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WATER-WAVES EQUATION AND FREE BOUNDARY PROBLEMS: INVERSE PROBLEMS AND CONTROL

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
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Resumen

En este trabajo se aborda el problema de existencia de algunos tipos de soluciones para las ecuaciones de ondas en el agua así como la relación que existe entre estas soluciones y la forma de un fondo impermeable sobre la que se desliza el fluido.

Empezamos por describir las ecuaciones que modelan el fenómeno físico a partir de las leyes de conservación; el modelo general de las ecuaciones de ondas en el agua, escrito para la restricción de la velocidad potencial a la superficie libre, es

$$\begin{cases} \partial_t\zeta - G(\zeta,b)\psi = 0, \\ \partial_t\psi + g\zeta + \dfrac{1}{2}|\nabla_X\psi|^2 - \dfrac{1}{2(1+|\nabla_X\zeta|^2)}(G(\zeta,b)\psi + \nabla_X\zeta\cdot\nabla_X\psi)^2 = 0, \end{cases}$$

donde $G = G(\zeta,b)\psi$ es el operador Dirichlet-Neumann, el cual contiene la información del fondo b,

$$G(\zeta,b)\psi := -\sqrt{1+|\nabla_X\zeta|^2}\partial_n\phi|_{y=\zeta(t,X)},$$

y

$$\begin{cases} \quad\Delta\phi = 0, \quad \mathbb{R}\times(b,\zeta), \\ \quad\phi|_{y=\zeta} = \psi, \quad \partial_n\phi|_{y=b(X)} = 0. \end{cases}$$

Después de describir las condiciones para un teorema de existencia y unicidad de soluciones de las ecuaciones de ondas en el agua, en espacios de Sobolev, nos preguntamos sobre el mínimo de datos necesarios, sobre la superficie libre, para identificar el fondo de manera única. Por la relación que existe entre el operador Dirichlet-Neumann y la velocidad dentro del fluido y utilizando la propiedad de continuación única de las funciones armónicas hemos probado que basta conocer el perfil, la velocidad potencial y la velocidad normal en un instante de tiempo dado y un abierto de $\mathbb{R}$, aún cuando nuestro sistema es de evolución.

En la segunda parte se estudia la existencia de soluciones en forma de salto hidráulico para las ecuaciones estacionarias de ondas en el agua, en dimensión dos y su relación con la velocidad aguas arriba, caracterizada por un parámetro adimensional, llamado el número de Froude, F, como consecuencia de la existencia de ramas de bifurcación de la solución trivial para el problema

$$\mathcal{F}(\eta,F) = \eta + F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}_{x'}^2 + \widetilde{\psi}_{y'}^2) - \epsilon^2\eta_x\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} + \frac{\epsilon^3}{2}\eta_x^2\widetilde{\psi}_{y'}^2;$$

donde

$$\begin{cases} \Delta\widetilde{\psi} = \epsilon G, & (-L,L)\times(0,1), \\ \widetilde{\psi}_{x'} = 0, & x' = -L, L, \\ \widetilde{\psi} = 0, & y' = 0, \\ \widetilde{\psi} = -F\eta, & y' = 1. \end{cases}$$

Abstract

This work is devoted to water-wave equations. The aim is to find some types of solutions to water-wave equations, as well as the relation between those solutions and the shape of a solid impermeable bottom.

We begin by describing the equations that model the physical phenomena from the conservation laws. The general water-wave model, written in terms of the velocity potential on the free surface, is given by

$$\begin{cases} \partial_t\zeta - G(\zeta,b)\psi = 0, \\ \partial_t\psi + g\zeta + \dfrac{1}{2}|\nabla_X\psi|^2 - \dfrac{1}{2(1+|\nabla_X\zeta|^2)}(G(\zeta,b)\psi + \nabla_X\zeta\cdot\nabla_X\psi)^2 = 0, \end{cases}$$

with $G = G(\zeta,b)\psi$ the Dirichlet-Neumann Operator, which contains the information of the bottom b,

$$G(\zeta,b)\psi := -\sqrt{1+|\nabla_X\zeta|^2}\partial_n\phi|_{y=\zeta(t,X)},$$

and

$$\begin{cases} \quad \Delta\phi = 0, \quad \mathbb{R}\times(b,\zeta), \\ \quad \phi|_{y=\zeta} = \psi, \quad \partial_n\phi|_{y=b(X)} = 0. \end{cases}$$

After describing the conditions for a well-posedness theorem of water-wave equations on Sobolev spaces, we asked about the minimal set of data, needed on the free surface, to identify uniquely the bottom shape on $\mathbb{R}$. By the existent relation between the Dirichlet-Neumann operator with the velocity inside the domain and by the unique continuation property for harmonic functions we have proved that the knowledge of the shape, the velocity potential and the normal velocity on a single time and an open subset of $\mathbb{R}$, is enough to identify the bottom, even though we are dealing with an evolution system.

The second part is focused on the existence of solutions in the form of a hydraulic jump for stationary two dimensional water-waves, and its relation with the upstream velocity, characterized by a nondimensional parameter, called Froude number, F; this types of solutions are derived as a consequence of the existence of bifurcation branches of the trivial solution of the problem

$$\mathcal{F}(\eta,F) = \eta + F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}^2_{x'} + \widetilde{\psi}^2_{y'}) - \epsilon^2\eta_x\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} + \frac{\epsilon^3}{2}\eta_x^2\widetilde{\psi}^2_{y'};$$

with

$$\begin{cases} \Delta\widetilde{\psi} = \epsilon G, & (-L,L)\times(0,1), \\ \widetilde{\psi}_{x'} = 0, & x' = -L, L, \\ \widetilde{\psi} = 0, & y' = 0, \\ \widetilde{\psi} = -F\eta, & y' = 1. \end{cases}$$

A mis padres y hermanos
Y en especial para Carolina Figueroa

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Contents

General introduction

This document is a compilation of the work done along two and a half years of supervised doctoral thesis at *Universidad de Chile* with the collaboration of the Instituto de Ciencias Matemáticas (ICMAT), *Universidad Autónoma de Madrid*. Is based on two submitted papers [17], [18] [1].

The work is framed on the water-waves equations and two main subjects are studied. In the first chapter we consider the inverse problem, namely, to recover the shape of the solid bottom boundary from measurements of a portion of the free surface of the fluid. At this point a natural question arise: How many different measurements are necessary to impose?; depending on the data, one is able to set different optimal control problems which can be studied, for example, using the shape differentiation methods. The second chapter is focused on the two dimensional stationary water-waves equations; it is concerned with the proof of the existence of bifurcations branches originated from a sequence of upstream velocities, and as a consequence, the existence of hydraulic jumps, a physical phenomenon easily observed in nature.

In the next paragraphs, we are going to settle the general water-waves model and some properties of interest related. A briefly discussion of the results shall be done as well, leaving the precise framework for the subsequence chapters.

Water-waves equations

The theory of water waves has been a source of many interesting, and often difficult, mathematical problems for the last two centuries. Indeed, linear problems provide a useful exemplar for simple descriptions of wave propagation, with nonlinearity adding an important level of complexity. It embodies the equations of fluid mechanics, the concepts of wave propagation, and the critically important role of boundary conditions. Furthermore, the results of a calculation provide a description that can be tested whenever an expanse of water is to hand: a river or pond, the ocean, or simply the household bath. Even more, many times the motivation for those who study water waves is to obtain information that will help to tame this beautiful phenomenon of nature.

The propagation of surface waves through an incompressible homogeneous inviscid fluid is described by the Euler equations combined with nonlinear boundary conditions at the free surface and at the solid bottom, which is not necessarily flat and would be dependent on time. This problem is difficult to solve, in particular, because the moving surface boundary is part of the solution. The complexity of this problem led physicists, oceanographers, and mathematicians to derive simpler sets of equations in some specific physical regimes; namely,

[1] Joint work with Marco Fontelos, Rodrigo Lecaros and Jaime Ortega.

shallow water models and deep water models [26, 30, 31].

To establish the specific physical setting in which we will work, we focus on a case of a fluid (water) ideal, incompressible, irrotational and under the influence of gravity; such simplifying assumptions are conventionally used in oceanography and can be easily justified [16, 27, 30]. An important consequence of these properties is that the water can not gain angular moment as waves pass and it is impossible for eddies or whirlpools to be created. We work on an Eulerian frame, assuming that surface and bottom are both parameterized by functions $\zeta(t,X)$ and $b(t,X) < 0$ respectively (avoiding wave-breaking phenomena), where t denotes the time variable and $(X,y) \in \mathbb{R}^{d+1}$; but the only physically relevant cases are of course $d=1$ and $d=2$. We denote by $\Omega_t = \{(X,y) \in \mathbb{R}^{d+1} : b(t,X) < y < \zeta(t,X)\}$ the fluid domain at time t.

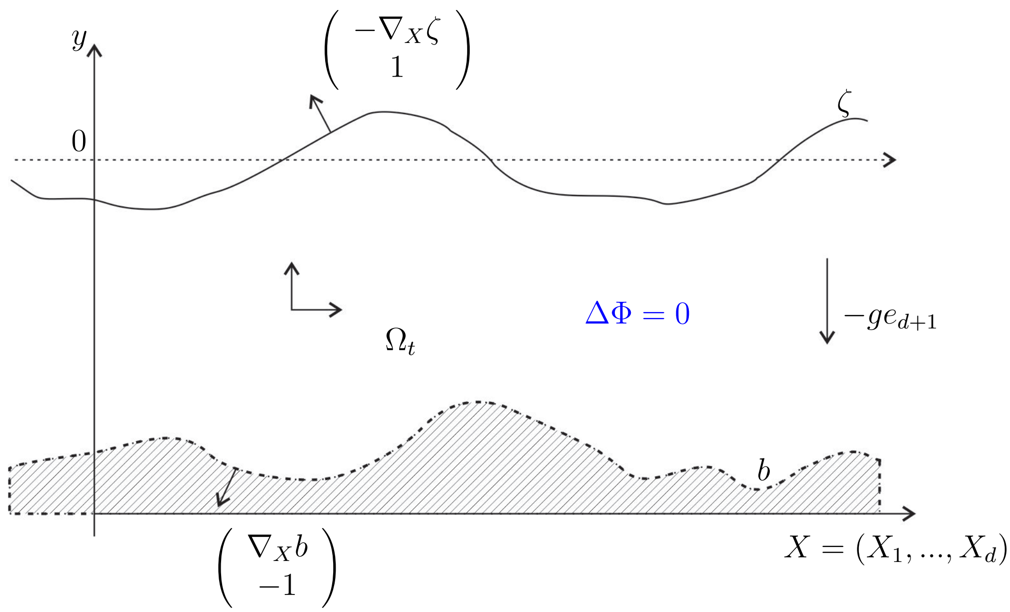

Figure 1: Scheme of water waves model

Remark 0.1. Throughout this document we shall use the following notation:

$$\nabla = \nabla_{X,y}, \qquad \text{and} \qquad \Delta = \Delta_{X,y}.$$

The incompressibility of the fluid is expressed by

$$\nabla \cdot V = 0 \text{ in } \Omega_t,\ t \geq 0, \tag{0.1}$$

where $V = (V_1, ..., V_d, V_{d+1})$ denotes the velocity field. Irrotationality means that

$$\nabla \times V = 0 \text{ in } \Omega_t,\ t \geq 0. \tag{0.2}$$

At the free surface, the boundary condition is kinematic and is given by

$$\partial_t \zeta - \sqrt{1+|\nabla_X \zeta|^2} V \cdot n^+|_{y=\zeta(t,X)} = 0, \text{ for } t \geq 0,\ X \in \mathbb{R}^d, \tag{0.3}$$

where $n^+ = \frac{1}{\sqrt{1+|\nabla_X\zeta|^2}}(-\nabla_X \zeta, 1)^T$ denotes the outward normal vector to the free surface.

Similarly, at the bottom we have a kinematic boundary condition [42, 51] given by

$$\partial_t b - \sqrt{1+|\nabla_X b|^2} V \cdot n^-|_{y=b(t,X)} = 0, \text{ for } t \geq 0, \ X \in \mathbb{R}^d, \tag{0.4}$$

where $n^- = \frac{1}{\sqrt{1+|\nabla_X b|^2}}(\nabla_X b, -1)^T$ denotes the outward normal vector to the lower boundary of Ω_t. Neglecting the effects of surface tension yields that the pressure P is constant at the interface. Up to normalization, we can assume that

$$P|_{y=\zeta(t,X)} = 0, \text{ for } t \geq 0, \ X \in \mathbb{R}^d. \tag{0.5}$$

Finally, the set of equations is closed with Euler's equation within the fluid,

$$\partial_t V + V \cdot \nabla V = -g e_{d+1} - \nabla P, \text{ in } \Omega_t, \ t \geq 0, \tag{0.6}$$

where $-ge_{d+1}$ is the acceleration due to gravity. The water-waves system can thus be written as

$$\begin{cases} \partial_t V + V \cdot \nabla V = -g e_{d+1} - \nabla P, & \Omega_t, \\ \nabla \cdot V = 0, & \Omega_t, \\ \nabla \times V = 0, & \Omega_t, \\ \partial_t \zeta - \sqrt{1+|\nabla_X \zeta|^2} V \cdot n^+ = 0, & y = \zeta(t,X), \\ P = 0, & y = \zeta(t,X), \\ \partial_t b - \sqrt{1+|\nabla_X b|^2} V \cdot n^- = 0, & y = b(t,X). \end{cases} \tag{0.7}$$

From the incompressibility and irrotationality assumptions, there exist a potential flow Φ such that $V = \nabla\Phi$ and

$$\Delta\Phi = 0 \text{ in } \Omega_t, \tag{0.8}$$

the boundary conditions (0.3) and (0.4) can also be expressed in terms of Φ:

$$\partial_t \zeta - \sqrt{1+|\nabla_X \zeta|^2} \partial_{n^+}\Phi = 0, \text{ for } t \geq 0, \ X \in \mathbb{R}^d, \tag{0.9}$$

and

$$\partial_t b - \sqrt{1+|\nabla_X b|^2} \partial_{n^-}\Phi = 0, \text{ for } t \geq 0, \ X \in \mathbb{R}^d, \tag{0.10}$$

where we used the notation $\partial_{n+} = n^+ \cdot \nabla$ and $\partial_{n-} = n^- \cdot \nabla$. Finally, Euler's equation (0.6), can be put into Bernoulli's form

$$\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + gy = -P, \text{ in } \Omega_t. \tag{0.11}$$

The set of equations (0.7) can thus be written under Bernoulli's formulation, in terms of the velocity potential ϕ defined on Ω_t:

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + gy = -P, & \Omega_t, \\ \Delta\Phi = 0, & \Omega_t, \\ \partial_t \zeta - \sqrt{1+|\nabla_X \zeta|^2} \partial_{n^+}\Phi = 0, & y = \zeta(t,X), \\ \partial_t b - \sqrt{1+|\nabla_X b|^2} \partial_{n^-}\Phi = 0, & y = b(t,X). \end{cases} \tag{0.12}$$

By separating the partial derivatives in X and y, and taking the trace of the first equation on the free surface, we finally obtain the system

$$\begin{cases} \Delta_X \Phi + \partial_y^2 \Phi = 0, & \Omega_t, \\ \partial_t \Phi + \frac{1}{2}\left(|\nabla_X \Phi|^2 + (\partial_y \Phi)^2\right) + g\zeta = 0, & y = \zeta(t,X), \\ \partial_t \zeta + \nabla_X \zeta \cdot \nabla_X \Phi = \partial_y \Phi, & y = \zeta(t,X), \\ \partial_t b + \nabla_X b \cdot \nabla_X \Phi = \partial_y \Phi, & y = b(t,X). \end{cases} \tag{0.13}$$

In his paper [53], Zakharov remarked that the knowledge of the free surface elevation ζ and the trace of the velocity potential at the surface $\psi = \Phi|_{y=\zeta}$ fully determine the flow, and Craig, Sulem, and Sulem [13, 14] gave a formulation of the equations involving the Dirichlet-Neumann operator. Indeed, the velocity potential $\Phi(t,\cdot,\cdot)$ is recovered by solving

$$\begin{cases} \Delta \Phi = 0, \quad \Omega_t, \\ \Phi|_{y=\zeta} = \psi, \quad \partial_n \Phi|_{y=b} = -\partial_t b. \end{cases} \tag{0.14}$$

Now, knowing Φ, one gets the velocity field V through the equation $V = \nabla\Phi$ and the pressure through the first equation in (0.12). The objective now is to find a set of two equations that determines ζ and ψ (and thus all the physical quantities relevant to the water waves problem). to this end one introduces the Dirichlet-Neumann operator:

$$G(\zeta, b) : \psi \mapsto \sqrt{1 + |\nabla_X \zeta|^2}\partial_n \phi|_{y=\zeta}, \tag{0.15}$$

where ϕ solves

$$\begin{cases} \Delta \phi = 0, \quad \Omega_t, \\ \phi|_{y=\zeta} = \psi, \quad \partial_n \phi|_{y=b} = 0. \end{cases} \tag{0.16}$$

This operator is linear with respect to ψ but highly nonlinear with respect to the surface and bottom parameterizations ζ and b, which play a role through the fluid domain Ω_t on which (0.14) and (0.16) are solved. In a similar way, let us consider the operator:

$$G^{NN}(\zeta, b) : \partial_t b \mapsto \sqrt{1 + |\nabla_X \zeta|^2}\partial_n \phi^N|_{y=\zeta}, \tag{0.17}$$

where ϕ^N solves

$$\begin{cases} \Delta \phi^N = 0, \quad \Omega_t, \\ \phi^N|_{y=\zeta} = 0, \quad \partial_n \phi^N|_{y=b} = -\partial_t b. \end{cases} \tag{0.18}$$

Then if we define the operator

$$J(\zeta, b, \partial_t b) : \psi \mapsto \sqrt{1 + |\nabla_X \zeta|^2}\partial_n \Phi|_{y=\zeta}, \tag{0.19}$$

we have that $J(\zeta, b, \partial_t b, \psi) = G(\zeta, b)\psi + G^{NN}(\zeta, b)\partial_t b$ with $\Phi = \phi + \phi^N$. By the third equation in (0.13) one has

$$\partial_t \zeta - J(\zeta, b, \partial_t b, \psi) = 0. \tag{0.20}$$

Since that $\psi(t,X) = \Phi(t,X,\zeta(t,X))$, a straightforward application of the chain rule then yields the relations

$$\begin{aligned} (\partial_t \Phi)|_{y=\zeta} &= \partial_t \psi - (\partial_y \Phi)|_{y=\zeta}\partial_t \zeta, \\ (\nabla_X \Phi)|_{y=\zeta} &= \nabla_X \psi - (\partial_y \Phi)|_{y=\zeta}\nabla_X \zeta, \\ (\partial_y \Phi)|_{y=\zeta} &= \frac{J + \nabla_X \zeta \cdot \nabla_X \psi}{1 + |\nabla_X \zeta|^2}. \end{aligned}$$

By the second equation in (0.13) one has

$$
\begin{aligned}
0 &= \partial_t\psi - \frac{J+\nabla_X\zeta\cdot\nabla_X\psi}{1+|\nabla_X\zeta|^2}J + \frac{1}{2}[|\nabla_X\phi|^2+(\partial_y\phi)^2] + g\zeta \\
&= \partial_t\psi + g\zeta + \frac{(|\nabla_X\phi|^2+(\partial_y\phi)^2)(1+|\nabla_X\zeta|^2) - 2(J+\nabla_X\zeta\cdot\nabla_X\psi)J}{2(1+|\nabla_X\zeta|^2)};
\end{aligned}
$$

but, from the expressions above

$$
|\nabla_X\psi|^2 = (1+|\nabla_X\zeta|^2)(|\nabla_X\Phi|^2+(\partial_y\Phi)^2) - (\partial_t\zeta)^2,
$$

therefore

$$
\begin{aligned}
0 &= \partial_t\psi + g\zeta + \frac{|\nabla_X\psi|^2 - J^2 - 2J\nabla_X\zeta\cdot\nabla_X\psi}{2(1+|\nabla_X\zeta|^2)} \\
&= \partial_t\psi + g\zeta + \frac{(1+|\nabla_X\zeta|^2)|\nabla_X\psi|^2 - J^2 - 2J\nabla_X\zeta\cdot\nabla_X\psi - |\nabla_X\zeta|^2|\nabla_X\psi|^2}{2(1+|\nabla_X\zeta|^2)} \\
&= \partial_t\psi + g\zeta + \frac{|\nabla_X\psi|^2}{2} - \frac{(J+\nabla_X\zeta\cdot\nabla_X\psi)^2}{2(1+|\nabla\zeta|^2)}.
\end{aligned}
$$

Then, the water waves problem can be reduced to the following system of two scalar evolution equations [25]:

$$
\begin{cases}
\partial_t\zeta - J(\zeta,b,\partial_t b,\psi) = 0, \\
\partial_t\psi + g\zeta + \frac{1}{2}|\nabla_X\psi|^2 - \frac{1}{2(1+|\nabla_X\zeta|^2)}(J(\zeta,b,\partial_t b,\psi)+\nabla_X\zeta\cdot\nabla_X\psi)^2 = 0.
\end{cases}
\tag{0.21}
$$

Notice that if the bottom does not vary in time, system (0.21) reduces to

$$
\begin{cases}
\partial_t\zeta - G(\zeta,b)\psi = 0, \\
\partial_t\psi + g\zeta + \frac{1}{2}|\nabla_X\psi|^2 - \frac{1}{2(1+|\nabla_X\zeta|^2)}(G(\zeta,b)\psi+\nabla_X\zeta\cdot\nabla_X\psi)^2 = 0.
\end{cases}
\tag{0.22}
$$

Well-posedness and energy estimates of water-waves equations (0.7), within a Sobolev class, have been widely studied; we refer, for instance, the works of Nalimov [35], Yosihara [52] and Craig [10] as far as 1D-surface waves and small data are concerned, S. Wu [49] and [50] in two and three dimensions respectively, locally in time, but in the case of a layer of fluid of infinite depth and without other restriction than smoothness for the initial data. Masmoudi and J. Shatah [22] in three dimensions obtained global in time solutions for small initial data in the case of infinite depth. Lannes [29] in d-dimensions proved the existence of local in time solutions, for system (0.22), without restrictions on the size of the initial data; Iguchi [25] studied system (0.21) in the case of a time dependent bottom with the goal of investigating tsunamis created by submarine earthquakes in the context of Sain-Venant equations.

Bottom detection through surface measurements

The direct problem of water-wave equations is the problem of determining the surface, and its velocity potential, in time $T > 0$, for a given initial profile and velocity potential, where the profile of the bottom, the bathymetry, is known. In chapter 1 we study the inverse problem of recovering the shape of the solid bottom boundary of an inviscid, irrotational, incompressible fluid from measurements of a portion of the free surface. In particular, given the water-wave height and its velocity potential on an open set, together with the first time derivative, on a single time, we addressed the identifiability problem. Moreover we compute the derivatives with respect to the shape of the bottom, which allow us to obtain the optimality conditions for this inverse problem. More precisely: *letting* ζ_0, ψ_0, *the corresponding initial conditions of* (0.22) *and assuming* $\zeta(t,X)$, $\psi(t,X)$ *and* $\partial_t\zeta(t,X)$ *are known on* $S\times\{t_0\}$, *where* S *is an open subset of* $\mathbb{R}^d$, *and* $t_0>0$ *is a single time, then the problem consists in determining the bottom* $b(X)$ *on* $\mathbb{R}^d$, that is, the bathymetry.

We remark that this inverse problem arises naturally in oceanography, and have both theoretical and practical interests; one of the most widely methods to study the problem is based upon the generation and subsequent detection of acoustic waves, which propagate down to the ocean floor and reflect back up to the surface [46]. From a rather different approach, concerning the fluid mechanics equations, and its nonlinear features, we mention the work of Nicholls and Taber [40], in which the authors considered an analytic expansion of the Dirichlet-Neumann operator (0.15) in powers of ζ, but with the restriction of working with standing wave profiles on the free surface; recently Vasan and Deconinck [48] addressed the numerical problem of recovering the bottom from the water wave height, and its first two time derivatives assuming that the velocity potential is periodic. We refer also the work of Ait-Yahia, Hernane and Teniou, [1], concerning the identifiability of bottoms in one dimension, in the stationary water-waves regime and assuming that the upstream velocity is constant and the bottom has a compact support.

Our approach does not make any assumption on the velocity and the domain, further than those needed in the well-posedness theorems [29]; it works equally well for any integer $d\geq 1$, even though the only physically relevant cases are $d=1$ and $d=2$. We are working here with the evolution system (0.22) which is an improvement of the work in [1], where the authors addressed the stationary case with restrictions on the velocity upstream and for compact supported bottoms. Since we assumed ζ and ψ to be known, our method relies heavily on system (0.22) and its relation with the harmonic velocity potential inside the domain through (0.15)-(0.16), where the last data $\partial_t\zeta$ plays a central role, because using the first equation in (0.22) allows to know the Neumann data on the surface in an open set $S\subseteq\mathbb{R}^d$ at time $t=t_0$.

Our first result concerns the identifiability of the bottom shape $b(X)$; it can be stated as:

Theorem 0.2. *Let* $T>0$ *and* $s\in\mathbb{R}^+$, *depending only on* d. *Assume that for* $j=1,2$, $(\zeta_j,\psi_j)\in C^1([0,T];H^s(\mathbb{R}^d)\times H^s(\mathbb{R}^d))$ *are solutions of* (0.22), *with* $b_1,b_2\in H^s(\mathbb{R}^d)$ *and there exists* $h_{min}>0$, *such that, for all* $X\in\mathbb{R}^d$ *and* $t\in(0,T)$,

$$\zeta_j(t,X)-b_j(X)\geq h_{min},\quad j=1,2.$$

Let S *be an open subset of* $\mathbb{R}^d$ *and* $t_0\in(0,T)$ *a given time. If* $\forall X\in S$,

$$\zeta_1(t_0,X)=\zeta_2(t_0,X),\quad \psi_1(t_0,X)=\psi_2(t_0,X)\quad \textit{and}\quad \partial_t\zeta_1(t_0,X)=\partial_t\zeta_2(t_0,X),$$

then

$$b_1(X)=b_2(X)\quad\forall X\in\mathbb{R}^d.$$

The proof of this injectivity is mainly based on some structural properties of the Laplace operator, such as unique continuation properties involves in the Dirichlet-Neumann operator (0.15).

This kind of geometric inverse problems, have been well studied for scalar elliptic and parabolic equations; for instance, we refer, among others, the works by Beretta and Vessella [7], Alessandrini et al. [2, 3] and the references therein for details.

In the second part of the work, having into account the identifiability problem and the quantity $\partial_t\zeta(t,X)|_{t=t_0}$ used to determine uniquely the bottom and given a target function $\tau(X)$, we consider the functional

$$F(b) = \frac{1}{2}\int_{\mathbb{R}^d} \left|\partial_t\zeta(X)|_{t=t_0} - \tau(X)\right|^2 dX, \tag{0.23}$$

and introduce the set of admissible bottom $\mathcal{B}_{ad} \subset H^s(\mathbb{R}^d)$,

$$\mathcal{B}_{ad} = \{f \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d) : \operatorname{supp}(f) \subset K, |f|_{H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)} \leq C\}, \tag{0.24}$$

with $s > t_0 > \frac{d}{2}$, in order to study the following optimization problem: *Find* $b^{min} \in \mathcal{B}_{ad}$, *such that,*

$$F(b^{min}) = \min_{b\in\mathcal{B}_{ad}} F(b). \tag{0.25}$$

To address this problem, we first recall some useful results about well-posedness and analyticity of the Dirichlet-Neumann operator with respect to variations of the domain [30]; these results could be adapted to many other functionals and control problems involving analytic dependence of the surface, velocity potential and Dirichlet-Neumann operator, with respect to bottom variations. In particular, as is established later, if we consider the data $\zeta(t,X)$ instead of $\partial_t\zeta(X)|_{t=t_0}$. As we will see, the existence of minimizers is easily established under some natural assumptions on the class of admissible data $\mathcal{B}_{ad}$, using well-posedness of the water-waves equations and well-known Sobolev embeddings.

Theorem 0.3. *Let* $\zeta \in H^{d_0+2}(\mathbb{R}^d)$ *and* $\psi \in \dot{H}^2(\mathbb{R}^d)$. *Assume that* $\tau(X) \in L^2(\mathbb{R}^d)$ *and* $\mathcal{B}_{ad}$ *defined in* (1.32). *Then the minimization problem,*

$$\min_{b\in\mathcal{B}_{ad}} F(b),$$

has one minimizer $b^m \in \mathcal{B}_{ad}$.

This existence theorem is not common in the context of water-waves equation in the sense that system (0.22) is a highly nonlinear, non-local, differential system.

Finally, the last part of the chapter concerns the computation of the shape derivative of the functional F with respect to bottom variations; these computations are useful if one wants to develop a numerical algorithm that allows to recover b from the surface measurements.

