional irreducible module M. Finally we present some open problems.

2. Representations

In this section we study representations of generalized almost-Jordan algebras.

Lemma 2.1. *Let A be a generalized almost-Jordan algebra and $\rho : A \to End(M)$ a linear map. Then ρ is a representation of A, if and only if for every $a, b \in A$ the following identities hold*

$$(\beta+\gamma)\rho_a^3 - \beta\rho_a\rho_{a^2} - \gamma\rho_{a^3} = 0 \tag{2}$$

$$(\beta+\gamma)(\rho_a\rho_{ab} + \rho_a^2\rho_b + \rho_{(ab)a}) - \beta(2\rho_a\rho_b\rho_a + \rho_{a^2b}) - \gamma(2\rho_b\rho_a^2 + \rho_b\rho_{a^2}) = 0 \tag{3}$$

where $\rho_a := \rho(a) \in End(M)$, and for every $a, b \in A$, $\rho_a \circ \rho_b$ will be denoted by $\rho_a\rho_b$.

Proof. ρ is a representation of A if and only if every $a+m, b+n \in A \oplus M$ satisfy the identity (1). Straightforward calculations give

$$[(a+m)^2(b+n)](a+m) = (a^2b)a + 2\rho_a(\rho_b(\rho_a(m))) + \rho_a(\rho_{a^2}(n)) + \rho_{a^2b}(m)$$

$$[(a+m)^3](b+n) = a^3b + 2\rho_b(\rho_a(\rho_a(m))) + \rho_b(\rho_{a^2}(m)) + \rho_{a^3}(n)$$

$$[[(a+m)(b+n)](a+m)](a+m) = ((ab)a)a + \rho_a(\rho_a(\rho_b(m))) + \rho_a(\rho_{ab}(m)) + \rho_a^3(n) + \rho_{(ab)a}(m)$$

Replacing $x = a+m, y = b+n$ in identity (1) we get

$\beta\{2\rho_a(\rho_b(\rho_a(m))) + \rho_a(\rho_{a^2}(n)) + \rho_{a^2b}(m) - \rho_a^2((\rho_b(m)) - \rho_a(\rho_{ab}(m)) - \rho_a^3(n) -$

$$\rho_{(ab)a}(m)\} + \gamma\{2\rho_b(\rho_a^2(m)) + \rho_b(\rho_{a^2}(m)) + \rho_{a^3}(n) - \rho_a^2((\rho_b(m)) - \rho_a(\rho_{ab}(m)) - \rho_a^3(n) - \rho_{(ab)a}(m)\} = 0$$

Now it is easy to see that this relation holds if and only if identities (2) and (3) hold in A. □

In the following suppose that A has an idempotent element $e \neq 0$. Taking $a = e$ in identity (2), we obtain

$$(\beta+\gamma)\rho_e^3 - \beta\rho_e^2 - \gamma\rho_e = 0 \tag{4}$$

Proposition 2.2. *Let A be a generalized almost-Jordan algebra and β, γ satisfying $0 \notin \{\gamma, \beta+\gamma, \beta+2\gamma\}$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be a representation of A. Then*

$$M = M_0 \oplus M_1 \oplus M_\lambda$$

where $M_i = \{m \in M | \rho_e(m) = im\}$, and $i \in \{0, 1, \lambda\}$.

Proof. Using identity (4) we see that ρ_e satisfies the polynomial $p(x) = (\beta+\gamma)x^3 - \beta x^2 - \gamma x = 0$. Since $\beta + \gamma \neq 0$ we define $\lambda = \frac{-\gamma}{\beta+\gamma} \in F$ and $p(x) = (\beta+\gamma)x(x-1)(x-\lambda)$. Moreover, since $\gamma \neq 0, \beta+2\gamma \neq 0$, we have that $\lambda \neq 0$ and $\lambda \neq 1$, therefore $p(x)$ has only simple roots. On the other hand, since the minimal polynomial of the operator ρ_e, is a divisor of $p(x)$ then ρ_e, also has simple roots. Then ρ_e is diagonalizable and

$$M = M_0 \oplus M_1 \oplus M_\lambda$$

□

In [2, Theorem 1] M. Arenas proves that the Peirce decomposition of a generalized almost-Jordan algebra A is given by $A = A_0 \oplus A_1 \oplus A_\lambda$, $A_i = \{a \in A | ea = ia\}$, where $\lambda = \frac{-\gamma}{\beta+\gamma}$. Moreover, when $0 \notin \{\gamma, \beta+\gamma, \beta+2\gamma\}$ we have the following relations among these subspaces

$$(A_0)^2 \subseteq A_0,\ (A_1)^2 \subseteq A_1,\ A_0A_1 = \{0\}$$
$$A_\lambda A_0 \subseteq A_\lambda,\ A_\lambda A_1 \subseteq A_\lambda,\ (A_\lambda)^2 \subseteq A_0 \oplus A_1$$

