



On fractional extensions of Barbalat Lemma



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HIGHLIGHTS

- Barbalat-like lemmas for fractional integral are posited.
- Qualitative analysis of asymptotic properties for some types of fractional differential equations is shown to indicate applicability of those lemmas.
- Error Model of Type I with fractional derivative is comparatively analyzed.

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ABSTRACT

This paper presents Barbalat-like lemmas for fractional order integrals, which can be used to conclude about the convergence of a function to zero, based on some conditions upon its fractional integral. Some examples in the context of asymptotic behaviour of solutions of fractional order differential equations, indicate the potential application of these lemmas in control theory.

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1. Introduction

Barbalat Lemma is a fundamental result in asymptotic analysis of differential equation solutions and thereby in control theory, relating the convergence of an integral with the convergence of its integrand. For instance, the major theorems in adaptive control (in the MRAC framework) rely on this Lemma and its Corollaries [1].

As expected, many publications have this Lemma as a main element of their developments. Among them, relevant to our work are those where a change in the hypothesis of original Lemma is proposed [2] or this is applied in a somehow different context [3].

On the other hand, many techniques to study the properties of solutions of integer order differential equations have recently been generalized to the fractional order case, such as Laplace Transform, Lyapunov functions and frequency methods [4].

Then, two natural questions arise; is there an analogue of Barbalat Lemma for fractional integrals? If so, has it any utility in the analysis of solutions of fractional order differential equations and in control theory? Partial responses to both questions were given in [5]. The present work studies these questions and displays the results in the following way: Section 2 provides the necessary background to understand the answers to the first question which are presented in Sections 3 and 4. Section 5 contains our answers to the second question, while Section 6 is devoted to general conclusions and to open questions arisen from precedent sections.

2. Preliminaries and notation

Some useful definitions and properties (taken mainly from [6] except where indicated) are presented in this section for $\alpha > 0$.

Definition 1 (*Fractional Integral*). The fractional integral of order α of function $f(t)$ on the half axis \mathbb{R}^+ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau. \quad (1)$$

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Also, we denote $I_T^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_T^t (t-\tau)^{\alpha-1} f(\tau) d\tau$ and $I_{[0,T]}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^T (t-\tau)^{\alpha-1} f(\tau) d\tau$ when $t > T$.

In the following, $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = [\alpha]$ otherwise, where $[\alpha]$ denotes the integer part of the real number α .

Definition 2 (Caputo Derivative). The Caputo derivative of order α of function $f(t)$ on the half axis \mathbb{R}^+ is defined as

$${}^C D^\alpha f(t) = I^{n-\alpha} f^{(n)}(t). \quad (2)$$

Definition 3 (Riemann–Liouville Derivative). The Riemann–Liouville fractional derivative of order α of function $f(t)$ on the half axis \mathbb{R}^+ is defined as

$${}^R D^\alpha f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t). \quad (3)$$

We will denote ${}^C D^\alpha f(t)$ as $D^\alpha f(t)$ since we will be using mainly the Caputo fractional derivative throughout the paper because of its properties which will simplify our analysis. Also, by those properties, equations with Caputo fractional derivative use initial conditions of the function and its integer order derivatives – with a clear physical meaning – and therefore, the majority of the results found in control applications use Caputo's derivative and we would like to contribute with results in that direction. In the cases where we use the Riemann–Liouville fractional derivative we will use the notation given in (3).

An analogue to the fundamental theorem of integer calculus is stated in the next two properties for Caputo fractional derivative:

Property 4. If f belongs to $C^n[a, b]$, the space of continuous functions that have continuous first n derivatives, then for all $t \in [a, b]$

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=1}^n \frac{f^{(k)}(0)}{k!} t^k. \quad (4)$$

Property 5. If f belongs to $\mathcal{L}^\infty[(a, b)]$, the Lebesgue space of bounded functions in the interval (a, b) , then for all $t \in [a, b]$

$$D^\alpha I^\alpha f(t) = f(t). \quad (5)$$

The next property allows to give a semi group structure to the set of operators I^α [7].

Property 6. Let $\alpha > 0$ and $\beta > 0$. If $f \in C[a, b]$ then for all $t \in [a, b]$

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t). \quad (6)$$

Since our object of study is the generalization to the fractional order case of the Barbalat Lemma and corollary [1], next we will explicitly state the integer order versions of them.

Lemma 7. If $f(t) : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a uniformly continuous function for $t \geq 0$, such that $\lim_{t \rightarrow \infty} \int_0^\infty |f(\tau)| d\tau$ exists and is finite, then $\lim_{t \rightarrow \infty} f(t) = 0$.

Corollary 8. If $f \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ and \dot{f} is bounded, then $\lim_{t \rightarrow \infty} f(t) = 0$.