Stationary shapes for 2-d water-waves and hydraulic jumps

A hydraulic jump is a physical phenomenon commonly observed in nature such as open channel flows or spillways and is dependent upon the relation between the initial upstream fluid speed and a critical speed characterized by a dimensionless number F known as the

Froude number. In Chapter 2 we prove the existence of hydraulic jumps for stationary water-waves as a consequence of the existence of bifurcation branches of non flat liquid interfaces originated from each of a sequence of upstream velocities $F_0 > F_1 > \cdots > F_r > \cdots$ ($F_r \to 0$ as $r \to \infty$). We further establish explicitly, for $F > 0$, $F \neq F_r$, $r \in \mathbb{N}$, the existence and uniqueness of the solution of a perfect, incompressible, irrotational free surface flow over a flat bottom, under the influence of gravity and neglecting surface tension; as well as the corresponding hydraulic jump.

As far as we know, existence of hydraulic jump has not been studied on water-waves equations; bifurcation problems have been studied for capillary gravity waves and capillary waves with stagnation points and constant vorticity by Matioc et al [23, 32, 33]. Among others we also mention the works by Craig, Nicholls and Reitich who addressed the existence, and parametric analyticity, of branches of capillary-gravity waves in the absence of resonance [11], [39], [44] and the work developed by Nicholls in the context of boundary perturbation method of the Dirichlet-Neumann operator [37].

The problem of existence of hydraulic jump has been studied in the context of conducting fluids and charged drops [19], [20] as well as some results established for system of elliptic partial differential equations with free boundaries arising in some biological models [8], [21].

Concerning water-waves we mention the work by Titri-Bouadjenak et al. [47], where the existence and uniqueness of solutions are addressed for $F > 1$, applying the implicit function theorem on Banach spaces. We also mention the book of Lannes [30] for a general review on non-stationary water-waves equations as well as some asymptotic regimes.

Since we are considering an incompressible, irrotational, two dimensional fluid we introduce an harmonic stream function ψ in the fluid domain, Ω, such that the velocity of the fluid is the orthogonal gradient of ψ, $V = \nabla^{\perp}\psi = (\partial_y\psi, -\partial_x\psi)^T$. The bottom and the free surface of Ω are streamlines so ψ is constant on $\partial\Omega$ and we choose $\psi = 0$ on the bottom; in addition, on the free surface we have an equilibrium condition given by the Bernoulli equation. Finally, we restrict our model to a bounded domain in the horizontal variables and make the assumption that the fluid stream lines are horizontal upstream and downstream. Putting all these facts together, after normalization, we obtain the system

$$\begin{cases} \Delta\psi = 0, & \Omega, \\ \psi_x = 0, & x = -L, L, \\ \psi = 0, & y = 0, \\ \psi = F, & y = 1 + \eta(x), \\ \dfrac{1}{2}|\nabla\psi|^2 = -y + \dfrac{F^2}{2} + 1, & y = 1 + \eta(x). \end{cases} \tag{0.26}$$

Later in chapter two we briefly show a deduction of this model based on the incompressibility, irrotationality and upstream assumptions on the fluid and its velocity. The system (2.1) has an explicit solution in the case that Ω is the flat strip $\mathbb{R} \times (0, 1)$,

$$\psi(x, y) = Fy;$$

our purpose is to establish the existence of bifurcation branches of solutions with non flat domains Ω. We choose as bifurcation parameter the Froude number F (in other words the initial velocity upstream). After establishing existence and uniqueness of solutions on Sobolev spaces for system (2.1) in the cases $F > 1$ and for $F < 1$ when $x \in [-L, L]$, we shall prove that there exist a sequence of bifurcation branches with

$$F = F_l + \epsilon F_{l1} + \epsilon^2 F_{l2} + \cdots \,, \ (l = 0, 1, 2, ...),$$

which, for $l = 0$ becomes

$$F = 1 + O(L^{-2}),$$

and free boundary

$$\eta = 1 + \epsilon \sin \frac{\pi x}{2L} + \cdots \,.$$

Since the first bifurcation point F_0 has special physical significance, we want to determine the shape of the bifurcation curve $F = F(\epsilon)$ near $F = F_0$, $\epsilon = 0$; following the ideas in [19], [21], we compute F_0 and, even more, we get that

$$F'(0) > 0,$$

and therefore, $F'(\epsilon) > 0$ near $\epsilon = 0$.

Chapter 1

Bottom detection through surface measurements

Given the water-waves system:

$$\begin{cases} \partial_t \zeta - G(\zeta, b)\psi = 0, \\ \partial_t \psi + g\zeta + \dfrac{1}{2}|\nabla_X \psi|^2 - \dfrac{1}{2(1+|\nabla_X \zeta|^2)}(G(\zeta, b)\psi + \nabla_X \zeta \cdot \nabla_X \psi)^2 = 0 \end{cases} \tag{1.1}$$

where $\zeta : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ ($T > 0$), $b : \mathbb{R}^d \to \mathbb{R}$ are the surface and bottom parameterizations of the free and solid boundaries of an inviscid, incompressible and irrotational fluid; we address in this chapter the problem of recovering the shape of the solid bottom boundary from the data: $\zeta(t_o, X)$, $\zeta_t(t_0, X)$ and $\psi(t_0, X)$ with $X \in S$, S an open subset of $\mathbb{R}^d$ and t_0 a given single time.

1.1 Introduction

The motion of a layer of incompressible, irrotational and inviscid fluid, delimited below by a solid (not necessarily flat) bottom, and above by a free surface under the influence of gravity, is described by means of conservation laws, together with a kinematic boundary condition on the surface, and impermeability condition on the bottom.

This problem corresponds to the so-called *water waves,* thus, given $(X, y) \in \mathbb{R}^{d+1}$, let $\zeta : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ ($T > 0$), $b : \mathbb{R}^d \to \mathbb{R}$ be the surface and bottom parameterizations respectively such that $\Omega_t = \{(X, y) \in \mathbb{R}^{d+1} : b(X) < y < \zeta(t, X)\}$; then, neglecting the effects of surface tension, the general water-waves model is given by (see [30])

$$\begin{cases} \Delta_X \phi + \partial_y^2 \phi &= 0, & \Omega_t, \\ \partial_t \zeta + \nabla_X \zeta \cdot \nabla_X \phi &= \partial_y \phi, & y = \zeta, \\ \partial_t \phi + \frac{1}{2}\left(|\nabla_X \phi|^2 + (\partial_y \phi)^2\right) + g\zeta &= 0, & y = \zeta, \\ \nabla_X b \cdot \nabla_X \phi - \partial_y \phi &= 0, & y = b, \end{cases} \tag{1.2}$$

where ϕ is the velocity potential, that is, $u = \nabla_{X,y}\phi$ is the velocity and g is the acceleration due to gravity (see Figure 1.1). Throughout this paper we shall use the following notation:

$$\nabla = \nabla_{X,y}, \qquad \text{and} \qquad \Delta = \Delta_{X,y}. \tag{1.3}$$

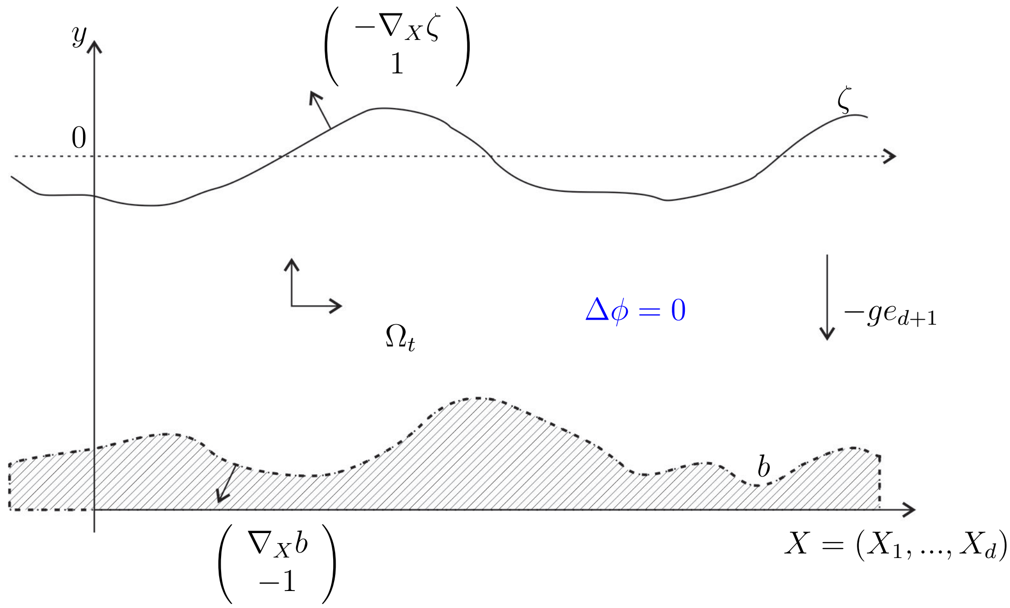

Figure 1.1: Scheme of water waves model

Well-posedness and energy estimates of water-waves equations (2.8), within a Sobolev class, have been widely studied; we refer, for instance, the works of Nalimov [35], Yosihara [52] and Craig [10] as far as 1D-surface waves and small data are concerned, S. Wu [49] and [50] in two and three dimensions respectively, locally in time, but in the case of a layer of fluid of infinite depth and without other restriction than smoothness for the initial data. Masmoudi and J. Shatah [22] in three dimensions obtained global in time solutions for small initial data in the case of infinite depth. Lannes [29] in d-dimensions proved the existence of local in time solutions, for system (2.8), without restrictions on the size of the initial data.

Following Craig and Sulem [13], the water wave problem (2.8) can be reduced from one posed inside the entire fluid domain, to one posed at the free surface alone; this fact was noticed by Zakharov [53]. Then the system takes the form

$$\left\{ \begin{aligned} \partial_t\zeta - G(\zeta,b)\psi &= 0, \\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla_X\psi|^2 - \frac{(G(\zeta,b)\psi + \nabla_X\zeta\cdot\nabla_X\psi)^2}{2(1+|\nabla_X\zeta|^2)} &= 0, \\ \zeta(0,X) = \zeta_0(X),\ \psi(0,X) &= \psi_0(X) \end{aligned} \right. \tag{1.4}$$

where all functions are evaluated only on the free boundary, $\psi(t,X) := \phi(t,X,\zeta(t,X))$, which is the trace of the velocity potential ϕ, and $G(\zeta,b)\psi$ is the called *Dirichlet-Neumann map*

$$G(\zeta,b)\psi := -\sqrt{1+|\nabla_X\zeta|^2}\ \ \partial_n\phi|_{y=\zeta(t,X)}, \tag{1.5}$$

where $\partial_n\phi$ is the normal derivative and ϕ is the solution of the problem

$$\left\{ \begin{array}{l} \Delta\phi = 0, \quad \Omega_t, \\ \phi|_{y=\zeta} = \psi, \quad \partial_n\phi|_{y=b(X)} = 0. \end{array} \right. \tag{1.6}$$

The problem addressed in this paper corresponds to a geometrical inverse problem, that is, we are interested in recovering geometric information of the bottom from a few measurements of the free surface, by using the water-waves formulation (2.9)-(1.6). More precisely: *letting* ζ_0, ψ_0, *the corresponding initial conditions of* (2.9) *and assuming* $\zeta(t,X)$, $\psi(t,X)$ *and* $\partial_t\zeta(t,X)$ *are known on* $S\times\{t_0\}$, *where* S *is an open subset of* $\mathbb{R}^d$, *and* $t_0>0$ *is a single time, then the problem consists in determining the bottom* $b(X)$ *on* $\mathbb{R}^d$, that is, the bathymetry.

Remark 1.1. It is worth to mention, in this work we consider a second approach of the problem, where $\zeta(t,X)$ is known on an open interval of time, instead of considering three pointwise measurements at a given time; this is relevant when the optimal control problem is studied, because it allows us to choose a different functional to be minimized. This result will be explained later. □

We remark that this inverse problem arises naturally in oceanography, and has both theoretical and practical interests; one of the most widely methods to study the problem is based upon the generation and subsequent detection of acoustic waves, which propagate down to the ocean floor and reflect back up to the surface [46]. From a rather different approach, concerning the fluid mechanics equations, and its nonlinear features, we mention the work of Nicholls and Taber [40], in which the authors considered an analytic expansion of the Dirichlet-Neumann operator (2.14) in powers of ζ, but with the restriction of working with standing wave profiles on the free surface; recently Vasan and Deconinck [48] addressed the numerical problem of recovering the bottom from the water wave height, and its first two time derivatives assuming that the velocity potential is periodic. We refer also the work of Ait-Yahia, Hernane and Teniou, [1], concerning the identifiability of bottoms in one dimension, in the stationary water-waves regime and assuming that the upstream velocity is constant and the bottom has a compact support.

Our approach does not make any assumption on the velocity and the domain, further than those needed in the well-posedness theorems [29]; it works equally well for any integer $d\geq 1$, even though the only physically relevant cases are $d=1$ and $d=2$. We are working here with the evolution system (2.8) which is an improvement of the work in [1], where the authors addressed the stationary case with restrictions on the velocity upstream and for compact supported bottoms. Since we assumed ζ and ψ to be known, our method relies heavily on system (2.9) and its relation with the harmonic velocity potential inside the domain through (2.14)-(1.6), where the last data $\partial_t\zeta$ plays a central role, because using the first equation in (2.9) allows to know the Neumann data on the surface in an open set $S\subseteq\mathbb{R}^d$ at time $t=t_0$.

Our first result concerns the identifiability of the bottom shape $b(X)$; it can be stated as:

Theorem 1.2. *Let* $T>0$ *and* $s\in\mathbb{R}^+$, *depending only on* d. *Assume that for* $j=1,2$, $(\zeta_j,\psi_j)\in C^1([0,T];H^s(\mathbb{R}^d)\times H^s(\mathbb{R}^d))$ *are solutions of* (2.9), *with* $b_1,b_2\in H^s(\mathbb{R}^d)$ *and there exists* $h_{min}>0$, *such that, for all* $X\in\mathbb{R}^d$ *and* $t\in(0,T)$,

$$\zeta_j(t,X)-b_j(X)\geq h_{min},\quad j=1,2.$$

Let S *be an open subset of* $\mathbb{R}^d$ *and* $t_0\in(0,T)$ *a given time. If* $\forall X\in S$,

$$\zeta_1(t_0,X)=\zeta_2(t_0,X),\quad \psi_1(t_0,X)=\psi_2(t_0,X)\quad \textit{and}\quad \partial_t\zeta_1(t_0,X)=\partial_t\zeta_2(t_0,X),$$

then

$$b_1(X)=b_2(X)\quad \forall X\in\mathbb{R}^d.$$

The proof of this injectivity is mainly based on some structural properties of the Laplace operator, such as unique continuation properties involves in the Dirichlet-Neumann operator (2.14).

This kind of geometric inverse problems, have been well studied for scalar elliptic and parabolic equations; for instance, we refer, among others, the works by Beretta and Vessella [7], Alessandrini et al. [2, 3] and the references therein for details.

In the second part of the work, having into account the identifiability problem and the quantity $\partial_t\zeta(t,X)|_{t=t_0}$ used to determine uniquely the bottom and given a target function $\tau(X)$, we consider the functional

$$F(b) = \frac{1}{2}\int_{\mathbb{R}^d} \Big|\partial_t\zeta(X)|_{t=t_0} - \tau(X)\Big|^2 dX, \tag{1.7}$$

and introduce the set of admissible bottom $\mathcal{B}_{ad} \subset H^s(\mathbb{R}^d)$, with $s > t_0 > \frac{d}{2}$, in order to study the following optimization problem: *Find $b^{min} \in \mathcal{B}_{ad}$, such that,*

$$F(b^{min}) = \min_{b\in\mathcal{B}_{ad}} F(b). \tag{1.8}$$

To address this problem, we first recall some useful results about well-posedness and analyticity of the Dirichlet-Neumann operator with respect to variations of the domain [30]; these results could be adapted to many other functionals and control problems involving analytic dependence of the surface, velocity potential and Dirichlet-Neumann operator, with respect to bottom variations. In particular, as was mentioned before, if we consider the data $\zeta(t,X)$ instead of $\partial_t\zeta(X)|_{t=t_0}$. As we will see, the existence of minimizers is easily established under some natural assumptions on the class of admissible data $\mathcal{B}_{ad}$, using well-posedness of the water-waves equations and well-known Sobolev embeddings.

Theorem 1.3. *Let $\zeta \in H^{d_0+2}(\mathbb{R}^d)$ and $\psi \in \dot{H}^2(\mathbb{R}^d)$. Assume that $\tau(X) \in L^2(\mathbb{R}^d)$ and $\mathcal{B}_{ad}$ defined in (1.32). Then the minimization problem,*

$$\min_{b\in\mathcal{B}_{ad}} F(b),$$

has one minimizer $b^m \in \mathcal{B}_{ad}$.

This existence theorem is not common in the context of water-waves equation in the sense that system (2.9) is a highly nonlinear, non-local, differential system.

Finally, the last part of the work concerns the computation of the shape derivative of the functional F with respect to bottom variations; these computations are useful if one wants to develop a numerical algorithm that allows to recover b from the surface measurements.

1.2 A general elliptic problem on a strip and shape derivative of the Dirichlet-Neumann operator

Before stating the identifiability problem of bottom in the general model of water-waves, we want to describe in this chapter an elliptic problem on Ω that, after a suitable diffeomorphism, becomes a general elliptic problem on a strip. This procedure is key because relates the connection between the corresponding velocities (potential velocities) on the free surface and inside the domain.

Next we want to state a general result of the shape derivative of the Dirichlet-Neumann operator J. This result is important in itself, but also because is the key in the linearization procedure of the general water-waves model. As we will comment later, it suggest a well-posedness theorem for the general water-waves model in the case of a bottom depending on time, following the existence results stated in [29], [30].

1.2.1 Elliptic boundary value problems on a strip

The main objective of this section is to formulate the elliptic boundary value problem (0.14) as a variable coefficients elliptic problem on a fixed domain through a simple diffeomorphism. The main idea is to get estimates with respect to the fluid parameterization following classical ideas of existence and regularity estimates of solutions to elliptic equations on regular domains.

Let us consider the domain Ω defined as

$$\Omega = \{(X,y) \in \mathbb{R}^{d+1} : b(X) < y < a(X)\},$$

where a and b satisfy the following condition:

$$\exists h_0 > 0 : \ \min\{-b, a-b\} \geq h_0 > 0 \quad \forall X \in \mathbb{R}^d. \tag{GC}$$

This assumption means that beaches or islands are excluded for the fluid domain; besides it allows to consider the simple diffeomorphism between Ω and the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1,0)$ given by

$$r(X,y) = \frac{y - a(X)}{a(X) - b(X)}.$$

Instead of working with the elliptic operator $-\Delta_{X,y}$, we consider a constant coefficients elliptic operator $\mathbf{P} = -\nabla_{X,y} \cdot P\nabla_{X,y}$, where P is a symmetric matrix satisfying the coercivity condition:

$$\exists p > 0 : \ P\theta \cdot \theta \geq p|\theta|^2, \ \forall \theta \in \mathbb{R}^{d+1}. \tag{1.9}$$

Finally we consider boundary value problems of the form

$$\begin{cases} \mathbf{P}\phi = h \quad \text{on } \Omega, \\ \phi|_{y=a(X)} = f, \quad \partial_n^P \phi|_{y=b(X)} = g, \end{cases} \tag{1.10}$$

where h is a function defined on Ω and f, g are functions defined on $\mathbb{R}^d$. Moreover, $\partial_n^P \phi|_{y=b(X)}$ denotes the conormal derivative associated to $\mathbf{P}$ of ϕ at the boundary $y = b(X)$,

$$\partial_n^P \phi|_{y=b(X)} = -n_- \cdot P\nabla_{X,y}\phi|_{\{y=b(X)\}}$$

where n_- denotes the outwards normal at the bottom.

Let R denotes any diffeomorphism between Ω and the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1,0)$, which we assume to be of the form

$$R : \begin{array}{ccc} \Omega & \to & \mathcal{S} \\ (X,y) & \mapsto & (X, r(X,y)), \end{array}$$

and we denote its inverse R^{-1} by S,

$$S: \begin{array}{ccc} \mathcal{S} & \to & \Omega \\ (\widetilde{X}, \widetilde{y}) & \mapsto & (\widetilde{X}, s(\widetilde{X}, \widetilde{y})) \end{array} \tag{1.11}$$

As was said before, the most simple diffeomorphism R between Ω and $\mathcal{S}$ is given by

$$r(X,y) = \frac{y - a(X)}{a(X) - b(X)},$$

and hence

$$s(\widetilde{X}, \widetilde{y}) = (a(\widetilde{X}) - b(\widetilde{X}))\widetilde{y} + a(\widetilde{X}) = -b(\tilde{X})\widetilde{y} + (1 + \widetilde{y})a(\tilde{X}). \tag{1.12}$$

This last diffeomorphism s, has the advantage that it can be written as a sum of two terms that depends from the bottom and surface parameterizations respectively, and motivates the following definition.

To any distribution Φ defined on Ω one can associate, using the diffeomorphism R and its inverse S, a distribution $\widetilde{\phi}$ defined on $\mathcal{S}$ as

$$\widetilde{\phi} = \Phi \circ S,$$

and vice-versa,

$$\Phi = \widetilde{\phi} \circ R.$$

By the chain rule

$$\begin{aligned} \nabla_X \Phi(X,y) &= \nabla_X (\widetilde{\phi} \circ R)(X,y) \\ &= \nabla_X \widetilde{\phi}(X, r(X,y)) \\ &= \nabla_{\widetilde{X}} \widetilde{\phi}(X, r(X,y)) + \partial_{\widetilde{y}} \widetilde{\phi}(X, r(X,y)) \nabla_X r(X,y) \\ &= \nabla_{\widetilde{X}} \widetilde{\phi}(R(X,y)) + \partial_{\widetilde{y}} \widetilde{\phi}(R(X,y)) \nabla_{\widetilde{X}} r(X,y), \end{aligned}$$

and similarly $\partial_y \Phi(X,y) = \partial_{\widetilde{y}} \widetilde{\phi}(X, r(X,y)) \partial_y r(X,y)$.

The following lemma shows that the constant coefficients elliptic equation $\mathbf{P}\phi = 0$ on Ω can equivalently be formulated as a variable coefficients elliptic equation $\widetilde{\mathbf{P}}\widetilde{\phi} = 0$ on $\mathcal{S}$. (Note that the Jacobian of the mapping $(\widetilde{X}, \widetilde{y}) \in \mathcal{S} \mapsto (\widetilde{X}, s(\widetilde{X}, \widetilde{y})) \in \Omega$ is equal to $|\partial_{\widetilde{y}} s|$).

Lemma 1.4. *([29], Lemma 2.5) Suppose that the mapping s, given by (1.11) satisfies Assumption (GC). Then the equation $\boldsymbol{P}\phi = h$ holds in $\mathcal{D}'(\Omega)$ if and only if the equation $\widetilde{\boldsymbol{P}}\widetilde{\phi} = (\partial_{\widetilde{y}} s)\widetilde{h}$ holds in $\mathcal{D}'(\mathcal{S})$, and $\widetilde{\boldsymbol{P}} := -\nabla_{\widetilde{X},\widetilde{y}} \cdot \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}}$, with*

$$\widetilde{P} = \frac{1}{\partial_{\widetilde{y}} s} \begin{pmatrix} \partial_{\widetilde{y}} s I_d & 0 \\ -\nabla_{\widetilde{X}} s^T & 1 \end{pmatrix} P \begin{pmatrix} \partial_{\widetilde{y}} s I_d & -\nabla_{\widetilde{X}} s \\ 0 & 1 \end{pmatrix}$$

Moreover, for all $\theta \in \mathbb{R}^{d+1}$,

$$\widetilde{P}\theta \cdot \theta \geq \widetilde{p}|\theta|^2, \quad \textit{with} \quad \widetilde{p} = Cp \frac{c_0^2}{\|\partial_{\widetilde{y}} s\|_\infty (1 + \|\nabla_{\widetilde{X},\widetilde{y}} s\|_\infty)^2}$$

and C a positive constant.

Proof. By definition, $\mathbf{P}\phi = h$ in $\mathcal{D}'(\Omega)$ if and only if

$$\int_\Omega \mathbf{P}\phi\omega = \int_\Omega h\omega, \quad \forall \omega \in \mathcal{D}(\Omega).$$

By definition of $\mathbf{P}$, one also has

$$\int_\Omega \mathbf{P}\phi\omega = -\int_\Omega \nabla_{X,y}\cdot P\nabla_{X,y}\phi\omega = \int_\Omega P\nabla_{X,y}\phi\cdot\nabla_{X,y}\omega$$

$$\begin{aligned}
&= \int_\Omega P\begin{pmatrix}(\nabla_{\widetilde{X}}\widetilde{\phi})\circ R + \nabla_X r(\partial_{\widetilde{y}}\widetilde{\phi})\circ R\\ \partial_y r(\partial_{\widetilde{y}}\widetilde{\phi})\circ R\end{pmatrix}\cdot\begin{pmatrix}(\nabla_{\widetilde{X}}\widetilde{\omega})\circ R + \nabla_X r(\partial_{\widetilde{y}}\widetilde{\omega})\circ R\\ \partial_y r(\partial_{\widetilde{y}}\widetilde{\omega})\circ R\end{pmatrix}\\
&= \int_{\mathcal{S}} |\partial_{\widetilde{y}}s|P\begin{pmatrix}\nabla_{\widetilde{X}} + (\nabla_X r)\circ S\partial_{\widetilde{y}}\\ (\partial_y r)\circ S\partial_{\widetilde{y}}\end{pmatrix}\widetilde{\phi}\cdot\begin{pmatrix}\nabla_{\widetilde{X}} + (\nabla_X r)\circ S\partial_{\widetilde{y}}\\ (\partial_y r)\circ S\partial_{\widetilde{y}}\end{pmatrix}\widetilde{\omega}\\
&= -\int_{\mathcal{S}} \widetilde{\omega}\begin{pmatrix}\nabla_{\widetilde{X}}\cdot + \partial_{\widetilde{y}}((\nabla_X r)\circ S\cdot)\\ \partial_{\widetilde{y}}((\partial_y r)\circ S\cdot)\end{pmatrix}\cdot|\partial_{\widetilde{y}}s|P\begin{pmatrix}\nabla_{\widetilde{X}} + (\nabla_X r)\circ S\partial_{\widetilde{y}}\\ (\partial_y r)\circ S\partial_{\widetilde{y}}\end{pmatrix}\widetilde{\phi}\\
&= -\int_{\mathcal{S}} \widetilde{\omega}\nabla_{\widetilde{X},\widetilde{y}}\cdot\begin{pmatrix}Id & 0\\ ((\nabla_X r)\circ S)^T & (\partial_y r)\circ S\end{pmatrix}|\partial_{\widetilde{y}}s|P\begin{pmatrix}Id & (\nabla_X r)\circ S\\ 0 & (\partial_y r)\circ S\end{pmatrix}\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}.
\end{aligned}$$

By definition of r and s, one has $r(\widetilde{X}, s(\widetilde{X},\widetilde{y})) = \widetilde{y}$ for all $(\widetilde{X},\widetilde{y}) \in \mathcal{S}$. Differentiating this identity with respect to $\widetilde{X}$ and $\widetilde{y}$ respectively, yields

$$(\nabla_X r)\circ S + (\partial_y r)\circ S\nabla_{\widetilde{X}}s = 0, \quad \partial_{\widetilde{y}}s(\partial_y r)\circ S = 1.$$

Using these expressions into the equalities above gives

$$\int_\Omega \mathbf{P}\phi\omega = \int_{\mathcal{S}} \widetilde{\mathbf{P}}\widetilde{\phi}\widetilde{\omega},$$

where $\widetilde{\mathbf{P}}$ is as given in the statement of the Lemma. Since one clearly has

$$\int_\Omega h\omega = \int_{\mathcal{S}} \partial_y s\widetilde{h}\widetilde{\omega},$$

one has the first claim of the Lemma.