Next we obtain relations among A_i and M_i, where $i \in \{0, 1, \lambda\}$. Linearising identity (2) we have

$$(\beta+\gamma)(\rho_a\rho_b\rho_a + \rho_b\rho_a^2 + \rho_a^2\rho_b) - 2\beta\rho_a\rho_{ab} - \beta\rho_b\rho_{a^2} - 2\gamma\rho_{(ab)a} - \gamma\rho_{a^2b} = 0 \tag{5}$$

Moreover sustracting identities (3) and (5), we obtain

$$(\gamma-\beta)\{\rho_a\rho_b\rho_a + \rho_a\rho_{ab} - \rho_b\rho_a^2 - \rho_{(ab)a}\} + (\beta+\gamma)\{2\rho_a^2\rho_b - \rho_b\rho_{a^2} - \rho_{a^2b}\} = 0 \tag{6}$$

Theorem 2.3. *Let A be a generalized almost-Jordan algebra and β, γ satisfying $0 \notin \{\gamma, \beta+\gamma, \beta+2\gamma\}$. Suposse that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be a representation of A. Then the action of A on M satisfies the following relations*

$$A_0 \cdot M_0 \subseteq M_0,\ A_0 \cdot M_1 = \{0\},\ A_0 \cdot M_\lambda \subseteq M_\lambda,$$

$$A_1 \cdot M_0 = \{0\},\ A_1 \cdot M_1 \subseteq M_1,\ A_1 \cdot M_\lambda \subseteq M_\lambda,$$
$$A_\lambda \cdot M_0 \subseteq M_\lambda,\ A_\lambda \cdot M_1 \subseteq M_\lambda,\ A_\lambda \cdot M_\lambda = M_0 \oplus M_1.$$

Moreover, if we assume that $\beta \neq 0$ *and* $\beta + 3\gamma \neq 0$*, then* $A_0 \cdot M_\lambda = A_\lambda \cdot M_0 = \{0\}$ *and* $A_\lambda \cdot M_\lambda = \{0\}$*, where* $\lambda = \frac{-\gamma}{\beta+\gamma}$*.*

Proof. Since ρ is a representation of A, we have that A and $S = A \oplus M$, are generalized almost-Jordan algebra for the same scalars $(\beta, \gamma) \in F \times F$. If e is an idempotent element in S, since $0 \notin \{\gamma, \beta + \gamma, \beta + 2\gamma\}$ then the Peirce decomposition of S relative to e is $S = S_0 \oplus S_1 \oplus S_\lambda$, where $S_i = \{a + m \in S \mid e(a + m) = i(a + m)\}$ for $i = 0, 1, \lambda$. Moreover, we have the following relations

$$(S_0)^2 \subseteq S_0, \quad (S_1)^2 \subseteq S_1, \quad (S_\lambda)^2 \subseteq S_0 \oplus S_1,$$
$$S_\lambda S_0 \subseteq S_\lambda, \quad S_\lambda S_1 \subseteq S_1, \quad S_0 S_1 = \{0\}.$$

On the other hand we have that $A_i = S_i \cap A$ and $M_i = S_i \cap M$, in fact $S_i \cap A = \{a + m \in S \mid e(a + m) = i(a + m)\} \cap A = \{a \in A \mid ea = ia\} = A_i$, similarly we have that $M_i = S_i \cap M$.

$A_0 \cdot M_0 = (S_0 \cap A)(S_0 \cap M) \subseteq (S_0 \cap M) = M_0$,
$A_1 \cdot M_1 = (S_1 \cap A)(S_1 \cap M) \subseteq (S_1 \cap M) = M_1$,
$A_\lambda \cdot M_\lambda = (S_\lambda \cap A)(S_\lambda \cap M) \subseteq (S_0 \cap M) \oplus (S_1 \cap M) = M_0 \oplus M_1$,
$(S_0 \cap A)(S_1 \cap M) = (S_0 \cap M)(S_1 \cap A) = \{0\}$, that is $A_0 \cdot M_1 = A_1 \cdot M_0 = \{0\}$.
$A_\lambda \cdot M_1 = (S_\lambda \cap A)(S_1 \cap M) \subseteq (S_\lambda \cap M) = M_\lambda$, similarly we have that $A_1 \cdot M_\lambda \subseteq M_\lambda$.
$A_\lambda \cdot M_0 = (S_\lambda \cap A)(S_1 \cap M) \subseteq (S_\lambda \cap M) = M_\lambda$. In a similar way we prove that $A_0 \cdot M_\lambda \subseteq M_\lambda$.