Perhaps, one of the most important applications of the Barbalat Lemma and Corollary in adaptive control theory is its use in proving the convergence of the identification error (in a identification scheme) and the convergence of the control error (in a direct model reference adaptive control scheme). See for instance Sections 3.3.1 and 3.3.2 in [1], respectively.

In what follows, the results are stated for initial condition at $t = 0$, but they can be easily generalized to initial condition at $t = a$ by a change of variable $t \rightarrow (t - a)$.

3. Results for $\alpha \geq 1$

In this section a generalization of the Corollary of Barbalat Lemma is presented for $\alpha \geq 1$ using conditions on the fractional order integral rather than on the integer integral.

Lemma 9. Let f be a non negative uniformly continuous function. If for all $t \geq 0$, $I^\alpha f(t) < C$ with C a positive constant and an integer $p \geq 1$, then f converges to zero.

Proof. We will proceed by contradiction. Let us suppose that f does not converge to zero. Then, by negation of the definition of convergence, $\exists \epsilon > 0$ and there exists an increasing divergent sequence $(t_i)_{i \in \mathbb{N}}$ such that $f(t_i) > \epsilon$. Since f is uniformly continuous $\exists \delta > 0$ such that $\forall i \in \mathbb{N}$ if $|t - t_i| < \delta$ then $|f(t) - f(t_i)| \leq \epsilon/2$. Therefore if $t \in ([t_i, t_i + \delta])$ then $|f(t)| = f(t) = |f(t_i) - f(t_i) + f(t)| \geq f(t_i) - |f(t_i) - f(t)| > \epsilon/2$.

Let $p(t)$ be a function null in every point except when $t \in ([t_i, t_i + \delta])$ where its value is $\epsilon/2$.

By definition $\Gamma(\alpha) I^\alpha f = \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$ and therefore

$$\Gamma(\alpha) I^\alpha f = \int_0^{t-1} (t-\tau)^{\alpha-1} f(\tau) d\tau + \int_{t-1}^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Since f is a positive function, we have

$$\Gamma(\alpha) I^\alpha f \geq \int_0^{t-1} (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Since $(t-\tau)^{\alpha-1} \geq 1$ for $0 \leq \tau \leq t-1$ and since $f(t) \geq p(t)$ for every t , we can write

$$\Gamma(\alpha) I^\alpha f \geq \int_0^{t-1} f(\tau) d\tau \geq \int_0^{t-1} p(\tau) d\tau \geq \sum_1^{[n_t]} \epsilon \delta / 2$$

where $n_t = \max\{i : t_i \leq t-1\}$. Taking limit when $t \rightarrow \infty$ we obtain

$$\infty > \Gamma(\alpha) C \geq \Gamma(\alpha) I^\alpha f \geq \int_0^{t-1} p(\tau) d\tau \rightarrow \infty$$

which contradicts the assumption that $I^\alpha f$ is bounded. Therefore, f converges to zero, as t tends to infinity. \square

Remark 10. For $\alpha = 1$ this Lemma implies Corollary 8 (and therefore is its generalization). In fact, since f is a bounded function with bounded derivative (by hypothesis of Corollary 8), f and f^2 are uniformly continuous functions, and since $f \in \mathcal{L}^2$ then $f^2 \in \mathcal{L}^1$, whereby applying Lemma 9, one have f^2 converges to zero, and in particular, f converges to zero. For $\alpha = 2$ Lemma holds only for $f \equiv 0$ because $I^2 f = I^1 I^1 f$ and since $I^1 f$ is an increasing function, $I^2 f$ is unbounded (f being uniformly continuous). For similar reason, the same conclusion can be derived for $\alpha > 2$.

Remark 11. Since $I^\alpha f$ can be expressed in terms of $t^{\alpha-1} * f$, where $*$ denotes the convolution operator, the Lemma can be thus extended: if the convolution $g * f$ is uniformly bounded where g is a monotone increasing non negative function ($g(t) = 0$ for $t < 0$) and f is a positive uniformly continuous function, then f converges to zero at infinity.

Corollary 12. Let us suppose that f is a bounded function in $C^{[\alpha]+1}(\mathbb{R}^+)$ and $f^{(k)}(0) = 0$ for $k = 0, 1, \dots, [\alpha] + 1$. If $D^\alpha f$ is positive and uniformly continuous then $D^\alpha f \rightarrow 0$.

Proof. It is a straightforward consequence of previous lemma by applying Property 4. \square

The next proposition is a mathematical observation.

Proposition 13. Let f be a positive continuous function. If for all $t \geq 0$, $I^\alpha f(t) < C$ with C a positive constant then $I^\alpha f$ converges as $t \rightarrow \infty$.