We now prove the coercivity of $\widetilde{\mathbf{P}}$. Firstly, we have that for all $\theta \in \mathbb{R}^{d+1}$,

$$\widetilde{P}\theta\cdot\theta = \frac{1}{\partial_{\widetilde{y}}s}PA\theta\cdot A\theta, \text{ with } A := \begin{pmatrix}\partial_{\widetilde{y}}sI_d & -\nabla_{\widetilde{X}}s\\ 0 & 1\end{pmatrix},$$

and owing to (1.9) we have therefore

$$\widetilde{P}\theta\cdot\theta \geq \frac{p}{\partial_{\widetilde{y}}s}|A\theta|^2. \tag{1.13}$$

The matrix A is invertible, and its inverse is given by

$$A^{-1} = \frac{1}{\partial_{\widetilde{y}}s}\begin{pmatrix}I_d & \nabla_{\widetilde{X}}s\\ 0 & \partial_{\widetilde{y}}s,\end{pmatrix}$$

so that if $\|A\|_\infty := \max_{1\le i\le d+1}\sum_{j=1}^{d+1}|a_{ij}|$ where a_{ij} are the components of the matrix, then $\theta = A^{-1}A\theta$ can be bounded as

$$\begin{aligned}|\theta| &\le C\|A^{-1}\|_\infty |A\theta|\\ &= \frac{C}{|\partial_{\widetilde{y}} s|}\max\{1+|\nabla_{\widetilde{X}} s|, |\partial_{\widetilde{y}} s|\}|A\theta|\\ &\le \frac{C}{c_0}(1+\|\nabla_{\widetilde{X},\widetilde{y}} s\|_\infty)|A\theta|.\end{aligned}$$

Together with (1.13), this estimate yields the result of the lemma. □

The next lemma complements the previous lemma in the sense that shows how the boundary conditions of problem (1.10) are transformed by the diffeomorphism S.

Lemma 1.5. *([29], Lemma 2.6) Suppose that the mapping s, given by (1.11), satisfies Assumption (GC). Then for all $\Phi \in C^1(\bar{\Omega})$, one has*

$$\Phi|_{y=a} = \widetilde{\phi}|_{\widetilde{y}=0} \quad \textit{and} \quad \partial_n^P \Phi|_{y=b} = \frac{1}{\sqrt{1+|\nabla_X b|^2}}\partial_n^{\widetilde{P}}\widetilde{\phi}|_{\widetilde{y}=-1}.$$

Proof. The first assertion of the lemma is straightforward from the equality $\widetilde{\phi} = \phi\circ S$. Now we will prove the second part. Using the notation of the last Lemma, by definition

$$\begin{aligned}\partial_n^{\widetilde{P}}\widetilde{\phi}|_{\widetilde{y}=-1} &= -(-e_{d+1})\cdot \widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}|_{\widetilde{y}=-1}\\ &= \frac{1}{\partial_{\widetilde{y}} s} e_{d+1}\cdot A^T P A \nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}|_{\widetilde{y}=-1}\\ &= \frac{1}{\partial_{\widetilde{y}} s} A e_{d+1}\cdot P A \nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}|_{\widetilde{y}=-1}.\end{aligned}$$

Using the definition of A, by the chain rule, we obtain

$$\begin{aligned}&\partial_n^{\widetilde{P}}\widetilde{\phi}|_{\widetilde{y}=-1} =\\ &\quad\begin{pmatrix}-\nabla_{\widetilde{X}} s|_{\widetilde{y}=-1}\\ 1\end{pmatrix}\cdot P\begin{pmatrix} I_d & -\frac{\nabla_{\widetilde{X}} s}{\partial_{\widetilde{y}} s}\Big|_{\widetilde{y}=-1}\\ 0 & \frac{1}{\partial_{\widetilde{y}} s|_{\widetilde{y}=-1}}\end{pmatrix}\begin{pmatrix}\nabla_X\phi|_{y=b} + \nabla_{\widetilde{X}} s|_{\widetilde{y}=-1}\partial_y\phi|_{y=b}\\ \partial_{\widetilde{y}} s|_{\widetilde{y}=-1}\partial_y\phi|_{y=b}\end{pmatrix}\\ &\quad= \begin{pmatrix}-\nabla_{\widetilde{X}} s|_{\widetilde{y}=-1}\\ 1\end{pmatrix}\cdot P\begin{pmatrix}\nabla_X\phi|_{y=b} + \nabla_{\widetilde{X}} s|_{\widetilde{y}=-1}\partial_y\phi|_{y=b} - \nabla_{\widetilde{X}} s|_{\widetilde{y}=-1}\partial_y\phi|_{y=b}\\ \partial_y\phi|_{y=b}\end{pmatrix},\end{aligned}$$

so that we get

$$\begin{aligned}\partial_n^{\widetilde{P}}\widetilde{\phi}|_{\widetilde{y}=-1} &= -\begin{pmatrix}\nabla_{\widetilde{X}} s|_{\widetilde{y}=-1}\\ -1\end{pmatrix}\cdot P\nabla_{X,y}\phi|_{y=b}\\ &= \sqrt{1+|\nabla_X b|^2}\,\partial_n^P\phi|_{y=b},\end{aligned}$$

which ends the proof of the lemma. □

The previous Lemmas show that the study of the boundary problems (1.10) in the domain Ω can be deduced from the study of elliptic boundary value problems on the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$ allowing to transfer the geometry of the problem in the domain Ω to the coefficients of the operator $\widetilde{\mathbf{P}}$; this will prove be very useful later when we use the shape derivative with respect to the surface parametrization to calculate the derivative of the operator J in the domain $\mathcal{S}$.

Summarizing the last two lemmas.

Proposition 1.6. *([29], Proposition 2.7) Suppose that the mapping s, given by (1.11), satisfies Assumption (GC). Then Φ is a (classical, variational) solution of (1.10) if and only if $\widetilde{\phi} = \Phi \circ S$ is a (classical, variational) solution of*

$$\begin{cases} \widetilde{\boldsymbol{P}}\widetilde{\phi} = (\partial_{\widetilde{y}} s)\widetilde{h} \quad \textit{on } \mathcal{S}, \\ \widetilde{\phi}|_{\widetilde{y}=0} = f, \quad \partial_n^{\widetilde{P}} \widetilde{\phi}|_{\widetilde{y}=-1} = \sqrt{1 + |\nabla_X b|^2} g, \end{cases}$$

where $\widetilde{\boldsymbol{P}}$ is as given in Lemma 1.4.

1.2.2 The Dirichlet-Neumann operator

The main objective of this section is to present some properties of the Dirichlet-Neumann operator. One notice from the equations (2.9) that the Dirichlet-Neumann operator, and its relations with the surface and bottom boundary conditions, plays an important role in the solution to the water-waves equations. Although many of the results are given for the Dirichlet-Neumann operator G from a Sobolev viewpoint, later we show that the operator J can be written as $J = G + G^{NN}$ and even more important that the derivative with respect to the surface parameterization of the operator J, can be written in terms of the operator G which allows to use the same methods developed by Lannes [29].

In this section we consider a particular case of the boundary value problem (1.10), namely, the case of a homogeneous source term, which is precisely the case when we study the water-waves equations (2.9). Let Φ be a solution of the problem

$$\begin{cases} \mathbf{P}\Phi = 0 \quad \text{on } \Omega, \\ \Phi|_{y=a(X)} = f, \quad \partial_n^P \Phi|_{y=b(X)} = g. \end{cases} \tag{1.14}$$

For all $k \in \mathbb{N}$, $f \in H^{k+3/2}(\mathbb{R}^d)$, $g \in H^{k+1/2}(\mathbb{R}^d)$ and provided that a and b are smooth enough, it is classic that $\phi \in H^{k+2}(\Omega)$ exists and is unique.

Definition 1.7. Let $k \in \mathbb{N}$. Assume that $g \in H^{k+1/2}(\mathbb{R}^d)$ and $a, b \in W^{k+2,\infty}(\mathbb{R}^d)$ satisfy condition (GC). We define the Dirichlet-Neumann operator to be the operator $J(a, b, f, g)$ given by

$$J(a, b, \cdot, g) : \begin{array}{ccc} H^{k+3/2}(\mathbb{R}^d) & \to & H^{k+1/2}(\mathbb{R}^d) \\ f & \mapsto & -\sqrt{1 + |\nabla_X a|^2} \partial_n^P \Phi|_{y=a(X)}. \end{array}$$

where Φ denotes the solution of (1.14).

Under the same hypothesis, let us also consider the operator

$$G(a, b) : \begin{array}{ccc} H^{k+3/2}(\mathbb{R}^d) & \to & H^{k+1/2}(\mathbb{R}^d) \\ f & \mapsto & -\sqrt{1 + |\nabla_X a|^2} \partial_n^P u|_{y=a(X)}, \end{array}$$

where u is the unique solution of the problem

$$\begin{cases} \mathbf{P}u = 0 \quad \text{on } \Omega, \\ u|_{y=a(X)} = f, \quad \partial_n^P u|_{y=b(X)} = 0. \end{cases} \tag{1.15}$$

Remark 1.8. If $g = 0$ in the Definition 1.7, the operator J becomes operator G; in other words $J(a, b, f, 0) = G(a, b)f$.

As before (remember Lemma 1.5), we can associate to (1.14) an elliptic boundary value problem on the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$,

$$\begin{cases} \widetilde{\mathbf{P}}\widetilde{\phi} = 0 \quad \text{on } \mathcal{S}, \\ \widetilde{\phi}|_{\widetilde{y}=0} = f, \quad \partial_n^{\widetilde{P}}\widetilde{\phi}|_{\widetilde{y}=-1} = \sqrt{1+|\nabla_X b|^2}g. \end{cases} \tag{1.16}$$

Remark 1.9. In the particular case of the water-wave equations, one has $g = -\frac{\partial_t b}{\sqrt{1+|\nabla_X b|2}}$, so that the Neumann condition for the elliptic problem on the flat strip will be $\partial_n^{\widetilde{P}}\widetilde{\phi}|_{\widetilde{y}=-1} = -\partial_t b$.

In the same way we associate to (1.15) the boundary value problem

$$\begin{cases} \widetilde{\mathbf{P}}\widetilde{u} = 0 \quad \text{on } \mathcal{S}, \\ \widetilde{u}|_{\widetilde{y}=0} = f, \quad \partial_n^{\widetilde{P}}\widetilde{u}|_{\widetilde{y}=-1} = 0. \end{cases} \tag{1.17}$$

Proceeding as in the proof of Lemma 1.5, $\partial_n^{\widetilde{P}}\widetilde{\phi}|_{\widetilde{y}=0} = \sqrt{1+|\nabla_X a|^2}\partial_n^P \Phi|_{y=a}$ and $\partial_n^{\widetilde{P}}\widetilde{u}|_{\widetilde{y}=0} = \sqrt{1+|\nabla_X a|^2}\partial_n^P u|_{y=a}$, hence one can define the Dirichlet-Neumann operator in terms of $\widetilde{\phi}$ and $\widetilde{u}$ respectively.

Proposition 1.10. *([29], Proposition 3.4) Under the same assumptions as in Def. 1.7, one has*

$$J(a, b, f, g) = -\partial_n^{\widetilde{P}}\widetilde{\phi}|_{\widetilde{y}=0}, \quad \forall f \in H^{3/2}(\mathbb{R}^d),$$

$$G(a, b)f = -\partial_n^{\widetilde{P}}\widetilde{u}|_{\widetilde{y}=0}, \quad \forall f \in H^{3/2}(\mathbb{R}^d).$$

The following theorem gives bounds for the Dirichlet-Neumann operator in Sobolev spaces. As was mentioned before, this bounds are useful in the well-posedness theorem for water-waves.

Theorem 1.11. *([29], Theorem 3.6) Let $m_0 = \lceil \frac{d+1}{2} \rceil$ and a, b be two continuous functions satisfying* (GC). *Then:*

i. For all $k \in \mathbb{N}$, if $a, b \in W^{k+2,\infty}(\mathbb{R}^d)$, then for all f such that $\nabla_X f \in H^{k+1/2}(\mathbb{R}^d)^2$, one has

$$|G(a, b)f|_{H^{k+1/2}} \leq C(|a|_{W^{k+2,\infty}}, |b|_{W^{k+2,\infty}})|\nabla_X f|_{H^{k+1/2}}.$$

ii. For all $k \in \mathbb{N}$, if $a \in H^{2m_0+1/2} \cap H^{k+3/2}(\mathbb{R}^d)$ and if $b \in W^{k+2,\infty}(\mathbb{R}^d)$, then

$$\begin{aligned} |G(a, b)f|_{H^{k+1/2}} \leq & C(|a|_{H^{2m_0+1/2}}, |b|_{W^{k+2,\infty}}, |b|_{W^{m_0+1,\infty}})(|\nabla_X f|_{H^{k+1/2}} \\ & + |a|_{H^{k+3/2}}|\nabla_X f|_{H^{m_0-1/2}}), \end{aligned}$$

for all f such that $\nabla_X f \in H^{k+1/2} \cap H^{m_0-1/2}$.

Proof. We just describe the second part of the proof; the first part is very similar. Owing to Proposition 1.10 and since $\widetilde{P}$ is symmetric, we have

$$|G(a,b)f|_{H^{k+1/2}} = \left|\partial_n^{\widetilde{P}}\widetilde{u}|_0\right|_{H^{k+1/2}} = |\widetilde{P}|_{\widetilde{y}=0}e_{d+1}\cdot\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}|_{\widetilde{y}=0}|_{H^{k+1/2}}.$$

By the trace theorem, this yields

$$|G(a,b)f|_{H^{k+1/2}} \le C\|\widetilde{P}e_{d+1}\cdot\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2}.$$

Using the decomposition $\widetilde{P} = \widetilde{P}_1 + \widetilde{P}_2$

$$|G(a,b)f|_{H^{k+1/2}} \le C\|\widetilde{P}_1 e_{d+1}\cdot\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} + C\|\widetilde{P}_2 e_{d+1}\cdot\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2}.$$

Since $a \in H^{2m_0+1/2}\cap H^{k+3/2}(\mathbb{R}^d)$, by the Sobolev embedding $H^{m_0}(\mathcal{S}) \subset L^\infty(\mathcal{S})$ one has $a \in L^\infty \cap H^{k+1}(\mathbb{R}^d)$; by the embedding $H^{m_0}(\mathcal{S}) \subset L^\infty(\mathcal{S})$ one has

$$\begin{aligned}
|G(a,b)f|_{H^{k+1/2}} &\le C\|\widetilde{P}_1\|_{k+1,\infty}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} + C\|\widetilde{P}_2\|_\infty\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} \\
&\quad + C\|\widetilde{P}_2\|_{k+1,2}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_\infty \\
&\le C\|\widetilde{P}_1\|_{k+1,\infty}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} + C\|\widetilde{P}_2\|_{m_0,2}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} \\
&\quad + C\|\widetilde{P}_2\|_{k+1,2}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{m_0,2}.
\end{aligned}$$

Now,

$$\|\widetilde{P}_1\|_{k+1,\infty} \le C(|b|_{W^{k+2,\infty}}), \tag{1.18}$$

and

$$\begin{aligned}
\|\widetilde{P}_2\|_{k+1,2} &\le C(\|s_1\|_{k+2,\infty}, \|s_2\|_{1,\infty})\|s_2\|_{k+2,2} \\
&\le C(|b|_{W^{k+2,\infty}}, \|s_2\|_{m_0+1})|a|_{H^{k+3/2}} \\
&\le C(|b|_{W^{k+2,\infty}}, |a|_{H^{m_0+1/2}})|a|_{H^{k+3/2}}.
\end{aligned}$$

Hence, one obtains

$$\begin{aligned}
|G(a,b)f|_{H^{k+1/2}} &\le C(|b|_{W^{k+2,\infty}}, |b|_{W^{m_0+1,\infty}}, |a|_{H^{m_0+1/2}})\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} \\
&\quad + C(|b|_{W^{k+2,\infty}}, |a|_{H^{m_0+1/2}})|a|_{H^{k+3/2}}\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{m_0,2}.
\end{aligned}$$

By hypothesis we have $a \in H^{2m_0+1/2}(\mathbb{R}^d)$. Then $\widetilde{P}_2 \in H^{2m_0}(\mathcal{S})^{(d+1)^2}$ and by the Sobolev embedding $H^{m_0}(\mathcal{S}) \subset L^\infty(\mathcal{S})$ one obtains $\widetilde{P}_2 \in W^{m_0,\infty}(\mathcal{S})^{(d+1)^2}$.

Therefore

$$\begin{aligned}
\|\nabla_{\widetilde{X},\widetilde{y}}\widetilde{u}\|_{k+1,2} &\le C(|a|_{H^{2m_0+1/2}}, |b|_{W^{k+2,\infty}})(|\nabla_{\widetilde{X}}f|_{H^{k+1/2}} \\
&\quad + |a|_{H^{k+3/2}}|\nabla_{\widetilde{X}}f|_{H^{m_0-1/2}});
\end{aligned}$$

if we define $C(-) := C(|a|_{H^{2m_0+1/2}}, |b|_{W^{k+2,\infty}}, |b|_{W^{m_0+1,\infty}})$, it follows that

$$\begin{aligned}
|G(a,b)f|_{H^{k+1/2}} &\le C(-)(|\nabla_{\widetilde{X}}f|_{H^{k+1/2}} + |a|_{H^{k+3/2}}|\nabla_{\widetilde{X}}f|_{H^{m_0-1/2}}) \\
&\quad + C(-)|a|_{H^{k+\frac{3}{2}}}(|\nabla_{\widetilde{X}}f|_{H^{m_0-1/2}} + |a|_{H^{m_0+1/2}}|\nabla_{\widetilde{X}}f|_{H^{m_0-1/2}}) \\
&\le C(-)(|\nabla_{\widetilde{X}}f|_{H^{k+1/2}} + |a|_{H^{k+3/2}}|\nabla_{\widetilde{X}}f|_{H^{m_0-1/2}}).
\end{aligned}$$

∎

Following the same ideas of the last proof it is possible to extend this results to the operator J.

Theorem 1.12. *Let $m_0 = \lceil \frac{d+1}{2} \rceil$ and a, b be two continuous functions satisfying (GC). Then:*

i. For all $k \in \mathbb{N}$, if $a, b \in W^{k+2,\infty}(\mathbb{R}^d)$, then for all f such that $\nabla_X f \in H^{k+1/2}(\mathbb{R}^d)^2$ and all $g \in H^{k+1/2}(\mathbb{R}^d)$, one has

$$|J(a,b,f,g)|_{H^{k+1/2}} \leq C(|a|_{W^{k+2,\infty}}, |b|_{W^{k+2,\infty}})(|\nabla_X f|_{H^{k+1/2}} + |g|_{H^{k+1/2}}).$$

ii. For all $k \in \mathbb{N}$, if $a \in H^{2m_0+1/2} \cap H^{k+3/2}(\mathbb{R}^d)$ and if $b \in W^{k+2,\infty}(\mathbb{R}^d)$, then

$$\begin{aligned}|J(a,b,f,g)|_{H^{k+1/2}} \leq & C(|a|_{H^{2m_0+1/2}}, |b|_{W^{k+2,\infty}}, |b|_{W^{m_0+1,\infty}})[|\nabla_X f|_{H^{k+1/2}} \\ & + |g|_{H^{k+1/2}} + |a|_{H^{k+3/2}}(|\nabla_X f|_{H^{m_0-1/2}} + |g|_{H^{m_0-1/2}})],\end{aligned}$$

for all f such that $\nabla_X f \in H^{k+1/2} \cap H^{m_0-1/2}$ and all $g \in H^{k+1/2} \cap H^{m_0-1/2}(\mathbb{R}^d)$.

Proof. For the second part, following the same method as in the above proof, one has

$$\begin{aligned}|J(a,b,f,g)|_{H^{k+1/2}} \leq C(|b|_{W^{k+2,\infty}}, |b|_{W^{m_0+1,\infty}}, |a|_{H^{m_0+1/2}}) \|\nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}\|_{k+1,2} \\ + C(|b|_{W^{k+2,\infty}}, |a|_{H^{m_0+1/2}})|a|_{H^{k+3/2}} \|\nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}\|_{m_0,2},\end{aligned}$$

where $\widetilde{\phi}$ is the solution of the b.v.p (1.16). Then

$$\begin{aligned}\|\nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}\|_{k+1,2} \leq & C(|a|_{H^{2m_0+\frac{1}{2}}}, |b|_{W^{k+2,\infty}}) \Big[|\nabla_X f|_{H^{k+\frac{1}{2}}} + |\sqrt{1+|\nabla_X b|^2} g|_{H^{k+\frac{1}{2}}} \\ & + |a|_{H^{k+3/2}} (|\nabla_X f|_{H^{m_0-1/2}} + |\sqrt{1+|\nabla_X b|^2} g|_{H^{m_0-1/2}}) \Big] \\ \leq & C(|a|_{H^{2m_0+1/2}}, |b|_{W^{k+2,\infty}}) \Big[|\nabla_X f|_{H^{k+1/2}} + |g|_{H^{k+1/2}} \\ & + |a|_{H^{k+3/2}} (|\nabla_X f|_{H^{m_0-1/2}} + |g|_{H^{m_0-1/2}}) \Big];\end{aligned}$$

therefore,

$$\begin{aligned}|J(a,b,f,g)|_{H^{k+1/2}} \leq & C(-)[|\nabla_X f|_{H^{k+1/2}} + |g|_{H^{k+1/2}} \\ & + |a|_{H^{k+3/2}} (|\nabla_X f|_{H^{m_0-1/2}} + |g|_{H^{m_0-1/2}})\big] \\ & + C(-)|a|_{H^{k+\frac{3}{2}}} [|\nabla_X f|_{H^{m_0-1/2}} + |g|_{H^{m_0-1/2}} \\ & + |a|_{H^{m_0+1/2}} (|\nabla_X f|_{H^{m_0-1/2}} + |g|_{H^{m_0-1/2}})],\end{aligned}$$

and thus we have

$$\begin{aligned}|J(a,b,f,g)|_{H^{k+1/2}} \leq & C(-)[|\nabla_X f|_{H^{k+1/2}} + |g|_{H^{k+1/2}} \\ & + |a|_{H^{k+3/2}} (|\nabla_X f|_{H^{m_0-1/2}} + |g|_{H^{m_0-1/2}})].\end{aligned}$$

□

Notation 1.13. We denote by f^b the solution of the b.v.p (1.17).

Proposition 1.14. *([29], Proposition 3.8) Let $a, b \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (GC). Then:*

i. The operator $G(a,b)$ is self-adjoint:

$$(G(a,b)f,g) = (f, G(a,b)g), \quad \forall f,g \in H^{3/2}(\mathbb{R}^d).$$

ii. The operator $G(a,b)$ is positive:

$$(G(a,b)f,f) \geq 0, \quad \forall f \in H^{3/2}(\mathbb{R}^d).$$

iii. We also have the estimates

$$|(G(a,b)f,g)| \leq C(|a|_{W^{2,\infty}}, |b|_{W^{2,\infty}})|f|_{H^{1/2}}|g|_{H^{1/2}} \quad \forall f,g \in H^{3/2}(\mathbb{R}^d),$$

and for all $\mu \geq \frac{2\widetilde{p}}{3}$, where $\widetilde{p}$ is given in Lemma 1.4, one has

$$|([G(a,b)+\mu]f,f)| \geq C\widetilde{p}|f|^2_{H^{1/2}}, \quad \forall f \in H^{3/2}(\mathbb{R}^d).$$

Proof. i. Multiplying by g^b and integrating by parts in (1.17) one has

$$\begin{aligned} 0 = \int_{\mathcal{S}} (\widetilde{\mathbf{P}} f^b) g^b &= -\int_{\mathcal{S}} (\nabla_{\widetilde{X},\widetilde{y}} \cdot \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b) g^b \\ &= \int_{\mathcal{S}} \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b \cdot \nabla_{\widetilde{X},\widetilde{y}} g^b - \int_{\partial\mathcal{S}} g^b (\widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b \cdot n) \\ &= \int_{\mathcal{S}} \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b \cdot \nabla_{\widetilde{X},\widetilde{y}} g^b - \int_{\mathbb{R}^d} g^b (\widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b \cdot n) \Big|_{\widetilde{y}=-1}^{\widetilde{y}=0} \\ &= \int_{\mathcal{S}} \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b \cdot \nabla_{\widetilde{X},\widetilde{y}} g^b - \int_{\mathbb{R}^d} g (\widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b|_{\widetilde{y}=0} \cdot e_{d+1}). \end{aligned} \tag{1.19}$$

Thus

$$(\partial_n^{\widetilde{P}} f^b|_{\widetilde{y}=0}, g) = -\int_{\mathcal{S}} \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b \cdot \nabla_{\widetilde{X},\widetilde{y}} g^b; \tag{1.20}$$

similarly for g^b one has

$$(\partial_n^{\widetilde{P}} g^b|_{\widetilde{y}=0}, f) = -\int_{\mathcal{S}} \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} g^b \cdot \nabla_{\widetilde{X},\widetilde{y}} f^b. \tag{1.21}$$

According to Proposition 1.10 one has $(G(a,b)f,g) = (-\partial_n^{\widetilde{P}} f^b|_{\widetilde{y}=0}, g)$. Since $\widetilde{P}$ is symmetric, from (1.20) and (1.21) we get the result.
ii. Proceeding as in (1.19), by Proposition 1.10 and the coercivity of $\widetilde{P}$ one has

$$\begin{aligned} (G(a,b)f,f) &= \int_{\mathcal{S}} \nabla_{\widetilde{X},\widetilde{y}} f^b \cdot \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} f^b \\ &\geq \widetilde{p} \|\nabla_{\widetilde{X},\widetilde{y}} f^b\|_2^2. \end{aligned} \tag{1.22}$$

iii. Proceeding as in (*ii*), one has

$$\begin{aligned} (G(a,b)f,g) &= \int_{\mathcal{S}} \nabla_{\widetilde{X},\widetilde{y}} f^b \cdot \widetilde{P} \nabla_{\widetilde{X},\widetilde{y}} g^b \\ &\leq \|\widetilde{P}\|_\infty \|\nabla_{\widetilde{X},\widetilde{y}} f^b\|_2 \|\nabla_{\widetilde{X},\widetilde{y}} g^b\|_2 \\ &\leq C(\|\widetilde{P}\|_\infty) |f|_{H^{1/2}} |g|_{H^{1/2}}. \end{aligned}$$

Therefore we have

$$(G(a,b)f,g) \le C(|a|_{W^{2,\infty}}, |b|_{W^{2,\infty}})|f|_{H^{1/2}}|g|_{H^{1/2}}.$$

To prove the second estimate, remark first that for $(\widetilde{X},\widetilde{y}) \in \mathcal{S}$, $f^b(\widetilde{X},\widetilde{y}) = \int_0^{\widetilde{y}} \partial_{\widetilde{y}} f^b(\widetilde{X},z)dz + f(\widetilde{X},0)$. Hence

$$\begin{aligned} |f^b(\widetilde{X},\widetilde{y}) - f(\widetilde{X},0)|^2 &= \left| \int_{\widetilde{y}}^0 \partial_{\widetilde{y}} f^b(\widetilde{X},z)dz \right|^2 \\ &\le |\widetilde{y}| \int_{-1}^0 |\partial_{\widetilde{y}} f^b(\widetilde{X},z)|^2 dz. \end{aligned}$$

Thus, integrating over $\widetilde{X}$, one has

$$\int_{\mathbb{R}^d} |f^b(\widetilde{X},\widetilde{y}) - f(\widetilde{X},0)|^2 d\widetilde{X} \le |\widetilde{y}| \|\partial_{\widetilde{y}} f^b\|_2^2,$$

and integrating again over $\widetilde{y}$, $-1 < \widetilde{y} < 0$, it follows that

$$\|f^b\|_2^2 - |f|_2^2 \le \frac{1}{2}\|\partial_{\widetilde{y}} f^b\|_2^2.$$

Therefore $\|\nabla_{\widetilde{X},\widetilde{y}} f^b\|_2^2 \ge 2\|f^b\|_2^2 - 2|f|_2^2$. Since $\|f^b\|_{1,2}^2 = \|f^b\|_2^2 + \|\nabla_{\widetilde{X},\widetilde{y}} f^b\|_2^2 \le \frac{3}{2}\|\nabla_{\widetilde{X},\widetilde{y}} f^b\|_2^2 + |f|_2^2$, we obtain $\|\nabla_{\widetilde{X},\widetilde{y}} f^b\|_2^2 \ge \frac{2}{3}\|f^b\|_{1,2}^2 - \frac{2}{3}|f|_2^2$. Using (1.22), by the last inequality and the estimate $|f|_{H^{1/2}}^2 \le C\|f^b\|_{1,2}^2$, one has

$$\begin{aligned} |([G(a,b)+\mu]f,f)| &\ge \widetilde{p}\|\nabla_{\widetilde{X},\widetilde{y}} f^b\|_2^2 + \mu|f|_2^2 \\ &\ge \frac{2\widetilde{p}}{3}\|f^b\|_{1,2}^2 - \frac{2\widetilde{p}}{3}|f|_{H^{1/2}}^2 + \mu|f|_2^2 \\ &\ge C\widetilde{p}|f|_{H^{1/2}}^2 - \frac{2\widetilde{p}}{3}|f|_2^2 + \mu|f|_2^2 \\ &\ge C\widetilde{p}|f|_{H^{1/2}}^2. \end{aligned}$$

□

1.2.3 Shape derivative of the Dirichlet-Neumann operator

Let us write the matrix P in the form $P = \begin{pmatrix} P_1 & \mathbf{p} \\ \mathbf{p}^T & p_{d+1} \end{pmatrix}$ where P_1 is a $d \times d$ symmetric matrix, $\mathbf{p} \in \mathbb{R}^d$ and $p_{d+1} \in \mathbb{R}$. We denote by $\widetilde{\phi}_a^b$ the solution of the boundary value problem

$$\begin{cases} \widetilde{\mathbf{P}}\widetilde{\phi} = 0 \quad \text{on } \mathcal{S}, \\ \widetilde{\phi}|_{\widetilde{y}=0} = f, \quad \partial_n^{\widetilde{P}} \widetilde{\phi}|_{\widetilde{y}=-1} = \sqrt{1+|\nabla_X b|^2}\, g. \end{cases} \tag{1.23}$$

Before stating the main result let us prove the following Lemma.

