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Transverse phase shielding solitons in the degenerated optical parametric oscillator

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ABSTRACT

Localized structures in optics have attracted attention for their potential applications in telecommunications and information storage. In the present work, localized structures with non-uniform phase structure in the degenerate optical parametrical oscillator are reported and elucidated. Dissipative solitons with non-uniform phase structures in parametrically driven systems have been already observed in prototype models being found in two typical shapes: symmetrical and asymmetrical. In contrast, the phase structure in degenerate optical parametrical oscillator is always symmetrical, showing a pronounced bell-shaped phase. We show that the nonlinear saturation present in this physical system is responsible of a relaxation dynamics and this symmetry. Probing that real physical systems exhibit always this unique type of localized structure with phase structure. An adequate analytical description for the phase, based on a simple model, is achieved showing that such structure is controlled by the interplay between the detuning, the external pump and the losses of the cavity. Numerical simulations present quite agreement with theoretical predictions.

1. Introduction

For decades, dissipative localized structures have been under intensive investigations. In the context of nonlinear optics, localized states are observed in cavities containing both active and passive media (cavity soliton), in liquid crystal light valves, in sodium vapor based systems, just to mention a few [1–3]. In these systems due to their ability to confine light, localized structures are good candidates for technological application in all optical storage devices. Among cavity solitons, those generated by means of optical parametric oscillation have the particularity to be excited by two different beams and also be tunable over a broadband spectral domain. In degenerated optical parametric oscillators (DOPO), the emitted pair of cavity solitons is completely undistinguishable. This feature is interesting in the domain of parallel optical technology and quantum information processing [4,5]. Hence, a better understanding of cavity solitons characteristics and dynamical behavior in DOPO is crucial from fundamental and applied point of view.

Optical parametric oscillators are obtained by filling an optical cavity with a nonlinear quadratic ($\chi^{(2)}$) medium. Thanks to this medium an intense electric field pump with a frequency ω_p can be converted into a signal and idler wave with a ω_s and a ω_i frequency, respectively, such that $\omega_p = \omega_s + \omega_i$ [6]. When $\omega_p = 2\omega_s$, the optical parametric oscillator is degenerated (DOPO). In this case the dynamical spatiotemporal evolution of the slowly varying envelope of the electric field inside the cavity is modelled by [6]

$$\partial_t A_p = \gamma_p[-(\nu_p + i\Delta_p)A_p + E - \epsilon_p A_s^2 + ia_p\nabla_\perp^2 A_p], \tag{1a}$$

$$\partial_t A_s = \gamma_s[-(\nu_s + i\Delta_s)A_s + \epsilon_s A_s^* A_p + ia_s\nabla_\perp^2 A_s]. \tag{1b}$$

Here, A_j with $j = \{p, s\}$ represent the slowly varying envelopes for the pump and signal wave. A_s^* accounts for the complex conjugate of A_s. The parameters Δ_j, γ_j, and a_j are the detunings, the cavity decay rates and the diffraction coefficients, respectively. E is the external pump and $\nu_{p,s}$ account for the cavity losses corresponding to the pump and the signal, respectively. The effective coupling parameters coming from the quadratic susceptibility $\chi^{(2)}$ are given by ϵ_p and ϵ_s. This model is characterized to exhibit a complex self-organization such as patterns [6], fronts [7,8] and localized structure [9].

Using a Madelung transformation, localized structures in parametrically driven systems can be expressed in terms of the magnitude and the phase of the envelope. Since the phase is

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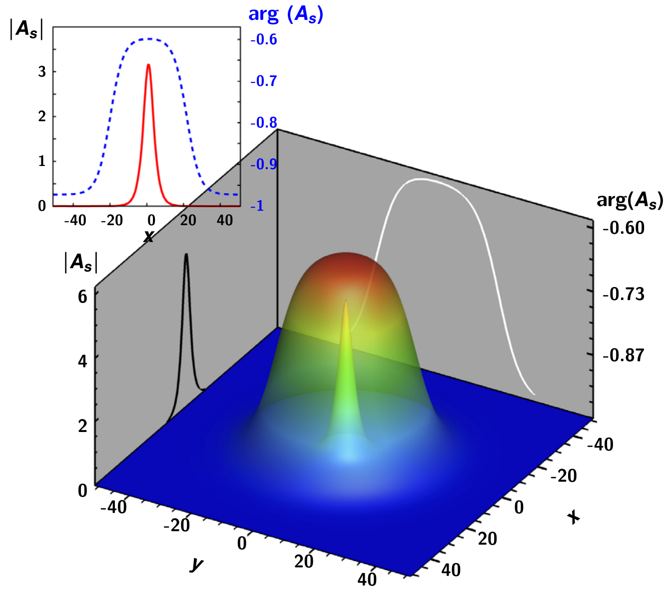

