A formulation of the wide partition conjecture using the atom problem in discrete tomography

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A B S T R A C T
The Wide Partition Conjecture (WPC) was introduced by Chow and Taylor as an attempt to prove inductively Rota’s Basis Conjecture, and in the simplest case tries to characterize partitions whose Young diagram admits a “Latin” filling. Chow et al. (2003) showed how the WPC is related to problems such as edge-list coloring and multi-commodity flow. As far as we know, the conjecture remains widely open.

We show that the WPC can be formulated using the k-atom problem in Discrete Tomography, introduced in Gardner et al. (2000). In this approach, the WPC states that the sequences arising from partitions admit disjoint realizations if and only if any combination of them can be realized independently. This realizability condition can be checked in polynomial time, although it is not sufficient in general [12]. Guíñez et al. (2011) showed that the problem is NP-hard for any fixed \( k > 2 \). A stronger condition, called the saturation condition, was introduced in Guíñez et al. (2011) to solve instances where the realizability condition fails. We prove that in our case, the saturation condition is implied by the realizability condition. Moreover, we show that the saturation condition can be obtained as the Lagrangian dual of the linear programming relaxation of a natural integer programming formulation of the k-atom problem.

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1. Introduction

A partition is a sequence of integers \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) ordered as \( \lambda_1 \geq \cdots \geq \lambda_\ell > 0 \). It can be described through the Young tableau \( Y_\lambda \), which is a collection of cells arranged in left justified rows, with \( \lambda_i \) cells in row \( i \). We say that a partition \( \lambda \) is Latin if the cells of \( Y_\lambda \) can be labeled such that row \( i \) contains the integers \( 1, \ldots, \lambda_i \) and such that no two squares in the same row or column have the same value. See Fig. 1 for an example.

The conjugate of \( \lambda \) is the partition \( \lambda^* \) with \( \lambda_j^* = |\{i : \lambda_i \geq j\}| \). Here \( Y_\lambda^* \) is obtained from \( Y_\lambda \) by interchanging rows and columns. A subpartition \( \mu \) of \( \lambda \), denoted \( \mu \subseteq \lambda \), is a partition obtained by deleting some parts of \( \lambda \). Equivalently, \( Y_\mu \) is obtained from \( Y_\lambda \) by deleting some rows and making the remaining rows adjacent.

We say that \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) dominates \( \mu = (\mu_1, \ldots, \mu_k) \), written \( \lambda \triangleright \mu \), if \( \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \) for each \( j = 1, \ldots, \ell \), and \( \sum_{i=1}^{\ell} \lambda_i = \sum_{i=1}^{k} \mu_i \). A partition \( \lambda \) is wide if \( \mu \triangleright \mu^* \) for each \( \mu \subseteq \lambda \).

As noticed in [3], every Latin partition is wide. The Wide Partition Conjecture states that this necessary condition is also sufficient.

Conjecture 1 (WPC, See [3]). \( \lambda \) is Latin if and only if it is wide.
The WPC was introduced by Chow and Taylor and originally motivated as an attempt to prove Rota's basis conjecture [14,3]. Chow et al. [3] showed how the WPC is related to some problems such as edge-list coloring, multi-commodity network flows and the Greene–Kleitman theorem. They also mentioned some connections with the invariant theory, although they did not make it explicit.

Let us denote \([n] = \{1, \ldots, n\}\) for convenience. The row projection of a subset \(M\) of the grid \([m] \times [n]\) is the vector \(r \in \mathbb{Z}_n^m\) such that \(r_j = |\{(i, j) \in M \text{ for some } j\}|\). The column projection \(s \in \mathbb{Z}_n^m\) is defined analogously. Given a pair \(b = (r, s) \in \mathbb{Z}_n^m \oplus \mathbb{Z}_n^m\), any subset \(M\) having \(r\) and \(s\) as row and column projection is a realization of \(b\). If \(b\) admits a realization then we say that \(b\) is realizable. Notice that using the analogy between the \(m \times n\) grid and the complete bipartite graph \(K_{m,n}\), realizations of \(b\) correspond to \(b\)-factors of \(K_{m,n}\) (see Section 3).

The \(k\)-atom problem consists in, given sequences \(b^i = (r^i, s^i), i = 1, \ldots, k\), finding pairwise disjoint realizations \(M^1, \ldots, M^k\) of \(b^1, \ldots, b^k\), respectively. If such realizations exist then we say that \((M^1, \ldots, M^k)\) is a \((b^1, \ldots, b^k)\)-packing. This problem is motivated by the reconstruction problem of a polyatomic structure organized on a grid, where \(b^i\) are the projections of the atoms of type \(i\). Gardner et al. [10] introduced it as a generalization of the binary matrix reconstruction problem under given row and columns sums, first studied by Ryser [16]. Many problems in discrete tomography can be modeled using the \(k\)-atom problem [4,5,11,1,8], which makes it a central problem in the area; see [12,13] for the foundations and recent advances in discrete tomography. Moreover, the \(k\)-atom problem can be seen as a \(k\)-commodity flow problem over a bipartite directed network and also as a problem of finding a 3-way consistency table of size \(m \times n \times (k+1)\) with specified line sums (or 2-margins as they are known in the statistical context); see [6,7] for definitions and complexity results and [17] for a summary of necessary conditions for the existence of such tables.