Lemma 1.15. *Let $m_0 = \lceil \frac{d+1}{2} \rceil$. For all $k \in \mathbb{N}$, if $a \in H^{2m_0+1/2} \cap H^{k+3/2}(\mathbb{R}^d)$ and $b \in W^{k+2,\infty}(\mathbb{R}^d)$ satisfy (GC), then*

$$\frac{\partial_{\widetilde{y}} \widetilde{\phi}_a^b|_{\widetilde{y}=0}}{a-b} = \frac{1}{N \cdot PN}\left(J(a,b,f,g) - N \cdot P \begin{pmatrix} \nabla_X f \\ 0 \end{pmatrix} \right),$$

where $N = (-\nabla_X a, 1)^T$.

Proof. By definition,

$$\begin{aligned}\partial_n^{\widetilde{P}}\widetilde{\phi}_a^b|_{\widetilde{y}=0} &= -e_{d+1}\cdot\widetilde{P}\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}_a^b|_{\widetilde{y}=0}\\ &= -\frac{1}{\partial_{\widetilde{y}}s}e_{d+1}\cdot A^T PA\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}_a^b|_{\widetilde{y}=0},\end{aligned}$$

with $A=\begin{pmatrix}\partial_{\widetilde{y}}sI_d & -\nabla_{\widetilde{X}}s\\ 0 & 1\end{pmatrix}$. Therefore

$$\begin{aligned}\partial_n^{\widetilde{P}}\widetilde{\phi}_a^b|_{\widetilde{y}=0} &= -\frac{1}{\partial_{\widetilde{y}}s}Ae_{d+1}\cdot P\begin{pmatrix}\partial_{\widetilde{y}}sI_d & -\nabla_{\widetilde{X}}s\\ 0 & 1\end{pmatrix}\begin{pmatrix}\nabla_{\widetilde{X}}\widetilde{\phi}_a^b\\ \partial_{\widetilde{y}}\widetilde{\phi}_a^b\end{pmatrix}\Bigg|_{\widetilde{y}=0}\\ &= -\frac{1}{\partial_{\widetilde{y}}s}\begin{pmatrix}-\nabla_{\widetilde{X}}s\\ 1\end{pmatrix}\cdot P\left[\partial_{\widetilde{y}}s\begin{pmatrix}\nabla_{\widetilde{X}}\widetilde{\phi}_a^b\\ 0\end{pmatrix}+\partial_{\widetilde{y}}\widetilde{\phi}_a^b\begin{pmatrix}-\nabla_{\widetilde{X}}s\\ 1\end{pmatrix}\right]\Bigg|_{\widetilde{y}=0}\\ &= -\frac{1}{a-b}\begin{pmatrix}-\nabla_X a\\ 1\end{pmatrix}\cdot P\left[(a-b)\begin{pmatrix}\nabla_X f\\ 0\end{pmatrix}+\partial_{\widetilde{y}}\widetilde{\phi}_a^b|_{\widetilde{y}=0}\begin{pmatrix}-\nabla_X a\\ 1\end{pmatrix}\right]\\ &= -N\cdot P\begin{pmatrix}\nabla_X f\\ 0\end{pmatrix}-\frac{N\cdot PN}{a-b}\partial_{\widetilde{y}}\widetilde{\phi}_a^b|_{\widetilde{y}=0};\end{aligned}$$

and since $J(a,b,f,g)=-\partial_n^{\widetilde{P}}\widetilde{\phi}_a^b|_{\widetilde{y}=0}$ then we have the Lemma. $\square$

If we write $\hat{Z}=\frac{1}{N\cdot PN}\left(J(a,b,f,g)-N\cdot P\begin{pmatrix}\nabla_X f\\ 0\end{pmatrix}\right)$, by the previous Lemma, $\hat{Z}=\frac{\partial_{\widetilde{y}}\widetilde{\phi}_a^b|_{\widetilde{y}=0}}{a-b}$.

Remark 1.16. For the water-waves equations, one has $P=Id_{(d+1)\times(d+1)}$, and $\hat{Z}$ is simply given by $\hat{Z}=\frac{1}{1+|\nabla_X a|^2}(J(a,f)+\nabla_X a\cdot\nabla_X f)$.

The following theorem gives an explicit expression of the shape derivative of the Dirichlet-Neumann operator J in terms of the operator G.

Theorem 1.17. *Let $m_0=\lceil\frac{d+1}{2}\rceil$ and $k\in\mathbb{N}$, $k\geq m_0$. Suppose that $\underline{a}\in H^{k+3/2}(\mathbb{R}^d)$ and $b\in W^{k+2,\infty}(\mathbb{R}^d)$ satisfy (GC). Then there exists a neighborhood $\mathcal{U}_{\underline{a}}$ of $\underline{a}$ in $H^{k+3/2}(\mathbb{R}^d)$ such that for all given $f\in H^{k+3/2}(\mathbb{R}^d)$, $g\in H^{k+1/2}(\mathbb{R}^d)$, the mappings*

$$a\in\mathcal{U}_{\underline{a}}\subset H^{k+3/2}(\mathbb{R}^d)\mapsto J(a,b,f,g)\in H^{k+1/2}(\mathbb{R}^d)$$

$$a\in\mathcal{U}_{\underline{a}}\subset H^{k+3/2}(\mathbb{R}^d)\mapsto G(a,b)f\in H^{k+1/2}(\mathbb{R}^d)$$

are well defined and differentiable. Moreover, for all $h\in H^{k+3/2}(\mathbb{R}^d)$, one has

$$d_{\underline{a}}J(\cdot,f)\cdot h=-G(\underline{a},b)(\underline{\hat{Z}}h)-\nabla_X\cdot(h(P_1\underline{\hat{\mathbf{v}}}+\underline{\hat{Z}}\boldsymbol{p})),$$

where

$$\underline{\hat{Z}}=\frac{1}{N\cdot PN}\left(J(\underline{a},b,f,g)-N\cdot P\begin{pmatrix}\nabla_X f\\ 0\end{pmatrix}\right),$$

with

$$N=(-\nabla_X\underline{a},1)^T\quad\text{and}\quad\underline{\hat{\mathbf{v}}}=\nabla_X f-\underline{\hat{Z}}\nabla_X\underline{a}.$$

Proof. We can choose a neighborhood $\mathcal{U}_{\underline{a}} \subset H^{k+3/2}$ of $\underline{a}$ such that for all $a \in \mathcal{U}_{\underline{a}}$, condition (GC) is satisfied (taking h_0 smaller if necessary). To each $a \in \mathcal{U}_{\underline{a}}$ it is therefore possible to associate a diffeomorphism $S_a(X,y) = (X, s_a(X,y))$. Taking $\mathcal{U}_{\underline{a}}$ smaller if necessary, we can assume that the mapping $a \mapsto s_a$ is affine. We denote by $d_{\underline{a}}s$ its derivative at $\underline{a}$. Since the matrix $\widetilde{P}_a$, given by Lemma 1.4 with $s = s_a$, has coefficients in $H^{k+1}(\mathcal{S})$, it follows that the mapping

$$a \in \mathcal{U}_{\underline{a}} \subset H^{k+3/2}(\mathbb{R}^d) \mapsto \widetilde{P}_a \in H^{k+1}(\mathcal{S})^{(d+1)^2}$$

is smooth. We denote by $d_{\underline{a}}\widetilde{P}$ its derivative at $\underline{a}$. Since $\widetilde{\phi}^b_a$ is the solution of the boundary value problem

$$\begin{cases} -\nabla_{\widetilde{X},\widetilde{y}} \cdot \widetilde{P}_a \nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}^b_a = 0 \quad \text{in } \mathcal{S}, \\ \widetilde{\phi}^b_a|_{\widetilde{y}=0} = f, \quad \partial_n^{\widetilde{P}_a} \widetilde{\phi}^b_a|_{\widetilde{y}=-1} = \sqrt{1+|\nabla_X b|^2} g, \end{cases} \tag{1.24}$$

we know that $\widetilde{\phi}^b_a \in H^{k+2}(\mathcal{S})$. It is easy to prove that the mapping $\mathcal{B}$ defined as

$$\mathcal{B} : a \in \mathcal{U}_{\underline{a}} \subset H^{k+3/2}(\mathbb{R}^d) \mapsto \widetilde{\phi}^b_a \in H^{k+2}(\mathcal{S})$$

is continuous. Differentiating (1.24) with respect to a, it is easy to show that $\mathcal{B}$ is differentiable at $\underline{a}$ and that for all $h \in H^{k+3/2}(\mathbb{R}^d)$, $\widetilde{v}_{\underline{a},h} := d_{\underline{a}}\mathcal{B} \cdot h$ solves

$$\begin{cases} -\nabla_{\widetilde{X},\widetilde{y}} \cdot \widetilde{P}_{\underline{a}} \nabla_{\widetilde{X},\widetilde{y}} \widetilde{v}_{\underline{a},h} = \nabla_{\widetilde{X},\widetilde{y}} \cdot d_{\underline{a}}\widetilde{P} \cdot h \nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \quad \text{in } \mathcal{S}, \\ \widetilde{v}_{\underline{a},h}|_{\widetilde{y}=0} = 0, \quad \partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}_{\underline{a},h}|_{\widetilde{y}=-1} = -e_{d+1} \cdot (d_{\underline{a}}\widetilde{P} \cdot h \nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}^b_{\underline{a}})|_{\widetilde{y}=-1}. \end{cases} \tag{1.25}$$

The following is a key lemma. It gives an explicit function solving (1.25) except for the Dirichlet condition at the surface (see [29], Lemma 3.22).

Lemma 1.18. *For all $h \in H^{k+3/2}(\mathbb{R}^d)$, the function $\widetilde{v}^b_{\underline{a},h} := \frac{d_{\underline{a}}s \cdot h}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}$ solves*

$$\begin{cases} -\nabla_{\widetilde{X},\widetilde{y}} \cdot \widetilde{P}_{\underline{a}} \nabla_{\widetilde{X},\widetilde{y}} \widetilde{v}^b_{\underline{a},h} = \nabla_{\widetilde{X},\widetilde{y}} \cdot d_{\underline{a}}\widetilde{P} \cdot h \nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \quad \text{in } \mathcal{S}, \\ \widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} = \dfrac{h}{\underline{a}-b} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a},h}|_{\widetilde{y}=0}, \quad \partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=-1} = -e_{d+1} \cdot (d_{\underline{a}}\widetilde{P} \cdot h \nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}^b_{\underline{a}})|_{\widetilde{y}=-1}. \end{cases}$$

Then, from (1.25) and the previous Lemma, $\widetilde{v}_{\underline{a},h} - \widetilde{v}^b_{\underline{a},h}$ solves

$$\begin{cases} -\nabla_{\widetilde{X},\widetilde{y}} \cdot \widetilde{P}_{\underline{a}} \nabla_{\widetilde{X},\widetilde{y}} (\widetilde{v}_{\underline{a},h} - \widetilde{v}^b_{\underline{a},h}) = 0 \\ (\widetilde{v}_{\underline{a},h} - \widetilde{v}^b_{\underline{a},h})|_{\widetilde{y}=0} = -\dfrac{h}{\underline{a}-b} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0}, \quad \partial_n^{\widetilde{P}_{\underline{a}}} (\widetilde{v}_{\underline{a},h} - \widetilde{v}^b_{\underline{a},h})|_{\widetilde{y}=-1} = 0. \end{cases}$$

By definition of the operator $G(\underline{a}, b)$, it follows that

$$G(\underline{a}, b)\Big(-\frac{h}{\underline{a}-b} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} \Big) = -\partial_n^{\widetilde{P}_{\underline{a}}} (\widetilde{v}_{\underline{a},h} - \widetilde{v}^b_{\underline{a},h})|_{\widetilde{y}=0}.$$

Then

$$-\partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}_{\underline{a},h}|_{\widetilde{y}=0} = -\partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} - G(\underline{a}, b)\Big(\frac{h}{\underline{a}-b} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} \Big),$$

and by Lemma 1.15 one has

$$-\partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}_{\underline{a},h}|_{\widetilde{y}=0} = -\partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} - G(\underline{a}, b) h \underline{\hat{Z}}.$$

To finish the proof, we write $d_{\underline{a}}J(\cdot,b,f,g)\cdot h$ in terms of $-\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}_{\underline{a},h}|_{\widetilde{y}=0}$.

One has $J(\underline{a},b,f,g) = -\partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} = e_{d+1}\cdot \widetilde{P}_{\underline{a}}\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0}$; hence, using the fact that $\widetilde{v}_{\underline{a},h}$ denotes the derivative of the mapping $a\mapsto \widetilde{\phi}^b_a$ at $\underline{a}$ applied to $h\in H^{k+3/2}(\mathbb{R}^d)$,

$$\begin{aligned} d_{\underline{a}}J(\cdot,b,f,g)\cdot h &= e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} + e_{d+1}\cdot \widetilde{P}_{\underline{a}}\nabla_{\widetilde{X},\widetilde{y}}\widetilde{v}_{\underline{a},h}|_{\widetilde{y}=0} \\ &= e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} - \partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}_{\underline{a},h}|_{\widetilde{y}=0} \\ &= e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} - \partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} - G(\underline{a},b)h\underline{\hat{Z}}. \end{aligned} \tag{1.26}$$

Lemma 1.19. *Under the assumptions and with the notation of the Theorem, one has*

$$e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} - \partial_n^{\widetilde{P}_{\underline{a}}}\widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} = -\begin{pmatrix}\nabla_X\\ 0\end{pmatrix}\cdot\left[hP\begin{pmatrix}\hat{\mathbf{v}}\\ \underline{\hat{Z}}\end{pmatrix}\right].$$

Proof. Since $P = \begin{pmatrix} P_1 & \mathbf{p}\\ \mathbf{p}^T & p_{d+1}\end{pmatrix}$ and by the Lemma 1.4 $\widetilde{P}$ has the form

$$\widetilde{P} = \frac{1}{\partial_{\widetilde{y}}s}\begin{pmatrix}\partial_{\widetilde{y}}sI_d & 0\\ -\nabla_{\widetilde{X}}s^T & 1\end{pmatrix} P \begin{pmatrix}\partial_{\widetilde{y}}sI_d & -\nabla_{\widetilde{X}}s\\ 0 & 1\end{pmatrix}.$$

Then the matrix $\widetilde{P}_{\underline{a}}$ can be written

$$\widetilde{P}_{\underline{a}} = \begin{pmatrix}\partial_{\widetilde{y}}s_{\underline{a}}P_1 & -P_1\nabla_{\widetilde{X}}s_{\underline{a}}+\mathbf{p}\\ (-P_1\nabla_{\widetilde{X}}s_{\underline{a}}+\mathbf{p})^T & \frac{1}{\partial_{\widetilde{y}}s_{\underline{a}}}(\nabla_{\widetilde{X}}s_{\underline{a}}\cdot P_1\nabla_{\widetilde{X}}s_{\underline{a}} + p_{d+1} - 2\mathbf{p}\cdot\nabla_{\widetilde{X}}s_{\underline{a}})\end{pmatrix},$$

and it follows that for any $h\in H^{k+3/2}(\mathbb{R}^d)$, the matrix $d_{\underline{a}}\widetilde{P}\cdot h$ is given by

$$\begin{pmatrix}\partial_{\widetilde{y}}(d_{\underline{a}}s\cdot h)P_1 & -P_1\nabla_{\widetilde{X}}(d_{\underline{a}}s\cdot h)\\ (-P_1\nabla_{\widetilde{X}}(d_{\underline{a}}s\cdot h))^T & \begin{matrix}\frac{1}{\partial_{\widetilde{y}}s_{\underline{a}}}(2\nabla_{\widetilde{X}}(d_{\underline{a}}s\cdot h)\cdot P_1\nabla_{\widetilde{X}}s_{\underline{a}} - 2\mathbf{p}\cdot\nabla_{\widetilde{X}}(d_{\underline{a}}s\cdot h))\\ -\frac{\partial_{\widetilde{y}}(d_{\underline{a}}s\cdot h)}{\partial_{\widetilde{y}}s_{\underline{a}}}\frac{\nabla_{\widetilde{X}}s_{\underline{a}}\cdot P_1\nabla_{\widetilde{X}}s_{\underline{a}}+p_{d+1}-2\mathbf{p}\cdot\nabla_{\widetilde{X}}s_{\underline{a}}}{\partial_{\widetilde{y}}s_{\underline{a}}}\end{matrix}\end{pmatrix}.$$

For simplicity, let us introduce the following notation: $\nabla_{\widetilde{X}}\equiv\nabla$, $E\equiv -P_1\nabla_X a+\mathbf{p}$. One has $d_{\underline{a}}s\cdot h|_{\widetilde{y}=0} = h$, $d_{\underline{a}}s\cdot h|_{\widetilde{y}=-1}=0$, $\partial_{\widetilde{y}}d_{\underline{a}}s\cdot h|_{\widetilde{y}=0} = h$ and $\partial_{\widetilde{y}}s_{\underline{a}}|_{\widetilde{y}=0} = \underline{a}-b$, therefore

$$\begin{aligned} e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} = \Big[&- P_1\nabla(d_{\underline{a}}s\cdot h)\cdot\nabla(\widetilde{\phi}^b_{\underline{a}}) \\ &+ \frac{1}{\partial_{\widetilde{y}}s_{\underline{a}}}(2\nabla(d_{\underline{a}}s\cdot h)\cdot P_1\nabla s_{\underline{a}} - 2\mathbf{p}\cdot\nabla(d_{\underline{a}}s\cdot h))\partial_{\widetilde{y}}\widetilde{\phi}^b_{\underline{a}} \\ &- \frac{\partial_{\widetilde{y}}(d_{\underline{a}}s\cdot h)}{\partial_{\widetilde{y}}s_{\underline{a}}}\frac{\nabla s_{\underline{a}}\cdot P_1\nabla s_{\underline{a}} + p_{d+1} - 2\mathbf{p}\cdot\nabla s_{\underline{a}}}{\partial_{\widetilde{y}}s_{\underline{a}}}\partial_{\widetilde{y}}\widetilde{\phi}^b_{\underline{a}}\Big]_{\widetilde{y}=0} \end{aligned}$$

and then

$$\begin{aligned} e_{d+1}\cdot d_{\underline{a}}\widetilde{P}\cdot h\nabla_{\widetilde{X},\widetilde{y}}\widetilde{\phi}^b_{\underline{a}}|_{\widetilde{y}=0} &= -P_1\nabla h\cdot\nabla f + (2\nabla h\cdot P_1\nabla\underline{a} - 2\mathbf{p}\cdot\nabla h)\underline{\hat{Z}} \\ &\quad - \frac{h}{\underline{a}-b}(PN\cdot P)\underline{\hat{Z}} \\ &= -P_1\nabla h\cdot\nabla f - 2E\cdot\nabla h\underline{\hat{Z}} - \frac{h}{\underline{a}-b}(PN\cdot P)\underline{\hat{Z}}. \end{aligned} \tag{1.27}$$

Now, recall that

$$\begin{aligned}
-\partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} &= e_{d+1} \cdot \widetilde{P}_{\underline{a}} \nabla_{\widetilde{X},\widetilde{y}} \widetilde{v}^b_{\underline{a},h}|_0 \\
&= (-P_1 \nabla_{\widetilde{X}} s_{\underline{a}} + \mathbf{p}) \nabla \widetilde{v}^b_{\underline{a},h}\Big|_0 + \frac{1}{\partial_{\widetilde{y}} s_{\underline{a}}|_0} (PN \cdot N) \partial_{\widetilde{y}} \widetilde{v}^b_{\underline{a},h}|_0 \\
&= E \nabla \widetilde{v}^b_{\underline{a},h}|_0 + \frac{1}{\underline{a} - b} (PN \cdot N) \partial_{\widetilde{y}} \widetilde{v}^b_{\underline{a},h}|_0.
\end{aligned}$$

Then, owing to Lemmas 1.15 and 1.18

$$\begin{aligned}
\nabla \widetilde{v}^b_{\underline{a},h}|_0 &= \nabla \Big(\frac{d_{\underline{a}} s \cdot h}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \Big) \Big|_0 \\
&= h \nabla \underline{\hat{Z}} + \underline{\hat{Z}} \nabla h,
\end{aligned}$$

and

$$\begin{aligned}
\partial_{\widetilde{y}} \widetilde{v}^b_{\underline{a},h}|_0 &= \partial_{\widetilde{y}} \Big(\frac{d_{\underline{a}} s \cdot h}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \Big) \Big|_0 \\
&= \Big[\partial_{\widetilde{y}} (d_{\underline{a}} s \cdot h) \frac{\partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}}{\partial_{\widetilde{y}} s_{\underline{a}}} \Big]_0 + \Big[d_{\underline{a}} s \cdot h \partial_{\widetilde{y}} \Big(\frac{1}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \Big) \Big]_0 \\
&= h \underline{\hat{Z}} + h \Big[\partial_{\widetilde{y}} \Big(\frac{1}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \Big) \Big]_0 .
\end{aligned}$$

Consequently

$$-\partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} = E h \nabla \underline{\hat{Z}} + E \underline{\hat{Z}} \nabla h + \frac{(PN \cdot N) h}{\underline{a} - b} \Big(\underline{\hat{Z}} + \Big[\partial_{\widetilde{y}} \Big(\frac{1}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \Big) \Big]_0 \Big).$$

According to $-\nabla_{\widetilde{X},\widetilde{y}} \cdot \widetilde{P}_a \nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}^b_a = 0$, solving for the term $\partial_{\widetilde{y}} \Big(\frac{1}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \Big)$ and evaluating at $\widetilde{y} = 0$ we get (for simplicity we write E and $PN \cdot N$ even if they are not evaluated at $\widetilde{y} = 0$)

$$\begin{aligned}
\Big[\partial_{\widetilde{y}} \Big(\frac{1}{\partial_{\widetilde{y}} s_{\underline{a}}} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}} \Big) \Big]_0 = &- \frac{1}{PN \cdot N} \nabla \cdot ((\underline{a} - b) P_1 \nabla f + E \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}|_0) \\
&- \frac{1}{PN \cdot N} \partial_{\widetilde{y}} (E \nabla \widetilde{\phi}^b_{\underline{a}})|_0 - \frac{1}{(\underline{a} - b)(PN \cdot N)} \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}|_0 \partial_{\widetilde{y}} (PN \cdot N)|_0,
\end{aligned}$$

therefore

$$\begin{aligned}
-\partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} = &E h \nabla \underline{\hat{Z}} + E \underline{\hat{Z}} \nabla h + \frac{1}{\underline{a} - b} (PN \cdot N) h \underline{\hat{Z}} \\
&- \frac{h}{\underline{a} - b} [\nabla \cdot ((\underline{a} - b) P_1 \nabla f + E \partial_{\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}|_0) + \partial_{\widetilde{y}} (E \nabla \widetilde{\phi}^b_{\underline{a}})|_0 + \underline{\hat{Z}} \partial_{\widetilde{y}} (PN \cdot N)|_0],
\end{aligned}$$

and from this last expression one has

$$-\partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^b_{\underline{a},h}|_{\widetilde{y}=0} = E \underline{\hat{Z}} \nabla h + \frac{1}{\underline{a} - b} (PN \cdot N) h \underline{\hat{Z}} - h \nabla \cdot P_1 \nabla f - h \nabla \cdot (E \underline{\hat{Z}}). \tag{1.28}$$

Finally, from (1.27) and (1.28) it follows

$$\begin{aligned} e_{d+1} \cdot d_{\underline{a}} \widetilde{P} \cdot h \nabla_{\widetilde{X},\widetilde{y}} \widetilde{\phi}^b_{\underline{a}}|_0 - \partial_n^{\widetilde{P}_{\underline{a}}} \widetilde{v}^b_{\underline{a},h}|_0 &= -E \cdot \nabla h \underline{\hat{Z}} - P_1 \nabla h \cdot \nabla f - h \nabla \cdot P_1 \nabla f \\ &\quad - h\nabla \cdot (E\underline{\hat{Z}}) \\ &= -\nabla \cdot (hE\underline{\hat{Z}} + hP_1 \nabla f) \\ &= -\nabla \cdot (h(P_1 \hat{\underline{\mathbf{v}}} + \underline{\hat{Z}}\mathbf{p})) \\ &= -\begin{pmatrix} \nabla \\ 0 \end{pmatrix} \cdot \left[hP \begin{pmatrix} \hat{\underline{\mathbf{v}}} \\ \underline{\hat{Z}} \end{pmatrix} \right]. \end{aligned}$$

□

The Theorem is then a consequence of (1.26) and Lemma 1.19. □

1.3 An Identifiability result for water waves

The purpose of this section is to prove Theorem 1.2; that is, given two different bottoms b_1, b_2, there correspond different surface measurements $\zeta, \psi, \partial_t \zeta$ on any open subset $S \subset \mathbb{R}^d$ and at a single time $t = t_0 > 0$. The proof relies heavily on the unique continuation principle for harmonic functions, which holds even when the domain is unbounded.

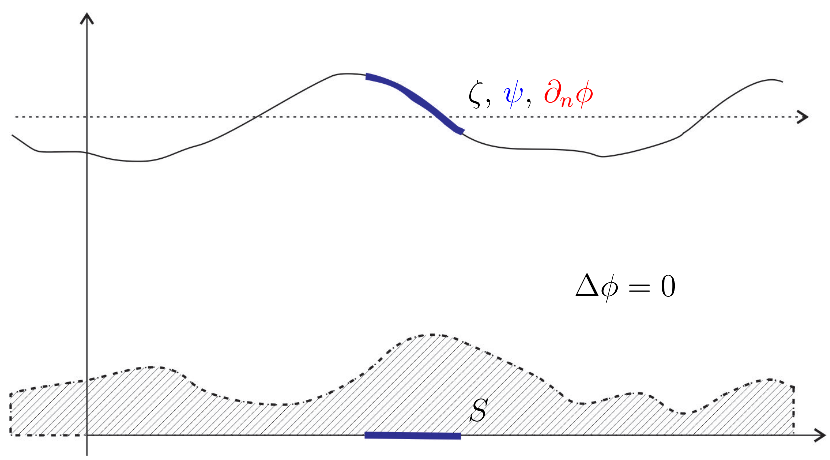

Figure 1.2: Scheme for the inverse problem

Assume first that we know the functions ζ, ψ and ζ_t on a region $S \times \{t_0\}$. The general water-waves model on the surface is given by system (2.9) and it is related to the velocity potential on the interior of the domain through the elliptic problem (1.6).

Throughout this section, we assume that ϕ, the velocity potential, is not constant.

Remark 1.20. This assumption excludes, among others, the still water case in which it is not possible to determine (uniquely) the bottom shape. □

In the statement of the theorem we have assumed the existence of a solution of the problem (2.9) in certain Sobolev spaces which we will present below in section 3. It is important to highlight that the measures are performed only on an open subset of $\mathbb{R}^d$ and a single time

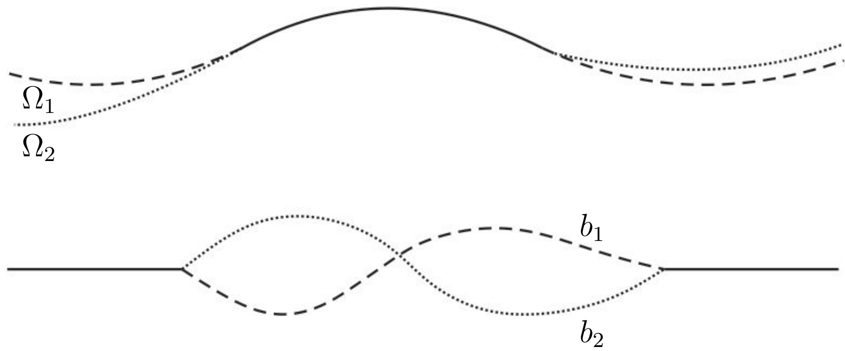

Figure 1.3: Two different bottoms geometry

t_0. As theorem shows, that information allows to identify the bottom uniquely in the whole space $\mathbb{R}^d$.

