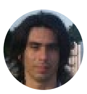
Rantzer's density functions for nonautonomous differential systems: a converse result

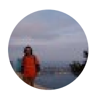
ARTICLE · JUNE 2014

Source: arXiv

2 AUTHORS:



Álvaro Castañeda
University of Chile



Gonzalo Robledo
University of Chile

DIFFERENTIABILITY OF PALMER'S LINEARIZATION THEOREM AND CONVERSE RESULT FOR DENSITY FUNCTIONS

ÁLVARO CASTAÑEDA AND GONZALO ROBLEDO

Abstract. We study differentiability properties in a particular case of the Palmer's linearization Theorem, which states the existence of an homeomorphism H between the solutions of a linear ODE system having exponential dichotomy and a quasilinear system. Indeed, if the linear system is uniformly asymptotically stable, sufficient conditions ensuring that H is a C^2 preserving orientation diffeomorphism are given. As an application, we generalize a converse result of density functions for a nonlinear system in the nonautonomous case.

1. Introduction

The seminal paper of K.J. Palmer [19] provides sufficient conditions ensuring the topological equivalence between the solutions of the linear system

$$y' = A(t)y, \tag{1.1}$$

and the solutions of the quasilinear one

$$x' = A(t)x + f(t,x), \tag{1.2}$$

where the bounded and continuous $n \times n$ matrix $A(t)$ and the continuous function $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy some technical conditions.

Roughly speaking, (1.1) and (1.2) are topologically equivalents if there exists a map $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu \mapsto H(t,\nu)$ is an homeomorphism for any fixed t. In particular, if $x(t)$ is a solution of (1.2), then $H[t,x(t)]$ is a solution of (1.1).

To the best of our knowledge, there are no results concerning the differentiability of the map H and the purpose of this paper is to find sufficient conditions ensuring that the map above is a preserving orientation diffeomorphism of class C^1 (Theorem 1) and C^2 (Theorem 2), both under the assumption that (1.1) is uniformly asymptotically stable.

As an application of our results, we will construct a density function for the system (1.2) when $f(t,0) = 0$ (Theorem 3), generalizing a converse result in the autonomous case presented in [17].

Date: November 2014.

1991 *Mathematics Subject Classification.* 34A34,34D20,34D23.

Key words and phrases. Palmer's linearization Theorem, Density functions, Preserving orientation diffeomorphism.

The first author was funded by FONDECYT Iniciación Project 11121122 and by the Center of Dynamical Systems and Related Fields DySyRF (Anillo Project 1103, CONICYT). The second author was funded by FONDECYT Regular Project 1120709.

1.1. **Palmer's linearization Theorem.** We are interested in the particular case:

Proposition 1 (Palmer [19]). *If the assumptions:*

- **(H1)** $|f(t,x)| \leq \mu$ *for any* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$.
- **(H2)** $|f(t,x_1) - f(t,x_2)| \leq \gamma |x_1 - x_2|$ *for any* $t \in \mathbb{R}$, *where* $|\cdot|$ *denotes a norm in* $\mathbb{R}^n$.
- **(H3)** *There exist some constants* $K \geq 1$ *and* $\alpha > 0$ *such that the transition matrix* $\Psi(t,s) = \Psi(t)\Psi^{-1}(s)$ *of* (1.1) *verifies*

$$||\Psi(t,s)|| \leq Ke^{-\alpha(t-s)}, \quad \text{for any} \quad t \geq s. \tag{1.3}$$

- **(H4)** *The Lipschitz constant of* f *verifies:*

$$\gamma \leq \alpha/4K, \tag{1.4}$$

are satisfied, there exists a unique function $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfying*

- i) $H(t,x) - x$ *is bounded in* $\mathbb{R} \times \mathbb{R}^n$,
- ii) *If* $t \mapsto x(t)$ *is a solution of* (1.2), *then* $H[t,x(t)]$ *is a solution of* (1.1).

Morevoer, H *is continous in* $\mathbb{R} \times \mathbb{R}^n$ *and*

$$|H(t,x) - x| \leq 4K\mu\alpha^{-1}$$

for any $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. *For each fixed* t, $H(t,x)$ *is a homeomorphism of* $\mathbb{R}^n$. $L(t,x) = H^{-1}(t,x)$ *is continous in* $\mathbb{R} \times \mathbb{R}^n$ *and if* $y(t)$ *is any solution of* (1.1), *then* $L[t,y(t)]$ *is a solution of* (1.2).

Remark 1. The original Palmer's result assumes that (1.1) has an exponential dichotomy property. The condition **(H3)** is a particular case considering the identity as projector, which implies that (1.1) is exponentially stable at $+\infty$. In addition, let us recall that uniform asymptotical and exponential stability are equivalent in the linear case (see [7] or Theorem 4.11 from [13]).

This result has been extended and generalized in several directions [4], [12], [15] [20], [21], [26] but there are no results about the differentiability of $x \mapsto H(t,x)$. In this article we provides sufficient conditions, described in term of $\Psi(t,s)$, Df and $D^2 f$, such that H is a C^p $(p = 1,2)$ preserving orientation diffeomorphism.

1.2. **Density functions.** Let us consider the system

$$z' = g(t,z) \quad \text{with} \quad g(t,0) = 0 \quad \text{for any } t \in \mathbb{R}, \tag{1.5}$$

where $g\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is such that the existence, uniqueness and unbounded continuation of the solutions is verified.