Fig. 1. Phase shielding soliton of degenerated optical parametric oscillator, Eqs. (1) by $\Delta_s = 0.100$, $\Delta_p = 3.000$, $\nu_s = 0.200$, $\nu_p = 0.005$, $\epsilon_p = 0.500$, $\epsilon_s = 0.215$, and E=3.000. The inset accounts for the equivalent phase shielding soliton in one spatial dimension.

assumed as uniform, the localized state is only described by means of the modulus. This is the case for DOPO systems, where in the limit of large pump detuning, it is expected that one dimensional localized structures possess a uniform phase profile [10]. Notwithstanding, it has recently been reported that quasi-reversible parametrically driven systems exhibit a type of dissipative soliton named phase shielding soliton (PSS) which present a non-uniform phase structure [11]. As a consequence of the quasi-reversibility nature of the system under study [12], this phase structure is characterized by a front propagation dynamics, which asymptotically converges to different equilibria. After this initial transient, the phase structure exhibits namely eight different stationary configurations which can be simply separated into two groups: symmetrical and asymmetrical. In Fig. 1, we show localized structures in one and two dimensions, obtained by numerical simulations of a degenerated optical parametric oscillator, Eqs. (1). Both modulus and phase of the envelope are displayed.

As expected, the modulus is spatially localized having a bell-like shape. However, a pronounced bell-shaped symmetrical structure surrounding the modulus is observed also for the phase. Note that systems with oscillatory instability, described by the Ginzburg–Landau equation, also exhibit dissipative solitons with bell-like phase structure [13]. However, these solutions are unstable [14]. Conversely to phase shielding solutions in prototype models, the DOPO phase structure does not exhibit front propagation. In addition its final phase size is remarkably small with an always symmetrical shape. This stationary configuration is also unique, i.e., there is no evidence of other (symmetrical or asymmetrical) configuration.

In this paper, we investigate theoretically and numerically the phase configuration and dynamical behavior of the localized structures in the degenerated optical parametric oscillator. Particularly, we examine the effect of nonlinear dissipation in the phase dynamical behavior. By means of a simple model, we show that the presence of nonlinear saturation leads to a relaxation dynamics. The phase structure is also controlled by such effect, being an interplay between the detuning and the losses in the cavity. Numerical simulations show a quite good agreement with the theoretical predictions.

2. Minimal description

To understand the existence, stability properties, dynamical evolution and bifurcation diagram of the localized state with shield-like phase structure, we consider the large pump detuning limit ($\gamma_p \gg 1$)—single resonant DOPO limit—and small cavity losses (ν_p/Δ_p and $a_p/\Delta_p \ll 1$). Considering this limit in Eq. (1a), we can adiabatically replace the dynamics of the pump envelope by

$$A_p = -i\frac{E}{\Delta_p} + i\frac{\epsilon_p}{\Delta_p}A_s^2 + \frac{\nu_p}{\Delta_p^2}E - \frac{\nu_p\epsilon_p}{\Delta_p^2}A_s^2 + i\frac{a_p\epsilon_p}{\Delta_p^2}\nabla^2 A_s^2. \tag{2}$$

Under these conditions the dynamics of the intra-cavity signal field reads

$$\partial_\tau\psi = i\Delta_s\psi - (\delta + i)|\psi|^2\psi - i\nabla^2\psi - \nu_s\psi + \gamma\psi^*, \tag{3}$$

where we have introduced the complex variable $\psi^* \equiv A_s\sqrt{\epsilon_p\epsilon_s/\Delta_p}\,e^{i\theta}$ with $\theta = \pi/4 - \arctan(\delta)/2$, $\delta \equiv \nu_p/\Delta_p \ll 1$, and scaling the space by $\sqrt{a_s}$. The effective intensity of parametric forcing is $\gamma \equiv \epsilon_s E\sqrt{1+\delta^2}/\Delta_p$. The above amplitude equation is valid in the scaling $\nu_s \sim \Delta_s \sim \gamma \ll 1$, and $\partial_\tau \sim \nabla \sim \nu_s^{1/2}$, $|\psi| \sim \nu_s^{1/2}$.

In the particular case, δ=0, Eq. (3) becomes the well-known parametrical driven and damped nonlinear Schrödinger equation (PDNLS). This is a universal model describing the dynamics of a chain of coupled oscillators with dissipation and parametric forcing, which has been derived in several parametrically driven physical contexts. Spatially extended systems described by this equation have been reported to exhibit periodic patterns (triangles, hexagons, rolls, and so forth) [15–17], localized structures without (dissipative soliton, fronts, kinks, localized states) and with [9,21,19,18,20] propagative domain walls [22] and many other dynamical dissipative structures. In the optical context, the PDNLS has been derived in a Kerr optical medium forced by modulated (spatially and temporally) injection [23].