The proof is as follows. Assume that $b_1 \neq b_2$ on an open subset of $\mathbb{R}^d$ as in Figure 1.3. If we denote $\Omega^i = \{(X, y) \in \mathbb{R}^{d+1} : b_i(X) < y < \zeta_i(X)\}$ then, inside the domains, if ϕ_1, ϕ_2 are the velocities on Ω^1, Ω^2, respectively, we have the following boundary value problems

$$\left\{ \begin{array}{ll} \Delta\phi_1 = 0, \quad \Omega^1, & \Delta\phi_2 = 0, \quad \Omega^2, \\ \phi_1|_{y=\zeta_1(t_0,S)} = \psi_1, & \phi_2|_{y=\zeta_2(t_0,S)} = \psi_2, \\ \partial_n\phi_1|_{y=b_1(S)} = 0, & \partial_n\phi_2|_{y=b_2(S)} = 0. \end{array} \right. \tag{1.29}$$

By the hypothesis of the theorem, if we write $\phi := \phi_1 - \phi_2$ and $\zeta = \zeta_1(t_0, S) = \zeta_2(t_0, S)$, on the intersection of the two domains we have the problem

$$\left\{ \begin{array}{rll} \Delta\phi & = 0, & \Omega^1 \cap \Omega^2 \\ \phi|_{y=\zeta(S)} & = 0, & \\ \partial_n\phi|_{b_1(S)\vee b_2(S)} & = -\partial_n\phi_2|_{b_1(S)} \text{ or } \partial_n\phi_1|_{b_2(S)}. & \end{array} \right. \tag{1.30}$$

Even more, by the first equation of (2.9) and the condition $\partial_t\zeta_1 = \partial_t\zeta_2$ on $t = t_0$ one has, on S, $G(\zeta, b_1)\psi = G(\zeta, b_2)\psi$. Thus

$$\sqrt{1 + |\nabla\zeta_1|^2}\partial_n\phi_1|_{\zeta_1} = \sqrt{1 + |\nabla\zeta_2|^2}\partial_n\phi_2|_{\zeta_2} \quad \forall X \in S.$$

Then for the problem (1.30) we also have the Neumann condition on $y = \zeta(X)$, $\forall X \in S$:

$$\partial_n\phi|_{y=\zeta(X)} = 0 \quad \forall X \in S.$$

By the unique continuation principle for Laplace equation one has $\phi = 0$ on $\Omega^1 \cap \Omega^2$.
Thus we have

$$\phi = 0 \quad \text{on } \partial(\Omega^1 \cap \Omega^2).$$

Let $D = (\Omega^1 \cup \Omega^2) \setminus (\overline{\Omega^1 \cap \Omega^2})$. Since we are assuming D is not empty, let D_C a connected component of D, say $D_C \subset \Omega^2 \setminus (\overline{\Omega^1 \cap \Omega^2})$ as in Figure 1.4. One has that $\partial D_C = O_1 \cup O_2$ with O_i open sets such that $O_i \subset \{(X, y) \in \mathbb{R}^{d+1} : y = b_i(X)\}$. From the argument above, it follows that, on O_1, $\partial_n\phi_2 = \partial_n\phi_1 = 0$ by the Neumann condition on the bottom. Moreover, on O_2, $\partial_n\phi_2 = 0$.

Then on D_C, $\Delta\phi_2 = 0$ and $\partial_n\phi_2|_{\partial D_C} = 0$. Therefore $\phi_2 = C_2$ on D_C, where C_2 is a constant, and then $\phi_2 = C_2$ on Ω^2.

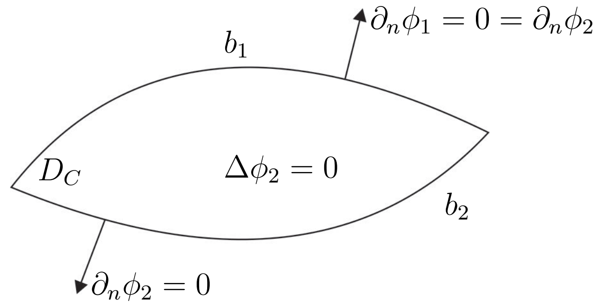

Figure 1.4: Open subset of $(\Omega^1 \cup \Omega^2) \smallsetminus (\overline{\Omega^1 \cap \Omega^2})$

Similarly, $\phi_1 = C_1$ on Ω^1, with C_1 a constant, and since that $\phi = 0$ on $\{(X, y) \in \mathbb{R}^{d+1} : y = \zeta_1(X) = \zeta_2(X),\ \forall X \in S\}$ one has $\phi_1 = \phi_2 = C$. Now, if $\phi_1 = \phi_2 = C$ then $u_1 = \nabla\phi_1 = 0 = \nabla\phi_2 = u_2$, a case that we are not considering because it corresponds to still water. Therefore $b_1 = b_2$. This completes the proof.

As was mentioned in the introduction, we can measure on the free surface the quantities $\zeta(t, X)$ on S and a small interval of time $(0, t^*)$, and $\psi(t_0, X)$ on S and $t_0 \in (0, t^*)$, instead $\psi, \zeta|_{t_0}, \partial_t\zeta|_{t_0}$ on S; indeed once we compute $\partial_t\zeta$, by the first equation in (2.9) we know the Neumann data on the free surface, namely $\partial_n\phi(X)$ on S. If one follows the same lines of the proof above, for $\phi = \phi_1 - \phi_2$ one has $\Delta\phi = 0$ on $\Omega^1 \cap \Omega^2$ together with homogeneous Dirichlet and Neumann conditions $\phi = 0$, $\partial_n\phi = 0$ on S and the same proof holds. The result can be stated as:

Theorem 1.21. *Let $T > 0$ and $s \in \mathbb{R}^+$, depending only on d. Assume that for $j = 1, 2$, $(\zeta_j, \psi_j) \in C^1([0, T]; H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d))$ are solutions of (2.9), with $b_1, b_2 \in H^s(\mathbb{R}^d)$ and there exists $h_{min} > 0$, such that, for all $X \in \mathbb{R}^d$ and $t \in (0, T)$,*

$$\zeta_j(t, X) - b_j(X) \geq h_{min}, \quad j = 1, 2.$$

Let S be an open subset of $\mathbb{R}^d$ and $t_0 \in (0, t^*) \subset (0, T)$ a given time. If $\forall X \in S$, $\forall t \in (0, t^*)$,

$$\zeta_1(t, X) = \zeta_2(t, X),$$

and, $\forall X \in S$,

$$\psi_1(t_0, X) = \psi_2(t_0, X),$$

then

$$b_1(X) = b_2(X) \quad \forall X \in \mathbb{R}^d.$$

1.4 Existence of minimizers

We note that from the result of identifiability, one can view our inverse problem as a minimization problem. In what follows, we will prove the existence of minimizers for this minimization problem; this is a classical issue, namely, to use the identification problem to prove the uniqueness of an optimal control problem.

In fact, given a target function $\tau(X) \in L^2(\mathbb{R}^d)$, if $s \in \mathbb{R}^+$ depends only on d, we consider the cost functional to be minimized $F : H^s(\mathbb{R}^d) \to \mathbb{R}$, defined by

$$F(b) = \frac{1}{2}\int_{\mathbb{R}^d} \Big|\partial_t \zeta(X)|_{t=t_0} - \tau(X)\Big|^2 dX. \tag{1.31}$$

Later in theorem 1.23 one actually has that $\tau \in H^{d_0+2}(\mathbb{R}^d)$, $d_0 > \frac{d}{2}$; the choice we have made here is just for giving a sense to the functional F.

The aim of this section is to prove that, under certain conditions on the set of admissible bottoms $\mathcal{B}_{ad}$, there exists at least one minimizer of the functional F; even more, this minimum turns out to be unique in a certain Sobolev space thanks to the identifiability result addressed in Section 2. Let us take $N \in \mathbb{N}$, $N \geq d_0 + max\{d_0, 2\} + 3/2$, with $d_0 > \frac{d}{2}$. Thus, we consider the following class of admissible bottoms $\mathcal{B}_{ad}$,

$$\mathcal{B}_{ad} = \{f \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d) : \mathrm{supp}(f) \subset K, |f|_{H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)} \leq C\}, \tag{1.32}$$

where $K \subset \mathbb{R}^d$ is a given compact set and $C > 0$ is a constant. Let us remark that it is possible to take $N \geq 5$ in the cases $d = 1, 2$.

The optimization problem can be stated as: *Find $b^{min} \in \mathcal{B}_{ad}$, such that,*

$$F(b^{min}) = \min_{b \in \mathcal{B}_{ad}} F(b).$$

We define the space $\dot{H}^2(\mathbb{R}^d) = \{f \in L^2_{loc}(\mathbb{R}^d) : \nabla_X f \in H^1(\mathbb{R}^d)^d\}$. Using the shape derivative method with respect to bottom variations as in [29, 30], it is possible to show the analytic dependence of ζ and ψ (and hence of the functional F) with respect to bottom variations of b, which will allow us to take the limit process of a sequence of bottoms inside the functional F.

Remark 1.22. Initially one defines the Dirichlet-Neumann operator $G(\zeta, b)$ from $\dot{H}^{3/2}(\mathbb{R}^d)$ into $H^{1/2}(\mathbb{R}^d)$; but, it can be shown that $G(\zeta, b)$ has a self-adjoint realization on $L^2(\mathbb{R}^d)$ with domain $H^1(\mathbb{R}^d)$ (See [30], Section 3.1). □

Before giving the proof of theorem (1.3), we are going to state some previous results in order to establish the analytic dependence of the Dirichlet-Neumann operator, and hence of the functional F, with respect to bottom variations; once we take a minimizing sequence, this dependence will allow us to perform the limit process inside of F.

1.4.1 Linearization of water-waves equations

Before stating the main theorem, let us recall some results related to the analyticity of the Dirichlet-Neumann operator and its continuous dependence with respect to bottom variations as well as the well-posedness of the water-waves equations (2.9).

Firstly note that the system (2.9) can be written in condensed form as

$$\partial_t U + \mathcal{F}(U) = 0, \tag{1.33}$$

with $U = (\zeta, \psi)^T$ and

$$\mathcal{F}(U) = \Big(- G(\zeta)\psi,\ g\zeta + \frac{1}{2}|\nabla_X \psi|^2 - \frac{(G(\zeta)\psi + \nabla_X\zeta \cdot \nabla_X \psi)^2}{2(1 + |\nabla_X \zeta|^2)}\Big)^T.$$

Let us write

$$\underline{Z} = \underline{Z}(U) = \frac{G(\zeta)\psi + \nabla_X\zeta \cdot \nabla_X\psi}{1 + |\nabla_X\zeta|^2} \quad \text{and} \quad \underline{V} = \underline{V}(U) = \nabla_X\psi - \underline{Z}\nabla_X\zeta.$$

Following Lannes [30], for $N \in \mathbb{N}$, one consider the following quantity

$$\mathcal{E}^N(U) = |B\psi|^2_{H^{d_0+3/2}} + \sum_{\alpha\in\mathbb{N}^d, |\alpha|\leq N} |\zeta_{(\alpha)}|_2^2 + |B\psi_{(\alpha)}|_2^2, \tag{1.34}$$

with

$$\zeta_{(\alpha)} = \partial^\alpha\zeta, \quad \psi_{(\alpha)} = \partial^\alpha\psi - \underline{Z}\partial^\alpha\zeta$$

and B the Fourier multiplier, $B = \dfrac{|D|}{(1+|D|)^{1/2}}$.

Associated to the energy (1.34), one denotes by E_T^N the functional space

$$E_T^N = \{U \in C([0,T]; H^{d_0+2} \times \dot{H}^2(\mathbb{R}^d)) : \mathcal{E}^N(U(\cdot)) \in L^\infty([0,T])\}$$

where one denotes by $\dot{H}^{s+1}(\mathbb{R}^d)$ the topological vector space

$$\dot{H}^{s+1}(\mathbb{R}^d) = \{f \in L^2_{loc}(\mathbb{R}^d) : \nabla_X f \in H^s(\mathbb{R}^d)^d\}$$

endowed with the (semi)norm

$$|f|_{\dot{H}^{s+1}(\mathbb{R}^d)} = |\nabla_X f|_{H^s(\mathbb{R}^d)}.$$

The following quantity plays a very important role in the well-posedness theorem below, see [30] Section 4.3.5 for a physical interpretation of this quantity,

$$\underline{\mathfrak{a}} = \mathfrak{a}(U) = g + \partial_t\underline{Z} + \underline{V}\cdot\nabla_X\underline{Z}.$$

Theorem 1.23. *(Local existence, [30], Theorem 4.16) Let $d_0 > \frac{d}{2}$ and $N \geq d_0 + max\{d_0, 2\} + 3/2$. Then let $U_0 = (\zeta_0, \psi_0) \in E_0^N$, $b \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$, and moreover assume that*

$$\exists h_{min} > 0,\ \exists a_0 > 0, \quad \zeta_0(X) - b(X) \geq h_{min} \text{ and } \mathfrak{a}(U_0) \geq a_0.$$

Then there exists $T > 0$ and a unique solution $U \in E_T^N$ to (2.9) with initial data U_0.

Remark 1.24. Notice that for $d = 2$ it is possible to take $d_0 = 3/2$ and $N = 5$ in the statement above.

Let us denote by $\mathbf{\Gamma}$ the set of all $(\zeta, b) \in H^{d_0+1}(\mathbb{R}^d)^2$ such that there exists $h_{min} > 0$ with $\zeta(X) - b(X) \geq h_{min}$ $\forall X \in \mathbb{R}^d$. Let $0 \leq s \leq d_0 + 1/2$ ($d_0 > d/2$). For $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$ one define the mapping

$$G(\cdot,\cdot) : \begin{array}{ccc} \mathbf{\Gamma} \subset H^{d_0+1}(\mathbb{R}^d)^2 & \to & H^{s-1/2}(\mathbb{R}^d) \\ \Gamma = (\zeta, b) & \mapsto & G(\zeta, b)\psi. \end{array}$$

□

The next theorem shows that de Dirichlet-Neumann operator G is analytic and gives a formula for the first-order partial derivative with respect to ζ.

Theorem 1.25. *([30], Theorem 3.21) Let $d_0 > d/2$. Then,*

1. *For all $0 \leq s \leq d_0 + 1/2$ and $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, the mapping $G(\cdot,\cdot)$ is analytic.*
2. *Let $\Gamma = (\zeta, b) \in \mathbf{\Gamma}$ and $\psi \in \dot{H}^{3/2}(\mathbb{R}^d)$. Then for all $h \in H^{d_0+1}(\mathbb{R}^d)$, one has*

$$d_\zeta G(\cdot, b)\psi \cdot h = -G(\zeta, b)(h\underline{Z}) - \nabla_X \cdot (h\underline{V}).$$

Remark 1.26. By definition, the linearized operator $\mathcal{L}$ associated to (1.33) is given by

$$\mathcal{L} = \partial_t + d_U\mathcal{F},$$

with

$$d_U\mathcal{F} = \begin{pmatrix} G(\zeta,b)(\underline{Z}\cdot) + \nabla_X \cdot (\cdot\underline{V}) & -G(\zeta,b)\cdot \\ \underline{Z}G(\zeta,b)(\underline{Z}\cdot) + \underline{Z}\nabla_X\cdot\underline{V} + g & \underline{V}\cdot\nabla_X\cdot - \underline{Z}G(\zeta,b)\cdot \end{pmatrix}. \tag{1.35}$$

If one considers the change of unknowns $V = (\zeta, \psi - \underline{Z}\zeta)^T$, and if $D = (D_1, D_2)^T$ then from the equation $\mathcal{L}U = D$, we obtain $\mathcal{M}V = H$ with

$$\mathcal{M} := \partial_t + \begin{pmatrix} \nabla_X \cdot (\cdot\underline{V}) & -G(\zeta,b)\cdot \\ \underline{\mathfrak{a}} & \underline{V}\cdot\nabla_X \end{pmatrix}, \quad H = \begin{pmatrix} D_1 \\ D_2 - \underline{Z}D_1 \end{pmatrix},$$

and one has the following proposition related to the linear water-waves equations. □

Proposition 1.27. *([29], Proposition 4.2) Let $T > 0$ and $D = (D_1, D_2)^T \in C^2([0,T]; H^{d_0+2} \times \dot{H}^2(\mathbb{R}^d))$.*
The following two assertions are equivalent:

(i) $U = (\zeta, \psi)^T$ solves $\mathcal{L}U = D$ on $[0,T] \times \mathbb{R}^d$;

(ii) V solves $\mathcal{M}V = H$ on $[0,T] \times \mathbb{R}^d$.

Finally we recall the proposition below that allows to linearize the water-waves equations (2.9). Let us introduce the matrix operators

$$\mathfrak{A}(U) = \begin{pmatrix} 0 & -G(\zeta,b)\cdot \\ \underline{\mathfrak{a}} & 0 \end{pmatrix}, \quad \mathfrak{B}(U) = \begin{pmatrix} \nabla_X\cdot(\cdot\underline{V}) & 0 \\ 0 & \underline{V}\cdot\nabla_X \end{pmatrix}.$$

Proposition 1.28. *([30], Proposition 4.10) Let $T > 0$, $d_0 > d/2$ and $N \geq d_0 + max\{d_0, 2\} + 3/2$. If $U = (\zeta, \psi) \in E_T^N$ and $b \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$ satisfy $\zeta(X) - b(X) \geq h_{min}$ uniformly on $[0,T]$ and solve (2.9), then for all $\alpha \in \mathbb{N}^d$ with $1 \leq |\alpha| \leq N$, the couple $V_{(\alpha)} = (\zeta_{(\alpha)}, \psi_{(\alpha)})$ solves*

$$\partial_t V_{(\alpha)} + \mathfrak{A}(U)V_{(\alpha)} = (R_\alpha, S_\alpha)^T, \quad (|\alpha| < N),$$
$$\partial_t V_{(\alpha)} + \mathfrak{A}(U)V_{(\alpha)} + \mathfrak{B}(U)V_{(\alpha)} = (R_\alpha, S_\alpha)^T, \quad (|\alpha| = N),$$

where the residuals R_α and S_α satisfy the estimates

$$|R_\alpha|_2 + |BS_\alpha|_2 \leq C\Big(\frac{1}{h_{min}}, |\zeta|_{H^{d_0+2}}, |b|_{H^{d_0+2}}, |b|_{H^{N+max\{d_0,1\}+1}}, \mathcal{E}^N(U)\Big)$$

uniformly on $[0,T]$.

We end this section recalling the Implicit Function Theorem in Banach spaces

Theorem 1.29. *Let X, Y, Z be Banach spaces. Let the mapping $f : X \times Y \to Z$ be continuously Frechet differentiable. If $(x_0, y_0) \in X \times Y$, $f(x_0, y_0) = 0$, and $y \mapsto Df(x_0, y_0)(0, y)$ is a Banach space isomorphism from Y onto Z, then there exist neighborhoods U of x_0 and V of y_0 and a Frechet differentiable function $g : U \to V$ such that $f(x, g(x)) = 0$ and $f(x, y) = 0$ if and only if $y = g(x)$, for all $(x, y) \in U \times V$.*

We have then the continuous dependence of the Dirichlet-Neumann operator with respect to bottom variations applying the Implicit function theorem, Theorems 1.23, 1.25 and Proposition 1.28, to the mapping

$$f: \begin{array}{cl} H^{N+\max\{d_0,1\}+1} \times C([0,T]; H^{d_0+2} \times \dot{H}^2(\mathbb{R}^d)) & \to C^2([0,T]; H^{d_0+2} \times \dot{H}^2(\mathbb{R}^d)) \\ (b, \zeta, \psi) & \mapsto f(b, \zeta, \psi) = \partial_t U + \mathcal{F}(U) \end{array}.$$

1.4.2 Existence of a minimizer

With the results obtained in the last two sections, we are going to prove the theorem of existence of a minimizer for the functional F in the set of admissible bottoms $\mathcal{B}_{ad}$; actually the proof shows that the set $\mathcal{B}_{ad}$ is a compact set.

Let $\{b_n\} \subset \mathcal{B}_{ad}$ be a minimizing sequence of F. Then $\{b_n\}$ is bounded in $H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$ and there exists a subsequence, still denoted by $\{b_n\}$, such that $b_n \rightharpoonup \bar{b}$ weakly in $H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$, with $\bar{b} \in \mathcal{B}_{ad}$. Even more, thanks to the injection $H^s(\mathbb{R}^d) \hookrightarrow B^k(\mathbb{R}^d) := \{f \in C^k(\mathbb{R}^d) : \lim_{|\alpha|\to\infty} |D^\alpha f| = 0, |\alpha| \le k\}$, with $s > \frac{d}{2} + k$ and by the compactness of the embedding from $W^{1,\infty}(K)$ into $L^\infty(K)$ one has, up to the extraction of a subsequence, $b_n \to \bar{b}$ strongly in $L^\infty(K)$.

Let ϕ_n and $\bar{\phi}$ be the solutions of (1.6) with bottom b_n and $\bar{b}$ respectively. By the analytic dependence of the Dirichlet-Neumann operator with respect to bottom variations (see Theorem (1.29)) we have

$$G_n(\zeta, b)\psi = G(\zeta, b_n)\psi \to G(\zeta, \bar{b})\psi, \quad \text{strongly in } L^2(\mathbb{R}^d),$$

thus

$$\inf_{b \in \mathcal{B}_{ad}} F(b) = \lim_{n\to\infty} F(b_n) = F(\bar{b}),$$

and we conclude that $\bar{b}$ is a minimizer of F. Uniqueness of this minimum follows immediately from the identifiability Theorem 1.2.

1.5 Shape differentiation

In this section we use the shape differentiation method [6, 45] to compute the shape derivative of the functional F with respect to the bottom. Let us assume ζ_0, b to be known and satisfy the condition

$$\exists h_{min} > 0, \forall X \in \mathbb{R}^d,\ \zeta_0(X) - b(X) \ge h_{min}. \tag{1.36}$$

We also use the notation

$$\Gamma_{\zeta_0} = \{(X, y) \in \mathbb{R}^{d+1} : y = \zeta_0(X)\},$$

and

$$\Gamma_b = \{(X,y) \in \mathbb{R}^{d+1} : y = b(X)\},$$

then we consider the functional F, defined by

$$F(b) = \frac{1}{2}\int_{\mathbb{R}^d} \left|\partial_t \zeta(X)|_{t=t_0} - \tau(X)\right|^2 dX, \tag{1.37}$$

where $\zeta(X,t)$ is the unique solution to the water-waves equation on $[0,T]$ (see [29])and $\tau \in L^2(\mathbb{R}^d)$ is a given target function. If we assume ψ_0 to be known, then by the first equation in (2.9) one has

$$F(b) = \frac{1}{2}\int_{\mathbb{R}^d} |G(\zeta_0, b)\psi_0 - \tau(X)|^2 dX,$$

with ϕ, the velocity potential, satisfying the following elliptic problem on the domain Ω_0:

$$\begin{cases} \Delta\phi = 0, \quad \Omega_0 \\ \phi|_{\zeta_0(X)} = \psi_0, \quad \partial_n \phi|_{b(X)} = 0, \end{cases} \tag{1.38}$$

and

$$G(\zeta_0, b)\psi_0 = \sqrt{1 + |\nabla_X \zeta_0|^2}\, \frac{\partial \phi}{\partial n}\Big|_{\zeta_0} \tag{1.39}$$

being the corresponding Dirichlet-Neumann operator.

Theorem 1.30. *Let $d_0 > \frac{d}{2}$ and $N \geq d_0 + max\{d_0, 2\} + 3/2$. Suppose that $\underline{b} \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$ and $(\zeta_0, \psi_0) \in E_0^N$ satisfy (1.36). Then there exists a neighborhood $V_{\underline{b}}$ of $\underline{b} \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$, such that, for all given $b \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$, the mapping*

$$b \in V_{\underline{b}} \subset H^{N+max\{d_0,1\}+1}(\mathbb{R}^d) \mapsto F(b) \in \mathbb{R}$$

is well defined and differentiable. Moreover, for all $h \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$, one has

$$F'(b) \cdot h = -\int_{\Gamma_b} h \frac{\nabla\phi \cdot \nabla\psi}{\sqrt{1 + |\nabla_X b|^2}}$$

with ψ being the solution of

$$\begin{cases} \Delta\psi = 0, \quad \Omega_0 \\ \psi|_{\zeta_0} = G - \tau, \\ \frac{\partial\psi}{\partial n}\Big|_b = 0. \end{cases} \tag{1.40}$$

Proof. Using the shape differentiation method (see [6],[45]), if $\delta > 0$, $u \in W^{1,\infty}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})$ and $\phi_\delta = \phi(\Omega + \delta u)$ is the solution of the problem

$$\begin{cases} \Delta\phi_\delta = 0, \quad \Omega(\delta) \\ \phi_\delta|_{\zeta_0} = \psi_0, \\ \dfrac{\partial\phi_\delta}{\partial n_\delta}\Big|_{b_\delta} = 0, \end{cases} \tag{1.41}$$

with $\Omega(\delta) = \Omega_0 + \delta u$, $b_\delta = b + \delta u$, then writing $\phi' = lim_{\delta\to 0}\dfrac{\phi_\delta - \phi_0}{\delta}$ one has

$$\begin{cases} \Delta\phi' = 0, \quad \Omega \\ \phi'|_{\zeta_0} = 0, \\ \left.\dfrac{\partial\phi'}{\partial n}\right|_b = \nabla_{\partial\Omega}(u_n)\cdot\nabla\phi - u_n\dfrac{\partial^2\phi}{\partial n^2}. \end{cases} \tag{1.42}$$

Now, if $u = (0, h(X))^T \in \mathbb{R}^{d+1}$, $b_\delta(X) = b(X) + \delta h(X)$, then one has

$$(X, b_\delta(X)) = (X, b(X)) + \delta(0, h(X)).$$

Thus

$$u_n = u\cdot n = \frac{1}{\sqrt{1+|\nabla_X b|^2}}\begin{pmatrix} 0 \\ h(X)\end{pmatrix}\cdot\begin{pmatrix}\nabla_X b \\ -1\end{pmatrix} = -\frac{h(X)}{\sqrt{1+|\nabla_X b|^2}}. \tag{1.43}$$

By (1.37),

$$\begin{aligned} F'(b)\cdot h &= \int_{\mathbb{R}^d}(G(\zeta_0,b)\psi_0 - \tau)\sqrt{1+|\nabla_X\zeta_0|^2}\left.\frac{\partial\phi'}{\partial n}\right|_{\zeta_0} dX \\ &= \int_{\Gamma_{\zeta_0}}(G-\tau)\frac{\partial\phi'}{\partial n}dA, \end{aligned} \tag{1.44}$$

then if one considers ψ be the solution of the boundary problem

$$\begin{cases} \Delta\psi = 0, \quad \Omega \\ \psi|_{\zeta_0} = G - \tau, \\ \left.\dfrac{\partial\psi}{\partial n}\right|_b = 0, \end{cases} \tag{1.45}$$

one has

$$\int_{\partial\Omega}\psi\frac{\partial\phi'}{\partial n}dA = \int_{\partial\Omega}\phi'\frac{\partial\psi}{\partial n}dA$$

and therefore

$$\int_{\Gamma_{\zeta_0}}(G-\tau)\frac{\partial\phi'}{\partial n}dA + \int_{\Gamma_b}\psi\frac{\partial\phi'}{\partial n}dA = 0.$$

Thus one finally has

$$\begin{aligned} F'(b)\cdot h &= -\int_{\Gamma_b}\psi\frac{\partial\phi'}{\partial n}dA \\ &= \int_{\Gamma_b}\psi\Big[\nabla_{\partial\Omega}\Big(\frac{h}{\sqrt{1+|\nabla_X b|^2}}\Big)\cdot\nabla_X\phi - \frac{h}{\sqrt{1+|\nabla_X b|^2}}\frac{\partial^2\phi}{\partial n^2}\Big]dA. \end{aligned}$$

By the chain rule

$$w := (\nabla_X\phi)|_b = \nabla_X(\phi|_b) - (\partial_y\phi)|_b\nabla_X b \tag{1.46}$$

and since $\frac{\partial\phi}{\partial n} = 0$ on Γ_b, integrating by parts and denoting $\theta = \theta(X) := \sqrt{1+|\nabla_X b|^2}$, one has

$$\begin{aligned} F'(b)\cdot h &= -\int_{\mathbb{R}^d}\frac{h}{\theta}\nabla_X\cdot(\psi w\theta)dX - \int_{\mathbb{R}^d}h\psi\left.\frac{\partial^2\phi}{\partial n^2}\right|_{\Gamma_b}dX \\ &= -\int_{\mathbb{R}^d}\frac{h}{\theta}w\cdot\nabla_X(\psi\theta)dX - \int_{\mathbb{R}^d}h\psi\Big(\nabla_X\cdot w + \left.\frac{\partial^2\phi}{\partial n^2}\right|_b\Big)dX; \end{aligned} \tag{1.47}$$

here we use the convention $\psi = \psi|_b$.

