MAGNETOHYDRODYNAMIC EQUILIBRIA IN BAROTROPIC STARS

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RESUMEN

Aunque la materia barotrópica no constituye un modelo realista para estrellas magnéticas, sería interesante confirmar una conjetura reciente que establece que las estrellas magnéticas con ecuación de estado barotrópica, serían dinámicamente inestables (Reisenegger 2009). En este trabajo construimos un conjunto de equilibrios barotrópicos, los cuales pueden ser finalmente testeados usando un criterio de estabilidad. Una descripción general de las ecuaciones de MHD ideal que gobiernan estos equilibrios es revisada, permitiendo tanto una componente poloidal, como una componente toroidal del campo magnético. Un nuevo código numérico en diferencia finita es desarrollado para resolver la llamada ecuación de Grad-Shafranov que describe el equilibrio de estas configuraciones, y algunas propiedades de los equilibrios obtenidos son brevemente discutidas.

ABSTRACT

Although barotropic matter does not constitute a realistic model for magnetic stars, it would be interesting to confirm a recent conjecture that states that magnetized stars with a barotropic equation of state would be dynamically unstable (Reisenegger 2009). In this work we construct a set of barotropic equilibria, which can eventually be tested using a stability criterion. A general description of the ideal MHD equations governing these equilibria is summarized, allowing for both poloidal and toroidal magnetic field components. A new finite-difference numerical code is developed in order to solve the so-called Grad-Shafranov equation describing the equilibrium of these configurations, and some properties of the equilibria obtained are briefly discussed.

Key Words: Stars: MHD — Stars: barotropic

1. BARO... WHAT?

Barotropic equations of state, where pressure is a function solely of density, are often assumed to describe the matter within magnetic stars in ideal magnetohydrodynamic (MHD) equilibrium (Yoshida & Eriguchi 2006; Haskell et al. 2008; Lander & Jones 2009). Barotropy strongly restricts the range of possible equilibrium configurations, and does not strictly represent the realistic stably stratified matter within these objects, which is likely to be an essential ingredient in the stability of magnetic fields in stars (Reisenegger 2009). With this in mind, it is interesting to carry out the pedagogical exercise of checking whether the unrealistic barotropic equilibria are really stable or not. This work is focused on obtaining a wide range of these equilibria and study their main properties, as a starting point to study their stability.

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2. MHD EQUILIBRIA: THE GRAD-SHAFRANOV EQUATION

In the ideal MHD approximation, a magnetic star may be considered as a perfectly conducting fluid in dynamical equilibrium described by the Euler equation,

$$\nabla P + \rho \nabla \Phi = \frac{1}{c}\mathbf{J} \times \mathbf{B}, \tag{1}$$

where the right-side is the Lorentz force per unit volume. Considered objects have a very large fluid pressure P ($P \sim GM^2/R^4$, M being the mass and R the radius), to magnetic pressure $B^2/8\pi$ ratio (B being a characteristic magnetic field strength), $8\pi P/B^2 \sim 10^6$ (Reisenegger 2009), which suggests that magnetic forces may be balanced by a slight perturbation of an unmagnetized spherical background equilibrium. This implies that, as an approximation, we can consider the star as spherical with negligible deformations due to the magnetic forces. In addition, if axial symmetry is assumed, and spherical coordinates (r, θ, ϕ) are used to describe the model, all scalar quantities are independent of the azimuthal coordinate, and the magnetic field may be expressed as the sum of a *poloidal* (meridional field lines) component, and a *toroidal* (azimuthal field lines) com-

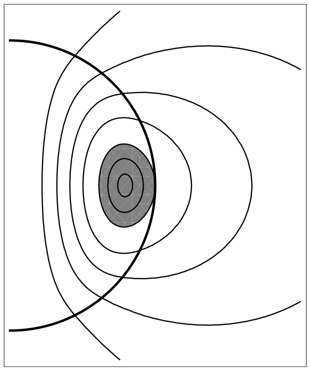

Fig. 1. Meridional cut of a star bearing an axisymmetric magnetic field. The bold curve is the surface of the star, while the thinner curves are poloidal field lines. The toroidal component of the magnetic field may lie only in regions where the poloidal field lines close inside the star (gray region).