Definition 1. *A density function of* (1.5) *is a* C^1 *function* $\rho\colon \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$, *integrable outside a ball centered at the origin that satisfies*

$$\frac{\partial \rho(t,z)}{\partial t} + \nabla \cdot [\rho(t,z)g(t,z)] > 0$$

almost everywhere with respect to $\mathbb{R}^n$ *and for every* $t \in \mathbb{R}$, *where*

$$\nabla \cdot [\rho g] = \nabla \rho \cdot g + \rho[\nabla \cdot g],$$

and $\nabla \rho$, $\nabla \cdot g$ *denote respectively the gradient of* ρ *and divergence of* g.

The density functions were introduced by Rantzer in 2001 [23] in order to obtain sufficient conditions for almost global stability of autonomous systems, we refer to [3], [8], [10], [14] and [16] for a deeper discussion and applications. The extension to the nonautonomous case has been proved in [18], [25]:

Proposition 2 (Theorem 4, [25])**.** *Consider the system* (1.5) *such that* $z = 0$ *is a locally stable equilibrium point. If there exists a density function associated to* (1.5)*, then for every initial time* t_0*, the sets of points that are not asymptotically attracted by the origin has zero Lebesgue measure.*

Converse results (*i.e.*, global asymptotic stability implies the existence of a density function) were presented simultaneously by Rantzer [24] and Monzón [17] in the autonomous case by using different methods. In particular, in [17] the author constructs a density function associated to the system

$$z' = g(z) \quad \text{with} \quad g(0) = 0, \tag{1.6}$$

where 0 is a globally asymptotically stable equlibrium and $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 function, whose jacobian matrix at $z = 0$ has eigenvalues with negative real part.

Such construction has two steps: i) As $u' = Dg(0)u$ is globally asymptotically stable, it is well known that there exists a density function $\rho(z)$ (we refer to Proposition 1 from [23] for details). ii) A C^2 preserving orientation diffeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is constructed, such that $\bar{\rho}(z)$ defined by

$$\bar{\rho}(z) = \rho(h(z)) \det Dh(z) \tag{1.7}$$

is a density function for (1.6).

To the best of our knowledge, there are few converse results in the nonautonomous framework. A first one was presented by Monzón in 2006:

Proposition 3 (Monzón, [18])**.** *If* (1.1) *is globally asymptotically stable, then there exists a* C^1 *density function* $\rho\colon \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$*, associated to* (1.1)*.*

If we assume that

(H5) The function f satisfies $f(t, 0) = 0$ for any $t \in \mathbb{R}$,

By Gronwall's inequality, combined with **(H4)**, it is easy to deduce that any solution $t \mapsto \phi(t, t_0, x_0)$ of (1.2) passing throuhg x_0 at $t = t_0$ verifies

$$|\phi(t, t_0, x_0)| \leq K e^{\{-\alpha + K\gamma\}(t - t_0)} |x_0|, \quad \text{for} \quad t \geq t_0$$

which implies the exponential stability of (1.2) at $t = +\infty$.

This property prompt us to extend the Monzon's converse result [17] to the nonlinear system (1.2) by constructing a density function in the same way as in (1.7), where ρ is defined by Proposition 3 and replacing h by $H(t, x)$ defined in Proposition 1.

The paper is organized as follows. The section 2 states our main results and its proofs. The section 3 is devoted to prove the converse result and to construct a density function for (1.2) and an illustrative example is presented in section 5.

2. Main Results

As usual, given a matrix $M(t) \in M_n(\mathbb{R})$, its trace will be denoted by $\operatorname{Tr} M(t)$ while its determinant by $\det M(t)$, the identity matrix is denoted by I.

The solution of (1.2) passing through ξ at t_0 will be denoted by $\phi(t, t_0, \xi)$. It will be interesting to consider the map $\xi \mapsto \phi(t, t_0, \xi)$ and its properties. Indeed, if f is C^1, it is well known (see *e.g.* [6, Chap. 2]) that $\partial\phi(t,t_0,\xi)/\partial\xi = \phi_\xi(t,t_0,\xi)$ satisfies the matrix differential equation

$$
(2.1) \qquad \left\{ \begin{array}{rcl} \dfrac{d}{dt}\phi_\xi(t,t_0,\xi) & = & \{A(t) + Df(t,\phi(t,t_0,\xi))\}\phi_\xi(t,t_0,\xi). \\ \\ \phi_\xi(t_0,t_0,\xi) & = & I. \end{array} \right.
$$

Moreover, it is proved that (see *e.g.*, Theorem 4.1 from [11, Ch.V]) if f is C^r with $r > 1$, then the map $\xi \mapsto \phi(t,t_0,\xi)$ is also C^r. In particular, if f is C^2, we can verify that the second derivatives $\partial^2\phi(s,t_0,\xi)/\partial\xi_j\partial\xi_i$ are solutions of the system of differential equations

$$
(2.2) \qquad \left\{ \begin{array}{rcl} \dfrac{d}{dt}\dfrac{\partial^2\phi}{\partial\xi_j\partial\xi_i} & = & \{A(t) + Df(t,\phi)\}\dfrac{\partial^2\phi}{\partial\xi_j\partial\xi_i} + D^2f(t,\phi)\dfrac{\partial\phi}{\partial\xi_j}\dfrac{\partial\phi}{\partial\xi_i} \\ \\ \dfrac{\partial^2\phi}{\partial\xi_j\partial\xi_i} & = & 0, \end{array} \right.
$$

with $\phi = \phi(t,t_0,\xi)$, for any $i,j = 1,\ldots,n$.