If we add the conditions $\beta \neq 0$ and $\beta + 3\gamma \neq 0$, we have that $(S_\lambda)^2 = S_0 S_\lambda = \{0\}$, and the relations $A_0 \cdot M_\lambda = A_\lambda \cdot M_0 = \{0\}$ and $A_\lambda \cdot M_\lambda = \{0\}$ follow. □

3. Exceptional cases

We now look at the three cases which arose as exception in Theorem 2.3.

3.1. **Case $\gamma = 0$.** Let A be a generalized almost-Jordan algebra and $\gamma = 0$. Then $\beta \neq 0$, and A satisfies the identity $(yx^2)x - ((yx)x)x = 0$, for every $x, y \in A$. Let $\rho : A \to End(M)$ be a representation of A. For these algebras the minimal polynomial of the operator $R_e : A \to A$, is the same of the operator ρ_e and it is given by $p(t) = t^2(t - 1)$, (see identity (4)). Then the Peirce decomposition of the algebra A is $A = A_0 \oplus A_1$, where $A_0 = \{x \in A \mid (ex)e = 0\}$ and $A_1 = \{x \in A \mid (ex) = x\}$. Similarly we have that $M = M_0 \oplus M_1$, where $M_0 = \{m \in M \mid \rho_e^2(m) = 0\}$ and $M_1 = \{m \in M \mid \rho_e(m) = m\}$.

Moreover, we have the relations (see [6]).

$$A_0 A_1 \subseteq A_0,\ (A_0)^2 \subseteq A_0.$$

Lemma 3.2. *Let A be a generalized almost-Jordan algebra and $\gamma = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be a representation of A. Then*

$$A_1 \cdot M_0 \subseteq M_0, A_0 \cdot M_1 \subseteq M_0,\ A_0 \cdot M_0 \subseteq M_0.$$

Proof. As in the proof of Theorem 2.3, if we consider the algebra $S = A \oplus M$, then S satisfies identity (1) for $\beta \neq 0$ and $\gamma = 0$. Moreover e is an idempotent element in S, so in this case the Peirce decomposition of S relative to e is $S = S_0 \oplus S_1$, where $S_0 = \{a+m \in S \mid e(e(a+m)) = 0\}$ and $S_1 = \{a+m \in S \mid e(a+m) = a+m\}$.

On the other hand we have that $A_i = S_i \cap A$ and $M_i = S_i \cap M$. In fact $S_1 \cap A = \{a+m \in S \mid e(a+m) = (a+m)\} \cap A = \{a \in A \mid ea = a\} = A_1$ and $S_0 \cap A = \{a \in a \mid e(ea) = 0\} = A_0$. Similarly we have that $M_i = S_i \cap M$. Moreover, the subspaces S_i satisfy the following relations

$$S_0 S_1 \subseteq S_0,\ (S_0)^2 \subseteq S_0,$$

and we obtain that

$$A_0 \cdot M_0 = (S_0 \cap A)(S_0 \cap M) \subseteq (S_0 \cap M) = M_0.$$
$$A_1 \cdot M_0 = (S_1 \cap A)(S_0 \cap M) \subseteq (S_0 \cap M) = M_0.$$

Similarly we obtain that $A_0 \cdot M_1 \subseteq M_0$ and Lemma 3.2 follows.

□

3.3. **Case $\beta + \gamma = 0$.** Let M be a vector space and $\rho : A \to End(M)$ a representation of A. We know by identity (4) that the minimal polynomial of ρ_e and R_e is $p(x) = x^2 - x$. Then the Peirce decomposition of the algebra A is $A = A_0 \oplus A_1$, where $A_i = \{a \in A \mid ea = ia\}$ for $i = 0, 1$. Similarly we have that $M = M_0 \oplus M_1$, where $M_i = \{m \in M \mid \rho_e(m) = im\}$ for $i = 0, 1$.

We know that $A_0 A_1 = \{0\}$ and $(A_1)^2 \subseteq A_1$, (see [6]).