Recently, in Ref. [11], we have characterized a family of dissipative localized solutions of PDNLS with a nonuniform phase

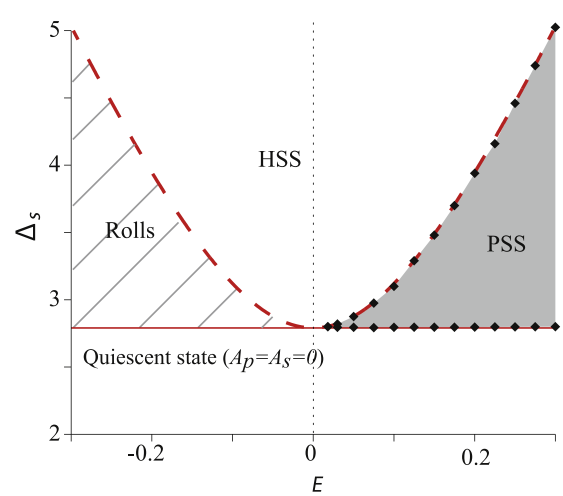

Fig. 2. Bifurcation diagram of degenerated optical parametric oscillator, Eqs. (1), in the external pump and detuning space $\{E, \Delta_s\}$ by $\Delta_p = 3.000$, $\nu_s = 0.200$, $\nu_p = 0.005$, $\epsilon_p = 0.500$, and $\epsilon_s = 0.215$. Dashed curve accounts for Arnold tongue. Inside this region is observed homogeneous stable state (HSS). The gray area stands for the region where phase shielding solitons (PSS) are observed using Eqs. (1) with $\delta \ll 1$. The dashed area accounts for the zone where roll patterns are observed. For $E < \nu_s\Delta_p/\epsilon_s\sqrt{1+\delta^2}$, the quiescent state ($A_p = A_s = 0$) is the only stable solution.

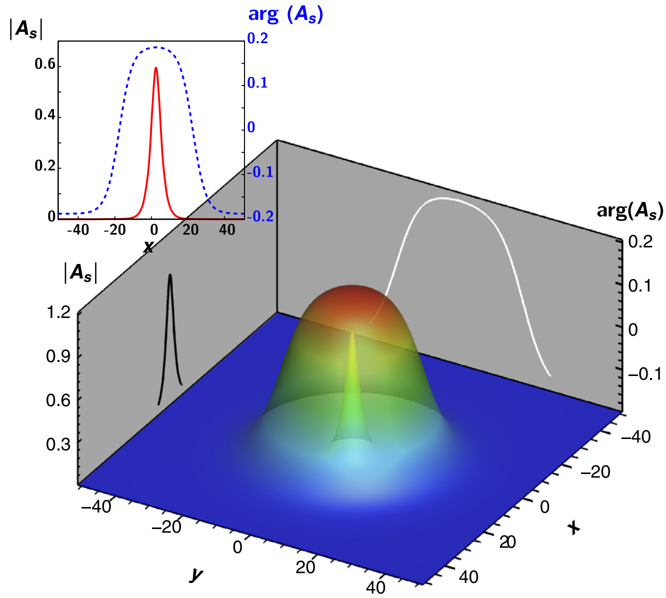

Fig. 3. Phase shielding soliton of nonlinear dissipative PDNLS, Eqs. (3) by $\Delta_s = 0.100$, $\nu_s = 0.200$, $\gamma=0.215$ and $\delta = 1.6\cdot 10^{-3}$. The inset accounts for the equivalent phase shielding soliton in one spatial dimension.

structure—the phase shielding solitons in one and two dimensions—and developed an analytical description for their structure and dynamics. Fig. 2 shows the bifurcation diagram in the $\{E, \Delta_s\}$ space of PDNLS model, Eq. (3) with $\delta \ll 1$. The bifurcation diagram of PDNLS has been widely studied in the literature [11,19,22]. Eq. (3) is a generalization of the PDNLS, where the effects of nonlinear dissipation are addressed by the term proportional to δ. Note that in the limit under consideration, the dissipative nonlinear term is perturbative. It is known that the PDNLS equation is structurally unstable, i.e., small perturbations can generate new solutions [21].

To verify the persistence of phase shielding solitons under the presence of a perturbative nonlinear dissipation, we have conducted numerical simulations of Eq. (3). Fig. 3 shows the typical observed dissipative localized structure. It is easy to see the correspondence between these structures and the dissipative solitons arising in the transversal section of DOPO system [see Fig. 1]. Note that among the entire family of phase shielding solitons, only those survive localized states possessing a phase structure with a symmetrical bell-shaped which tends to form very close to the soliton core.