Now, let us introduce the following conditions

(D1) $f(\cdot,\cdot)$ is C^2 and, for any fixed t, its first derivative is such that

$$\int_{-\infty}^{t} ||\Psi(t,r)Df(r,\phi(r,0,\xi))\Psi(r,t)||_\infty \, dr < 1.$$

(D2) For any fixed t, $A(t)$ and $Df(t,\phi(t,0,\xi))$ are such that

$$\liminf_{s\to-\infty} -\int_s^t \operatorname{Tr} A(r)\, dr > -\infty \quad \text{and} \quad \liminf_{s\to-\infty} -\int_s^t \operatorname{Tr}\{A(r)+Df(r,\phi(r,0,\xi))\}\, dr > -\infty.$$

(D3) For any fixed t and $i,j = 1,\ldots,n$, the following limit exists

$$(2.3) \qquad \lim_{s\to-\infty} \frac{\partial Z(s,x(t))}{\partial x_j(t)} e_i,$$

where $x(t) = (x_1(t),\ldots,x_n(t)) = \phi(t,0,\xi)$, e_i is the i–th component of the canonical basis of $\mathbb{R}^n$ and $Z(s,x(t))$ is a fundamental matrix of the $x(t)$–parameter dependent system

$$(2.4) \qquad z' = F(s,x(t))z$$

satisfying $Z(t,x(t)) = I$, where $F(r,x(t))$ is defined as follows

$$(2.5) \qquad F(r,x(t)) = \Psi(t,r)Df(r,\phi(r,t,x(t)))\Psi(r,t),$$

Remark 2. We will see that the construction of the homeomorphism H considers the behavior of $\phi(t,0,\xi)$ for any $t \in (-\infty,\infty)$. In particular, to prove that H is a C^2 preserving orientation diffeomorphism will require to know the behavior on $(-\infty,t]$. Indeed, notice that:

(D1) is a technical assumption, introduced to ensure that the homeomorphism $H(t,x)$ stated in Proposition 1 is a C^1 diffeomorphism. It is interesting to point out that, by using a result of p. 11 of [7], we know that $|H[s,\phi(s,0,\xi)]| \to +\infty$ when $s \to -\infty$, this fact combined with statement ii) from Proposition 1 implies that $|\phi(s,0,\xi)| \to +\infty$. In consequence, **(D1)** suggest that the asymptotic behavior of $Df(s,x)$ (when $|x| \to +\infty$ and $s \to -\infty$) ensures integrability.

Moreover, let us note that appears in some results about asymptotical equivalence (see *e.g.*, [2],[22]).

(D2) is introduced in order to assure that H is a preserving orientation diffeomorphism. We emphasize that this asumption is related to Liouville's formula and is used in the asymptotic integration literature (see *e.g.*, [9]).

(D3) is introduced to ensure that H is a C^2 diffeomorphism. Notice that, as stressed by Palmer [19, p.757], the uniqueness of the solution of (1.2) implies the identity

$$\phi(s,t,\phi(t,0,\xi)) = \phi(s,0,\xi), \tag{2.6}$$

and this fact combined with $x(t) = \phi(t,0,\xi)$ allows us to see that $F(r,x(t))$ is the same function of **(D1)**.

Theorem 1. *If* **(H1)**–**(H4)** *and* **(D1)**–**(D2)** *are satisfied, then, for any fixed t, the function $x \mapsto H(t,x)$ is a preserving orientation diffeomorphism. In particular, if $t \mapsto x(t)$ is a solution of* (1.2)*, then, for any fixed t, $x(t) \mapsto H[t,x(t)]$ is a preserving orientation diffeomorphism.*

Proof. In order to make the proof self contained, we will recall some facts of the Palmer's proof [19, Lemma 2] tailored for our purposes.

Firstly, let us consider the system

$$z' = A(t)z - f(t,\phi(t,\tau,\nu)),$$

where $t \mapsto \phi(t,\tau,\nu)$ is the unique solution of (1.2) passing through ν at $t = \tau$. Moreover, it is easy to prove that the unique bounded solution of the above system is given by

$$\chi(t,(\tau,\nu)) = -\int_{-\infty}^{t} \Psi(t,s)f(s,\phi(s,\tau,\nu))\,ds.$$

The map H is constructed as follows:

$$H(\tau,\nu) = \nu + \chi(\tau,(\tau,\nu)) = \nu - \int_{-\infty}^{\tau} \Psi(\tau,s)f(s,\phi(s,\tau,\nu))\,ds.$$

It will be essential to note that the particular case $(\tau,\nu) = (t,\phi(t,0,\xi))$, leads to

$$\chi(t,(t,\phi(t,0,\xi))) = -\int_{-\infty}^{t} \Psi(t,s)f(s,\phi(s,t,\phi(t,0,\xi)))\,ds. \tag{2.7}$$

In addition, (2.6) allows us to reinterpret (2.7) as

$$\chi(t,(t,\phi(t,0,\xi))) = -\int_{-\infty}^{t} \Psi(t,s)f(s,\phi(s,0,\xi)))\,ds. \tag{2.8}$$

In consequence, when $(\tau,\nu) = (t,\phi(t,0,\xi))$, we have:

$$H[t,\phi(t,0,\xi)] = \phi(t,0,\xi) + \chi(t,(t,\phi(t,0,\xi))) \tag{2.9}$$

and the reader can notice that the notation $H[\cdot,\cdot]$ is reserved to the case where H is defined on a solution of (1.2).