A necessary condition for the existence of a \((b^1, \ldots, b^k)\)-packing is that \(b^i = (r^i, s^i)\) is individually realizable for each \(J \subseteq [k]\), where \(r^i = \sum_{j \in i} r^j\) and \(s^i = \sum_{j \in i} s^j\). We refer to this as the realizability condition. This condition can be checked in polynomial time using Ryser’s algorithm [16,11]. Although this condition ensures the existence of \(k\)-packings for some special instances, it does not suffice in general, even for \(k = 2\) [2]. In fact, the \(k\)-atom problem is NP-hard for any fixed \(k \geq 2\) [8].

In [11], a stronger necessary condition was introduced and proved to be sufficient for a family of instances of the 2-atom problem. For \(A \subseteq [m] \times [n]\) and a given sequence \(b = (r, s)\), let \(\min_b(A) = \min\{|M \cap A| : M \text{ is a realization of } b\}\). Guiñez et al. [11] showed how to calculate \(\min_b(A)\) in polynomial time for any set \(A\) using the minimum-weight max-flow algorithm. Observe that if \((M^1, \ldots, M^k)\) is a \((b^1, \ldots, b^k)\)-packing then \(\sum_{i=1}^{k} \min_{b^i}(A) \leq \sum_{i=1}^{k} |M^i \cap A| \leq |A|\). Then we say that \((b^1, \ldots, b^k)\) satisfy the saturation condition if \(b^i\) is realizable for each \(i \in [k]\), and for every set \(A \subseteq [m] \times [n]\)

\[
\min_{b^1} \cdots \min_{b^k}(A) \leq |A|.
\]

Notice that the saturation condition requires to check a number of inequalities which is exponential in the size of the grid. It is still unknown if we can check it for a fixed number \(k\) of sequences in polynomial time.

As first contribution, we present an equivalent formulation of the WPC using the \(k\)-atom problem, which might import new tools and ideas to solve it. This formulation is presented in Section 2, where we define an instance \(b^\lambda = (b^1, \ldots, b^k)\) for each partition \(\lambda\) such that there exists a \(b^\lambda\)-packing of a given grid if and only if \(\lambda\) is Latin. We also show that the wideness of \(\lambda\) is equivalent to the realizability condition of \(b^\lambda\). In Section 3 we show that the saturation condition is strictly stronger than the realizability condition for the general \(k\)-atom problem. The main result of this section is however that for instances arising from partitions, these two conditions are equivalent. Finally, in Section 4 we discuss how the saturation condition can be obtained as a combinatorial analog of the dual of the relaxation of a Linear Program formulation \((P)\) of the \(k\)-atom problem. The following scheme outlines our main contributions.

| \(\lambda\) is Latin | \(\rightarrow\) | \(|(P)\) is feasible | \(\leftrightarrow\) | \(b^\lambda\) satisfies saturation | \(\Rightarrow\) | \(b^\lambda\) satisfies realizability |
|------------------------|-----------------|----------------------|-----------------|----------------------------------------|-----------------|----------------------------------------|
| \(\exists b^\lambda\)-packing | \(\rightarrow\) | \(\rightarrow\) | \(\leftrightarrow\) | \(\rightarrow\) | \(\rightarrow\) |
WPC and the k-atom problem

Let \( \lambda = (\lambda_1, \dots, \lambda_\ell) \) be a partition and let \( n = \lambda_1 \). For \( t \in [\ell] \), let \( b^t = (r^t, s^t) \), where \( r^t = s^t = (i, \lambda_1, 0, \dots, 0) \). We denote \( b^k = (b^1, b^2) \). The following result shows that a partition \( \lambda \) is Latin if and only if there exists pairwise disjoint realizations of the sequences \( b^1, \dots, b^k \). The idea of the proof is to map the \( k \)-th row in a Latin assignment of \( Y_\lambda \) to a realization \( M^k \) of \( b^k \) in the \( n \times n \) grid.

**Theorem 1.** \( \lambda \) is Latin if and only if there is a \( b^k \)-packing in \([n] \times [n]\).

**Proof.** Let \( f \) be a Latin assignment in \( Y_\lambda \). For each \( k = 1, \ldots, \ell \), let \( M^k = \{(i, j) \in [n] \times [n] : f(k, i) = j \} \). In other words, for each \( i = 1, \ldots, \lambda_k \), \( (i, f(k, i)) \in M^k \). Since the assignments in row \( k \) of \( Y_\lambda \) are precisely the integers in \([\lambda_k]\), there is no column \( i \) such that \( f(k, i) > \lambda_k \). This shows that \( M^k \) is a realization of \( b^k \).

Let us prove that these assignments are pairwise disjoint. Assume that \( (i, j) \in M^k \cap M^{k'} \). Then \( f(k, i) = j = f(k', i) \). But the assignments in column \( i \) of \( Y_\lambda \) are all distinct, and then \( k = k' \). It follows that \( M^1, \ldots, M^\ell \) are pairwise disjoint and then \((M^1, \ldots, M^\ell)\) is a \( b^k \)-packing.