To simplify the last expression let us write $\dfrac{\partial^2\phi}{\partial n^2}$ on Γ_b in terms of w as follows:

$$\frac{\partial^2\phi}{\partial n^2} = \frac{1}{\theta^2}(\nabla_X b^T H_X\phi\nabla_X b - 2\nabla_X(\partial_y\phi)\cdot\nabla_X b + \partial_y^2\phi) \tag{1.48}$$

where $H\phi$ represents the Hessian matrix of ϕ. By (1.46) and the chain rule, on Γ_b, one has

$$\begin{aligned}\nabla_X\cdot w &= \nabla_X\cdot(\nabla_X\phi|_b)\\ &= (\Delta_X\phi)|_b + (\partial_y\nabla_X\phi)|_b\cdot\nabla_X b\\ &= -(\partial_y^2\phi)|_b + (\partial_y\nabla_X\phi)|_b\cdot\nabla_X b.\end{aligned} \tag{1.49}$$

Also, $\left.\dfrac{\partial\phi}{\partial n}\right|_b = 0$ means that

$$(\nabla_X\phi)|_b\cdot\nabla_X b = (\partial_y\phi)|_b,$$

so that

$$\nabla_X[(\nabla_X\phi)|_b\cdot\nabla_X b] = \nabla_X[(\partial_y\phi)|_b]$$

and therefore

$$((H_X\phi)|_b + (\partial_y\nabla_X\phi)|_b\cdot\nabla_X b)\nabla_X b + H_X b(\nabla_X\phi)|_b = (\nabla_X\partial_y\phi)|_b + (\partial_y^2\phi)|_b\nabla_X b.$$

If one considers the dot product of this last equality with $\nabla_X b$ one gets

$$\nabla_X b^T H_X\phi\nabla_X b - \nabla_X b\cdot(\nabla_X\partial_y\phi)|_b = -\left[(\nabla_X\partial_y\phi)|_b\cdot\nabla_X b - (\partial_y^2\phi)|_b\right]|\nabla_X b|^2 - \nabla_X b^T H_X b w.$$

Thus, by (1.49) one has

$$\nabla_X b^T H_X\phi\nabla_X b - \nabla_X b\cdot(\nabla_X\partial_y\phi)|_b = -\nabla_X\cdot w|\nabla_X b|^2 - \nabla_X b^T H_X b w \tag{1.50}$$

and finally replacing (1.49) and (1.50) in (1.48)

$$\begin{aligned}\frac{\partial^2\phi}{\partial n^2} &= \tfrac{1}{\theta^2}(-\nabla_X\cdot w\theta^2 - \nabla_X b^T H_X b w)\\ &= -\nabla_X\cdot w - \tfrac{1}{\theta}\nabla_X\theta\cdot w.\end{aligned} \tag{1.51}$$

Then, by (1.47)

$$\begin{aligned}F'(b)\cdot h &= -\int_{\mathbb{R}^d}\frac{h}{\theta}w\cdot\nabla_X(\psi\theta) + \int_{\mathbb{R}^d}\frac{h\psi}{\theta}w\cdot\nabla_X\theta\\ &= -\int_{\mathbb{R}^d} hw\cdot\nabla_X\psi\\ &= \int_{\mathbb{R}^d}\psi\cdot\nabla_X(hw)\end{aligned} \tag{1.52}$$

where $\psi = \psi|_b$, and the proof is complete. □

As a final remark and using that $\left.\dfrac{\partial\phi}{\partial n}\right|_b = 0 = \left.\dfrac{\partial\psi}{\partial n}\right|_b$, by (1.46)

$$\begin{aligned} w\cdot\nabla_X(\psi|_b) &= w\cdot((\nabla_X\psi)|_b + (\partial_y\psi)|_b\nabla_X b) \\ &= \nabla_X\phi|_b\cdot\nabla_X\psi|_b + (\partial_y\psi)|_b(\nabla_X\phi)|_b\cdot\nabla_X b \\ &= \nabla_X\phi|_b\cdot\nabla_X\psi|_b + (\partial_y\psi)|_b(\partial_y\phi)|_b \\ &= [\nabla\phi\cdot\nabla\psi]_b\,. \end{aligned} \tag{1.53}$$

Therefore, one gets an expression that provides a way to compute a descent direction for the functional F,

$$F'(b)\cdot h = -\int_{\Gamma_b} h\frac{\nabla\phi\cdot\nabla\psi}{\sqrt{1+|\nabla_X b|^2}}.$$

Chapter 2

Stationary shapes for 2-d water-waves and hydraulic jumps

This chapter is focused on the existence of bifurcation branches of solutions for the problem

$$\mathcal{F}(\eta, F) = \eta + F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}^2_{x'} + \widetilde{\psi}^2_{y'}) - \epsilon^2\eta_x\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} + \frac{\epsilon^3}{2}\eta^2_x\widetilde{\psi}^2_{y'},$$

where

$$\begin{cases} \Delta\widetilde{\psi} = \epsilon G, & (-L, L) \times (0, 1), \\ \widetilde{\psi}_{x'} = 0, & x' = -L, L, \\ \widetilde{\psi} = 0, & y' = 0, \\ \widetilde{\psi} = -F\eta, & y' = 1. \end{cases}$$

2.1 Introduction

The water-waves problem for an ideal, incompressible, irrotational fluid under the influence of gravity consists of describing the motion of the free surface and the velocity field of a layer of fluid delimited below by a solid, not necessary flat, bottom; is described by means of the Bernoulli equation, together with an impermeability condition on the bottom. In this paper we prove the existence of bifurcations for two dimensional stationary water-waves in the case of a flat bottom. In fact, assuming upstream uniform horizontal velocity $U \geq 0$ and depth H, we were able to prove existence of bifurcations when a nondimensional parameter, called Froude number [27],

$$F = \frac{U}{\sqrt{gH}},$$

appearing in the Bernoulli equation and characterizing the flow, is such that $F < 1$, (in fact $F = 1 + O(L^{-2})$, with $2L$ the length of the interval in the x variable), see Figure 2.1.

As far as we know, existence of hydraulic jump has not been studied on water-waves equations; bifurcation problems have been studied for capillary gravity waves and capillary waves with stagnation points and constant vorticity by Matioc et al [23, 32, 33]. Among others we also mention the works by Craig, Nicholls and Reitich who addressed the existence, and parametric analyticity, of branches of capillary-gravity waves in the absence of resonance [11], [39], [44] and the work developed by Nicholls in the context of boundary perturbation method of the Dirichlet-Neumann operator [37].

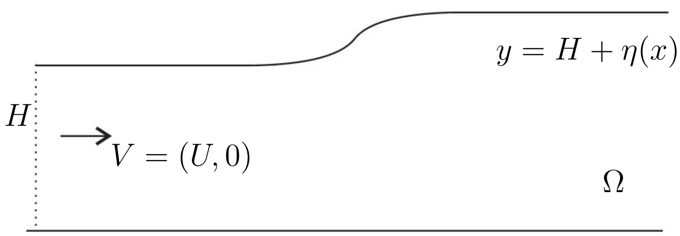

Figure 2.1: Main notation

The problem of existence of hydraulic jump has been studied in the context of conducting fluids and charged drops [19], [20] as well as some results established for system of elliptic partial differential equations with free boundaries arising in some biological models [8], [21].

Concerning water-waves we mention the work by Titri-Bouadjenak et al. [47], where the existence and uniqueness of solutions are addressed for $F > 1$, applying the implicit function theorem on Banach spaces. We also mention the book of Lannes [30] for a general review on non-stationary water-waves equations as well as some asymptotic regimes.

Since we are considering an incompressible, irrotational, two dimensional fluid we introduce an harmonic stream function ψ in the fluid domain, Ω, such that the velocity of the fluid is the orthogonal gradient of ψ, $V = \nabla^{\perp}\psi = (\partial_y\psi, -\partial_x\psi)^T$. The bottom and the free surface of Ω are streamlines so ψ is constant on $\partial\Omega$ and we choose $\psi = 0$ on the bottom; in addition, on the free surface we have an equilibrium condition given by the Bernoulli equation. Finally, we restrict our model to a bounded domain in the horizontal variables and make the assumption that the fluid stream lines are horizontal upstream and downstream. Putting all these facts together, after normalization, we obtain the system (see Figure 2.1)

$$\begin{cases} \Delta\psi = 0, & \Omega, \\ \psi_x = 0, & x = -L, L, \\ \psi = 0, & y = 0, \\ \psi = F, & y = 1 + \eta(x), \\ \frac{1}{2}|\nabla\psi|^2 = -y + \frac{F^2}{2} + 1, & y = 1 + \eta(x). \end{cases} \tag{2.1}$$

Later in chapter two we briefly show a deduction of this model based on the incompressibility, irrotationality and upstream assumptions on the fluid and its velocity. The system (2.1) has an explicit solution in the case that Ω is the flat strip $\mathbb{R} \times (0, 1)$,

$$\psi(x, y) = Fy.$$

Our purpose is to establish the existence of bifurcation branches of solutions with non flat domains Ω. We choose as bifurcation parameter the Froude number F (in other words the initial velocity upstream). After establishing existence and uniqueness of solutions on Sobolev spaces for system (2.1) in the cases $F > 1$ and for $F < 1$ when $x \in [-L, L]$, we shall prove that there exist a sequence of bifurcation branches with

$$F_l = F_{l0} + \epsilon F_{l1} + \epsilon^2 F_{l2} + \cdots, \ (l = 0, 1, 2, \ldots),$$

which, for $l = 0$ becomes

$$F = 1 + O(L^{-2}),$$

and free boundary

$$\eta = 1 + \epsilon \sin \frac{\pi x}{2L} + \cdots .$$

Since the first bifurcation point F_0 has special physical significance, we want to determine the shape of the bifurcation curve $F = F(\epsilon)$ near $F = F_0$, $\epsilon = 0$. Following the ideas in [19], [21], we compute F_0 and, even more, we get that

$$F_{01} = F'(0) > 0,$$

and therefore, $F'(\epsilon) > 0$ near $\epsilon = 0$.

2.2 Elliptic boundary problem on a strip

In this section, we state the stationary two dimensional water-waves problem on a general domain. By the incompressible and irrotational assumptions on the fluid, one is able to write the velocity of the fluid as the orthogonal gradient of an harmonic function ψ. Then, assuming that the free surface variations are small, it is possible to transform this elliptic problem for ψ on Ω, into an elliptic problem on a flat strip. This allows us to compute explicitly the solution of (2.1) and to relate this solution to the free boundary through the Bernoulli equation.

Let $H > 0$ be a constant reference depth and for a given $L > 0$, let $\eta : [-L, L] \to \mathbb{R}$ be the free surface parametrization. Throughout this section, we work on a domain Ω defined as

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -L < x < L,\ 0 < y < H + \eta(x)\}.$$

The motion of the fluid is governed by the (stationary) incompressible Euler equations inside the fluid domain, with constraints on the divergence and curl of the velocity field V:

$$\begin{aligned}
(V \cdot \nabla)V &= -\frac{1}{\rho}\nabla P - g e_2, && \Omega,\\
\operatorname{div} V &= 0, && \Omega,\\
\operatorname{curl} V &= 0, && \Omega,
\end{aligned}$$

where ρ is the constant density of the fluid and $-ge_2$ is the constant acceleration of gravity, with $g > 0$ and e_2 the unit upward vector in the vertical direction. The free surface Bernoulli equation is another formulation of the free surface Euler equation based on the representation of the velocity field V in terms of the stream function ψ such that

$$\begin{aligned}
V &= \nabla^{\perp}\psi, && \Omega,\\
\Delta\psi &= 0, && \Omega,\\
\frac{1}{2}|\nabla\psi|^2 &= -\frac{1}{\rho}(P - P_{atm}) - gy + C, && \Omega,
\end{aligned} \tag{2.2}$$

where C is a constant, P_{atm} is the constant atmospheric pressure and $\nabla^{\perp}\psi = (\partial_y\psi, -\partial_x\psi)^T$.

The bottom and the free surface of Ω are streamlines so ψ is constant on $y = 0$ and $y = \eta(x)$. Without loss of generality we are going to assume homogeneous condition on the bottom

$$\psi = 0, \quad \text{on } y = 0. \tag{2.3}$$

At $x = \pm L$, we assume zero vertical component of the velocity, namely

$$\psi_x = 0, \;\; x = -L, L. \tag{2.4}$$

This assumption is physically reasonable in the sense that we are setting an observation point at $x = -L$, in which the streamlines are horizontal if the bottom is flat and the velocity is constant, then we want to determine if there are certain Froude numbers at $x = -L$, generating hydraulic jumps.

Assuming there is no surface tension and the external pressure is constant, on the free surface we have the Bernoulli equation

$$\frac{1}{2}|\nabla\psi|^2 + gy = C \quad \text{on } y = H + \eta(x). \tag{2.5}$$

From (2.4) and (2.5) it is possible to determine the value of the constant C; indeed, at $x = -L$, one has $\eta = 0$ and $V = (U, 0) = (\partial_y\psi, -\partial_x\psi)$ with $U \geq 0$. Then

$$\frac{1}{2}U^2 + gH = C.$$

Therefore, on the free surface, equation (2.5) becomes

$$\frac{1}{2}|\nabla\psi|^2 = gH\left(-\frac{y}{H} + \frac{1}{2}\frac{U^2}{gH} + 1\right).$$

After the rescaling $y' = y/H$, $\psi' = \psi/(gH^3)^{1/2}$, $\eta' = \eta/H$, dropping the prime notation, (2.5) can be written as

$$\frac{1}{2}|\nabla\psi|^2 = -y + \frac{F^2}{2} + 1, \quad \text{on } y = 1 + \eta(x), \tag{2.6}$$

where $F = \frac{U}{\sqrt{gH}}$ is the Froude number [27].

By the assumptions above and proceeding as we did to determine the constant C, it is possible to determine the exact value of the stream function on the free surface; indeed, at $x = -L$, one has $\eta = 0$ and $U = \frac{\partial\psi}{\partial y} \geq 0$. Therefore, from (2.6)

$$\frac{\partial\psi}{\partial y} = F,$$

which implies, using (2.3),

$$\psi = F \quad \text{on } y = 1 + \eta(x). \tag{2.7}$$

Putting together equations (2.2)-(2.7) we have that ψ verifies the following system (see Figure 2.2)

$$\begin{cases} \Delta\psi = 0, & \Omega, \\ \psi_x = 0, & x = -L, L, \\ \psi = 0, & y = 0, \\ \psi = F, & y = 1 + \eta(x), \\ \dfrac{1}{2}|\nabla\psi|^2 = -y + \dfrac{F^2}{2} + 1, & y = 1 + \eta(x). \end{cases} \tag{2.8}$$

Our free boundary problem will be to determine η such that (2.8) is satisfied.

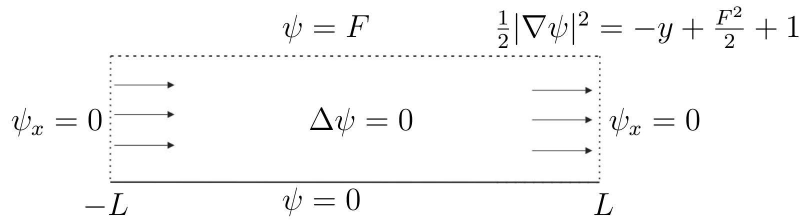

Figure 2.2: Free boundary value problem conditions

2.2.1 Small perturbation of the flat domain

If we had $\eta = 0$ then we know that $\psi = Fy$ is the solution of (2.8). For a given $\epsilon > 0$, let us assume that the free surface has the form $1 + \epsilon\eta$ and

$$\psi(x, y) = Fy + \epsilon\widetilde{\psi}(x, y).$$

We are interested in to quantify the relation between η and the initial velocity upstream characterized by F. To do that let us write a system of equations for $\widetilde{\psi}$. From the last equation in (2.8)

$$\begin{gathered}
\frac{1}{2}(\psi_x^2 + \psi_y^2) = -\epsilon\eta + \frac{F^2}{2} \\
\frac{1}{2}(\epsilon^2\widetilde{\psi}_x^2 + (F + \epsilon\widetilde{\psi}_y)^2) = -\epsilon\eta + \frac{F^2}{2} \\
F\widetilde{\psi}_y + \frac{\epsilon}{2}\widetilde{\psi}_x^2 + \frac{\epsilon}{2}\widetilde{\psi}_y^2 = -\eta.
\end{gathered}$$

Then, it is easy to see that $\widetilde{\psi}$ verifies the following system:

$$\begin{cases}
\Delta\widetilde{\psi} = 0, & \Omega, \\
\widetilde{\psi}_x = 0, & x = -L, L, \\
\widetilde{\psi} = 0, & y = 0, \\
\widetilde{\psi} = -F\eta, & y = 1 + \epsilon\eta, \\
F\widetilde{\psi}_y + \frac{\epsilon}{2}(\widetilde{\psi}_x^2 + \widetilde{\psi}_y^2) = -\eta, & y = 1 + \epsilon\eta.
\end{cases} \tag{2.9}$$

2.2.2 Hanzawa transformation

We are going to reduce system (2.9) to an elliptic equation on a flat domain an establish the link between the corresponding solutions through a suitable diffeomorphism. This method allow us to compute explicitly the solutions.

Let us consider the transformation

$$\begin{aligned}
x' &= x, \\
y' &= y - \epsilon\phi(x, y),
\end{aligned}$$

with

$$\phi(x, y) \in C^\infty([-L, L] \times \mathbb{R}), \;\; \phi(x, y) = \begin{cases} \eta(x), & |y - 1| < \delta_0, \\ 0, & |y - 1| > 2\delta_0, \end{cases}$$

where δ_0 is positive and small; which maps the domain Ω into the rectangle $\mathcal{S} = [-L, L] \times (0, 1)$.

By the chain rule

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} = \frac{\partial}{\partial x'} - \epsilon\phi_x \frac{\partial}{\partial y'}, \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y} = (1-\epsilon\phi_y)\frac{\partial}{\partial y'}
\end{aligned} \tag{2.10}$$

and then

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x'} - \epsilon\phi_x\frac{\partial}{\partial y'}\right) \\
&= \frac{\partial^2}{\partial x'^2} - \epsilon\frac{\partial}{\partial x'}\left(\phi_x\frac{\partial}{\partial y'}\right) - \epsilon\phi_x\left(\frac{\partial^2}{\partial x'\partial y'} - \epsilon\phi_x\frac{\partial^2}{\partial y'^2}\right) \\
&= \frac{\partial^2}{\partial x'^2} - \epsilon\frac{\partial}{\partial x'}\left(\phi_x\frac{\partial}{\partial y'}\right) - \epsilon\phi_x\frac{\partial^2}{\partial x'\partial y'} + \epsilon^2\phi_x^2\frac{\partial^2}{\partial y'^2},
\end{aligned}$$

and in a similar way

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y'^2} - \epsilon\left(2\phi_y\frac{\partial^2}{\partial y'^2} + \phi_{yy}\frac{\partial}{\partial y'}\right) + \epsilon^2\phi_y^2\frac{\partial^2}{\partial y'^2}.$$

Therefore, from the first equation in (2.9), on $\mathcal{S}$

$$\Delta\widetilde{\psi} = \epsilon\left(\frac{\partial}{\partial x'}\left(\phi_x\frac{\partial\widetilde{\psi}}{\partial y'}\right) + \phi_x\frac{\partial^2\widetilde{\psi}}{\partial x'\partial y'} + 2\phi_y\frac{\partial^2\widetilde{\psi}}{\partial y'^2} + \phi_{yy}\frac{\partial\widetilde{\psi}}{\partial y'}\right) - \epsilon^2\frac{\partial^2\widetilde{\psi}}{\partial y'^2}(\phi_x^2+\phi_y^2). \tag{2.11}$$

Even more, the Dirichlet boundary conditions are the same

$$\widetilde{\psi}|_{y'=0} = 0 \quad \text{and} \quad \widetilde{\psi}|_{y'=1} = -F\eta. \tag{2.12}$$

From the last equation in (2.9) and relations in (2.10) we have

$$F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}_{x'} - \epsilon\eta_{x'}\widetilde{\psi}_{y'})^2 + \frac{\epsilon}{2}\widetilde{\psi}_{y'}^2 = -\eta$$

$$\widetilde{\psi}_{y'} = -\frac{\eta}{F} - \frac{\epsilon}{2F}(\widetilde{\psi}_{x'}^2 + \widetilde{\psi}_{y'}^2) + \frac{\epsilon^2}{F}\eta_{x'}\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} - \frac{\epsilon^3}{2F}\eta_{x'}^2\widetilde{\psi}_{y'}^2. \tag{2.13}$$

Thus, from (2.11)-(2.13), $\widetilde{\psi}$ satisfies the following system on $\mathcal{S}$:

$$\begin{cases}
\Delta\widetilde{\psi} = \epsilon G, & \mathcal{S}, \\
\widetilde{\psi}_{x'} = 0, & x' = -L, L, \\
\widetilde{\psi} = 0, & y' = 0, \\
\widetilde{\psi} = -F\eta, & y' = 1, \\
\widetilde{\psi}_{y'} = -\frac{\eta}{F} - \frac{\epsilon}{2F}(\widetilde{\psi}_{x'}^2 + \widetilde{\psi}_{y'}^2) + \frac{\epsilon^2}{F}\eta_{x'}\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} - \frac{\epsilon^3}{2F}\eta_{x'}^2\widetilde{\psi}_{y'}^2, & y' = 1,
\end{cases} \tag{2.14}$$

with
$$G = \frac{\partial}{\partial x'}\left(\phi_x \frac{\partial \widetilde{\psi}}{\partial y'}\right) + \phi_x \frac{\partial^2 \widetilde{\psi}}{\partial x' \partial y'} + 2\phi_y \frac{\partial^2 \widetilde{\psi}}{\partial y'^2} + \phi_{yy}\frac{\partial \widetilde{\psi}}{\partial y'} - \epsilon \frac{\partial^2 \widetilde{\psi}}{\partial y'^2}(\phi_x^2 + \phi_y^2). \tag{2.15}$$

Notice the presence of five terms at the right hand side of (2.15) so that we define
$$G = G_1 + \ldots + G_5, \tag{2.16}$$
accordingly.

2.3 Solution of the problem on the rectangle

In this section we use the Banach fixed point theorem to prove existence and uniqueness of the solution of problem (2.14). The idea is to compute an explicit solution of the homogeneous part and to get proper estimates in certain Sobolev norms of its non homogeneous part depending on the norm of η. More precisely, if we consider the operator
$$\mathcal{F}(\eta, F) := \eta + F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}_{x'}^2 + \widetilde{\psi}_{y'}^2) - \epsilon^2 \eta_{x'} \widetilde{\psi}_{x'} \widetilde{\psi}_{y'} + \frac{\epsilon^3}{2}\eta_{x'}^2 \widetilde{\psi}_{y'}^2,$$
by the regularity mentioned above we were able to prove that, for $\alpha \geq 1$,
$$\mathcal{F} : H^\alpha \times (0, +\infty) \to H^{\alpha-1}.$$

If $\widetilde{\psi} = \widetilde{\psi}^h + \widetilde{\psi}^p$ then $\widetilde{\psi}^h$ and $\widetilde{\psi}^p$, respectively, are the solutions of the problems
$$\begin{cases} \Delta \widetilde{\psi}^h = 0, & \Delta \widetilde{\psi}^p = \epsilon G, \\ \widetilde{\psi}^h_{x'}|_{x'=\pm L} = 0, & \widetilde{\psi}^p_{x'}|_{x'=\pm L} = 0, \\ \widetilde{\psi}^h|_{y'=0} = 0, & \widetilde{\psi}^p|_{y'=0} = 0, \\ \widetilde{\psi}^h|_{y'=1} = -F\eta, & \widetilde{\psi}^p|_{y'=1} = 0. \end{cases}$$

2.3.1 Solution of the homogeneous problem

If $\widetilde{\psi}^h(x', y') = f(x')g(y')$ then, since $\Delta \widetilde{\psi}^h = 0$,
$$\frac{f_{x'x'}(x')}{f(x')} = \lambda = -\frac{g_{y'y'}(y')}{g(y')}.$$

Therefore
$$f(x') = ae^{\sqrt{\lambda}x'} + be^{-\sqrt{\lambda}x'},$$
$$g(y') = ce^{\sqrt{-\lambda}y'} + de^{-\sqrt{-\lambda}y'}.$$

$\widetilde{\psi}^h_{x'}(\pm L, y') = 0$ implies
$$\begin{aligned} a\sqrt{\lambda}e^{\sqrt{\lambda}L} - b\sqrt{\lambda}e^{-\sqrt{\lambda}L} &= 0, \\ a\sqrt{\lambda}e^{-\sqrt{\lambda}L} - b\sqrt{\lambda}e^{\sqrt{\lambda}L} &= 0, \end{aligned} \tag{2.17}$$

thus $e^{3\sqrt{\lambda}L} = e^{-\sqrt{\lambda}L}$ and then $4\sqrt{\lambda}L = 2k\pi i$ or $\sqrt{\lambda} = \frac{k\pi}{2L}i$. One also has $\sqrt{-\lambda} = \frac{k\pi}{2L}$.

$\widetilde{\psi}^h(x',0) = 0$ implies $c + d = 0$. Hence

$$g(y') = c(e^{\frac{k\pi}{2L}y'} - e^{-\frac{k\pi}{2L}y'}) = 2c\sinh(\frac{k\pi}{2L}y').$$

From (2.17), $a = be^{2\sqrt{\lambda}L}$; then,

$$f(x') = b(e^{k\pi i}e^{\frac{k\pi x'}{2L}i} + e^{-\frac{k\pi x'}{2L}i}) = \begin{cases} 2b\cos(\dfrac{k\pi x'}{2L}), \ k \text{ even}, \\ -2bi\sin(\dfrac{k\pi x'}{2L}), \ k \text{ odd}. \end{cases}$$

Finally

$$\widetilde{\psi}^h(x',y') = \sum_{n=0}^{\infty} \widetilde{a_n}\sinh(\frac{(2n+1)\pi}{2L}y')\sin(\frac{(2n+1)\pi}{2L}x') + \sum_{n=1}^{\infty}\widetilde{b_n}\sinh(\frac{n\pi}{L}y')\cos(\frac{n\pi}{L}x').$$

Multiplying by $\sin(k_m x')$, with $k_m = \frac{(2m+1)\pi}{2L}$, $m \geq 0$, and integrating over x'

$$\int_{-L}^{L}\widetilde{\psi}^h(x',y')\sin(k_n x')dx' = \widetilde{a_n}\sinh(k_n y')L.$$

By the boundary condition $\widetilde{\psi}^h(x',1) = -F\eta(x')$,

$$\widetilde{a_n} = -\frac{F}{L\sinh(k_n)}\int_{-L}^{L}\eta(x')\sin(k_n x')dx'.$$

Similarly, if we denote $l_n = \frac{n\pi}{L}$, $n \geq 1$,

$$\widetilde{b_n} = -\frac{F}{L\sinh(l_n)}\int_{-L}^{L}\eta(x')\cos(l_n x')dx'.$$

Summarizing

$$\widetilde{\psi}^h(x',y') = -F\sum_{n=0}^{\infty} a_n \frac{\sinh(k_n y')}{\sinh(k_n)}\sin(k_n x') - F\sum_{n=1}^{\infty} b_n\frac{\sinh(l_n y')}{\sinh(l_n)}\cos(l_n x'), \tag{2.18}$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L}\eta(x')\sin(k_n x')dx', \quad b_n = \frac{1}{L}\int_{-L}^{L}\eta(x')\cos(l_n x')dx'.$$

Remark 2.1. If $y' = 1$, by the last expression and the boundary condition $\widetilde{\psi}^h|_{y'=1} = -F\eta(x')$ one has

$$\eta(x') = \sum_{n=0}^{\infty} a_n\sin(k_n x') + \sum_{n=1}^{\infty} b_n\cos(l_n x').$$

Note that we are not using the term b_0 in the expression for η because we are looking for a nontrivial η such that $\eta \in L^2(\mathbb{R}) \cap \{1\}^{\perp}$. □

The following lemma gives the regularity in $H^{\alpha+\frac{1}{2}}$, $\alpha \geq 0$, of the explicit solution $\widetilde{\psi}^h$ calculated in (2.18), in terms of the regularity of η. It will be used later together with the estimates of the non homogeneous part to prove the existence an uniqueness of solutions of system (2.14).