ponent, each determined by a single scalar function,

$$\mathbf{B} = \mathbf{B}_{\rm pol} + \mathbf{B}_{\rm tor} = \boldsymbol{\nabla}\alpha(r,\theta) \times \boldsymbol{\nabla}\phi + \beta(r,\theta)\boldsymbol{\nabla}\phi, \quad (2)$$

which turn out to be constant along their respective field lines (Chandrasekhar & Prendergast 1956). Under this symmetry, the azimuthal component of the magnetic force per unit volume must vanish, which implies a functional relation between these scalar functions, $\beta(r,\theta) = \beta(\alpha(r,\theta))$. In this way, both α and β are constant along field lines and, if vacuum is assumed outside the star, the toroidal field may lie only in regions where the poloidal field lines close within the star (Figure 1). On the other hand, if a barotropic equation of state, $P = P(\rho)$, is assumed, the Lorentz force per unit mass must be the gradient of some arbitrary function $\chi(r,\theta)$, which turns out to be a function of α as well, $\chi(r,\theta) = \chi(\alpha(r,\theta))$. Using all this formalism, a non-linear elliptic partial differential equation is found to be the master equation governing the equilibrium of a barotropic MHD equilibrium, the so-called Grad-Shafranov (GS) equation,

$$\frac{\partial^2\alpha}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\alpha}{\partial\theta}\right) + \beta\beta' + r^2\sin^2\theta\rho\chi' = 0 \quad (3)$$

(Grad & Rubin 1958; Shafranov 1966) where primes stand for derivative with respect to the argument, and both $\beta = \beta(\alpha)$ and $\chi = \chi(\alpha)$ are two arbitrary functions, whose form may be chosen depending on the particular magnetic configuration of interest. Under the assumption of weak magnetic field discussed in §1, the density ρ appearing in the GS equation may be replaced by its non-magnetic background counterpart, $\rho = \rho(r)$, such that we solve for the magnetic functions for a *given* density profile, instead of considering the more difficult problem of solving self-consistently for the magnetic functions *and* for the fluid quantities.

3. NUMERICAL SOLUTIONS

Outside the star, α corresponds to an infinite superposition of multipoles, which is the general solution of the GS equation with both $\beta = 0$ and $\rho = 0$. We have implemented a finite-difference code to solve numerically the GS equation inside the star, for arbitrary choices of $\beta(\alpha)$ and $\chi(\alpha)$. Solutions are matched to the exterior expansion by demanding continuity of α and its derivatives (related to the magnetic field components), in order to avoid surface currents. After testing our code, we obtained barotropic equilibria for the particular case $\chi(\alpha) = \alpha$, $\rho(r) = \rho_c(1 - r^2/R^2)$ and

$$\beta(\alpha) = \begin{cases} s(\alpha-\alpha_s)^{1.1} & \alpha_s \leq \alpha \\ 0 & \alpha < \alpha_s, \end{cases} \quad (4)$$

where s is a free parameter accounting for the relative strength between the poloidal and the toroidal component. In the definition below, an exponent larger than 1 was chosen in order to prevent a discontinuous β' at the layer between regions with and without toroidal field; it is found that larger exponents than 1.1 give a smaller toroidal field strength. Also, $\alpha_s \equiv \alpha(R,\pi/2)$ stands for the value of α along the largest poloidal field line closing within the star, being R the stellar radius, so the toroidal field lies in the region inside the curve $\alpha_s = \alpha(r,\theta)$ only. Figure 2 shows some numerical results for the particular $\beta(\alpha)$ in Eq. (4). Black lines correspond to poloidal field lines with $0.2\alpha_s$, $0.4\alpha_s$, $0.6\alpha_s$, $0.8\alpha_s$, $1.0\alpha_s$, $1.08\alpha_s$ and $1.13\alpha_s$, respectively, whereas the color map accounts for the strength of the toroidal field. In turn, Figures 3-4 show the strength of the magnetic field along the axis and the equatorial line for these equilibria.