The converse can be proved using a similar idea, since all the arguments above are in fact equivalences (see **Fig. 2** for an example of the construction). \( \square \)

For any \( \mu \subseteq \lambda \), let \( j_\mu \) be the indices of the parts of \( \mu \). Observe that \( r^i_\mu = s^i_\mu = |\{j : \mu_j \geq i\}| = \mu_i^* \) and then \( r^i_\mu = s^i_\mu = \mu^* \). Then by the Gale–Ryser condition, \( b^\mu \) is realizable if and only if \( \mu = (\mu^*)^* \supset \mu^* \) [9,16], That is,

**Theorem 2.** \( \lambda \) is wide if and only if \( b^\lambda \) satisfies the realizability condition.

Written in these terms, the WPC states that the sequences arising from partitions admit a packing provided they satisfy the realizability condition. As we mention in the introduction, the realizability condition does not suffice in general. An example by Chen and Shastri [2] is illustrated in **Fig. 3**.

However, we remark that the instance \( b^\lambda = (b^1, \ldots, b^\ell) \) arising from partition \( \lambda \) holds two special properties:

(i) it is binary: for each \( c \in [\ell] \), the row and column projections \( r^c, s^c \) are 0, 1-vectors. That is, the realizations of \( b^c = (r^c, s^c) \), if they exist, are sub-matrices of a \( n \times n \) permutation matrix;
Lemma 2. It is non-increasing in rows and columns: both $r^c$ and $s^c$ are non-increasing, simultaneously for each $c \in [\ell]$.

Observe that there exists a realization of the sequence $b^i$ in the $n \times n$ grid if and only if the number of ones in the row projection is the same than in the column projection. If this is the case for each $b^1, \ldots, b^k$, then properties (i) and (ii) characterize the instances of the $k$-atom problem arising from partitions. Moreover, we observe that the instance $b^i$ is symmetric, that is, $r^i_c = s^i_c$ for each $i \in [n]$ and $c \in [\ell]$. Theorem 3.1 in [8] can be easily modified to show that the $k$-atom problem restricted to symmetric instances is NP-hard for each $k \geq 2$. Considering this, and assuming that the WPC is true, an interesting question is to determine if any of these properties suffices for the $k$-atom problem to become tractable.

3. The saturation condition

We first prove that the saturation condition is stronger than the realizability condition. The proof is just an extension of Theorem 7 in [11] that uses a well-known characterization of bipartite $b$-factors by Ore [15]. Any bipartite graph $G$ with color classes of size $m$ and $n$ can be embedded in an $m \times n$ grid by assigning to each edge a grid cell. Thus, we can think of $G$ as a set $E_G \subseteq [m] \times [n]$. For sets $I \subseteq [m], J \subseteq [n]$ and $F \subseteq E_G$, we denote $F(I, J) = |F \cap (I \times J)|$. Observe that, given a sequence $b = (r, s)$ with $r \in \mathbb{Z}_m^m$ and $s \in \mathbb{Z}_n^n$, $F \subseteq E_G$ is a realization of $b$ on the $m \times n$ grid if $|F((i, [n]))| = r_i$ for each $i \in [m]$ and $|F([m], (j))| = s_j$ for each $j \in [n]$. If we think $b$ as a function on the vertices of $G$, then $F \subseteq E_G$ corresponds to a $b$-factor of $G$. Using the grid notation, Ore’s characterization says there exists a realization of $b$ included in $E_G$ (i.e. $G$ admits a $b$-factor) if and only if $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$ and $\sum_{i\in I} r_i \leq \sum_{j\in [n]\setminus J} s_j + |E_G(I, J)|$, for each $I \subseteq [m]$ and $J \subseteq [n]$.

**Lemma 1.** If $b^1, \ldots, b^k$ satisfy the saturation condition on the $m \times n$ grid, then $b^k$ is realizable for each $L \subseteq [k]$.

**Proof.** First, observe that $\sum_{i=1}^m r^i_1 = \sum_{j=1}^n s^j_1$ is straightforward since for each $I \subseteq L$, $\sum_{i=1}^m r^i_1 = \sum_{j=1}^n s^j_1$ by the realizability of $b^1$. Also notice that for every $I \subseteq [m]$ and $J \subseteq [n]$, each realization $F_I$ of $b^i$ is a $b^i$-factor of the bipartite graph $H$ such that $E_H = F_I$. Then by Ore’s characterization, it satisfies the inequality $\sum_{i\in I} r^i_1 \leq \sum_{j\in [n]\setminus J} s^j_1 + |F_I(I, J)|$. In particular, if we take the realization $F_I$ that minimizes the intersection with $I \times J$, we conclude that $\sum_{i\in I} r^i_1 \leq \sum_{j\in [n]\setminus J} s^j_1 + \min_s |(I \times J)|$. Then

$$\sum_{i\in I} r^i_1 = \sum_{i\in I} \sum_{j\in [n]\setminus J} r^i_j \leq \sum_{i\in I} \sum_{j\in [n]\setminus J} s^j_1 + \min_s |(I \times J)| \leq \sum_{j\in [n]\setminus J} s^j_1 + |I \times J|,$$

where the last inequality follows from the saturation inequality (1) applied to set $A = I \times J$. If we denote by $G$ the bipartite graph such that $E_G = [m] \times [n]$, then $E_G(I, J) = I \times J$. Since this holds for each $I \subseteq [m]$ and $J \subseteq [n]$, Ore’s characterization shows that $b^k$ is realizable in the $m \times n$ grid. $\square$