Proof. Since f is a positive function, $I^{\alpha-1}f$ is also positive function. By Property 6 we can write $I^\alpha f = I^1 I^{\alpha-1}f$ and therefore $I^\alpha f$ becomes an increasing function. Finally, since $I^{\alpha-1}f$ is bounded, it will converge as $t \rightarrow \infty$. \square

4. Results for $0 < \alpha < 1$

This section presents results obtained for $\alpha < 1$. In this case a direct generalization of the Corollary of Barbalat Lemma is not true as it is shown in the next proposition (the technical proof is given at the Appendix).

Proposition 14. There exists a non negative uniformly continuous function f such that $I^\alpha f(t) < C$, for all $t \geq 0$, with C a positive constant, but f does not converge to zero as t goes to infinity.

Nevertheless, one can assure at least the statement given in the next proposition.

Proposition 15. If f is a non negative bounded function and $I^\alpha f < C \in \mathbb{R}$, then $\liminf_{t \rightarrow \infty} f = 0$.

Proof. We prove it by contradiction. Let us assume that $\liminf_{t \rightarrow \infty} f = l > 0$ which exists because f is bounded, then there exist an arbitrary small $\epsilon > 0$ and $T = T(\epsilon) > 0$ such that $f \geq l - \epsilon > 0$ for all $t > T$. Therefore for $t > T$, $C > I^\alpha f \geq I^\alpha g = (l - \epsilon)(t - T)^\alpha + B_T \rightarrow \infty$ as $t \rightarrow \infty$ (where g takes the value of $l - \epsilon$ for all $t > T$ and the value of $f(t)$ for each $t < T$, and $B_T = I_{[0,T]}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^T (t - \tau)^{\alpha-1} g(\tau) d\tau$ which is positive when $t > T$), that is a contradiction. \square

Even for \mathcal{L}^p functions, since $\lim_{\alpha \rightarrow 0} I^\alpha f = f$ for almost every point [7], if $\lim_{t \rightarrow \infty} I^\alpha f = L \in \mathbb{R}$ and L is not zero, f does not converge to zero when f is a continuous function. Hence, if L is not zero, one cannot expect a general result for every $0 < \alpha < 1$ with hypotheses independent of α .

Thereby, we will make a stronger assumption (Proposition 13 does not necessary apply for $\alpha < 1$ because fractional integral of a non negative function can eventually decrease, so the convergence of the fractional integral cannot be assured from its boundedness) namely f a positive function and $\lim_{t \rightarrow \infty} I^\alpha f = L \in \mathbb{R}$, and we will seek for extra conditions assuring $\lim_{t \rightarrow \infty} f = 0$.

Also, it should be noted that if $L \neq 0$, $I f = I^{1-\alpha} I^\alpha f \rightarrow \infty$ (by Property 6). Thereby if $\lim_{t \rightarrow \infty} I^\alpha f = L$ and f is positive then $f \notin \mathcal{L}^1$. Thus this case effectively will extend the class of functions with bounded integer integral such that the function vanish at infinity to functions not in \mathcal{L}^1 but having convergent α -integral such that the function vanish at infinity.

Next, we remind the Lemma proved in [5]

Lemma 16. If f is a uniformly continuous function from \mathbb{R}^+ to \mathbb{R}^+ and $\lim_{t \rightarrow \infty} I^\alpha f(t) = 0$ then $\lim_{t \rightarrow \infty} f(t) = 0$ for $0 < \alpha \leq 1$. The case $\alpha = 1$ is only possible if $f(t) = 0$ for all t .

4.1. Extensions of Lemma 16

For the proofs of the lemmas of this subsection, we will first state and prove two interesting properties.

Property 17. If f is a bounded function that vanishes for all $t > T$ then $I^\alpha f \rightarrow 0$ and $D^\alpha f \rightarrow 0$ as $t \rightarrow \infty$. Moreover, $I^\alpha f$ will be a uniformly continuous function and if $D^\alpha f$ is continuous, $D^\alpha f$ will be a uniformly continuous function.

Proof. Since f is a bounded function vanishing for all $t > T$, we can write

$$|I^\alpha f| \leq \|f_{\max}\| I_{[0,T]}^\alpha 1. \quad (7)$$

Resolving the integral we get $|I^\alpha f| \leq C(t^\alpha - (t - T)^\alpha) = Ct^\alpha(1 - (1 - \frac{T}{t})^\alpha)$ where C is a constant defined in terms of $\|f_{\max}\|$ and $\Gamma(\alpha)$. Rewriting t^α as $1/t^{-\alpha}$ to use L'Hôpital's rule, the asymptotic behaviour when t goes to infinity is given by $|I^\alpha f| \leq Ct^{\alpha-1}$ which implies that $|I^\alpha f| \rightarrow 0$ and therefore $I^\alpha f \rightarrow 0$ as $t \rightarrow \infty$.

For the Caputo derivative part, note that f is constant after time T and therefore its derivative vanishes after time T . Then we can use the same argument as above to show that $D^\alpha f$ goes to zero as t goes to infinity.