Lemma 3.4. *Let A be a generalized almost-Jordan algebra and $\beta + \gamma = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be a representation of A. Then*

$$A_0 \cdot M_1 = A_1 \cdot M_0 = \{0\},\ A_1 \cdot M_1 \subseteq M_1.$$

Proof. The algebra $S = A \oplus M$, is a generalized almost-Jordan algebra so S satisfies identity (1) for β and γ satisfying $\beta + \gamma = 0$. Moreover the Peirce decomposition of S relative to the idempotent is $S = S_0 \oplus S_1$, where $S_i = \{a+m \in S \mid e(a+m) = i(a+m)\}$ for $i = 0, 1$.

As the above case we have that $A_i = S_i \cap A$ and $M_i = S_i \cap M$, and in this case we have the following relations

$$S_0 S_1 = \{0\},\ (S_1)^2 \subseteq S_1.$$

Therefore, we have that

$$A_1 \cdot M_1 = (S_1 \cap A)(S_1 \cap M) \subseteq (S_1 \cap M) = M_1,$$
$$A_1 \cdot M_0 = (S_1 \cap A)(S_0 \cap M) = \{0\},$$

Similarly we have that $A_0 \cdot M_1 = \{0\}$. □

3.5. **Case** $\beta + 2\gamma = 0$**.** Let $\rho : A \to End(M)$ be a representation of a generalized almost-Jordan algebra and $\beta + 2\gamma = 0$. For these algebras the minimal polynomial of the operator $R_e : A \to A$, is the same of the operator ρ_e and it is given by $p(t) = t(t-1)^2$, (see identity (4)). Then the Peirce decomposition of the algebra A is $A = A_0 \oplus A_1$, where $A_0 = \{x \in A \mid (ex) = 0\}$ and $A_1 = \{x \in A \mid (ex)e - 2(ex) + x = 0\}$. Similarly we have that $M = M_0 \oplus M_1$, where $M_0 = \{m \in M \mid \rho_e(m) = 0\}$ and $M_1 = \{m \in M \mid (\rho_e - id)^2(m) = 0\}$.

Moreover we have the following relations (see [6]).

$$A_0A_1 = \{0\},\ (A_0)^2 \subseteq A_0.$$

Lemma 3.6. *Let A be a generalized almost-Jordan algebra and $\beta + 2\gamma = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be a representation of A. Then*

$$A_0 \cdot M_1 = A_1 \cdot M_0 = \{0\},\ A_0 \cdot M_0 \subseteq M_0.$$

Proof. The algebra $S = A \oplus M$, is a generalized almost-Jordan algebra so S satisfies identity (1) for β and γ satisfying $\beta + 2\gamma = 0$. Moreover the Peirce decomposition of S relative to the idempotent is $S = S_0 \oplus S_1$, where $S_0 = \{a + m \in S \mid e(a + m) = 0\}$ and $S_1 = \{a + m \in S \mid e(e(a + m)) - 2e(a + m) + (a + m) = 0\}$.

As in the above Lemmas we have that $A_i = S_i \cap A$ and $M_i = S_i \cap M$. Moreover we have the following relations

$$S_0S_1 = \{0\},\ (S_0)^2 \subseteq S_0.$$

Therefore we have

$$\begin{aligned} A_0 \cdot M_0 &= (S_0 \cap A)(S_0 \cap M) \subseteq (S_0 \cap M) = M_0, \\ A_1 \cdot M_0 &= (S_1 \cap A)(S_0 \cap M) = \{0\}. \end{aligned}$$

Similarly we obtain that $A_0 \cdot M_1 = \{0\}$ and Lemma 3.6 follows. □

4. Irreducible Modules

Let A be an algebra over K and $\rho : A \to End(M)$ a representation of A.

Definition 4.1. *Let N be a subspace of M, we will say that N is a submodule of M if and only if $A \cdot N \subseteq N$.*

Definition 4.2. *We will say that M is an irreducible module or that ρ is an irreducible representation, if $M \neq 0$, and M has no proper submodules or equivalently there is no proper subspace of M which are invariant under all the transformations $\rho(a), a \in A$.*

Proposition 4.3. *Let A be a generalized almost-Jordan algebra and β, γ satisfying $0 \notin \{\gamma, \beta + \gamma, \beta + 2\gamma\}$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be an irreducible representation of A. Then one of the following conditions hold.*

$$(i)\ \ M = M_\lambda \ \ or\ \ (ii)\ \ M = M_0 \ \ or\ \ (iii)\ \ M = M_1$$

Proof. Using Theorem 2.3, we obtain that M_0 and M_λ are submodules of M. Since M is irreducible, then $M = M_0$ or $M_0 = \{0\}$. On the other hand, $M = M_\lambda$ or $M_\lambda = \{0\}$ and the Proposition follows. □