Observe that in the example in Fig. 3, the saturation inequality is not satisfied for the set $A = \{(1, 1)\}$ (the top left cell). Since $(b^1, b^2)$ satisfies the realizability condition, we conclude that the saturation is a strictly stronger condition, even for $k = 2$.

The saturation condition was introduced in a weaker version in [11], where it is proved to be sufficient for a class of instances of the 2-atom problem. Unfortunately, from the proof of NP-hardness in [8], one can construct examples showing that it is not sufficient. We prove that for the special instances arising from partitions they are equivalent. The proof goes in several steps. First, we prove that the saturation inequality is satisfied for some sets of the cells of the grid we refer to as rectangles. Then we use this to prove it for the union of rectangles, which we call as tableau sets. Finally, we show how to derive the saturation inequality for each set of cells from the result for tableau sets.

3.1. The saturation inequality for rectangles

We first prove that the saturation inequality (1) is satisfied for every set $A = [p] \times [q]$, where $1 \leq p, q \leq n$. We refer to any of these sets as a rectangle. Without any loss of generality, we can assume that $p \leq q$.

For each $x = 1, \ldots, n$, let $b(x) = (r, s)$ with $r = s = (\overbrace{1, \ldots, 1}^x, 0, \ldots, 0)$. Observe that the realizations of $b(x)$ are permutation matrices of $[x] \times [x]$.

**Lemma 2.** The value $\min_{b(x)}(A)$ is given by the function

$$f(x) = \begin{cases} x, & \text{if } x \leq p \\ p, & \text{if } p < x \leq q \\ p + q - x, & \text{if } q < x \leq p + q \\ 0, & \text{otherwise}, \end{cases}$$
This proves that \( \sum_{i=1}^{\ell} f(\lambda_i) = \sum_{j=1}^{p+q} \lambda_j^* - \sum_{j=q+1}^{p+q} \lambda_j^* \).

### Lemma 3

**Lemma 3.** Let \( f \) be the function defined in Lemma 2. For any partition \( \lambda \),

\[
\sum_{i=1}^{\ell} f(\lambda_i) = \sum_{j=1}^{p+q} \lambda_j^* - \sum_{j=q+1}^{p+q} \lambda_j^*.
\]
3.2. The saturation inequality for tableaux sets

The following is a well-known equality satisfied by the conjugate
\[ \sum_{j=1}^{p} \sum_{i=1}^{\ell} \lambda_{j}^{*} = \sum_{i=1}^{\ell} \min\{a, \lambda_{i}\} \]

for any integer \( a \geq 1 \). By applying this equality for values \( a = p, q \) and \( p + q \), we obtain that
\[ \sum_{j=1}^{p} \lambda_{j}^{*} - \sum_{j=q+1}^{p+q} \lambda_{j}^{*} = \sum_{j=1}^{p} \lambda_{j}^{*} + \sum_{j=p+q}^{q} \lambda_{j}^{*} = \sum_{j=1}^{\ell} \min\{p, \lambda_{i}\} - \sum_{i=1}^{\ell} \min\{p + q, \lambda_{i}\} + \sum_{i=1}^{\ell} \min\{q, \lambda_{i}\} \]
\[ = \sum_{i=1}^{\ell} (\min\{p, \lambda_{i}\} - \min\{p + q, \lambda_{i}\} + \min\{q, \lambda_{i}\}). \]

Then the result follows easily by observing that \( f(x) = \min\{p, x\} + \min\{q, x\} - \min\{p + q, x\} \) for each \( x \geq 0 \). \( \square \)

Notice that
\[ \sum_{j=1}^{p} \lambda_{j}^{*} + \sum_{j=p+1}^{p+q} \lambda_{j}^{*} = \sum_{j=p+1}^{p+q} \lambda_{j}^{*} + \sum_{j=p+1}^{q} \lambda_{j}^{*} = \sum_{j=1}^{q} \lambda_{j}^{*} + \sum_{j=q+1}^{p+q} \lambda_{j}^{*}, \]
and then
\[ \sum_{i=1}^{\ell} f(\lambda_{i}) = \sum_{j=1}^{q} \lambda_{j}^{*} - \sum_{j=q+1}^{p+q} \lambda_{j}^{*}. \]

That is, the roles of \( p \) and \( q \) are exchangeable in Lemma 3, as expected. In particular, \( \min_{\{0, \lambda\}}(\{p\} \times \{q\}) = \min_{\{0, \lambda\}}(\{q\} \times \{p\}) \) for each value \( x \).