Furthermore, since f is bounded, then $I^\alpha f$ is uniformly continuous (in fact, it is Hölder- α [7]). Since $D^\alpha f$ is a continuous time function and it is convergent to zero at infinity, then $D^\alpha f$ is uniformly continuous. \square

Property 18. Let f be a bounded function, if $I^\alpha f \rightarrow L$ as $t \rightarrow \infty$ then $I_T^\alpha f \rightarrow L$ as $t \rightarrow \infty$, for any $T > 0$.

Proof. According to notations given in Definition 1, $I^\alpha f$ can be written as

$$I^\alpha f - I_{[0,T]}^\alpha f = I_T^\alpha f. \quad (8)$$

Then the result will follow from Property 17 and algebra of limits. \square

The next Lemma is a simple extension of Lemma 16

Lemma 19. Let f be a bounded function and, after a time T , it is uniformly continuous and non negative (non positive) function. If $I^\alpha f \rightarrow 0$ as $t \rightarrow \infty$, then $f \rightarrow 0$ as $t \rightarrow \infty$.

Proof. Using Property 18 the result is straightforward, since Lemma 16 is also valid when initial time is $T \neq 0$ [5]. \square

Corollary 20. Let $D^\alpha f(t)$ be a uniformly continuous function in $C^1(\mathbb{R}^+)$ that changes sign up to time $T \geq 0$. If $f(t) \rightarrow f(0)$ as $t \rightarrow \infty$, then $D^\alpha f(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. By Property 4, we can write $I^\alpha D^\alpha f(t) = f(t) - f(0)$. Since $f(t)$ converges to $f(0)$, then $I^\alpha D^\alpha f(t)$ converges to zero as $t \rightarrow \infty$. Thus, $D^\alpha f(t)$ satisfies the hypothesis of last Lemma and therefore $D^\alpha f(t) \rightarrow 0$. \square

Now we will state the main extension of Lemma 16

Lemma 21. Let f be a bounded uniformly continuous function in $C^1(\mathbb{R}^+)$, the space of continuous functions that have continuous first derivative, such that $I^\alpha f \rightarrow L$ as $t \rightarrow \infty$. Then we can state the following:

(a) $I^\alpha f$ is decomposable as $I^\alpha f = \delta_T + \delta$ with $\delta(0) = 0$, $\delta(\infty) = 0$ and δ_T is a differentiable function with $\delta_T(0) = 0$ and $\delta_T(t > T) = L$.

(b) If $D^\alpha \delta$ does not change sign after a finite time (or vanishes after a finite time) then f converges to zero as $t \rightarrow \infty$

(c) If f is a positive function then f converges to zero as t goes to infinity.

Proof. (a) Since f is bounded and $I^\alpha f$ goes to L at infinity, there exist a continuous function $\bar{\delta}$ [7] so that $I^\alpha f = \bar{\delta}$ with $\bar{\delta}(0) = 0$ and $\bar{\delta}(\infty) = L$. Then, if we define $\delta = \bar{\delta} - \delta_T$ the result follows.

(b) Since f is bounded, by α -differentiating $I^\alpha f = \bar{\delta}$ and using Property 5, we can write

$$f = D^\alpha \delta_T + D^\alpha \delta. \quad (9)$$

By Property 17, we can assure that $D^\alpha \delta_T$ converges to zero and that it is uniformly continuous. Since f and $D^\alpha \delta_T$ are uniformly continuous functions in $C^1(\mathbb{R}^+)$, then $D^\alpha \delta$ is uniformly continuous

in $C^1(\mathbb{R}^+)$. Since $\delta(0) = 0$ and $\delta(\infty) = 0$, by using [Corollary 20](#) it can be concluded that $D^\alpha \delta$ goes to zero as $t \rightarrow \infty$ and therefore $f \rightarrow 0$ as $t \rightarrow \infty$.

(c) We define

$$(D^\alpha \delta)_+(t) = \begin{cases} D^\alpha \delta & \text{if } D^\alpha \delta \geq 0 \\ 0 & \text{if } D^\alpha \delta < 0 \end{cases}. \quad (10)$$