Linearizing identity (3) we obtain

$$(\beta+\gamma)(\rho_c\rho_{ab} + \rho_a\rho_{cb} + \rho_a\rho_c\rho_b + \rho_c\rho_a\rho_b + \rho_{(ab)c} + \rho_{(cb)a}) - 2\beta(\rho_a\rho_b\rho_c + \rho_c\rho_b\rho_a + \rho_{(ac)b}) - 2\gamma(\rho_b\rho_a\rho_c + \rho_b\rho_c\rho_a + \rho_b\rho_{ac}) = 0 \tag{7}$$

Theorem 4.4. *Let A be a generalized almost-Jordan algebra with β and γ satisfying $0 \notin \{\beta, \gamma, \beta+\gamma, \beta+2\gamma, \beta+3\gamma\}$. Suppose that A has an idempotent element $e \neq 0$, and M be an irreducible module. If $M = M_1$, then M is an associative module.*

Proof. M is associative if and only if

$$(a,b,m) = 0 \qquad \forall a,b \in A,\ m \in M \tag{8}$$

$$(a,m,b) = 0 \qquad \forall a,b \in A,\ m \in M \tag{9}$$

Since $M = M_1$ we have $\rho_e = id_M$. We must prove relations (8) and (9). Let $a, b \in A$ and $m \in M$, then $a = a_0 + a_1 + a_\lambda$ and $b = b_0 + b_1 + b_\lambda$ with $a_i, b_i \in A_i$ $i = 0, 1, \lambda$. We have that

$$(a,b,m) = (a_0 + a_1 + a_\lambda, b_0 + b_1 + b_\lambda, m) = (a_1, b_1, m)$$

Similarly we have that $(a,m,b) = (a_1, m, b_1)$. Therefore for proving that M is associative, we must verify relations (8) and (9) for all $a, b \in A_1$ and $m \in M = M_1$.

$$(a,b,m) = ab\cdot m - a\cdot(b\cdot m) = \rho_{ab}(m) - \rho_a(\rho_b(m)) = (\rho_{ab} - \rho_a\rho_b)(m)$$

and

$$(a,m,b) = (a\cdot m)\cdot b - a\cdot(m\cdot b) = b\cdot(a\cdot m) - a\cdot(b\cdot m) = (\rho_b\rho_a - \rho_a\rho_b)(m).$$

Therefore we need to prove that $\rho_b\rho_a = \rho_a\rho_b = \rho_{ab}$.

Replacing $a, b \in A_1$ and $c = e$ in identity (7) we have

$$(\beta+\gamma)(3\rho_{ab} + 3\rho_a\rho_b) - 2\beta(\rho_a\rho_b + \rho_b\rho_a + \rho_{ab}) - 6\gamma(\rho_b\rho_a) = 0.$$

Reordering the terms we have

$$(\beta+3\gamma)\rho_{ab} + (\beta+3\gamma)\rho_a\rho_b - 2(\beta+3\gamma)\rho_b\rho_a = 0.$$

Since $\beta + 3\gamma \neq 0$ we obtain

$$\rho_{ab} + \rho_a\rho_b - 2\rho_b\rho_a = 0 \tag{10}$$

Interchanging a and b in identity (10) we obtain

$$\rho_{ab} + \rho_b\rho_a - 2\rho_a\rho_b = 0 \tag{11}$$

Finally subtracting identity (10) and identity (11) and using that $\mathrm{char}(F) \neq 3$, we obtain $\rho_a\rho_b = \rho_b\rho_a$. Then we have that $\rho_{ab} = \rho_a\rho_b$. That is (8) and (9) are valid for all $a, b \in A_1$, and M is an associative module. □

In the case $M = M_\lambda$ we have the following result

Theorem 4.5. *Let A be a generalized almost-Jordan algebra with β and γ satisfaying $0 \notin \{\beta, \gamma, \beta+\gamma, \beta+2\gamma, \beta+3\gamma, \beta-\gamma\}$. Suppose that A has an idempotent element $e \neq 0$, and M be an irreducible module. If $M = M_\lambda$, then the following relations hold*

(i) $(a, m, b) = 0 \qquad \forall\ a, b \in A,\ m \in M$
(ii) $(ab)m = \lambda^{-1} a(bm) \qquad \forall\ a, b \in A,\ m \in M.$