Lemma 4. If \( \lambda \) is a wide partition, then \( \sum_{j=1}^{p} \lambda_{j}^{*} - \sum_{j=q+1}^{p+q} \lambda_{j}^{*} \leq pq \) for each \( p \) and \( q \).

Proof. Let \( \mu \) be the partition obtained from \( \lambda \) by deleting its first \( t = \lambda_{p+q}^{*} \) parts. Let \( \ell' = \ell - t \) be the length of \( \mu \) and \( n' = \mu_{1} = \lambda_{t+1} \). It is clear from the definitions of \( t \) and \( \mu \) that \( \lambda_{j}^{*} = \mu_{j}^{*} - t \) for each \( 1 \leq j \leq n' \). In particular,
\[ \sum_{j=1}^{p} \lambda_{j}^{*} = pt + \sum_{j=1}^{p} \mu_{j}^{*} \quad \text{and} \quad \sum_{j=q+1}^{p+q} \lambda_{j}^{*} = pt + \sum_{j=q+1}^{n'} \mu_{j}^{*} \]
(2)

Observe that
\[ \sum_{j=1}^{p} \mu_{j}^{*} = \sum_{i=1}^{\ell'} \min\{p, \mu_{i}\} \leq pq + \sum_{i=q+1}^{\ell'} \mu_{i}. \]
(3)

From (2) and (3), it remains to show that \( \sum_{i=1}^{\ell'} \mu_{i} \leq \sum_{j=q+1}^{n'} \mu_{j}^{*} \). But since \( \sum_{i=1}^{\ell'} \mu_{i} = \sum_{j=1}^{n'} \mu_{j}^{*} \) and \( \mu \succ \mu^{*} \),
\[ \sum_{i=q+1}^{n'} \mu_{i} - \sum_{j=q+1}^{n'} \mu_{j}^{*} = \sum_{j=q+1}^{n'} \mu_{j}^{*} - \sum_{i=1}^{q} \mu_{i} \leq 0. \]
\( \square \)

Lemma 5. Let \( b^{k} = (b^{1}, \ldots, b^{k}) \) be as defined in Section 2. If \( \lambda \) is wide then \( \sum_{j=1}^{\ell} \min_{b^{j}}(A) \leq |A| \), for each rectangle \( A \).

Proof. Let \( A = \{p\} \times \{q\} \) be a rectangle. By Lemmas 2–4, we have that for any wide partition \( \lambda \)
\[ \sum_{i=1}^{\ell} f(\lambda_{i}) = \sum_{j=1}^{p} \lambda_{j}^{*} - \sum_{j=q+1}^{p+q} \lambda_{j}^{*} \leq pq = |A|. \]
\( \square \)

3.2. The saturation inequality for tableaux sets

We say that \( A \subseteq [n] \times [n] \) is a tableau set if it can be represented as the union of rectangles \( A_{j} = \{p_{j}\} \times \{q_{j}\}, j = 1, \ldots, t \), for some \( t \). Let us assume that the rectangles are numbered such that \( p_{1} < p_{2} \leq \cdots \leq p_{t} \) and \( q_{1} > q_{2} > \cdots > q_{t} \); otherwise, there exist a pair of rectangles such that one is contained
in the other, so we can omit the smaller one in the definition. Let \(b(x)\) be defined as in Section 3.1, \(f(x) = \min_{b(x)}(A)\) and \(f_j(x) = \min_{b(x)}(A_j)\), for each \(j = 1, \ldots, t\). We recall that the value \(f_j(x)\) is given by the function in Lemma 2 with parameters \(p_j\) and \(q_j\). We denote by \(M^t = M^t_1 \cup M^t_2\) the realization of \(b(x)\) defined in that lemma that attains the value \(f_j(x)\).

**Lemma 6.** For each \(x, f(x) = \max(f_1(x), \ldots, f_t(x))\). Moreover, if \(j \in \arg\max f_j(x)\) for a fixed integer \(x\), then \(M^t\) is such that \(f(x) = |M^t\cap A|\).

**Proof.** Let \(j \in \arg\max f_j(x)\). We clearly have that \(f_j(x) \leq f(x)\), since \(A_j \subseteq A\). We claim that \(M^t\) is such that \(|M^t \cap A| \leq f_j(x)|\). Observe that this suffices because \(f(x) \leq |M^t \cap A|\). We claim that for each \(j' = 1, \ldots, t\), \(M^t_{j'} \cap A_{j'} = \emptyset\). By contradiction, assume that \((i, f_j(x) + x + 1 - i) \in A_{j'}\) for some \(i > f_j(x)\). Since \(A_{j'}\) is a rectangle and \(f_j(x) < f_j(x)\), we have that \((i, f_j(x) + x + 1 - i) \in A_{j'}\). This contradicts the fact that \(M^t_{j'} \cap A_{j'} = \emptyset\). Because \(A = A_1 \cup \cdots \cup A_t\), we deduce that \(M^t_{j'} \cap A = \emptyset\). Then \(|M^t \cap A| = |M^t_{j'} \cap A| \leq |M^t_{j'}| = f_j(x)\).