Similarly we define $(D^\alpha \delta)_-(t)$. Therefore we can write

$$D^\alpha \delta(t) = D^\alpha \delta_+(t) + D^\alpha \delta_-(t). \quad (11)$$

Since f is a positive function then $D^\alpha \delta_T + D^\alpha \delta > 0$. And since $D^\alpha \delta_T$ goes to zero as t goes to infinity, $(D^\alpha \delta)_-(t)$ also converges to zero. We will prove that $(D^\alpha \delta)_+(t)$ will also converge to zero as t goes to infinity. In fact, since $D^\alpha \delta$ is uniformly continuous, it follows that $(D^\alpha \delta)_+(t)$ is uniformly continuous and since $\delta(0) = 0$ and $\delta(\infty) = 0$ we can define a function $\hat{\delta}$ such that $(D^\alpha \delta)_+(t) = D^\alpha \hat{\delta}(t)$, in particular we have $D^\alpha[\delta - \hat{\delta}] = 0$ or $D^\alpha \hat{\delta} = 0$ with the initial condition $\hat{\delta}(0) = 0$ and, by integrating and applying [Proposition 14](#), the solution $\hat{\delta}$ decay to δ or to zero respectively, whereby $\hat{\delta}(\infty) = 0$, and applying part (b) $(D^\alpha \delta)_+ = D^\alpha \hat{\delta}$ converges to zero. Therefore, $D^\alpha \delta(t)$ converges to zero and, by [Eq. \(9\)](#), f converges to zero asymptotically. \square

Remark 22. If $\alpha = 1$ then $D^\alpha \delta_T = 0$ for $t > T$. Thus f converges to zero asymptotically by Part (b). Consequently, [Lemma 21](#) includes the case of the original Barbalat Lemma [1].

4.2. Alternative hypothesis

We will work with a Hölder continuity condition on the functions, which seems more natural than uniform continuity for fractional integrals as can be inferred from the many results in chapter 3 of [7]. In particular, we will need the following subspace.

Definition 23. We define H_α^λ as the subset of the Hölder- λ space of functions with asymptotic behaviour faster than $t^{-\alpha}$.

Let function g be defined by $I^\alpha f = g$ with $g(0) = 0$ and $g(\infty) = L$. If f is bounded then $f = D^\alpha g$. Since $D^\alpha g(t) = I^{1-\alpha} \dot{g}(t)$, f converges to zero if and only if $\dot{g}(t)$ has a vanishing $1 - \alpha$ fractional integral. An instance of the last condition is $\dot{g} = O(\exp(-t))$. The following lemma gives a sufficient condition for the convergence of f .

Lemma 24. Let us assume that $I^\alpha f = g$ with $g(0) = 0$ and $g(\infty) = L$. If f is bounded and $\dot{g} \in H_\alpha^\lambda$ with $0 < \lambda < \alpha$ then f converges to zero as t goes to infinity.

Proof. Since by hypothesis g converges to L as t goes to infinity and \dot{g} is uniformly continuous because it is Hölder- λ , by applying the integer Barbalat Lemma in its differential form [1], we conclude that \dot{g} converges to zero as t goes to infinity. It was proven in [7] that the behaviour of a function in H^λ when t goes to infinity is

$$I^{1-\alpha} \dot{g} \sim \dot{g}(0) \frac{t^\alpha}{1+t} + \dot{g}(\infty) \frac{t^{1+\alpha}}{1+t}. \quad (12)$$

Therefore, since $\dot{g}(\infty)$ goes to zero faster than $t^{-\alpha}$, f converges to zero as t goes to infinity. \square

Remark 25. A more familiar condition on g for the convergence of f is that $\dot{g} \in H^\lambda \cap \mathcal{L}^1$, because if the previous integral (18) does not converge to zero then $\dot{g} \notin \mathcal{L}^1$. Also, the statement of Lemma can be restated in terms of ${}^R D^{1-\alpha} f$ instead of \dot{g} . Hence, an equivalent condition on f will be that ${}^R D^{1-\alpha} f \in H_\alpha^\lambda$, besides the convergence of its fractional integral.

Remark 26. For the case when $\alpha = 1$, the differential version of the original Barbalat Lemma implies that \dot{g} goes to zero under the assumption of uniform continuity and, together with the assumption that $\lim_{\alpha \rightarrow 1} I^{1-\alpha} \dot{g} = \dot{g}$, f will go to zero asymptotically.

Corollary 27. Let us suppose that f converges to L as t goes to infinity, and for $\lambda < \alpha$, $\dot{f} \in H_\alpha^\lambda$. If $D^\alpha f$ is a uniformly continuous function in $C^1(\mathbb{R}^+)$ then $D^\alpha f$ goes to zero when t goes to infinity.

Proof. It follows from the fact that when t goes to infinity $I^\alpha D^\alpha f = f(t) - f(0) \rightarrow L - f(0)$. Then [Lemma 24](#) can be readily applied. \square

4.3. Unbounded case

So far, we have restricted our analysis to bounded functions. The next proposition relaxes this constraint and gives convergence conditions for functions in the following space:

Definition 28. Let $C(\mathbb{R}^+, \mathbb{R}^+)$ the space of continuous functions from \mathbb{R}^+ to \mathbb{R}^+ . We define the $C_{1-\alpha}(\mathbb{R}^+, \mathbb{R}^+)$ space as $C_{1-\alpha}(\mathbb{R}^+, \mathbb{R}^+) = \{f \in C(\mathbb{R}^+, \mathbb{R}^+) | f(t)t^\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)\}$.