Proof. Since $M = M_\lambda$ we have that $\rho_e = \lambda id$. We must prove (i) and (ii) for all $a, b \in A_1$ an $m \in M$. Replacing $a, b \in A_1$ and $c = e$ in identity (7) we have that

$$(\beta+\gamma)(\lambda\rho_{ab} + \rho_a\rho_b + 2\lambda\rho_a\rho_b + 2\rho_{ab}) - 2\beta(\lambda\rho_a\rho_b + \lambda\rho_b\rho_a + \rho_{ab})$$

$$-2\gamma(2\lambda\rho_b\rho_a + \rho_b\rho_a) = 0$$

Reordering in term of $\rho_{ab}, \rho_a\rho_b$ and $\rho_b\rho_a$ we have

$$((\beta+\gamma)(\lambda+2) - 2\beta)\rho_{ab} + ((\beta+\gamma)(2\lambda+1) - 2\beta\lambda)\rho_a\rho_b$$

$$-2(2\gamma\lambda + \gamma + \beta\lambda)\rho_b\rho_a = 0$$

Developing each coefficient and in the case of the coefficient of ρ_{ab} we use the value of λ, to get the identity

$$\gamma\rho_{ab} + (\beta+\gamma+2\gamma\lambda)\rho_a\rho_b - 2\gamma\lambda\rho_b\rho_a = 0 \tag{12}$$

Interchanging a and b in identity (12) we have

$$\gamma\rho_{ab} + (\beta+\gamma+2\gamma\lambda)\rho_b\rho_a - 2\gamma\lambda\rho_a\rho_b = 0 \tag{13}$$

Subtracting identity (12) and identity (13) we obtain

$$(\beta+\gamma+4\gamma\lambda)(\rho_a\rho_b - \rho_b\rho_a) = 0$$

Replacing the value of λ we obtain

$$\frac{(\beta+3\gamma)(\beta-\gamma)}{\beta+\gamma}(\rho_a\rho_b - \rho_b\rho_a) = 0 \tag{14}$$

Since $(\beta+3\gamma) \neq 0$ and $(\beta-\gamma) \neq 0$, we have $\rho_a\rho_b = \rho_b\rho_a$. Therefore we have (i). Using (12) we have

$$\gamma\rho_{ab} + (\beta+\gamma)\rho_a\rho_b = 0$$

Since $\beta+\gamma \neq 0$, and using the value of λ we obtain

$$-\lambda\rho_{ab} + \rho_a\rho_b = 0 \tag{15}$$

Therefore $\rho_{ab} = \lambda^{-1}\rho_a\rho_b$, we prove (ii), and the Theorem follows. $\square$

5. Exceptional cases

We now look at five cases that arose as exception in Theorem 4.4 and in Theorem 4.5

5.1. **Case $\beta = 0$.** In this case, since $\beta = 0$ using Theorem 2.3 we have that M_0 is submodule and we have the following result

Lemma 5.2. *Let A be a generalized almost-Jordan algebra with $\beta = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be an irreducible representation of A. Then $M = M_0$ or $M = M_1 \oplus M_{-1}$, where $M_i = \{m \in M \mid \rho_e(m) = im\}$ para $i = 0, 1, -1$.*

5.3. **Case $\gamma = 0$.** In this case, using Lemma 3.2 we have that M_0 is submodule and we have:

Lemma 5.4. *Let A be a generalized almost-Jordan algebra with $\gamma = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be an irreducible representation of A. Then $M = M_0$ or $M = M_1$, where $M_0 = \{m \in M \mid \rho_e^2(m) = 0\}$ y $M_1 = \{m \in M \mid \rho_e(m) = m\}$.*

Proposition 5.5. *Let A be a generalized almost-Jordan algebra with $\gamma = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be an irreducible representation of A. If $(A_1)^2 \subseteq A_1$ and $M = M_1$, then M is an associative module.*

Proof. Suppose $\gamma = 0, (A_1)^2 \subseteq A_1$ and $M = M_1$, that is $\rho_e = id_M$. We need to prove that $(a, b, m) = (a, m, b) = 0$ for all $a, b \in A_1,\ m \in M$. But

$$(a, b, m) = ab \cdot m - a \cdot (b \cdot m) = \rho_{ab}(m) - \rho_a(\rho_b(m)) = (\rho_{ab} - \rho_a\rho_b)(m)$$

and

$$(a, m, b) = (a \cdot m) \cdot b - a \cdot (m \cdot b) = b \cdot (a \cdot m) - a \cdot (b \cdot m) = (\rho_b\rho_a - \rho_a\rho_b)(m)$$