4. DISCUSSION

All the equilibria found consist of a mixed poloidal-toroidal field with a dominant poloidal component in the magnetic energy $E_{\rm mag}$. For the cases studied so far, the energy stored in the toroidal component $E_{\rm tor}$ is only a few percent of the total magnetic energy, even in cases where the maximum strength of the toroidal field is comparable to that

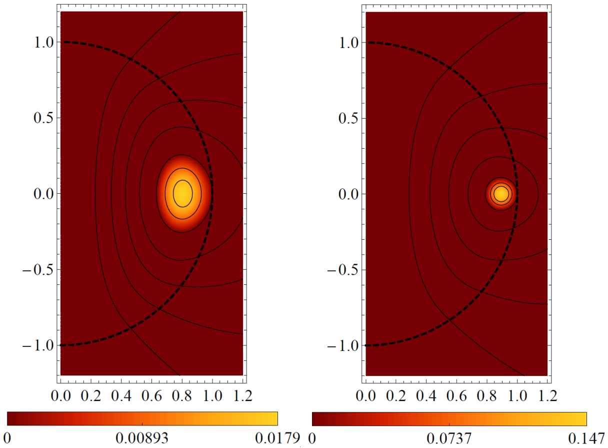

Fig. 2. Numerical equilibria found with our code. Left: $s = 10$, with $E_{\rm tor}/E_{\rm mag} \approx 0.5\%$. Right: $s = 35$, with $E_{\rm tor}/E_{\rm mag} \approx 3.2\%$.

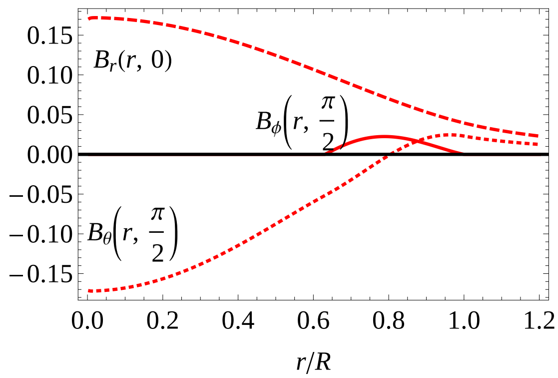

Fig. 3. Magnitude of the magnetic field for the equilibria shown in Figure 2 with $s = 10$. The maximum toroidal strength is about one order of magnitude smaller than the poloidal one.

of the poloidal component: the larger the toroidal field, the smaller the volume where it lies. This small contribution to the energy has already been reported in the literature, but assuming a purely dipolar magnetic field outside the star (Lander & Jones 2012). Our code, allowing an arbitrary number of multipoles, seems to indicate that higher multipoles do not contribute significantly to the energy of these equilibria, at least not for small to moderate values of s. It is desirable to study this fact in more detail and confirm, for instance, whether a global maximum for $E_{\rm tor}/E_{\rm mag}$ exists, already reported using general-relativistic MHD (Ciolfi et al. 2009). Once we obtain a wide range of relevant equilibria with consistent physical choices of the arbitrary magnetic functions, their dynamical stability may be analyzed using either a perturbative analysis or numerically solving the time-evolution

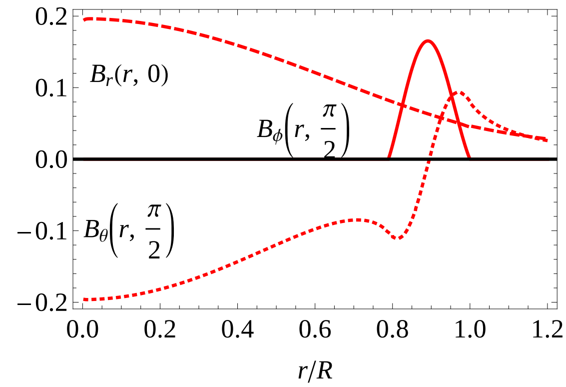

Fig. 4. Magnitude of the magnetic field for the equilibria shown in Figure 2 with $s = 35$. Both the poloidal and the toroidal maximum strength are of the same order of magnitude.

of such configurations.

5. ACKNOWLEDGMENTS

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