Fig. 6 shows a tableau set \(A\) obtained as the union of rectangles \(A_1, A_2, \ldots, A_t\). By the previous lemma, if we denote \(f(x) = \min_{b(x)}(A)\) and \(f_i(x) = \min_{b(x)}(A_i)\), for \(i = 1, \ldots, t\), then \(f(x) = \max(f_1(x), \ldots, f_t(x))\).

From Lemma 6, and since the function and realization for rectangle \([p_j] \times [q_j]\) coincide with those for rectangle \([q_j] \times [p_j]\), we can assume that \(p_j \leq q_j\) for each \(j \in [t]\). Observe that by assuming this, the size of \(A\) cannot increase. So if the saturation inequality holds under this assumption, it certainly does in all the cases.

In order to prove the saturation inequality for tableau set \(A = A_1 \cup \cdots \cup A_t\) we proceed by induction on the number \(t\) of rectangles that form \(A\). By Lemma 5, it holds for \(t = 1\). So let us assume the saturation inequality holds for any tableau set formed by at most \(t - 1\) rectangles and let us prove for tableau set \(A\). We recall that \(f(x) = \min_{b(x)}(A)\) and \(f_i(x) = \min_{b(x)}(A_i)\), for each \(i = 1, \ldots, t\). By Lemma 6, \(f(x) = \max(f_1(x), \ldots, f_t(x))\), for each \(x\).

We now consider two cases. First, assume that for every \(x, f_{i-1}(x) \leq f_i(x)\). Let \(A' = A_1 \cup \cdots \cup A_{t-2} \cup A_t\) and \(f'(x) = \min_{b(x)}(A')\). By Lemma 6, we have that \(f'(x) = \max(f_1(x), \ldots, f_{t-2}(x), f_t(x))\), for every \(x\). Then \(f' = f\) and we have

\[
\sum_{i=1}^{t} \min(A_i) = \sum_{i=1}^{t} f(A_i) = \sum_{i=1}^{t} f'(A_i) \leq |A'| \leq |A|,
\]

where the first inequality follows by the induction hypothesis.

By Lemma 2 and because \(p_{t-1} \leq p_t\), the previous case occurs if and only if \(p_{t-1} + q_{t-1} \leq p_t + q_t\). Thus, we can assume that \(p_{t-1} + q_{t-1} > p_t + q_t\). Let \(A' = A_1 \cup \cdots \cup A_{t-1} \cup \tilde{A} = [p_t - p_{t-1}] \times [q_t]\). We denote \(f' = \min_{b(x)}(A')\) and \(\tilde{f} = \min_{b(x)}(\tilde{A})\).

**Claim 1.** For every \(x, f(x) \leq f'(x) + \tilde{f}(x)\).

**Proof.** Let \(x\) be such that \(f(x) > f'(x)\). Observe that \(f(x) = f_i(x)\); this is immediate since \(f'(x) = \max(f_1(x), \ldots, f_{t-1}(x))\) by Lemma 6. By Lemma 2 applied to rectangles \(A_{t-1}\) and \(A_t, f_i(x) = f_{i-1}(x)\) for \(x \leq p_t\) and \(f_i(x) \leq f_{i-1}(x)\) for \(x \geq p_t + q_t - p_{t-1}\). Thus, we can assume that \(p_{t-1} < x < p_t + q_t - p_{t-1}\). If \(x \leq p_t\) then \(f(x) = x \leq p_t - p_{t-1} + \min\{x, p_t - p_{t-1}\} = f_{i-1}(x) + \tilde{f}(x) \leq f'(x) + \tilde{f}(x)\). Otherwise, \(f(x) = \min\{p_t, p_t + q_t - x\} = p_t - p_{t-1} + \min\{p_t - p_{t-1}, p_t - p_{t-1} + q_t - x\} = p_t + f(x)\). By a previous assumption, \(p_t + q_t - p_{t-1} < q_{t-1}\). Then \(x < q_{t-1}\) and so \(f_{i-1}(x) = p_t - p_{t-1}\). We conclude that \(f(x) \leq f_{i-1}(x) + \tilde{f}(x) \leq f'(x) + \tilde{f}(x)\), which ends the proof of the claim. □

![Fig. 6](image-url)
Fig. 7. \( Q \) is obtained from \( P \) by a left-compression from column \( j_1 \) to column \( j_0 \). Observe that only three cells are shifted (in dark gray), the ones in row \( i \) such that \((i, j_1) \in P \) and \((i, j_0) \notin P \).

**Lemma 7.** Let \( b^\lambda = (b^1, \ldots, b^\ell) \) be as defined in Section 2. If \( \lambda \) is wide then \( \sum_{i=1}^{\ell} \min_{b^i}(A) \leq |A| \), for each tableau set \( A \).

**Proof.** By a previous observation, we only need to prove the inductive step when \( \ell = 0 \), which follows from the claim that \( \sum_{i=1}^{\ell} \min_{b^i}(A') \leq |A'| \), and by Lemma 5, \( \ell = 0 \).

3.3 Reduction to tableau sets

So far, we have proved that the saturation inequality holds for any tableau set. In this section we show that these sets are the worst case, in the sense that the left-hand side of (1) is maximal for tableau sets in a certain partial order on the grid.