Lemma 29. Let f be a positive function in $C_{1-\alpha}(\mathbb{R}^+, \mathbb{R}^+)$. If $I^\alpha f$ monotonically converges to a limit in \mathbb{R}^+ , i.e. $I^\alpha f = g$ is such that $\dot{g} \leq 0$ for all $t > t_0$, then f converges to zero asymptotically.

Proof. Since f is positive and $\dot{g} \leq 0$, then $I^\alpha f = g$ is bounded from below and not increasing. Therefore $I^\alpha f$ will converge as t goes to infinity.

On the other hand, since \dot{g} is less than or equal to zero, we can write $\frac{d}{dt} I^\alpha f \leq 0$ or equivalently

$${}^R D^{1-\alpha} f \leq 0. \quad (13)$$

Applying Lemma 2.12 in [8] (in this case $h \equiv 0$ and therefore hypothesis of Lemma 2.12 trivially hold) we can write

$$0 \leq f(t) \leq \frac{f_0}{\Gamma(1-\alpha)} t^{-\alpha} \quad (14)$$

with $f_0 = \Gamma(1-\alpha) \lim_{t \rightarrow 0} f(t)t^\alpha$. Therefore f converges to zero, as t goes to infinity. \square

5. Applications

In this section we will give several examples to illustrate the application of the proposed results.

5.1. Example 1

Let us consider the following integral equation (with degenerate kernel) defined by

$$I^\alpha f = g(t, f). \quad (15)$$

Let us assume that $g(t, f)$ converges to limit L as t goes to infinity. If $\alpha \geq 1$ and f is a positive uniformly continuous function, then f converges to zero as t goes to infinity by [Lemma 9](#). If $0 < \alpha < 1$ and g as required in [Lemma 19](#) or [Lemma 21](#) Part (b) or (c), then f goes to zero as t goes to infinity. For instance, we can choose $g(t, f) = f(0) - g(t)f(t)$ with $g(t)$ vanishing at infinite, $g(0) = 1$ and such that f is positive and uniformly continuous from an arbitrary finite time $t_1 \geq 0$ and on.

On the other hand, by taking $g(t) = A + Ct^{-\beta}$ with $0 < \alpha < 1$, where C is a real positive constant and A is a real constant, the equation has convergent solutions to zero if f is positive and belongs to $C_{1-\alpha}(\mathbb{R}^+, \mathbb{R}^+)$. This is because the derivative of $t^{-\beta}$ is negative for positive β . The rest follows from [Lemma 29](#).

5.2. Example 2

Let the fractional equation be defined as

$${}^R D^\alpha f = h \quad (16)$$

with $0 < \alpha < 1$. Eq. (16) can be rewritten as $\frac{d}{dt} I^{1-\alpha} f = h$. By the notation of Lemma 24 we have $\dot{g} = h$. If h is a bounded function such that the integral $I^{1-\alpha} f$ converges as t goes to infinity (if h changes sign up to time T and the integral $I^{1-\alpha} f$ is bounded, g will be monotone). If $h \in H^\lambda \cap L^1$ with $\lambda < \alpha$ then f vanishes as t goes to infinity.

5.3. Example 3

Let us consider the fractional order equation $D^\beta x = f(x, t)$ with $f(0, t) = 0$.

Let $W(x(t))$ be a uniformly continuous function with respect to time and let us assume that there exists a positive constant C such that (locally) holds:

$$W(x) \geq C \|x\|^2 \quad (17)$$

for all $t > T > 0$, for some T . Let us suppose that $W(x(t))$ satisfies for some $0 < \alpha < 1$

$${}^R D^\alpha [W(x)](t) \leq 0. \quad (18)$$

If ${}^R D^\alpha [W(x)](t) \in H^\lambda_\alpha$ then the trajectory $x(t)$ starting from any $x(0)$ in a vicinity of the origin converges to zero as t goes to infinity.

Proof. Since $D^\alpha_R [W(x)](t) \equiv \frac{d}{dt} I^{1-\alpha} W \leq 0$ and $0 \leq W$, $I^{1-\alpha} W$ is bounded from below and not increasing, so it converges as t goes to infinity. Applying Lemma 24, W converges to zero as t goes to infinity. Since $0 \leq C \|x\|^2 \leq W(t)$ then it follows that $\|x\|^2 \rightarrow 0$ i.e. $x \rightarrow 0$. \square

5.4. Example 4

It is shown in [1] that many adaptive problems can in general be expressed by an error equation of the type $e = \phi^T w$ with the output error being $e = y - y^*$ where y and y^* are the actual and the desired output, respectively; $\phi = \theta - \theta^*$ is the parametric error with θ an estimate of the unknown true parameter θ^* and w is the information signal. That is the so called Error Model of Type I equation. By means of simulations and based on the gradient method for the objective function e^2 when $\alpha = 1$, it was proposed in [9], to adjust parameters using the following adaptive law

$$D^\alpha \phi = D^\alpha \theta = -e w. \quad (19)$$

Though this scheme has become relatively popular in recent applications, there is actually no analytic proofs of its effectiveness. We will show that in the set of uniformly continuous bounded functions, the set of functions w that makes e converge to zero, is more restrictive when $\alpha < 1$ with respect to the case when $\alpha = 1$.