Replacing $a, b \in A_1$ and $c = e$ en relation (7) we obtain $\rho_a\rho_b + \rho_{ab} - 2\rho_b\rho_a = 0$. Interchanging a and b in the above identity we obtain $\rho_b\rho_a + \rho_{ab} - 2\rho_a\rho_b = 0$. Subtracting both identities we have that $\rho_a\rho_b = \rho_b\rho_a$. So $\rho_{ab} = \rho_a\rho_b$ and M is an associative module. □

5.6. **Case $\beta + \gamma = 0$.** In this case using Lema 3.4 we have that M_1 is submodule of M and we have the following result

Lemma 5.7. *Let A be a generalized almost-Jordan algebra with $\beta + \gamma = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be an irreducible representation of A. Then $M = M_0$ ó $M = M_1$.*

Proposition 5.8. *Let A be a generalized almost-Jordan algebra with $\beta + \gamma = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be an irreducible representation of A. If $(A_0)^2 \subseteq A_0$ and $\rho_e \neq 0$, then M is an associative module.*

Proof. As the above results we need to prove that $(a, b, m) = (a, m, b) = 0 \ \forall \ a, b \in A, m \in M$. Since $\rho_e \neq 0$ Lema 5.7 implies that $\rho_e = id_M$. Moreover we have that $(A_0)^2 \subseteq A_0$, so if $a = a_0 + a_1$ and $b = b_0 + b_1$ with $a_i, b_i \in A_i$, we have that $(a, b, m) = (a_1, b_1, m)$ y $(a, m, b) = (a_1, m, b_1)$ and we only need to take $a, b \in A_1$. With the same argument using in the proof of Teorema 4.4 we prove that M is an associative module. □

The next example shows an irreducible module of dimension 2, in the case $\beta + \gamma = 0$

Example 5.9. *Let us consider the algebra A of base $\{e, a\}$ and multiplication table $e^2 = e, ea = ae = 0, a^2 = e$' given in Example 1.1. Let M be a 2-dimensional $\mathbb{R}$ - vector space M and $\{v, w\}$ a base of M. We define a linear map $\rho : A \to End(M)$ by $\rho_e = 0$ and $\rho_a(\lambda_1 v + \lambda_2 w) = (2\lambda_2 - \lambda_1)v + (\lambda_2 - \lambda_1)w$. Then ρ satisfies (2) and (3), so ρ is a representation of A. Suppose that M is not irreducible, that is, there exists a submodule $N = \mathbb{R}m$ for some $m \in M - \{0\}$. Let $m = \lambda_1 v + \lambda_2 w \neq 0$, since N is a submodule of M, we have that $\rho_x(m) = b_x m$ for some $b_x \in \mathbb{R}$, and for all $x \in A$. Taking $x = a$ we have that $\rho_a(m) = b_a m$, and we obtain that*

$$(2\lambda_2 - \lambda_1) = b_a\lambda_1, \ \ (\lambda_2 - \lambda_1) = b_a\lambda_2$$

From the first identity we have $\lambda_2 = \frac{(b_a+1)}{2}\lambda_1$, and replacing this value in the second identity we have

$$\begin{array}{rcl} \dfrac{(b_a+1)}{2}\lambda_1 - \lambda_1 & = b_a\frac{(b_a+1)}{2}\lambda_1 & \\ (b_a+1)\lambda_1 - 2\lambda_1 & = & b_a(b_a+1)\lambda_1 \\ ((b_a)^2 + b_a - b_a - 1 + 2)\lambda_1 & = & 0 \\ ((b_a)^2 + 1)\lambda_1 & = & 0 \end{array}$$

Since the polynomial $x^2 + 1 = 0$ is irreducible in $\mathbb{R}[x]$, we obtain that $\lambda_1 = 0$, and then $\lambda_2 = 0$. A contradiction since $m \neq 0$. Therefore M is a 2-dimensional irreducible module.

5.10. $\beta + 2\gamma = 0$**.** Lema 3.6 implies that M_0 is un submodule of M, and we have

Lemma 5.11. *Let A be a generalized almost-Jordan algebra with $\beta+2\gamma = 0$. Suppose that A has an idempotent element $e \neq 0$. Let $\rho : A \to End(M)$ be an irreducible representation of A. Then $M = M_0$ or $M = M_1$, where $M_0 = \{m \in M \mid \rho_e(m) = 0\}$ and $M_1 = \{m \in M \mid (\rho_e - id)^2(m) = 0\}$.*

5.12. $\beta + 3\gamma = 0$**.** These algebras are the almost-Jordan algebras and it is known that for this kind of algebras every irreducible module is a Jordan module (see [26]).