Lemma 2.2. *The solution $\widetilde{\psi}^h(x', y')$ given by (2.18) satisfies*

$$\|\widetilde{\psi}^h\|_{H^{\alpha+\frac{1}{2}}} \leq C|\eta|_{H^\alpha}, \ \ \alpha \geq 0.$$

Proof. We write

$$\alpha + \frac{1}{2} = \left[\alpha + \frac{1}{2}\right] + s,$$

with $[a]$ denoting the entire part of a and $0 \leq s < 1$. By taking $\left[\alpha + \frac{1}{2}\right]$ derivatives, we obtain terms such as

$$a_n k_n^{\left[\alpha+\frac{1}{2}\right]} \frac{\sinh(k_n y')}{\sinh(k_n)} \sin(k_n x')$$

(or with cosh instead of sinh, cos instead of sin or l_n instead of k_n). By expanding $\sinh(k_n y')$ in Fourier series, we obtain

$$\sinh(k_n y') = \frac{2}{\pi} \sum_m \frac{(-1)^{m+1} L^2}{\pi} \frac{m \sinh \frac{1}{2}\frac{\pi}{L}(2n+1)}{L^2 m^2 + n^2 + n + \frac{\pi}{4}} \sin(m\pi y),$$

with analogous expansions for $\cosh(k_n y')$, $\sinh(l_n y')$, $\cosh(l_n y')$. Hence,

$$\|\widetilde{\psi}^h\|_{H^{\alpha+\frac{1}{2}}} \leq C \sum_{m,n} \left(\frac{m}{L^2 m^2 + n^2 + n + \frac{\pi}{4}}\right)^2 n^{2\left[\alpha+\frac{1}{2}\right]} (n^{2s} + m^{2s}) a_n^2.$$

Since

$$\begin{aligned}
&\sum_m \left(\frac{m}{L^2 m^2 + n^2 + n + \frac{\pi}{4}}\right)^2 (n^{2s} + m^{2s}) \\
&\qquad = \frac{1}{n^{1-2s}} \sum_m \left(\frac{\frac{m}{n}}{L^2 \left(\frac{m}{n}\right)^2 + 1 + \frac{1}{n} + \frac{\pi}{4n^2}}\right)^2 \left(1 + \left(\frac{m}{n}\right)^{2s}\right) \frac{1}{n} \\
&\qquad \leq \frac{C}{n^{1-2s}} \int_0^\infty \left(\frac{x}{L^2 x^2 + 1}\right)^2 (1 + x^{2s}) dx \leq \frac{C}{n^{1-2s}},
\end{aligned}$$

where C is a generic constant, we have

$$\|\widetilde{\psi}^h\|_{H^{\alpha+\frac{1}{2}}} \leq C \sum_n n^{2\alpha} a_n^2 = C|\eta|_{H^\alpha}.$$

□

□

Remark 2.3. It is possible to consider also the case with nontrivial bottom $B \neq 0$, although we were not able to prove the existence of bifurcations in this case because, as shown in chapter 4, the condition $\mathcal{F}(0, F) = 0$ in the hypothesis of the Crandall-Rabinowitz theorem is not satisfied. Nevertheless this could be useful if one transform the system into a general elliptic problem.

Let $\bar{\psi}^h$ be the solution of

$$\begin{cases} \Delta\bar{\psi}^h = 0, & \mathcal{S}, \\ \bar{\psi}^h_{x'} = 0, & x' = -L, L, \\ \bar{\psi}^h = -FB, & y' = 0, \\ \bar{\psi}^h = 0, & y' = 1, \end{cases}$$

proceeding as before, one gets

$$\bar{\psi}^h(x', y') = -F\sum_{n=0}^{\infty} c_n \frac{\sinh(k_n(1-y'))}{\sinh(k_n)} \sin(k_n x') - F\sum_{n=1}^{\infty} d_n \frac{\sinh(l_n(1-y'))}{\sinh(l_n)} \cos(l_n x'),$$

with

$$c_n = \frac{1}{L}\int_{-L}^{L} B(x') \sin(k_n x') dx', \quad d_n = \frac{1}{L}\int_{-L}^{L} B(x') \cos(l_n x') dx'.$$

Note that, if $y' = 0$,

$$B(x') = \sum_{n=0}^{\infty} c_n \sin(k_n x') + \sum_{n=1}^{\infty} d_n \cos(l_n x').$$

Finally, if we denotes Ψ^h as the solution of

$$\begin{cases} \Delta\Psi^h = 0, & \mathcal{S}, \\ \Psi^h = -FB, & y' = 0, \\ \Psi^h = -F\eta, & y' = 1, \\ \Psi^h_{x'} = 0, & x' = -L, L, \end{cases}$$

$$\begin{aligned} \Psi^h(x', y') = &- F\sum_{n=0}^{\infty} \left(a_n \frac{\sinh(k_n y')}{\sinh(k_n)} + c_n \frac{\sinh(k_n(1-y'))}{\sinh(k_n)} \right) \sin(k_n x') \\ &- F\sum_{n=1}^{\infty} \left(b_n \frac{\sinh(l_n y')}{\sinh(l_n)} + d_n \frac{\sinh(l_n(1-y'))}{\sinh(l_n)} \right) \cos(l_n x'), \end{aligned}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \eta(x') \sin(k_n x') dx', \quad b_n = \frac{1}{L}\int_{-L}^{L} \eta(x') \cos(l_n x') dx',$$

$$c_n = \frac{1}{L}\int_{-L}^{L} B(x') \sin(k_n x') dx', \quad d_n = \frac{1}{L}\int_{-L}^{L} B(x') \cos(l_n x') dx'.$$

□

Later in chapter 4, we will see that the existence an uniqueness of solutions of problem (2.14), can be established for all $F > 0$ if one avoids certain bifurcation points, even in the case of a non flat bottom $B \neq 0$; which is a generalization of the result given in [47] where the existence and uniqueness is established for $F > 1$.

2.3.2 Estimates on the non-homogeneous associated problem

In this subsection we complement the calculations of the last section. The aim is to bound $\|\widetilde{\psi}^p\|_{H^{\alpha+\frac{1}{2}}(S)}$ in terms of a right hand side depending on $|\eta|_{H^\alpha}$ and a given function $\widetilde{\psi}_{giv}$ such that for ϵ sufficiently small it is possible to apply the fixed point theorem to prove the existence of solutions to (2.14).

$$\begin{cases} \Delta\widetilde{\psi}^p = \epsilon G, & \mathcal{S}, \\ \widetilde{\psi}^p_{x'} = 0, & x' = -L, L, \\ \widetilde{\psi}^p = 0, & y' = 0, \\ \widetilde{\psi}^p = 0, & y' = 1. \end{cases} \tag{2.19}$$

We start calculating the eigenvalues and eigenfunctions of the Laplacian on $\mathcal{S}$:

$$\Delta\psi = \lambda\psi.$$

If $\psi(x', y') = X(x')Y(y')$ then

$$\frac{X''(x')}{X(x')} + \frac{Y''(y')}{Y(y')} = \lambda.$$

$$\frac{X''(x')}{X(x')} = \alpha, \quad \frac{Y''(y')}{Y(y')} = \beta, \quad \lambda = \alpha + \beta.$$

As before,

$$X(x') = ae^{\sqrt{\alpha}x'} + be^{-\sqrt{\alpha}x'},$$
$$Y(y') = ce^{\sqrt{\beta}y'} + de^{-\sqrt{\beta}y'}.$$

$\psi_{x'}(L, y') = \psi_{x'}(-L, y') = 0$ implies $\sqrt{\alpha} = \frac{k\pi}{2L}i$, $k \in \mathbb{N}$, and then

$$X(x') = e^{k\pi i}e^{\frac{k\pi x'}{2L}i} + e^{-\frac{k\pi x'}{2L}i} = \begin{cases} \cos(\dfrac{k\pi x'}{2L}), & k \text{ even}, \\ -2i\sin(\dfrac{k\pi x'}{2L}), & k \text{ odd}. \end{cases}$$

$\psi(x', 0) = \psi(x', 1) = 0$ implies $\sqrt{\beta} = k\pi i$, $k \in \mathbb{N}$, and then

$$Y(y') = \sin(k\pi y').$$

Therefore, for $l, m = 1, 2, \ldots$, one has the family of eigenvalues

$$\lambda^1_{l,m} = \alpha_l + \beta_m = -\left(\frac{l\pi}{L}\right)^2 - (2m\pi)^2,$$

$$\lambda^2_{l,m} = -\left(\frac{(2l+1)\pi}{2L}\right)^2 - ((2m+1)\pi)^2,$$

with associated eigenfunctions

$$\psi^1_{l,m} = \cos\left(\frac{l\pi}{L}x'\right)\sin(2m\pi y'), \tag{2.20}$$
$$\psi^2_{l,m} = \sin\left(\frac{(2l+1)\pi}{2L}x'\right)\sin((2m+1)\pi y').$$

Since that $\psi^1_{l,m}(x',y')$, $\psi^2_{l,m}(x',y')$, $l,m \geq 1$, represents a complete orthogonal system, then for $G \in L^2(\mathcal{S})$ one has

$$G = \sum_{l,m=1}^{\infty} \left(\theta^1_{l,m}\psi^1_{l,m} + \theta^2_{l,m}\psi^2_{l,m}\right),$$

with

$$\theta^1_{l,m} = \frac{1}{\|\psi^1_{l,m}\|^2}\langle G, \psi^1_{l,m}\rangle = \frac{2}{L}\int_{-L}^{L}\int_0^1 G(x',y')\cos\left(\frac{l\pi}{L}x'\right)\sin(2m\pi y')dy'dx',$$

$$\theta^2_{l,m} = \frac{1}{\|\psi^2_{l,m}\|^2}\langle G, \psi^2_{l,m}\rangle = \frac{2}{L}\int_{-L}^{L}\int_0^1 G(x',y')\sin\left(\frac{(2l+1)\pi}{2L}x'\right)\sin((2m+1)\pi y')dy'dx'.$$

Therefore, if

$$\widetilde{\psi}^p = \sum_{l,m=1}^{\infty} \left(\gamma^1_{l,m}\psi^1_{l,m} + \gamma^2_{l,m}\psi^2_{l,m}\right), \tag{2.21}$$

by the equation $\Delta\widetilde{\psi}^p = \epsilon G$,

$$\sum_{l,m=1}^{\infty} \left(\gamma^1_{l,m}\lambda^1_{l,m}\psi^1_{l,m} + \gamma^2_{l,m}\lambda^2_{l,m}\psi^2_{l,m}\right) = \epsilon \sum_{l,m=1}^{\infty} \left(\theta^1_{l,m}\psi^1_{l,m} + \theta^2_{l,m}\psi^2_{l,m}\right).$$

Thus,

$$\gamma^1_{l,m} = \epsilon\frac{\theta^1_{l,m}}{\lambda^1_{l,m}}, \quad \gamma^2_{l,m} = \epsilon\frac{\theta^2_{l,m}}{\lambda^2_{l,m}}.$$

Notice that, for all $\alpha \in \mathbb{R}$, $\alpha > \frac{1}{2}$,

$$\|\widetilde{\psi}^p\|^2_{H^{\alpha+\frac{1}{2}}(\mathcal{S})} \sim \sum_{l,m=1}^{\infty} \left((l^2+m^2)^{\alpha+\frac{1}{2}}|\gamma^1_{l,m}|^2 + (l^2+m^2)^{\alpha+\frac{1}{2}}|\gamma^2_{l,m}|^2\right).$$

By the linearity of (2.16) and (2.19) we can write

$$\widetilde{\psi}^p = \widetilde{\psi}^p_1 + \ldots + \widetilde{\psi}^p_5.$$

Remark 2.4. As was mentioned at the beginning of this section, we want an estimate of the type, $\|\widetilde{\psi}^p\|_{H^{\alpha+\frac{1}{2}}} \leq C(|\eta|_{H^\alpha})\|\widetilde{\psi}_{giv}\|_{H^{\alpha+\frac{1}{2}}}$. However if we proceed directly we get $|\eta|_{H^{\alpha+\frac{1}{2}}}$ instead of $|\eta|_{H^\alpha}$. The reason is the term ϕ_{xx} appearing in G_1; that is why we perform a sort of integration by parts as explained below. □

For $i = 2,3,4,5$ we have

$$\begin{aligned}
\|\widetilde{\psi}^p_{i,y'}\|^2_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} &\leq C\sum_{l,m=1}^{\infty}\left((l^2+m^2)^{\alpha-\frac{1}{2}}m^2|\gamma^1_{i,lm}|^2 + (l^2+m^2)^{\alpha-\frac{1}{2}}m^2|\gamma^2_{i,lm}|^2\right) \\
&= C\epsilon^2\sum_{l,m=1}^{\infty}\left(\frac{(l^2+m^2)^{\alpha-\frac{1}{2}}m^2|\theta^1_{i,lm}|^2}{\left(\left(\frac{l\pi}{L}\right)^2 + (2m\pi)^2\right)^2} + \frac{(l^2+m^2)^{\alpha-\frac{1}{2}}m^2|\theta^2_{i,lm}|^2}{\left(\left(\frac{(2l+1)\pi}{2L}\right)^2 + ((2m+1)\pi)^2\right)^2}\right) \\
&\leq C\epsilon^2\sum_{l,m=1}^{\infty}\left((l^2+m^2)^{\alpha-\frac{5}{2}}m^2|\theta^1_{i,lm}|^2 + (l^2+m^2)^{\alpha-\frac{5}{2}}m^2|\theta^2_{i,lm}|^2\right) \\
&= C\epsilon^2\|G_i\|^2_{H^{\alpha-\frac{3}{2}}(\mathcal{S})}.
\end{aligned}$$

From (2.15), one has

$$\|G_i\|_{H^{\alpha-\frac{3}{2}}(\mathcal{S})} \le C(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha})\|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}$$

and therefore, for $i = 2,3,4,5$

$$\|\widetilde{\psi}^p_{i,y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha})\|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}. \tag{2.22}$$

Concerning the estimate of $\widetilde{\psi}^p_1$, we obtain

$$\|\widetilde{\psi}^p_{1,y'}\|^2_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C\epsilon^2 \sum_{l,m=1}^{\infty} \left((l^2+m^2)^{\alpha-\frac{5}{2}} l^2 \frac{m^2|\theta^1_{1,lm}|^2}{l^2} + (l^2+m^2)^{\alpha-\frac{5}{2}} l^2 \frac{m^2|\theta^2_{1,lm}|^2}{l^2}\right).$$

By (2.15), $G_1 = \frac{\partial}{\partial x'} g_1$ with $g_1 = \phi_x \widetilde{\psi}_{y'}$, therefore we can estimate

$$\|\widetilde{\psi}^p_{1,y'}\|^2_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C\epsilon^2 \left\|\frac{\partial}{\partial y'} g_1\right\|^2_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}. \tag{2.23}$$

We observe now that $\phi_{x'y'}$ involves only first order derivatives of η to conclude

$$\left\|\frac{\partial}{\partial y'} g_1\right\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C|\eta|_{H^{\alpha-\frac{1}{2}}} \|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}. \tag{2.24}$$

Putting together (2.22), (2.23) and (2.24) we get

$$\|\widetilde{\psi}^p_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha})\|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}.$$

Remark 2.5. In the calculations above it is possible to replace right hand side G by $\breve{G} = \breve{G}(\breve{\psi})$ so that

$$\|\widetilde{\psi}^p_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha})\|\breve{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}. \tag{2.25}$$

□

On the other hand, going back to Lemma 2.2 we find $\|\widetilde{\psi}^h\|_{H^{\alpha+\frac{1}{2}}} \le C|\eta|_{H^\alpha}$ and, taking $\breve{\psi} = \widetilde{\psi}^h + \breve{\psi}^p$ we estimate

$$\|\breve{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C|\eta|_{H^\alpha} + \|\breve{\psi}^p_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}. \tag{2.26}$$

From (2.25), (2.26) follows

Lemma 2.6. *Let $\alpha \in \mathbb{R}$, $\alpha > \frac{1}{2}$, and $\eta \in H^\alpha(-L,L)$. If $\widetilde{\psi}^p_{y'} \in H^{\alpha-\frac{1}{2}}(\mathcal{S})$ solves the non-homogeneous problem (2.19), then, for all $\breve{\psi}^p_{y'} \in H^{\alpha-\frac{1}{2}}(\mathcal{S})$ one has*

$$\|\widetilde{\psi}^p_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C\epsilon(|\eta|^2_{H^\alpha} + |\eta|^3_{H^\alpha}) + C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha})\|\breve{\psi}^p_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}.$$

Theorem 2.7 (Solution of the hydrodynamical problem on $\mathcal{S}$)**.** *Let $\alpha \in \mathbb{R}$, $\alpha > \frac{1}{2}$ and $\eta \in H^\alpha(-L,L)$ such that $|\eta|_{H^\alpha} < 1$. Then, for ϵ sufficiently small, there exists a unique solution $\widetilde{\psi}_{y'} \in H^{\alpha-\frac{1}{2}}(\mathcal{S})$ to (2.14). Moreover, there exist a constant $C > 0$ such that*

$$\|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \le C|\eta|_{H^\alpha}.$$

Proof. Let us consider the mapping

$$T: \begin{array}{ccc} H^{\alpha-\frac{1}{2}} & \to & H^{\alpha-\frac{1}{2}} \\ \breve{\psi}_{y'} & \mapsto & T(\breve{\psi}_{y'}) = \widetilde{\psi}^{h}_{y'} + \widetilde{\psi}^{p}_{y'}. \end{array}$$

Then, by lemmas 2.2 and 2.6 we have

$$\|T(\breve{\psi}_{y'})\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \leq C|\eta|_{H^\alpha} + C\epsilon(|\eta|^2_{H^\alpha} + |\eta|^3_{H^\alpha}) + C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha})\|\breve{\psi}^{p}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}.$$

Therefore, if ϵ is sufficiently small, T maps $B_{2s}(0)$ into itself where $s = C|\eta|_{H^\alpha}$. Furthermore, due to the linearity of T (since the PDE and the boundary conditions are linear), we can estimate

$$\|T(\breve{\psi}_{1,y'} - \breve{\psi}_{2,y'})\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \leq C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha})\|\breve{\psi}_{1,y'} - \breve{\psi}_{2,y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})},$$

implying that T is a contraction for ϵ small enough. Thus, by the Banach fixed point theorem, there exists a unique $\widetilde{\psi}_{y'} \in H^{\alpha-\frac{1}{2}}$ such that $T(\widetilde{\psi}_{y'}) = \widetilde{\psi}_{y'}$. Moreover,

$$\begin{aligned} \|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} &\leq \frac{C}{1 - C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha})}|\eta|_{H^\alpha} \\ &\leq 2C|\eta|_{H^\alpha}. \end{aligned}$$

□

Remark 2.8. As a consequence of lemma 2.6 and theorem 2.7 one has, for all $\alpha \in \mathbb{R}$, $\alpha > \frac{1}{2}$ and ϵ sufficiently small

$$\|\widetilde{\psi}^{p}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \leq C\epsilon(|\eta|^2_{H^\alpha} + |\eta|^3_{H^\alpha}).$$

Therefore, by Poincaré's inequality,

$$\|\widetilde{\psi}^{p}\|_{H^{\alpha+\frac{1}{2}}(\mathcal{S})} \leq C\epsilon(|\eta|^2_{H^\alpha} + |\eta|^3_{H^\alpha}).$$

□

Now we are going to make use of the unique solution of system (2.14) provided by theorem 2.7 and the corresponding estimate in $H^{\alpha-\frac{1}{2}}(\mathcal{S})$, together with the Bernoulli equation to prove that the linear operator $\mathcal{F}$ defined by

$$\mathcal{F}(\eta, F) = \eta + F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}^2_{x'} + \widetilde{\psi}^2_{y'}) - \epsilon^2\eta_{x'}\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} + \frac{\epsilon^3}{2}\eta^2_{x'}\widetilde{\psi}^2_{y'},$$

is such that, for $\alpha \geq 1$,

$$\mathcal{F} : H^\alpha \times (0, +\infty) \to H^{\alpha-1}.$$

Indeed, by the trace theorem and remark 2.8

$$\begin{aligned} \left|\widetilde{\psi}^{p}_{y'}|_{y'=1}\right|_{H^{\alpha-1}(-L,L)} &\leq C\|\widetilde{\psi}^{p}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \\ &\leq C\epsilon(|\eta|^2_{H^\alpha} + |\eta|^3_{H^\alpha}) \end{aligned}$$

and the same is true for $\widetilde{\psi}^h$ instead of $\widetilde{\psi}^p$. Notice that $\widetilde{\psi}_{y'} = \widetilde{\psi}^h_{y'} + \widetilde{\psi}^p_{y'}$ while $\widetilde{\psi}_{x'} = \widetilde{\psi}^h_{x'}$. Then, using the estimate

$$\|fg\|_{H^\alpha(\mathbb{R})} \leq C(\|f\|_\infty\|g\|_{H^\alpha(\mathbb{R})} + \|g\|_\infty\|f\|_{H^\alpha(\mathbb{R})}) \leq C\|f\|_{H^\alpha(\mathbb{R})}\|g\|_{H^\alpha(\mathbb{R})}, \quad \forall\alpha > 1/2,$$

and the inequalities above one gets the result.

2.4 Solution of the problem on the strip

The purpose of this section is to perform similar calculations than in the last section but in the case of a flat strip ($L = +\infty$); i.e we are going to calculate explicitly the solutions of the hydrodynamical problem, including the case of a non-flat bottom. Through this section we keep the same notation of the last two sections, namely

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -\infty < x < \infty,\ b(x) < y < H + \eta(x)\},$$

$$\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : -\infty < x < \infty,\ 0 < y < 1\}.$$

As before, after flatting the domain through the Hanzawa transformation, for $b = \epsilon B$,

$$\begin{cases} \Delta\widetilde{\psi} = \epsilon G, & \mathcal{S}, \\ \widetilde{\psi} = -FB, & y' = 0, \\ \widetilde{\psi} = -F\eta, & y' = 1, \\ \widetilde{\psi}_{y'} = -\dfrac{\eta}{F} - \dfrac{\epsilon}{2F}(\widetilde{\psi}^2_{x'} + \widetilde{\psi}^2_{y'}) + \dfrac{\epsilon^2}{F}\eta_x\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} - \dfrac{\epsilon^3}{2F}\eta_x^2\widetilde{\psi}^2_{y'}, & y' = 1. \end{cases} \tag{2.27}$$

Taking the Fourier transform, with respect to x, in (2.27)

$$\begin{cases} \widehat{\Delta\psi} = -|k|^2\widehat{\psi} + \dfrac{d^2\widehat{\psi}}{dy'^2} = \epsilon\widehat{G}, & \mathcal{S}, \\ \widehat{\psi} = -F\widehat{B}, & y' = 0, \\ \widehat{\psi} = -F\widehat{\eta}, & y' = 1, \end{cases} \tag{2.28}$$

where, for simplicity, we use the notation $\widehat{\psi} = \widehat{\widetilde{\psi}}$. Then $\widehat{\psi} = \widehat{\psi}^h + \widehat{\psi}^p$ where $\widehat{\psi}^h$ and $\widehat{\psi}^p$, respectively, are the solutions of the problems

$$\begin{cases} -|k|^2\widehat{\psi}^h + \dfrac{d^2\widehat{\psi}^h}{dy'^2} = 0, & -|k|^2\widehat{\psi}^p + \dfrac{d^2\widehat{\psi}^p}{dy'^2} = \epsilon\widehat{G}, \\ \widehat{\psi}^h|_{y'=0} = -F\widehat{B}, & \widehat{\psi}^p|_{y'=0} = 0, \\ \widehat{\psi}^h|_{y'=1} = -F\widehat{\eta}, & \widehat{\psi}^p|_{y'=1} = 0. \end{cases}$$

2.4.1 Solution of the homogeneous problem

Let us assume that $\widehat{\psi}^h$ has the form

$$\widehat{\psi}^h(k, y') = \beta_1(k)e^{|k|y'} + \beta_2(k)e^{-|k|y'},$$

with β_1, β_2 two real valued functions.

By the boundary conditions

$$\beta_1(k) + \beta_2(k) = -F\widehat{B},$$
$$\beta_1(k)e^{|k|} + \beta_2(k)e^{-|k|} = -F\widehat{\eta}.$$

Then

$$-(\beta_2(k) + \widehat{B}F)e^{|k|} + \beta_2(k)e^{-|k|} = -F\widehat{\eta},$$

$$\beta_2(k)(e^{|k|} - e^{-|k|}) = -F\widehat{B}e^{|k|} + F\widehat{\eta},$$

$$\beta_2(k) = \frac{-F}{2\sinh|k|}(\widehat{B}e^{|k|} - \widehat{\eta}).$$

Replacing in the equation above

$$\begin{aligned}\beta_1(k) &= \frac{F}{2\sinh|k|}(\widehat{B}e^{|k|} - \widehat{\eta}) - F\widehat{B}\\ &= \frac{F}{2\sinh|k|}\left(\widehat{B}(e^{|k|} - 2\sinh|k|) - \widehat{\eta}\right)\\ &= \frac{F}{2\sinh|k|}(\widehat{B}e^{-|k|} - \widehat{\eta}).\end{aligned}$$

Therefore

$$\begin{aligned}\widehat{\psi}^h(k,y') &= \frac{F}{2\sinh|k|}(\widehat{B}e^{-|k|} - \widehat{\eta})e^{|k|y'} - \frac{F}{2\sinh|k|}(\widehat{B}e^{|k|} - \widehat{\eta})e^{-|k|y'} \\ &= \widehat{B}\left(F\frac{\sinh|k|(y'-1)}{\sinh|k|}\right) - \widehat{\eta}\left(F\frac{\sinh|k|y'}{\sinh|k|}\right).\end{aligned} \tag{2.29}$$

Notice that for all $\alpha \geq 1$

$$|k|^{2\alpha}|\widehat{\psi}^h|^2 \leq C(|k|^{2\alpha}|\widehat{B}|^2 + |k|^{2\alpha}|\widehat{\eta}|^2),$$

which implies

$$\|\widetilde{\psi}^h\|_{H^\alpha(\mathcal{S})} \leq C(|B|_{H^{\alpha-\frac{1}{2}}} + |\eta|_{H^{\alpha-\frac{1}{2}}}). \tag{2.30}$$

2.4.2 Estimates on the non-homogeneous associated problem

$$\begin{cases} -|k|^2\widehat{\psi}^p + \dfrac{d^2\widehat{\psi}^p}{dy'^2} = \epsilon\widehat{G}, & \mathcal{S}\\ \widehat{\psi}^p = 0, & y' = 0,\\ \widehat{\psi}^p = 0, & y' = 1.\end{cases} \tag{2.31}$$

Taking the $\alpha - 3/2$ order derivative with respect to y',

$$-|k|^2 D_{y'}^{\alpha-\frac{3}{2}}\widehat{\psi}^p + D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p = \epsilon D_{y'}^{\alpha-\frac{3}{2}}\widehat{G}.$$

Multiplying by $|k|^{2\alpha-1}D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p$ and integrating on $\mathcal{S}$

$$-\int_{\mathcal{S}}|k|^{2\alpha+1}D_{y'}^{\alpha-\frac{3}{2}}\widehat{\psi}^p D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p + \int_{\mathcal{S}}|k|^{2\alpha-1}D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p = \epsilon\int_{\mathcal{S}}|k|^{2\alpha-1}D_{y'}^{\alpha-\frac{3}{2}}\widehat{G}D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p;$$

namely,

$$\int_{\mathcal{S}}|k|^{2\alpha+1}|D_{y'}^{\alpha-\frac{1}{2}}\widehat{\psi}^p|^2 + \int_{\mathcal{S}}|k|^{2\alpha-1}|D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p|^2 = \epsilon\int_{\mathcal{S}}|k|^{2\alpha-1}D_{y'}^{\alpha-\frac{3}{2}}\widehat{G}D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p. \tag{2.32}$$

Let us estimate the right-hand side of this last expression in terms of $\|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(x,y')}$:

$$\epsilon\int|k|^{2\alpha+1}D_{y'}^{\alpha-\frac{3}{2}}\mathcal{F}[2(\phi_x\widetilde{\psi}_{y'})_x-\phi_{xx}\widetilde{\psi}_{y'}+2\phi_{y'}\widetilde{\psi}_{y'y'}+\phi_{y'y'}\widetilde{\psi}_{y'}]D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p$$
$$-\epsilon^2\int\int|k|^{2\alpha-1}D_{y'}^{\alpha-\frac{3}{2}}\mathcal{F}[(\phi_x^2+\phi_{y'}^2)\widetilde{\psi}_{y'y'}]D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p$$

$$\begin{aligned}=&\epsilon\int\left[|k|^{2\alpha}D_{y'}^{\alpha-\frac{1}{2}}(\widehat{-2\phi_x\widetilde{\psi}_{y'}})+|k|^{2\alpha-1}D_{y'}^{\alpha-\frac{1}{2}}(\widehat{\phi_{xx}\widetilde{\psi}_{y'}})\right]D_{y'}^{\alpha-\frac{1}{2}}\widehat{\psi}^p\\&+\epsilon\int|k|^{2\alpha-1}D_{y'}^{\alpha-\frac{3}{2}}(2\widehat{\phi_{y'}\widetilde{\psi}_{y'y'}}+\widehat{\phi_{y'y'}\widetilde{\psi}_{y'}})D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p\\&-\epsilon^2\int|k|^{2\alpha-1}D_{y'}^{\alpha-\frac{3}{2}}(\widehat{\phi_x^2\widetilde{\psi}_{y'y'}}+\widehat{\phi_{y'}^2\widetilde{\psi}_{y'y'}})D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p\end{aligned}$$

$$\begin{aligned}=&\epsilon\int\left[|k|^{\alpha-\frac{1}{2}}D_{y'}^{\alpha-\frac{1}{2}}(\widehat{-2\phi_x\widetilde{\psi}_{y'}})+|k|^{\alpha-\frac{3}{2}}D_{y'}^{\alpha-\frac{1}{2}}(\widehat{\phi_{xx}\widetilde{\psi}_{y'}})\right]|k|^{\alpha+\frac{1}{2}}D_{y'}^{\alpha-\frac{1}{2}}\widehat{\psi}^p\\&+\epsilon\int|k|^{\alpha-\frac{1}{2}}D_{y'}^{\alpha-\frac{3}{2}}(2\widehat{\phi_{y'}\widetilde{\psi}_{y'y'}}+\widehat{\phi_{y'y'}\widetilde{\psi}_{y'}})|k|^{\alpha-\frac{1}{2}}D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p\\&-\epsilon^2\int|k|^{\alpha-\frac{1}{2}}D_{y'}^{\alpha-\frac{3}{2}}(\widehat{\phi_x^2\widetilde{\psi}_{y'y'}}+\widehat{\phi_{y'}^2\widetilde{\psi}_{y'y'}})|k|^{\alpha-\frac{1}{2}}D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p;\end{aligned}$$

from this and (2.32), for ϵ sufficiently small

$$\begin{aligned}\int_{\mathcal{S}}|k|^{2\alpha-1}|D_{y'}^{\alpha+\frac{1}{2}}\widehat{\psi}^p|^2\le&\, C\epsilon(\|\phi_x\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}+\|\phi_{xx}\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(H^{\alpha-\frac{3}{2}}(\mathcal{S}))})\\&+C\epsilon(\|\phi_x^2\widetilde{\psi}_{y'y'}+\phi_{y'}^2\widetilde{\psi}_{y'y'}\|_{H^{\alpha-\frac{3}{2}}(H^{\alpha-\frac{1}{2}}(\mathcal{S}))})\\\le&\, C\epsilon(|\eta|_{H^\alpha}+|\eta|_{H^\alpha}^2+|B|_{H^\alpha}+|B|_{H^\alpha}^2)\|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}.\end{aligned}$$