Having in mind the double representation (2.7)–(2.8) of $\chi(t,(t,\phi(t,0,\xi)))$, the map $H[t,\phi(t,0,\xi)]$ can be written as

$$H[t,\phi(t,0,\xi)] = \phi(t,0,\xi) - \int_{-\infty}^{t} \Psi(t,s)f(s,\phi(s,0,\xi))\,ds, \tag{2.10}$$

or

$$H[t,\phi(t,0,\xi)] = \phi(t,0,\xi) - \int_{-\infty}^{t} \Psi(t,s) f(s,\phi(s,t,\phi(t,0,\xi)))\,ds. \tag{2.11}$$

The proof that $\phi(t,0,\xi) \mapsto H[t,\phi(t,0,\xi)]$ is an homeomorphism for any fixed t is given by Palmer in [19, pp.756–757]. In addition, it is straightforward to verify that (2.10) is a solution of (1.1) passing through $H[0,\xi]$ at $t=0$.

Turning now to the proof that $\phi(\tau,\nu) \mapsto H(\tau,\nu)$ is a preserving orientation diffeomorphism for any fixed τ, we will only consider the case when $(\tau,\nu) = (t,\phi(t,0,\xi))$ by using (2.11). The general case can be proved analogously and is left to the reader.

The proof is decomposed in several steps.

Step 1: Differentiability of the map $\phi(t,0,\xi) \mapsto H[t,\phi(t,0,\xi)]$.

Let us denote $x(t) = \phi(t,0,\xi)$ for any fixed t. By using the fact that f is C^1 combined with (2.1) and

$$\frac{d}{dt}\Psi(t,s) = A(t)\Psi(t,s) \quad \text{and} \quad \frac{d}{ds}\Psi(t,s) = -\Psi(t,s)A(s), \tag{2.12}$$

we can deduce that:

$$\begin{aligned} \frac{\partial H[t,x(t)]}{\partial x(t)} &= I - \int_{-\infty}^{t} \Psi(t,s) Df(s,\phi(s,t,x(t))) \frac{\partial \phi(s,t,x(t))}{\partial x(t)}\,ds \\ &= I - \int_{-\infty}^{t} \frac{d}{ds}\Big\{\Psi(t,s)\frac{\partial \phi(s,t,x(t))}{\partial x(t)}\Big\}\,ds \\ &= \lim_{s\to-\infty} \Psi(t,s)\frac{\partial \phi(s,t,x(t))}{\partial x(t)}. \end{aligned} \tag{2.13}$$

In consequence, the differentiability of $x(t) \mapsto H[t,x(t)]$ follows if and only if the limit above exists.

Step 2: (2.13) is well defined.

By (2.1), we know that $\partial\phi(s,t,x(t))/\partial x(t)$ is solution of the equation:

$$\left\{ \begin{array}{rcl} Y'(s) &=& \{A(s) + Df(s,\phi(s,t,x(t)))\}Y(s) \\ Y(t) &=& I. \end{array} \right. \tag{2.14}$$

By (2.12),(2.6) and (2.14), the reader can verify that $Z(s,x(t)) = \Psi(t,s)\dfrac{\partial \phi(s,t,x(t))}{\partial x(t)}$ is solution of the $x(t)$–parameter dependent matrix differential equation

$$\left\{ \begin{array}{rcl} \dfrac{dZ}{ds} &=& \big\{\Psi(t,s)Df(s,\phi(s,0,\xi))\Psi(s,t)\big\}Z(s) \\ Z(t) &=& I. \end{array} \right. \tag{2.15}$$

A well known result of sucessive approximations (see *e.g.*, [1],[5]) states that

$$Z(s,x(t)) = I - \int_{s}^{t} F(r,\xi)\,dr + \sum_{k=2}^{+\infty}(-1)^k \left(\int_{s}^{t} F(r_1,\xi)\,dr_1 \cdots \int_{r_{k-1}}^{t} F(r_k,\xi)\,dr_k \right),$$

where $F(r,\xi)$ is defined by (2.5). Moreover, we also know that

$$||Z(s,x(t))|| \leq \exp\Big(\Big|\int_{s}^{t} ||F(r,\xi)||\,dr\Big|\Big)$$

and **(D1)** implies that (2.13) is well defined.
Step 3: $x \mapsto H[t, x(t)]$ is a preserving orientation diffeomorphism.

Notice that the continuity of $\partial\phi(s,t,x(t))/\partial x(t)$ for any $s \leq t$ (ensured by Theorem 4.1 from [11, Ch.V]) implies the continuity of $\partial H[t,x(t)]/\partial x(t)$ and we conclude that $H[t,x(t)]$ is a diffeomorphism.

The Liouville's formula (see *e.g.*, Theorems 7.2 and 7.3 from [6, Ch.1]) says that

$$\det \Psi(t,s) > 0 \quad \text{and} \quad \det \frac{\partial\phi(s,t,x(t))}{\partial x(t)} > 0 \quad \text{for any} \quad s \leq t$$

and **(D2)** implies that these inequalities are preserved at $s = -\infty$, and we conclude that $x(t) \mapsto H[t,x(t)]$ is a preserving orientation diffeomorphism. □

Remark 3. As $t \mapsto H[t,x(t)]$ is solution of (1.1), the uniqueness of the solution implies that

$$H[t,\phi(t,0,\xi)] = \Psi(t,0)H[0,\xi]. \tag{2.16}$$

Remark 4. The matrix differential equation (2.14) can be seen as a perturbation of the matrix equation

$$\left\{ \begin{array}{rcl} X'(s) & = & A(s)X(s) \\ X(t) & = & I \end{array} \right. \tag{2.17}$$

related to (1.1). In addition, (2.17) has a solution $s \mapsto X(s) = \Psi(s,t)$.