In the scalar case, Eq. (19) takes the form $D^\alpha \phi = -\phi w^2$. Based on Theorems of Chapter 6 of [10], it is easily proved that the solutions for $\alpha < 1$ can be expressed as $\phi(t) = \phi(0) - I^\alpha [\phi w^2](t)$ and that they do not change sign. Without loss of generality, by the linearity of Eq. (19), we will suppose that $\phi(0) > 0$. Then it follows that for all $0 < \phi(t) < \phi(0)$, thereby $I^\alpha [\phi w^2](t) < \phi(0) I^\alpha [w^2](t) < 1/2 \phi(0)$ where we have chosen, by using the proof of Proposition 14, a uniformly continuous function w that does not converge to zero such that $I^\alpha w^2(t) < 1/2$ for all $t > 0$ (specifically, $w^2 := (f/(2C))$ with f and C given by Proposition 14). Whereby, $\phi(t) \geq \phi(0)(1 - I^\alpha w^2)$. Since $\phi(t) \geq 1/2 \phi(0)$ we have that $e = \phi w$ does not converge to zero. In the vector case, we take $\bar{w}^T = (w, 0, 0, \dots, 0)$, whereby $e = \phi_1 w$ and the reasoning is similar to the given above.

On the other hand, for $\alpha = 1$, by defining $2W = \phi^T \phi$ it follows that $\dot{W} = -e^2 \leq 0$. Then, by integrating, it follows that the solution and its derivative are bounded (thus ϕ is uniformly continuous) and $\int_0^t e^2 d\tau$ is bounded. Since w is bounded and uniformly continuous by hypothesis, this implies that e is bounded and uniformly continuous and then e^2 is uniformly continuous, and by applying Lemma 9, it follows that e converges to zero.

By using the same function (and assuming differentiability of the information signal) $2W = \phi^T \phi$ for $\alpha < 1$ and applying properties of Caputo derivative given in [11], it follows that $D^\alpha W \leq -e^2 \leq 0$, thereby we have bounded solutions and $I^\alpha e^2$ turns out to be also bounded. But, by Proposition 14, this is not enough to conclude convergence of e even if it were uniformly continuous. However, by applying Proposition 15, we have that $\liminf e^2 = 0$.

Since $\phi = \phi(0) - I^\alpha \phi w^2$, if ϕ converges, $I^\alpha \phi w^2$ necessarily converges. In order to apply Lemma 21 to conclude that $e^2 = \phi w^2$ converges to zero, e should be differentiable and uniformly continuous; the latter is not hard to prove it but the former requires an extra differentiability condition upon the input w .

Therefore, in order to assure the convergence of the error to zero (effectiveness of the adaptive scheme (9)), a stronger condition must be imposed on the set of signals $w()$ than just uniform continuity, when adaptation with $\alpha < 1$ is used.

6. Conclusions

Some extensions for the fractional order case of the traditional Barbalat Lemma for the integer case, have been discussed throughout the paper. The analysis was separated for the cases when $\alpha \geq 1$ and when $0 < \alpha < 1$.

For $\alpha \geq 1$ the extensions are stated only for the Corollary of Barbalat Lemma. To consider a generalization of the original Barbalat Lemma one needs to work with functions of arbitrary sign but whose integrals have limits at infinity. This would lead to a proper generalization of the Barbalat Lemma.

For $0 < \alpha < 1$ there is no a straightforward extension for the Corollary of Barbalat Lemma. We derived conditions on the integrand and on the convergence of its fractional integral to assure convergence of the integrand. However, these conditions seem to be quite restrictive for general applicability. The study of the converse of fractional Barbalat Lemma, namely given a function f vanishing at infinity, to find conditions on f such that its fractional integral converges at infinity or to impose conditions on the integrand other than uniform continuity, could provide some keys in searching more general extensions.

In this study we mainly used the Caputo derivative in our developments, because there are many results of stability for fractional order dynamical systems that employ this definition and because it simplifies Property 4 for the case $\alpha < 1$. Nevertheless other definitions of fractional derivative could be considered in the Corollaries and similar results could be also obtained.