Let \( P, Q \subseteq [m] \times [n] \). We say that \( Q \) is obtained from \( P \) by a left-compression if there exist columns \( j_0 < j_1 \) such that:

(i) for each row \( i \) such that \((i, j_0) \notin P \) and \((i, j_1) \in P \), \((i, j_0) \in Q \) and \((i, j_1) \notin Q \), and

(ii) for any other row \( i \), \((i, j) \in P \) if and only if \((i, j) \in Q \).

Fig. 7 shows an example of a set \( Q \) obtained from a set \( P \) by a left-compression. We say that \( Q \) is obtained from \( P \) by an up-compression if \( P^T \) is obtained from \( Q^T \) by a left-compression. Let \( b = (r, s) \) be realizable in \([m] \times [n]\).

**Lemma 8.** Assume that \( s \) is non-increasing. If \( Q \) is obtained from \( P \) by a left-compression, then \( \min_b(P) \leq \min_b(Q) \).

**Proof.** Let \( j_0 < j_1 \) be the columns used in the left-compression from \( P \) to \( Q \). Let us consider the set \( I = \{ i : (i, j_1) \in P \setminus Q \} \), that is, the indexes of the rows that are modified. We remark that \((i, j_0) \in Q \setminus P \) for each \( i \in I \). Let \( F \) be a realization of \( b \). For \( k = 0, 1 \), let \( R_k = \{ i : (i, j_k) \in F, (i, j_{1-k}) \notin F \} \) and \( I_k = I \cap R_k \). Observe that \( F \cap Q \) and \( F \cap P \) only differ in the rows in \( I_0 \cup I_1 \). In fact:

\[
|F \cap Q| - |F \cap P| = |I_0| - |I_1|.
\]

We claim that there exists a realization \( F' \) of \( b \) such that

\[
|F' \cap P| \leq |F \cap P| - |I_1| + |I_0|.
\]

The proof is constructive. Since \( s_{j_0} \geq s_{j_1} \), \(|R_0| \geq |R_1| \). In particular, \(|R_0| \geq |I_1| \) and then we can choose \( S \subseteq R_0 \) such that \(|S| = |I_1| \). We define

\[
F' = F \cup \{ (i, j_0), (i', j_1) : i \in I_1, i' \in S \} \setminus \{ (i', j_0), (i, j_1) : i \in I_1, i' \in S \}.
\]

It is not difficult to check that \( F' \) is a realization of \( b \). Moreover, observe that \(|F' \cap P| \leq |F \cap P| - |I_1| + |I \cap S| \). But \( I \cap S \subseteq I \cap R_0 = I_0 \), which proves the inequality (5).

From (4) and (5), \( \min_b(P) \leq |F' \cap P| \leq |F \cap Q| \). Then the lemma follows by taking a realization \( F \) such that \( \min_b(Q) = |F \cap Q| \). □

Clearly the same result holds when \( r \) is non-increasing and \( Q \) is obtained from \( P \) by an up-compression.

**Lemma 9.** Each \( A \subseteq [m] \times [n] \) can be transformed into a tableau set \( \tilde{A} \subseteq [m] \times [n] \) by a sequence of left- and up-compressions.
Proof. Let us first show that $A$ can be transformed into a set $A'$ that is left-justified, that is, such that there are no cells $(i, j_0) \not\in A'$ and $(i, j_1) \in A'$ with $j_0 < j_1$, through a sequence of left-compressions.

For each $A' \subseteq [m] \times [n]$, we denote $\sigma(A')$ the number of cells $(i, j_0) \not\in A'$ such that there exists $(i, j_1) \in A$ with $j_0 < j_1$. We also denote $\sigma(A) = \sum_{i=1}^{m} \sigma_i(A')$. Observe that if $A$ is left-justified then $\sigma(A) = 0$. Otherwise consider the smallest $i$ such that $\sigma_i(A) > 0$ and let $j_0$ be the smallest integer for which $(i, j_0) \not\in A$ and there exists $(i, j_1) \in A$ with $j_0 < j_1$. We choose $j_1$ as the largest integer greater than $j_0$ having this property. We now perform a left-compression using columns $j_0$ and $j_1$. Let us call $A'$ the set we obtain. We claim that $\sigma(A') < \sigma(A)$. Let $I$ be the set of rows that were modified by the left-compression. Clearly $\sigma_i(A') = \sigma_i(A)$ for each $i \not\in I$. Also, $\sigma_i(A') \leq \sigma_i(A)$ for each $i \in I$ and by the choice of $j_1$, we have that $\sigma_i(A') < \sigma_i(A)$. We conclude that $\sigma(A') < \sigma(A)$. By repeating this process we will end up with a left-justified set $A'$ obtained from $A$ by a sequence of left-compressions.

Using an analogous method, we can use up-compressions to push cells up to obtain the desired tableau set $\tilde{A}$. \qed

Theorem 3. If $\lambda$ is wide, then $b^\lambda$ satisfies the saturation condition.