Notice that $\Psi(t,s)X(s) = I$ while Theorem 1 says that $s \mapsto \Psi(t,s)Y(s)$ exists at $s = -\infty$. This fact prompt us that the behavior of (2.14) and (2.17) at $s \to -\infty$ has some relation weaker than asymptotic equivalence. Indeed, in [2],[22] it is proved that **(D1)** is a necessary condition for asymptotic equivalence between a linear system and a linear perturbation.

Theorem 2. *If* **(H1)–(H4)** *and* **(D1)–(D3)** *are satisfied, then, for any fixed t, the function $x \mapsto H(t,x)$ is a C^2 preserving orientation diffeomorphism. In particular, if $t \mapsto x(t)$ is a solution of* (1.2)*, then, for any fixed t, $x(t) \mapsto H[t,x(t)]$ is a C^2 preserving orientation diffeomorphism.*

Proof. Let us denote $x(t) = \big(x_1(t),\ldots,x_n(t)\big) = \phi(t,0,\xi)$. As in the previous result, the proof will be decomposed in several steps:

Step 1: About $\partial^2 H[t,x(t)]/\partial x_j(t)\partial x_i(t)$.
For any $i,j \in \{1,\ldots,n\}$, we can verify that

$$\begin{array}{rcl} \dfrac{\partial^2 H}{\partial x_j \partial x_i}[t,x(t)] & = & -\displaystyle\int_{-\infty}^{t} \Psi(t,s)D^2 f(s,\phi(s,t,x(t)))\frac{\partial\phi(s,t,x(t))}{\partial x_j}\frac{\partial\phi(s,t,x(t))}{\partial x_i}\,ds \\ & & -\displaystyle\int_{-\infty}^{t} \Psi(t,s)Df(s,\phi(s,t,x(t)))\frac{\partial^2\phi(s,t,x(t))}{\partial x_j \partial x_i}\,ds, \end{array}$$

where $x_i = x_i(t)$ and $x_j = x_j(t)$.

Now, by using (2.2) and (2.12), the reader can verify that

$$\begin{array}{rcl} \dfrac{\partial^2 H}{\partial x_j \partial x_i}[t,x(t)] & = & -\displaystyle\int_{-\infty}^{t} \frac{d}{ds}\Big\{\Psi(t,s)\frac{\partial^2\phi(s,t,x(t))}{\partial x_j \partial x_i}\Big\}\,ds \\ & = & \displaystyle\lim_{s\to-\infty} \Psi(t,s)\frac{\partial^2\phi(s,t,x(t))}{\partial x_j \partial x_i} \end{array}$$

and the existence of $\partial^2 H[t,x(t)]/\partial x_j(t)\partial x_i(t)$ follows if and only if the limit above exists.
Step 2: $\partial^2 H[t,x(t)]/\partial x_j(t)\partial x_i(t)$ is well defined.

By using (2.1) and (2.12), we can see that $s \mapsto \Psi(t,s)\partial\phi(s,t,x(t))/\partial x_i$ is a solution of (2.4) passing through e_i at $s=t$. In consequence, we can deduce that

$$\Psi(t,s)\frac{\partial\phi(s,t,x(t))}{\partial x_i} = Z(s,x(t))e_i$$

and

$$\Psi(t,s)\frac{\partial^2\phi(s,t,x(t))}{\partial x_j \partial x_i} = \frac{\partial Z}{\partial x_j}(s,x(t))e_i.$$

By **(D3)**, the last identity has limit when $s\to-\infty$ and $\partial^2 H[t,x(t)]/\partial x_j \partial x_i$ is well defined and continuous with respect to $x(t)$. □

Remark 5. A careful reading of the results above, shows that our methods can be generalized in order to prove that H is a C^r diffeomorphism with $r \geq 2$.

3. Density function

As we pointed out in subsection 1.2, **(H1)** implies that (1.1) is uniformly asymptotically stable, which is a particular case of global asymptotical stability. Now, by Proposition 3, there exists a density function $\rho \in C^1(\mathbb{R}\times\mathbb{R}^n \setminus \{0\},[0,+\infty))$ associated to (1.1). By following the ideas for the autonomous case studied by Monzón [17, Prop. III.1] combined with the function ρ, we state the following result:

Theorem 3. *If* **(H1)**–**(H5)** *and* **(D1)**–**(D3)** *are satisfied, then there exists a density function* $\bar{\rho} \in C(\mathbb{R}\times\mathbb{R}^n\setminus\{0\},[0,+\infty))$ *associated to* (1.2)*, defined by*

$$\bar{\rho}(t,x) = \rho(t,H(t,x))\Big|\frac{\partial H(t,x)}{\partial x}\Big|, \tag{3.1}$$

where $H(\cdot,\cdot)$ *is the* C^2 *preserving orientation diffeomorphism defined before,* x *is any initial condition of* (1.2) *and* $|\cdot|$ *denotes a determinant.*

Proof. In spite that in the previous sections, the initial condition and the determinant were respectively denoted by ξ and $\det(\cdot)$, the notation of (3.1) is classical in the density function literature. The reader will not be disturbed by this fact.

We shall prove that (3.1) satisfies the properties of Definition 1 with $g(t,x) = A(t)x+f(t,x)$. Indeed, $\bar{\rho}$ is non–negative since ρ is non–negative and H is preserving orientation. In addition, $\bar{\rho}$ is C^1 since H is C^2.