To prove convergence of fractional integrals of an arbitrary positive function is harder than in the integer case, where it is enough to show that the function belongs to \mathcal{L}^1 . This is because its fractional integral does not have the monotony property as in the integer case. One way to prove convergence in this general case is to have information about the sign of the $1 - \alpha$ Riemann–Liouville derivative of the function and about the boundedness of the integral.

For applications in control theory, the examples discussed in the paper give conditions for guaranteeing convergence of variables in a fractional controlled system, for which the controller is designed to modify the original equations in order to fulfil the required conditions.

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Appendix

A.1. Proof of Proposition 14

Proof. Let $p(t)$ be a function null at every point except in the intervals $[t_i, t_i + \delta]$ where it takes the value 1 (aperiodic pulse) and $(t_i)_{i \in \mathbb{N}}$ is a divergent increasing sequence to be specified and δ is a fixed positive real to be specified.

Note that for each t and for all $\tau \leq t$, $p(\tau)$ can be written as

$$p(\tau) = \sum_{i=1}^n p_i(\tau) \quad (\text{A.1})$$

where $n = \max\{i : t \geq t_i\}$ and $p_i(t)$ is a function 0 at every point except in the interval $[t_i, t_i + \delta]$ where it takes the value 1.

For every i , we have $I^\alpha p_i(t) < C_1$ where C_1 is a positive real constant. In fact, if $t < t_i$ then $I^\alpha p_i(t) = 0$; if $t_i \leq t \leq t_{i+1}$ then $\alpha \Gamma(\alpha) I^\alpha p_i(t) = (t - t_i)^\alpha \leq \delta^\alpha$ and if $t \geq t_{i+1}$ then $\alpha \Gamma(\alpha) I^\alpha p_i(t) = (t - t_i)^\alpha - (t - t_i - \delta)^\alpha \leq \delta^\alpha$, because for every i and for $t \geq t_{i+1}$, $I^\alpha p_i(t)$ is strictly monotone decreasing since $\frac{d}{dt} [(t - t_i)^\alpha - (t - t_i - \delta)^\alpha] < 0$.

By Property 17, for every i , $I^\alpha p_i(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, by definition of convergence, there exists T_i such that for all $t > T_i$, $I^\alpha p_i(t) < \frac{1}{2}$. We will define t_{i+1} such that $t_{i+1} > T_i$. In this way, we construct a divergent increasing sequence $(t_i)_{i \in \mathbb{N}}$. For instance, let t_1 be arbitrary positive real number and $t_{i+1} = T_i + 1$ and a fixed δ so that $\delta^\alpha \alpha^{-1} \Gamma(\alpha)^{-1} = 1$, then necessarily $T_i + 1 < T_{i+1}$ because $T_{i+1} > t_{i+1}$ since the condition for T_{i+1} is $I^\alpha p_{i+1}(t) < \frac{1}{(i+1)^2} \leq 1$ but $I^\alpha p_{i+1}(t_{i+1} + \delta) = 1$. For the same reason, the p_i do not overlap each other. Therefore $t_{i+1} - t_i > 1$ and thus $(t_i)_{i \in \mathbb{N}}$ is an increasing sequence. Moreover, $t_i \rightarrow \infty$ as $i \rightarrow \infty$.

Hence, for any $t \in \mathbb{R}^+$ there exists $n \in \mathbb{N}$ so that $t_{n+1} \leq t \leq t_{n+2}$. Thus, by linearity of the integral operator, we can write

$$\begin{aligned} I^\alpha p(t) &= I^\alpha \left(\sum_{i=1}^n p_i(t) + p_{n+1}(t) \right) \\ &= \sum_{i=1}^n I^\alpha p_i(t) + I^\alpha p_{n+1}(t). \end{aligned} \quad (\text{A.2})$$

Then because of the construction of t_i it follows that

$$I^\alpha p(t) \leq \sum_{i=1}^n \frac{1}{i^2} + I^\alpha p_{n+1}(t) \leq 2 + C_1. \quad (\text{A.3})$$

Thereby, we have a bounded, positive, not vanishing at infinite function p , whose fractional integral remains bounded.

Let $f(t)$ be a positive triangular function so that for all $t > 0$, $f(t) \leq p(t)$. For instance, let f be null at every point except in the intervals $[t_i, t_i + \delta/2]$ where the function is $2\delta^{-1}(t - t_i)$ and in the intervals $[t_i + \delta/2, t_i + \delta]$ where the function is $2\delta^{-1}(t_i + \delta - t)$. Then, f is a uniformly continuous function because it is Lipschitz of constant $2\delta^{-1}$. Therefore f is a bounded, positive, not vanishing and uniformly continuous function such that

$$I^\alpha f \leq I^\alpha p \leq 2 + C_1 < C \quad (\text{A.4})$$

which proves Proposition 14. \square

Remark 30. From the proof, a stronger condition than uniform continuity, namely f a Lipschitz function, is not enough to assure convergence of f to zero.

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