Proof. Let $A \subseteq [m] \times [n]$ and let $\tilde{A}$ be a tableau set obtained from $A$ by a sequence of left- and up-compressions. Since for each $b^t = (r^t, s^t)$, $t = 1, \ldots, \ell$, both $r^t$ and $s^t$ are non-increasing, by Lemma 8 we have that $\min_{b^t} (A) \leq \min_{b^t} (\tilde{A})$. Then using Lemma 7, and the fact that left- and up-compressions preserve the cardinality of sets, we conclude that

$$\sum_{t=1}^{\ell} \min_{b^t} (A) \leq \sum_{t=1}^{\ell} \min_{b^t} (\tilde{A}) \leq |\tilde{A}| = |A|.$$

4. Final comments and open problems

In this work we have related the Wide Partition Conjecture to the $k$-atom problem in discrete tomography. In these terms, the WPC is equivalent to the existence of solutions for the instances of the $k$-atom problem arising from partitions provided they satisfy the necessary condition of realizability. This condition is sufficient only in very restricted cases, and it has been applied mainly for $k = 2$ [11]. Then, if the conjecture is true, we obtain a whole family of instances for which the realizability condition suffices.

Our main result shows the equivalence of the realizability and saturation conditions for the instances arising from partitions. This implies that the saturation condition can be checked in polynomial time for these instances, which is still open in general even for $k = 2$. Also, it is known that each instance of the 2-atom problem where at least one of the sequences is binary is realizable provided it satisfies the realizability condition [2]. On the other hand, the 2-atom problem remains NP-hard for symmetric instances as we mentioned in Section 2. Thus, an interesting open question is to determine the computational complexity of the problem restricted to non-increasing instances.

We finish this section showing an approach to the $k$-atom problem using a linear programming (LP) formulation. As we will see below, the saturation condition appears naturally when we interpret combinatorially the Lagrangian dual of this LP. For each $t = 1, \ldots, k$, let us consider sequences $b^t = (r^t, s^t)$, where $r^t \in \mathbb{Z}_+^m$ and $s^t \in \mathbb{Z}_+^n$, and let $P_t$ be the convex-hull of characteristic vectors of all realizations of $b^t$ in the $[m] \times [n]$ grid. We also consider $P = P_1 \times \cdots \times P_k$ and $f : P \to \mathbb{R}$ a linear function. Let us denote $q = mn$ and consider the following optimization problem

$$\begin{align*}
\text{max} & \quad f(x) \\
n & \quad \text{s.t. } x_1^i + \cdots + x_k^i \leq 1, \quad i = 1, 2, \ldots, q \\
& \quad x \in P.
\end{align*}$$

(P)

Observe that $(b^1, \ldots, b^k)$-packings are the integral feasible solutions of (P). We define the Lagrangian function for each $\pi \in \mathbb{R}^q$ as

$$L(x, \pi) = f(x) + \sum_{i=1}^{q} \pi_i (1 - x_1^i - \cdots - x_k^i),$$

and the Lagrangian dual function as $h(\pi) = \max_{x \in P} L(x, \pi)$.

By weak duality, we know that if $x^*$ is an optimal solution for (P), then $h(\pi) \geq f(x^*)$ for each $\pi \geq 0$. That is, for each nonnegative Lagrangian multipliers, the value of the dual function provides an upper bound on the optimal objective value of (P). To obtain the best upper bound we consider the dual problem

$$\min_{\pi \geq 0} h(\pi).$$

(D)

Since we are only interested in the feasibility of (P), we can take $f \equiv 0$. Because $h(0) = 0$, the optimum of (D) is non-positive. In fact, it is zero if and only if (P) is feasible. Observe that

$$h(\pi) = \max_{x \in P} \sum_{i=1}^{q} \pi_i (1 - x_1^i - \cdots - x_k^i) = \sum_{i=1}^{q} \pi_i - \sum_{j=1}^{k} \min_{x \in P} \sum_{i=1}^{q} \pi_i x_j^i.$$
Because $\mathcal{P}_j$ is integral for each $j$, if we take $A \subseteq [m] \times [n]$ and $\pi \in \{0, 1\}^q$ such that $X_A = \pi$, then $\min_{y \in \mathcal{P}_j} \sum_{i=1}^q \pi_i x_i$. Then $(b^1, \ldots, b^k)$ satisfies the saturation condition if and only if $h(\pi) \geq 0$ for every $\pi \in \{0, 1\}^q$. That is, (D) can be thought as a weighted version of the saturation condition.

By a continuity argument, in order to check the infeasibility of (P), that is, to find $\pi \geq 0$ such that $h(\pi) < 0$, we can assume that $\pi \in \mathbb{Z}_+^q$. However, it is an open question to determine if we can restrict our attention only to vectors $\pi \in \{0, 1\}^q$. Observe that if this is true, it would imply that the saturation condition could be checked in polynomial time through the previous linear feasibility problem. Otherwise, it would be interesting to find larger families of instances for which the restriction to binary vectors is sufficient.

We remark that from the NP-hardness proof for the 2-atom problem in [8], we can construct instances that are feasible for (P), but that are not integral. Because of this, any proof of integrality for the instances arising from partitions must rely on some of their special properties that they satisfy.

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