The rest of the proof will be decomposed in several steps:
Step 1: $\bar{\rho}(t,x)$ is integrable outside any ball centered in the origin.

Let B be an open ball centered at the origin. By using $H(t,0)=0$ and statement (i) from Proposition 1, we can conclude that $H(t,B)$ is an open and bounded set containing the origin. In consequence, for any fixed t, the outside of B is mapped in the outside of another ball centered at the origin and contained in $H(t,B)$.

Let $\mathcal{Z}$ be a measurable set whose closure does not contain the origin. The property stated above implies that $H(t,\mathcal{Z})$ is outside of some ball centered at the origin. Now, by the change of variables theorem, we can see that

$$\int_{\mathcal{Z}}\bar{\rho}(t,x)\,dx = \int_{\mathcal{Z}}\rho(t,H(t,x))\Big|\frac{\partial H(t,x)}{\partial x}\Big|\,dx = \int_{H(t,\mathcal{Z})}\rho(t,y)\,dy.$$

Finally, as $\rho(t,\cdot)$ is integrable outside any open ball centered at the origin, the same follows for $\bar{\rho}(t,\cdot)$.
Step 2: $\bar{\rho}(t,x)$ verifies

$$\frac{\partial\bar{\rho}}{\partial t}(t,x)+\nabla\cdot(\bar{\rho}g)(t,x)>0 \quad \text{a.e. in } \mathbb{R}^n. \tag{3.2}$$

Firstly, by using the Liouville's formula (see *e.g.*, [11, Corollary 3.1]), we know that

$$\frac{\partial}{\partial\eta}\left|\frac{\partial\phi(\tau+t,t,x)}{\partial x}\right|\Big|_{\tau=0}=\nabla\cdot g(t,x),$$

where $\eta=\tau+t$. Now, it is easy to verify that:

$$\begin{aligned}\frac{\partial\bar{\rho}}{\partial t}(t,x)+\nabla\cdot(\bar{\rho}g)(t,x) &= \frac{\partial}{\partial\eta}\Big\{\bar{\rho}(\tau+t,\phi(\tau+t,t,x))\Big|\frac{\partial\phi(\tau+t,t,x)}{\partial x}\Big|\Big\}\Big|_{\tau=0}\\ &= \frac{\partial}{\partial\eta}\Big\{\rho(\tau+t,H[\tau+t,\phi(\tau+t,t,x)])\\ &\quad \Big|\frac{\partial H[\tau+t,\phi(\tau+t,t,x)]}{\partial\phi(\tau+t,t,x)}\Big|\Big|\frac{\partial\phi(\tau+t,t,x)}{\partial x}\Big|\Big\}\Big|_{\tau=0}\\ &= \frac{\partial}{\partial\eta}\Big\{\rho(\tau+t,H[\tau+t,\phi(\tau+t,t,x)])\\ &\quad \Big|\frac{\partial H[\tau+t,\phi(\tau+t,t,x)]}{\partial x}\Big|\Big\}\Big|_{\tau=0}.\end{aligned}$$

Secondly, a consequence of (2.16) is

$$H[\tau+t,\phi(\tau+t,t,x)]=\Psi(\tau+t,t)H(t,x),$$

which implies:

$$\begin{aligned}\frac{\partial\bar{\rho}}{\partial t}(t,x)+\nabla\cdot(\bar{\rho}g)(t,x) &= \frac{\partial}{\partial\eta}\Big\{\rho(\tau+t,\Psi(\tau+t,t)H(t,x))\Big|\frac{\partial\Psi(\tau+t,t)H(t,x)}{\partial x}\Big|\Big\}\Big|_{\tau=0}\\ &= A_1(\tau+t,x)+A_2(\tau+t,x)\Big|_{\tau=0},\end{aligned}$$

where $A_1(\cdot,\cdot)$ and $A_2(\cdot,\cdot)$ are respectively defined by

$$\begin{aligned}A_1(\tau+t,x) &= \frac{\partial}{\partial\eta}\Big\{\rho(\tau+t,\Psi(\tau+t,t)H(t,x))\Big\}\Big|\frac{\partial\Psi(\tau+t,t)H(t,x)}{\partial x}\Big|\\ &= \Big\{\frac{\partial\rho}{\partial\eta}(\tau+t,\Psi(\tau+t,t)H(t,x))+\\ &\quad \nabla\rho\Big(\tau+t,\Psi(\tau+t,t)H(t,x)\Big)A(\tau+t)\Psi(\tau+t,t)H(t,x)\Big\}\\ &\quad \Big|\frac{\partial\Psi(\tau+t,t)H(t,x)}{\partial x}\Big|\end{aligned}$$

and

$$\begin{aligned} A_2(\tau+t,x) &= \rho(\tau+t,\Psi(\tau+t,t)H(t,x))\frac{\partial}{\partial\eta}\Big\{\Big|\frac{\partial\Psi(\tau+t,t)H(t,x)}{\partial x}\Big|\Big\} \\ &= \rho(\tau+t,\Psi(\tau+t,t)H(t,x)) \\ &\quad \frac{\partial}{\partial\eta}\Big\{\Big|\frac{\partial\Psi(\tau+t,t)H(t,x)}{\partial H(t,x)}\Big|\Big|\frac{\partial H(t,x)}{\partial x}\Big|\Big\} \end{aligned}$$