Open problems: We do not know which is the situation with an irreducible module M,

(1) In the case $M = M_0$
(2) In the case $\beta - \gamma = 0$, that is A satisfies the identity, $(yx^2)x + yx^3 - 2((yx)x)x = 0$.
(3) In the case $\beta = 0$, that is A satisfies the identity, $yx^3 - ((yx)x)x = 0$.
(4) In the case $\beta+2\gamma = 0$, that is A satisfies the identity, $yx^3 - 2(yx^2)x + ((yx)x)x = 0$.

In the last two cases we only know that M_0 is a submodule of M.

References

[1] Arenas, M., Labra, A. (2008). On nilpotency of generalized Almost-Jordan right-nilalgebras. *Algebra Colloquium* 15 (1):69-82.

[2] Arenas, M. (2007). The Wedderburn principal theorem for generalized almost-Jordan algebras. *Comm. Algebra* 35 (2): 675-688.

[3] Behn, A., Labra, A., Reyes C. (2013). Representations of Power-Associative Train Algebras. *Algebra Colloq.* Accepted in September.

[4] Benkart, G., Labra, A. (2006). Representations of algebras of rank 3. *Communications in Algebra,* 34 (8): 2867-2877.

[5] Bernard, J., Iltyacov, A., Martnez, C. (1994). *Bernstein representations*, Proceedings of the 3rd. Int. Conference on non associative algebras. (S. Gonzlez, Ed.) Kluwer Academic Publisher, Dordrech. pp.39-45.

[6] Carini, L., Hentzel, I. R., Piacentini-Cattaneo, J. M. (1988). Degree four Identities not implies by commutativity. *Comm. in Algebra* 16 (2):339-357.

[7] Elduque, A., Shestakov, I. P. (1995). Irreducible Malcev modules. *J. of Algebra* 173:622-637.

[8] Elduque, A., Shestakov, I. P (2001). Irreducible non-Lie modules for Malcev superalgebras. *J. of Algebra* 246:897-914

[9] Eilenberg, S. (1948). Extensions of general algebras. *Ann. Soc. Polon. Math.* 21:125-134.

[10] Hentzel, I. R., Labra, A. (2007). On left nilalgebras of left nilindex four satisfying an identity of degree four. *Internat. J. Algebra Comput.* 17 (1):27-35.

[11] Hentzel, I. R., Peresi, L. A. (1988). Almost Jordan Rings. *Proc. of A. M. S.* 104 (2):343-348.

[12] Humphreys, J. E. (1972). *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag New York Inc.

[13] Jacobson, N. (1951). General representations theory of Jordan algebras. *Trans. Amer. Math. Soc.* 70:509-530.

[14] Jacobson, N. (1954). Structure of alternative and Jordan bimodules. *Osaka Math. Journal* 6:1-71

[15] Jacobson, N. (1968). *Structure and representations of Jordan algebras.* Amer. Math. Soc. Colloc. Publ. XXXIX: X + 453 pp.

[16] Labra, A. Reyes, C. (2005). Representations on train algebras of rank 3. *Linear Algebra and its Applic.* 400:91-97.

[17] McCrimmon, K. (1966). Bimodules for composition algebras. *Proc. Amer. Math. Soc.* 17:480-486.

[18] Osborn, J. M. (1965). Commutative algebras satisfying an identity of degree four. *Proc. Amer. Math. Soc.* 16:1114-1120.

[19] Osborn, J. M. (1965). Identities of non-associative algebras. *Canad. J. Math.* 17:78-92.

[20] Osborn, J. M. (1995) Modules for Novikov algebras. *Contemporary Mathematics* 184:327-338.

[21] Osborn, J. M. (1971). Representations and radicals of Jordan algebras. *Scripta Mathematica,* Vol. **29**, No. 3-4: (1971), 297-329.

[22] Petersson, H. (1997). Zur Theorie der Lie-Tripel-Algebren. *Math. Z.* 97:1-15.

[23] Schafer, R. (1952). Representations of alternatives algebras. *Trans. Am. Math. Soc.* 72 (1):1-17.

[24] Schafer, R. (1966). Introduction to nonassociative algebras. Academic Press N. York, London.

[25] Shestakov, I. P. (1979). Irreducible representations of Alternative algebras. Translated from *Mathematicheskie Zametki* 26(5): 673-686.

[26] Sidorov, A. V. (1981). Lie triple algebras. Translated from *Algebra i Logika* 20 (1):101-108.

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Valparaíso. Errázuriz 1834, Valparaíso, Chile.
E-mail address: mfloreshenriquez@gmail.com

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile. Casilla 653, Santiago, Chile
E-mail address: alimat@uchile.cl