Performing similar calculations at lower orders we finally obtain

$$\|\widetilde{\psi}_{y'}^p\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}\le C\epsilon(|\eta|_{H^\alpha}+|\eta|_{H^\alpha}^2+|B|_{H^\alpha}+|B|_{H^\alpha}^2)\|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}.$$

Remark 2.9. As we did before, it is possible to replace right hand side G by $\breve{G}=\breve{G}(\breve{\psi})$ so that

$$\|\widetilde{\psi}_{y'}^p\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}\le C\epsilon(|\eta|_{H^\alpha}+|\eta|_{H^\alpha}^2+|B|_{H^\alpha}+|B|_{H^\alpha}^2)\|\breve{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}. \tag{2.33}$$

□

By (2.30), we find $\|\widetilde{\psi}^h\|_{H^{\alpha+\frac{1}{2}}}\le C(|B|_{H^\alpha}+|\eta|_{H^\alpha})$ and, taking $\breve{\psi}=\widetilde{\psi}^h+\breve{\psi}^p$ we estimate

$$\|\breve{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}\le C(|B|_{H^\alpha}+|\eta|_{H^\alpha})+\|\breve{\psi}_{y'}^p\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}. \tag{2.34}$$

From (2.33), (2.34) follows

Lemma 2.10. *Let $\alpha\in\mathbb{R}$, $\alpha>\frac{1}{2}$, and $\eta,b\in H^\alpha(\mathbb{R})$. If $\widetilde{\psi}_{y'}^p\in H^{\alpha-\frac{1}{2}}(\mathcal{S})$ solves the non-homogeneous problem (2.31), then, for all $\breve{\psi}_{y'}^p\in H^{\alpha-\frac{1}{2}}(\mathcal{S})$ one has*

$$\begin{aligned}\|\widetilde{\psi}_{y'}^p\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}\le&\, C\epsilon(|\eta|_{H^\alpha}^2+|\eta|_{H^\alpha}^3+|B|_{H^\alpha}^2+|B|_{H^\alpha}^3)\\&+C\epsilon(|\eta|_{H^\alpha}+|\eta|_{H^\alpha}^2+|B|_{H^\alpha}+|B|_{H^\alpha}^2)\|\breve{\psi}_{y'}^p\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}.\end{aligned}$$

Theorem 2.11 (Solution of the hydrodynamical problem on $\mathcal{S}$)**.** *Let $\alpha \in \mathbb{R}$, $\alpha > \frac{1}{2}$ and $\eta, b \in H^{\alpha}(\mathbb{R})$ such that $|\eta|_{H^\alpha}, |b|_{H^\alpha} < 1$. Then, for ϵ sufficiently small, there exists a unique solution $\widetilde{\psi}_{y'} \in H^{\alpha-\frac{1}{2}}(\mathcal{S})$ to (2.27). Moreover, there exist a constant $C > 0$ such that*

$$\|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \leq C(|\eta|_{H^\alpha} + |B|_{H^\alpha}).$$

Proof. Let us consider the mapping

$$T : \begin{array}{ccc} H^{\alpha-\frac{1}{2}} & \to & H^{\alpha-\frac{1}{2}} \\ \breve{\psi}_{y'} & \mapsto & T(\breve{\psi}_{y'}) = \widetilde{\psi}^h_{y'} + \widetilde{\psi}^p_{y'}. \end{array}$$

Then, by (2.30) and lemma 2.10 we have

$$\begin{aligned} \|T(\breve{\psi}_{y'})\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} &\leq C(|\eta|_{H^\alpha} + |B|_{H^\alpha}) + C\epsilon(|\eta|^2_{H^\alpha} + |\eta|^3_{H^\alpha} + |B|^2_{H^\alpha} + |B|^3_{H^\alpha}) \\ &\quad + C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha} + |B|_{H^\alpha} + |B|^2_{H^\alpha})\|\breve{\psi}^p_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})}; \end{aligned}$$

therefore, if ϵ is sufficiently small, T maps $B_{2s}(0)$ into itself where $s = C(|\eta|_{H^\alpha} + |B|_{H^\alpha})$. Furthermore, due to the linearity of T (since the PDE and the boundary conditions are linear), we can estimate

$$\|T(\breve{\psi}_{1,y'} - \breve{\psi}_{2,y'})\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \leq C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha} + |B|_{H^\alpha} + |B|^2_{H^\alpha})\|\breve{\psi}_{1,y'} - \breve{\psi}_{2,y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})},$$

implying that T is a contraction for ϵ small enough. Thus, by the Banach fixed point theorem, there exists a unique $\widetilde{\psi}_{y'} \in H^{\alpha-\frac{1}{2}}(\mathcal{S})$ such that $T(\widetilde{\psi}_{y'}) = \widetilde{\psi}_{y'}$. Moreover,

$$\begin{aligned} \|\widetilde{\psi}_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} &\leq \frac{C}{1 - C\epsilon(|\eta|_{H^\alpha} + |\eta|^2_{H^\alpha} + |B|_{H^\alpha} + |B|^2_{H^\alpha})}(|\eta|_{H^\alpha} + |B|_{H^\alpha}) \\ &\leq 2C(|\eta|_{H^\alpha} + |B|_{H^\alpha}). \end{aligned}$$

□

Remark 2.12. As a consequence of lemma 2.10 and theorem 2.11 one has, for all $\alpha \in \mathbb{R}$, $\alpha > \frac{1}{2}$ and ϵ sufficiently small

$$\|\widetilde{\psi}^p_{y'}\|_{H^{\alpha-\frac{1}{2}}(\mathcal{S})} \leq C\epsilon(|\eta|^2_{H^\alpha} + |\eta|^3_{H^\alpha} + |B|^2_{H^\alpha} + |B|^3_{H^\alpha});$$

therefore, by Poincaré's inequality,

$$\|\widetilde{\psi}^p\|_{H^{\alpha+\frac{1}{2}}(\mathcal{S})} \leq C\epsilon(|\eta|^2_{H^\alpha} + |\eta|^3_{H^\alpha} + |B|^2_{H^\alpha} + |B|^3_{H^\alpha}).$$

□

Finally, proceeding as before, one has that the linear operator

$$\mathcal{F}(\eta, F) = \eta + F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}^2_{x'} + \widetilde{\psi}^2_{y'}) - \epsilon^2\eta_x\widetilde{\psi}_{x'}\widetilde{\psi}_y + \frac{\epsilon^3}{2}\eta_x^2\widetilde{\psi}^2_{y'},$$

is such that, for $\alpha \geq 1$,

$$\mathcal{F} : H^\alpha \times (0, +\infty) \to H^{\alpha-1}.$$

2.5 Existence of bifurcation branches

As before, we are working on the domain Ω defined by

$$\Omega = \{(x,y) \in \mathbb{R}^2 : -L < x < L,\ 0 < y < 1 + \eta(x)\},$$

where $\eta(x)$ belongs to the set $H^\alpha(-L,L)$, $\alpha \geq 1$. Recall from section 2 that, for small variations of the free surface, it is possible to perform a transformation such that the Bernoulli equation can be written on a flat domain as

$$\mathcal{F}(\eta, F) = \eta + F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}^2_{x'} + \widetilde{\psi}^2_{y'}) - \epsilon^2 \eta_{x'}\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} + \frac{\epsilon^3}{2}\eta^2_{x'}\widetilde{\psi}^2_{y'} = 0.$$

In formulating the bifurcation problem we first solve, for given $\eta(x)$, the elliptic problem

$$\begin{cases} \Delta\psi = 0, & \Omega, \\ \psi_x = 0, & x = -L, L, \\ \psi = 0, & y = 0, \\ \psi = F, & y = 1 + \eta(x) \end{cases}$$

and then implement the boundary condition (the Bernoulli equation after flatting),

$$\widetilde{\psi}_{y'} = -\frac{\eta}{F} - \frac{\epsilon}{2F}(\widetilde{\psi}^2_{x'} + \widetilde{\psi}^2_{y'}) + \frac{\epsilon^2}{F}\eta_{x'}\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} - \frac{\epsilon^3}{2F}\eta^2_{x'}\widetilde{\psi}^2_{y'}.$$

That is,

$$\mathcal{F}(\eta, F) = 0. \tag{2.35}$$

Notice that for all $F > 0$

$$\mathcal{F}(0, F) = 0.$$

As in [20], we shall solve the bifurcation problem (2.35) using the Crandall-Rabinowitz theorem.

Theorem 2.13 (See [15])**.** *Let X, Y be real Banach spaces and $\mathcal{F}(\eta, F)$ a C^p map, $p \geq 3$, of a neighborhood of $(0, F_0)$ in $X \times \mathbb{R}$ into Y with $\mathcal{F}(0, F) = 0$. Suppose*

(i) $\mathcal{F}_F(0, F_0) = 0$,

(ii) $\ker \mathcal{F}_\eta(0, F_0)$ *is one dimensional, spanned by* η_0,

(iii) $\operatorname{Im} \mathcal{F}_\eta(0, F_0) = Y_1$ *has codimension* 1,

(iv) $\mathcal{F}_{FF}(0, F_0) \in Y_1$ *and* $\mathcal{F}_{F\eta}(0, F_0)\eta_0 \notin Y_1$.

Then $(0, F_0)$ is a bifurcation point of the equation $\mathcal{F}(\eta, F) = 0$ in the following sense: In a neighborhood of $(0, F_0)$ the set of solutions of (2.35) consists of two C^{p-2} smooth curves Γ_1 and Γ_2 which intersect only at the point $(0, F_0)$; Γ_1 is the curve $(0, F)$ and Γ_2 can be parametrized as follows:

$$\Gamma_2 : (\eta(\epsilon), F(\epsilon)),\ |\epsilon| \text{ small},$$

$$(\eta(0), F(0)) = (0, F_0),\ \eta'(0) = \eta_0.$$

We shall prove the existence of bifurcation branches of solutions at

$$F^2 = \frac{\tanh(k_n)}{k_n} = \frac{\tanh \frac{2n+1}{2}\frac{\pi}{L}}{\frac{2n+1}{2}\frac{\pi}{L}}, \quad n \geq 0,$$
$$F^2 = \frac{\tanh(l_n)}{l_n} = \frac{\tanh \frac{n\pi}{L}}{\frac{n\pi}{L}}, \quad n \geq 1.$$

2.5.1 The Bernoulli equation and functional related

In this section we proceed to compute and analyze the linear operator $\mathcal{F}_\eta(0,F)$. On $y'=1$, let

$$\mathcal{F}(\eta, F) = \eta + F\widetilde{\psi}_{y'} + \frac{\epsilon}{2}(\widetilde{\psi}^2_{x'} + \widetilde{\psi}^2_{y'}) - \epsilon^2 \eta_{x'}\widetilde{\psi}_{x'}\widetilde{\psi}_{y'} + \frac{\epsilon^3}{2}\eta^2_{x'}\widetilde{\psi}^2_{y'}.$$

By the results above, for $\alpha \geq 1$

$$\mathcal{F} : H^\alpha \times (0, +\infty) \to H^{\alpha-1},$$
$$\mathcal{F}(0, F) = 0.$$

We are going to compute now the first order derivative of functional $\mathcal{F}$ with respect to η.

$$\begin{aligned}
\mathcal{F}(\eta, F) - \mathcal{F}(0,F) &= \eta + F\widetilde{\psi}^h_{y'}|_{y'=1} + O(\epsilon) \\
&= \eta + F\left(-F\sum_{n=0}^{\infty}\frac{k_n}{\tanh(k_n)}a_n\sin(k_n x') - F\sum_{n=1}^{\infty}\frac{l_n}{\tanh(l_n)}b_n\cos(l_n x')\right) \\
&\quad + O(\epsilon) \\
&= \eta - F^2\left(\sum_{n=0}^{\infty}\frac{k_n}{\tanh(k_n)}a_n\sin(k_n x') + \sum_{n=1}^{\infty}\frac{l_n}{\tanh(l_n)}b_n\cos(l_n x')\right) + O(\epsilon) \\
&= (I - F^2\mathcal{A}_F)\eta + O(\epsilon),
\end{aligned}$$

with

$$\mathcal{A}_F : \begin{pmatrix} a_0 \\ b_1 \\ a_1 \\ b_2 \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{k_0}{\tanh(k_0)} & & & & \\ & \frac{l_1}{\tanh(l_1)} & & & \\ & & \frac{k_1}{\tanh(k_1)} & & \\ & & & \frac{l_2}{\tanh(l_2)} & \\ & & & & \ddots \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ b_1 \\ a_1 \\ b_2 \\ \vdots \end{pmatrix}.$$

Then, $\mathcal{F}_\eta|_{(0,F)} : H^\alpha \to H^{\alpha-1}$, is such that

$$\mathcal{F}_\eta|_{(0,F)}(a_0, b_1, a_1, b_2, ...)^T = (a_0, b_1, a_1, b_2, ...)^T - F^2(\frac{k_0}{\tanh(k_0)}a_0, \frac{l_1}{\tanh(l_1)}b_1, ...)^T. \quad (2.36)$$

If we take $F^2 = \frac{\tanh(k_0)}{k_0} < 1$, then the elements $(a_0, 0, 0, ...) \in H^\alpha$, $a_0 \in \mathbb{R}$, belong to $\ker \mathcal{F}_\eta|_{(0,F)}$; namely, $\mathcal{F}_\eta|_{(0,F)}$ has a non-null kernel.

Similarly for $F^2 = \frac{\tanh(l_1)}{l_1}$, $F^2 = \frac{\tanh(k_1)}{k_1}$ and so on. Then one has, possibly, infinitely many bifurcation branches for F^2 given by

$$F^2 = \frac{\tanh(k_n)}{k_n} = \frac{\tanh \frac{2n+1}{2}\frac{\pi}{L}}{\frac{2n+1}{2}\frac{\pi}{L}}, \quad n \geq 0, \quad (2.37)$$
$$F^2 = \frac{\tanh(l_n)}{l_n} = \frac{\tanh \frac{n\pi}{L}}{\frac{n\pi}{L}}, \quad n \geq 1.$$

Remark 2.14. In the case of a nontrivial bottom, as in remark 2.3,

$$\mathcal{F} : H^{\alpha} \times (0, +\infty) \to H^{\alpha-1}.$$

Since that $\Psi^p = O(\epsilon)$ and $\bar{\psi}^p = O(\epsilon)$,

$$\begin{aligned}
\mathcal{F}(\eta, F) - \mathcal{F}(0, F) &= \eta + F\Psi^h_{y'}|_{y'=1} - \bar{\psi}^h_{y'}|_{y'=1} + O(\epsilon) \\
&= \eta + F\widetilde{\psi}^h_{y'}|_{y'=1} + O(\epsilon) \\
&= \eta - F^2 \left(\sum_{n=0}^{\infty} \frac{k_n}{\tanh(k_n)} a_n \sin(k_n x') + \sum_{n=1}^{\infty} \frac{l_n}{\tanh(l_n)} b_n \cos(l_n x') \right) + O(\epsilon) \\
&= (I - F^2 \mathcal{A}_F)\eta + O(\epsilon).
\end{aligned}$$

Then, if one avoids the bifurcation points, $\ker \mathcal{F}_\eta|_{(0,F)} = \{0\}$. Therefore, by the implicit function theorem, for all $F > 0$, $F \neq \frac{\tanh(k_n)}{k_n}$, $F \neq \frac{\tanh(l_n)}{l_n}$, $n \in \mathbb{N}$, and if $B \in H^\alpha(\mathbb{R})$, $|B|_{H^\alpha} < 1$, there exist open neighborhoods U of F and V of $\eta = 0$ and a differentiable function $g : U \to V$ such that $\eta = g(F)$. $\square$

First bifurcation point: $(F_0)^2 = \frac{\tanh \frac{\pi}{2L}}{\frac{\pi}{2L}} = 1 + O(L^{-2})$.

Next, we are going to verify the assumptions (i)-(iv) of the Crandall-Rabinowitz theorem for the real Banach spaces $X = H^\alpha(-L, L)$, $Y = H^{\alpha-1}(-L, L)$, $\alpha \geq 1$. Let us to use the following notation:

$$A_{F_0} = \mathcal{F}_\eta|_{(\eta=0, F=F_0)}.$$

We are going to verify the four conditions of the Crandall-Rabinowitz theorem.

(i) $\mathcal{F}_F|_{(0,F_0)} = \widetilde{\psi}^h_{y'}|_{(0,F_0)} + \widetilde{\psi}^p_{y'}|_{(0,F_0)} = 0.$

(ii) It is clear that $\ker A_{F_0} = \{\sin(k_0 x')\} = \{\eta_0\}$ with $\eta_0 = \sin \frac{\pi x'}{2L}$. Furthermore, $\operatorname{Im} A_{F_0} = H^{\alpha-1} \ominus \{\sin \frac{\pi x'}{2L}\}$. So that $\dim(\ker A_{F_0}) = \operatorname{codim}(\operatorname{Im} A_{F_0}) = 1$.

(iii) $\mathcal{F}_F|_{(0,F_0)} = \widetilde{\psi}^h_{y'}|_{(0,F_0)} + \widetilde{\psi}^p_{y'}|_{(0,F_0)} = 0$ and then $\mathcal{F}_{FF}|_{(0,F_0)} = 0 \in \operatorname{Im} A_{F_0}$.

(iv) $\mathcal{F}_{F\eta}|_{(0,F_0)}(a_0, b_1, a_1, b_2, ...) = -2F_0(\frac{k_0}{\tanh(k_0)} a_0, \frac{l_1}{\tanh(l_1)} b_1, ...)$. Then, $\mathcal{F}_{F\eta}|_{(0,F_0)}(1, 0, 0, ...) = -2F_0(\frac{k_0}{\tanh(k_0)}, 0, 0, ...) \in \ker A_{F_0}$; namely, $\mathcal{F}_{F\eta}|_{(0,F_0)}(1, 0, 0, ...) \notin \operatorname{Im} A_{F_0}$.

Applying Crandall-Rabinowitz theorem, for each F as in (2.37), there exists a bifurcation branch of solutions of the free boundary value problem (2.8). Since $\eta_0 = \sin \frac{\pi x'}{2L}$, the assertion $\eta'(0) = \eta_0$ of the Crandall-Rabinowitz theorem implies that the free boundary has the form

$$y = 1 + \epsilon \sin \frac{\pi x'}{2L} + O(\epsilon^2).$$

2.5.2 Nature of the bifurcation in the point $(\eta, F) = (0, F_0)$

Since $F = F_0 = F_{00} + \epsilon F_{01} + ...$, and the bifurcation point F_0 is the one most relevant to the physical problem, it is of interest to determine $F'(0) = F_{01}$ and, in particular, its sign.

Let us write the free boundary as $y = 1 + \eta$. Along the free boundary

$$\eta = \epsilon \sin \frac{\pi x'}{2L} + \epsilon^2 \eta_2 + O(\epsilon^3), \quad \eta_2 \in \{\sin \frac{\pi x'}{2L}\}^\perp.$$

By Taylor's expansion in η,

$$0 = \mathcal{F}(\eta, F) = \mathcal{F}(0, F) + \mathcal{F}_\eta(0, F)\eta + \frac{1}{2}\eta\mathcal{F}_{\eta\eta}(0, F)\eta + O(\|\eta\|^3) \qquad (2.38)$$
$$= \mathcal{F}_\eta(0, F)\eta + \frac{1}{2}\eta\mathcal{F}_{\eta\eta}(0, F)\eta + O(\|\eta\|^3).$$

Since $F = F_0 + \epsilon F'(0) + O(\epsilon^2)$,

$$\begin{aligned}\mathcal{F}_\eta(0, F)\eta &= \mathcal{F}_\eta(0, F_0)\eta + (\epsilon F'(0) + O(\epsilon^2))\mathcal{F}_{\eta F}(0, F_0)\eta + O(\epsilon^3)\\ &= \epsilon\mathcal{F}_\eta(0, F_0)\sin\frac{\pi x'}{2L} + \epsilon^2\mathcal{F}_\eta(0, F_0)\eta_2 + \epsilon^2 F'(0)\mathcal{F}_{\eta F}(0, F_0)\sin\frac{\pi x'}{2L} + O(\epsilon^3)\\ &= \epsilon^2\mathcal{F}_\eta(0, F_0)\eta_2 + \epsilon^2 F'(0)\mathcal{F}_{\eta F}(0, F_0)\sin\frac{\pi x'}{2L} + O(\epsilon^3),\end{aligned}$$

because $\sin\frac{\pi x'}{2L} \in \ker A_{F_0}$.

From the $O(\epsilon^2)$ terms in (2.38) we get an equation for $F'(0)$:

$$\mathcal{F}_\eta(0, F_0)\eta_2 + F'(0)\mathcal{F}_{\eta F}(0, F_0)\sin\frac{\pi x'}{2L} + \frac{1}{2}\sin\frac{\pi x'}{2L}\mathcal{F}_{\eta\eta}(0, F)\sin\frac{\pi x'}{2L} = 0;$$

now, using that η_2 is orthogonal to $\sin\frac{\pi x'}{2L}$,

$$F'(0) = \frac{-\frac{1}{2}\left(\sin\frac{\pi x'}{2L}, \sin\frac{\pi x'}{2L}\mathcal{F}_{\eta\eta}(0, F)\sin\frac{\pi x'}{2L}\right)_{L^2}}{\left(\sin\frac{\pi x'}{2L}, \mathcal{F}_{\eta F}(0, F_0)\sin\frac{\pi x'}{2L}\right)_{L^2}}.$$

From (2.36) one has

$$\mathcal{F}_{F\eta}|_{(0,F_0)}(a_0, b_1, a_1, b_2, ...) = -2F(\frac{k_0}{\tanh(k_0)}a_0, \frac{l_1}{\tanh(l_1)}b_1, ...).$$

Therefore,

$$\mathcal{F}_{F\eta}|_{(0,F_0)}\sin\frac{\pi x'}{2L} = -2F\frac{k_0}{\tanh(k_0)}\sin\frac{\pi x'}{2L},$$

and then

$$\left(\sin\frac{\pi x'}{2L}, \mathcal{F}_{\eta F}(0, F_0)\sin\frac{\pi x'}{2L}\right)_{L^2} = -2F\frac{k_0}{\tanh(k_0)}\int_{-L}^{L}\sin^2\frac{\pi x'}{2L}dx' = -2LF\frac{k_0}{\tanh(k_0)}.$$

We are going to compute now $\mathcal{F}_{\eta\eta}(0, F)$.

There exists a unique solution of the problem

$$\begin{cases}\Delta\psi = 0, & \Omega,\\ \psi_x = 0, & x = -L, L,\\ \psi = 0, & y = 0,\\ \psi = F, & y = 1 + \eta,\\ \frac{1}{2}|\nabla\psi|^2 = -y + \frac{F^2}{2} + 1, & y = 1 + \eta.\end{cases}$$

$$\psi = Fy + \epsilon\psi_1 + \epsilon^2\psi_2 + O(\epsilon^3).$$

On the surface ($y = 1 + \eta$),

$$F = F(1 + \eta) + \epsilon\psi_1 + \epsilon^2\psi_2 + O(\epsilon^3).$$

Then

$$F\eta + \epsilon\psi_1 + \epsilon^2\psi_2 + O(\epsilon^3) = 0. \tag{2.39}$$

We already know that $\eta = \epsilon \sin \frac{\pi x}{2L} + \epsilon^2 \eta_2$ with $\eta_2 \in \{\sin \frac{\pi x}{2L}\}^\perp$. If we impose $\psi_1 = -F \sin \frac{\pi x}{2L}$ on $y = 1 + \eta$, then

$$\psi_1(x, y) = -F \sin\left(\frac{\pi x}{2L}\right) \frac{\sinh \frac{\pi y}{2L}}{\sinh \frac{\pi}{2L}}, \quad \text{for all}(x, y) \in \Omega'.$$

From (2.39), on $y = 1 + \eta$,

$$F(\epsilon \sin \frac{\pi x}{2L} + \epsilon^2\eta_2) - \epsilon F \sin\left(\frac{\pi x}{2L}\right) \frac{\sinh \frac{\pi(1+\epsilon \sin \frac{\pi x}{2L} + \epsilon^2\eta_2)}{2L}}{\sinh \frac{\pi}{2L}} + \epsilon^2\psi_2 + O(\epsilon^3) = 0$$

$$F(\epsilon \sin \frac{\pi x}{2L} + \epsilon^2\eta_2) - \epsilon F \sin\left(\frac{\pi x}{2L}\right) \frac{\sinh \frac{\pi}{2L} + \epsilon\frac{\pi}{2L} \sin \frac{\pi x}{2L} \cosh \frac{\pi}{2L} + O(\epsilon^2)}{\sinh \frac{\pi}{2L}} + \epsilon^2\psi_2 + O(\epsilon^3) = 0$$

$$\epsilon^2 F\eta_2 - \epsilon^2 F \frac{\pi}{2L} \coth\left(\frac{\pi}{2L}\right) \sin^2\left(\frac{\pi x}{2L}\right) + \epsilon^2\psi_2 + O(\epsilon^3) = 0;$$

namely,

$$\psi_2|_{y=1+\eta} = -F\eta_2 + F\frac{\pi}{2L} \coth\left(\frac{\pi}{2L}\right) \sin^2\left(\frac{\pi x}{2L}\right).$$

Therefore

$$\psi_2(x, y) = -F\eta_2 \frac{\sinh \frac{\pi y}{2L}}{\sinh \frac{\pi}{2L}} + F\frac{\pi}{2L} \coth\left(\frac{\pi}{2L}\right) \sin^2\left(\frac{\pi x}{2L}\right) \frac{\sinh \frac{\pi y}{2L}}{\sinh \frac{\pi}{2L}},$$

and

$$(\psi_2)_y|_{y=1+\eta} = -F\frac{\pi}{2L}\eta_2 \coth \frac{\pi}{2L} + O(\epsilon) + F\left(\frac{\pi}{2L}\right)^2 \coth^2\left(\frac{\pi}{2L}\right) \sin^2\left(\frac{\pi x}{2L}\right) + O(\epsilon),$$

thus one has

$$\begin{aligned}\left(\sin \frac{\pi x}{2L}, (\psi_2)_y \sin \frac{\pi x}{2L}\right)_{L^2} &= F\left(\frac{\pi}{2L}\right)^2 \coth^2\left(\frac{\pi}{2L}\right) \int_{-L}^{L} \sin^4\left(\frac{\pi x}{2L}\right) \\ &= \frac{3}{4}LF\left(\frac{\pi}{2L}\right)^2 \coth^2\left(\frac{\pi}{2L}\right).\end{aligned}$$

Finally

$$F'(0) = \frac{-\frac{3}{8}LF\left(\frac{\pi}{2L}\right)^2 \coth^2\left(\frac{\pi}{2L}\right)}{-2LF\frac{k_0}{\tanh(k_0)}} = \frac{3}{16}\frac{k_0}{\tanh(k_0)} > 0.$$

This characterizes the bifurcation as a trans-critical one. ϵ increases as F increases from the bifurcation point. This implies that, at least for ϵ small enough, the positive hydraulic jump, with η increasing in the direction of the flow (stable regime) occurs for $F > F_0$.

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