As

$$A_1(t,x) = \Big\{\frac{\partial\rho}{\partial\eta}(t,H(t,x)) + \triangledown\rho(t,H(t,x))A(t)H(t,x)\Big\}\Big|\frac{\partial H(t,x)}{\partial x}\Big|$$

and

$$A_2(t,x) = \rho(t,H(t,x))\,\mathrm{Tr}\,A(t)H(t,x)\Big|\frac{\partial H(t,x)}{\partial x}\Big|,$$

we can conclude that

$$\begin{aligned} \frac{\partial\bar\rho}{\partial t}(t,x) + \triangledown\cdot(\bar\rho g)(t,x) &= A_1(t,x) + A_2(t,x) \\ &= \Big\{\frac{\partial\rho}{\partial\eta}(t,H(t,x)) + \triangledown\cdot\rho(t,H(t,x))A(t)H(t,x)\Big\}\Big|\frac{\partial H(t,x)}{\partial x}\Big|, \end{aligned}$$

which is positive since is the product of two positive terms. The positiveness of the first one is ensured by Proposition 3, while the second follows by Theorem 1.
Step 3: End of proof.

As we commented before, the existence of density function associated to (1.2) is based on the homeomorphism H constructed by Palmer (Proposition 1) and the existence of the density function $\rho(t,x)$ associated to (1.1) constructed by Monzón (Proposition 3). Proposition 1 and Theorem 2 ensure that H is a C^2 preserving orientation diffeomorphism while the previous steps state that (3.1) is indeed a density function associated to (1.2) and the result follows. □

3.1. **An application to nonlinear systems.** Let us consider the nonlinear system

$$x' = g(t,x) \tag{3.3}$$

where g is a C^2 function satisfying

- **(H1')** $g(t,0)=0$ and $|g(t,x)|\leq\tilde\mu$ for any $t\in\mathbb{R}$ and $x\in\mathbb{R}^n$.
- **(H2')** $|g(t,x_1)-g(t,x_2)|\leq L|x_1-x_2|$ for any $t\in\mathbb{R}$.

Corollary 1. *If:*

- **(G1)** *The linear system $y'=Dg(t,0)y$ is exponentially stable and its transition matrix satisfy*

$$||\Phi(t,s)||\leq Ke^{-\alpha(t-s)} \quad \text{for some} \quad K\geq 1 \quad \text{and} \quad \alpha>0.$$

- **(G2)** *The Lipschitz constant L satisfies*

$$L+||Dg(t,0)||\leq \alpha/4K \quad \text{for any} \quad t\in\mathbb{R},$$

(G3) *The first derivative of g is such that*

$$\int_{-\infty}^{t} ||\widetilde{F}(r,\xi)||_{\infty}\, dr < 1$$

for any fixed t, with

$$\widetilde{F}(r,\xi) = \Phi(t,r)\{Dg(r,\varphi(r,0,\xi)) - Dg(r,0)\}\Phi(r,t),$$

where $\varphi(r,0,\xi)$ is the solution of (3.3) *passing through ξ at $r=0$.*

(G4) *For any fixed t, $Dg(t,0)$ and $Dg(t,\varphi(t,0,\xi))$ are such that*

$$\liminf_{s\to-\infty} -\int_{s}^{t} \mathrm{Tr}\, Dg(r,0)\, dr > -\infty \quad \text{and} \quad \liminf_{s\to-\infty} -\int_{s}^{t} \mathrm{Tr}\, Dg(r,\varphi(r,0,\xi))\, dr > -\infty$$

for any initial condition ξ.

(G5) *For any fixed t and $i,j=1,\ldots,n$, the following limit exists*

$$\lim_{s\to-\infty} \frac{\widetilde{Z}(s,x(t))}{\partial x_j(t)} e_i,$$

where $x(t)=\varphi(t,0,\xi)$ and $\widetilde{Z}(s,x(t))$ is a fundamental matrix of

$$\widetilde{Z}' = \widetilde{F}(s,x(t))\widetilde{Z}.$$

then there exists a density function $\bar{\rho} \in C(\mathbb{R}\times\mathbb{R}^n\setminus\{0\},[0,+\infty))$ associated to (3.3).

4. Illustrative Example

Let us consider the scalar equation

$$x' = -ax + h(t)\arctan(x), \tag{4.1}$$

where $a>0$ and $h\colon \mathbb{R}\to\mathbb{R}_{+}$ is bounded and continuous. In addition, we will suppose that

$$r \mapsto h(r)e^{-ar} \quad \text{is integrable on} \quad (-\infty,\infty). \tag{4.2}$$

It is easy to see that **(H1)**–**(H2)** are satisfied with $\mu = ||h||_{\infty}\pi/2$ and $\gamma = ||h||_{\infty}$.

Notice that that **(H3)** is satisfied since $\Psi(t,s)=e^{-a(t-s)}$ and **(H4)** is satisfied if and only if $4||h||_{\infty} \leq a$.

Moreover, **(D1)** is satisfied if for any solution $r\mapsto \phi(r,0,\xi)$ of (4.1)

$$\int_{-\infty}^{\infty} \frac{h(r)}{1+\phi^2(r,0,\xi)}\, dr < 1. \tag{4.3}$$

It is interesting to point out $\phi(t,0,\xi)$ is unbounded and have exponential growth at $t=-\infty$. Now, it is easy to note that

$$\lim_{s\to-\infty} -as = +\infty,$$

which implies that

$$\liminf_{s\to-\infty} -\int_{s}^{t} \Big\{ -a + \frac{h(r)}{1+\phi^2(r,0,\xi)} \Big\}\, dr > -\infty$$

for any fixed t, and **(D2)** is satisfied.

Letting $f(t,x)=h(t)\arctan(x)$ and noticing that

$$Z(s,x(t)) = \exp\Big\{ -\int_{s}^{t} Df(u,\phi(u,t,x(t)))\, du \Big\},$$

and

$$\frac{\partial \phi(s,t,x(t))}{\partial x(t)} = \exp\Big\{a(t-s) - \int_s^t Df(u,\phi(u,t,x(t)))\,du\Big\}$$

with $x(t) = \phi(t,0,\xi)$. Consequently, a straigthforward computation shows that **(D3)** is satisfied if and only if

$$Z(s,x(t))\Big[\int_s^t \exp\Big(a\{t-u\} - \int_u^t Df(r,\phi(r,t,x(t)))\,dr\Big) D^2 f(u,\phi(u,t,x(t)))\,du\Big],$$

has limit when $s \to -\infty$.

Finally, (4.2) and (4.3) imply that **(D3)** is satisfied since the function

$$u \mapsto \frac{h(u)e^{-a(u-t)}\phi(u,t,x(t))}{(1+\phi^2(u,t,x(t)))^2} = \frac{h(u)e^{-a(u-t)}\phi(u,0,\xi)}{(1+\phi^2(u,0,\xi))^2}$$

is integrable on $(-\infty, t]$ for any $t \in \mathbb{R}$.

References

[1] L. Ya Adrianova. Introduction to Linear Systems of Differential Equations. Translations of Mathematical Monographs. American Mathematical Society, Providence RI, 1995.

[2] M. Akhmet, M. A. Tleubergenova, A. Zafer. Asymptotic equivalence of differential equations and asymptotically almost periodic solutions. Nonlinear Analysis, 67 (2007) 1870–1877.

[3] D. Angeli. Some remarks on density functions for dual Lyapunov methods. Proceedings of the 42st IEEE Conference on Decision and Control. pages 5080–5082, 2003.

[4] Barreira L., Valls C. A simple proof of the Grobman–Hartman theorem for the nonuniformly hyperbolic flows. Nonlinear Anal. 74 (2011) 7210–7225.

[5] R. Bellman. Stability Theory of Differential Equations. Mc Graw–Hill, New York, 1953.

[6] E. Coddington and N. Levinson. Theory of Ordinary Differential Equations. Mc Graw–Hill, New York, 1955.

[7] W. Coppel. Dichotomies in Stability Theory. Lecture notes in mathematics 629, Springer, Berlin, 1978.

[8] D.V. Dimarogonas, K.J. Kyriakopoulos. An application of Rantzer's dual Lyapunov theorem to decentralized navigation, in: Mediterranean Conference on Control and Automation. Athens, 2007.

[9] M.S.P. Eastham. The asymptotic solution of linear differential systems. Applications of the Levinson theorem. London Mathematical Society Monographs. New Series, 4. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1989.

[10] J. Fernandez Vasconcelos, A. Rantzer, C. Silvestre, P.J. Oliveira. Combination of Lyapunov and density functions for stability of rotational motion. IEEE Trans. Aut. Cont., 56 (2011) 2599–2607.

[11] P. Hartman. Ordinary Differential Equations. SIAM, Philadelphia, 2002.

[12] Jiang L. Generalized exponential dichotomy and global linearization. J. Math. Anal. Appl., 315 (2006) 474–490.

[13] H. Khalil. Nonlinear Systems. Prentice Hall, Upper Saddle River NJ, 1996.

[14] S.G. Loizou. Density functions for navigation–function based systems. IEEE Trans. Aut. Cont., 53 (2008) 612–617.

[15] J. Lopez–Fenner, M. Pinto. On a Hartman linearization theorem for a class of ODE with impulse effect. Nonlinear Anal. Ser.A, 38 (1999) 307–325.

[16] G. Meinsma. On Rantzer's density function, in: Proc. 25th Benelux meeting, Heeze, The Netherlands, Mar. 2006.

[17] P. Monzón. On necessary conditions for almost global stability. IEEE Trans. Aut. Cont., 48 (2003) 631–634.

[18] P. Monzón. Almost Global Stability of Time-Varying Systems, in: Congresso Brasileiro de Automática. Bahia, Brasil, 2006, pp. 198–201.

[19] K.J. Palmer. A generalization of Hartman's linearization theorem. J. Math. Anal. Appl., 41 (1973) 753–758.

[20] K.J. Palmer. A characterization of exponential dichotomy in terms of topological equivalence. J. Math. Anal. Appl., 69 (1979) 8–16.
[21] K.J. Palmer. The structurally stable linear systems on the half-line are those with exponential dichotomies. J. Differential Equations, 33 (1979) 16–25.
[22] M. Ráb. Note sur les formules asymptotiques pour les solutions d'un systéme d'équations différentielles linéaires. Czechoslovak Mathematical Journal., 16 (1966) 127–129.
[23] A. Rantzer. A dual to Lyapunov's Stability Theorem. Syst.Cont.Lett., 42 (2001) 161–168.
[24] A. Rantzer. An converse theorem for density functions, in: Proceedings of the 41st IEEE Conference on Decision and Control, 2002, pp. 1890–1891.
[25] R. Schlanbusch, A. Loria, P.J. Nicklasson. On the stability and stabilization of quaternion equilibria of rigid bodies. Automatica, 48 (2012) 3135–3141.
[26] Xia Y., Cao J., Han M. A new analytical method for the linearization of dynamic equation on measure chains. Journal of Differential Equations, 235 (2007) 527–543.

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile
E-mail address: castaneda@u.uchile.cl,grobledo@u.uchile.cl