3. Phase shielding structure

To understand the dynamics of the non-uniform phase solitons exhibited in DOPO, we must take into account the effect of the nonlinear dissipation δ over the phase shielding solitons. The dissipative solitons of Eq. (3) have the same shape structure of the phase shielding solitons of PDNLS [11]. The main difference between them is the final size of the phase structure and its relaxation dynamics. Hence, the effect of the nonlinear dissipation is only manifest in the transient and the final equilibrium position of the phase structure.

Introducing the polar representation for the field $\psi = Re^{i\varphi}$ (Madelung transformation) in Eq. (3), we obtain a set of equations for the modulus and the phase:

$$\partial_t R = 2\nabla R \nabla\varphi - \nu_s R + R\nabla^2\varphi + \gamma R\cos(2\varphi) - \delta R^3, \tag{4a}$$

$$R\partial_t\varphi = \Delta_s R - R^3 - \nabla^2 R + R(\nabla\varphi)^2 - \gamma R\sin(2\varphi). \tag{4b}$$

The stationary solution $\{R_s, \phi_s\}$ is obtained by setting $\partial_t\varphi = \partial_t R = 0$ in the above set of equations. Note that the nonlinear dissipation does not affect the equation that determines the shape of the modulus, Eq. (4b). Indeed, in the one dimensional case, the phase shielding soliton profile when the nonlinear dissipation is ignored, is given by [11,24,20]

$$R_s = \sqrt{2\sigma}\ \mathrm{sech}(\sqrt{\sigma}[x - x_0]) \tag{5}$$

for the modulus, and

$$\phi_s \approx \arctan\left[\sqrt{\frac{\gamma \pm \nu_s}{\gamma \mp \nu_s}}\tanh\frac{\sqrt{\gamma^2 - \nu_s^2}(x - x_f)}{2\sqrt{\sigma}}\right] \tag{6}$$

for the phase front. Here, the characteristic parameter is defined as $\sigma \equiv \Delta_s + \sqrt{\gamma^2 - \nu_s^2}$ and $\{x_0, x_f\}$ account for the localized structure and phase jump position, respectively. As a consequence of the translation invariance symmetry, the position x_0 is not determined. However, once x_0 is fixed, the position of the phase jump is also fixed. Then, the PSS solution is characterized by the size of the phase structure x_f. For $\delta=0$, this size is considerably larger than the modulus width $1/\sqrt{\sigma}$. Thus, the shape of the modulus near the phase front position can be approximated by

$$R_s(x) \simeq 2\sqrt{2\sigma}e^{-\sqrt{\sigma}x}, \tag{7}$$

where we have set the origin of the system at the modulus position, $x_0 = 0$. Introducing the above approximation in Eq. (4a) and multiplying it by $\partial_x\phi_s$, after straightforward calculations, we obtain the equilibrium position for the phase front:

$$x_f^* = -\frac{1}{2\sqrt{\sigma}}\log\left(\frac{-A}{B}\right), \tag{8}$$

with $A \equiv \langle(-\Delta_s + \sigma + \gamma\sin(2\varphi_s) - (\partial_\zeta\varphi_s)^2)|\partial_\zeta\varphi_s\rangle/C$, $B \equiv 8\sigma\langle e^{-2\sqrt{\sigma}\zeta}|\partial_\zeta\varphi_s\rangle/C$, and $C \equiv \langle\partial_\zeta\varphi_s|\partial_\zeta\varphi_s\rangle$ where the inner product is defined by $\langle f|g\rangle \equiv \int f(\zeta)g(\zeta)\,d\zeta$ with $\zeta \equiv x - x_f$. Note that the parameter A is controlled by the detuning, the external pump, and the losses of the cavity. In contrast, the B parameter is determined by the exponential decay of the modulus. Therefore, the phase front position is determined by the energy difference between the steady phase states and the force induced by the asymptotic decay of the modulus.

The above scenario is modified in the presence of the nonlinear dissipation ($\delta \neq 0$ and $\delta \ll 1$), the phase front correction can be obtained by introducing the following ansatz $\varphi(x) = \varphi_s(x) + \theta(x)$ in Eq. (4a), where θ accounts for the disturbance on the phase front shape ($\theta \sim O(\delta)$), which satisfies the linear equation

$$-8\delta\sigma e^{-2\sqrt{\sigma}x} = 2\sqrt{\sigma}\partial_x\theta + 2\gamma\sin(2\varphi_s)\theta. \tag{9}$$

After straightforward calculus, we obtain

$$\theta(x) = -4\delta\sqrt{\sigma}\partial_x\varphi_s\int\frac{e^{-2\sqrt{\sigma}x}}{\partial_x\varphi_s}dx + C_1\partial_x\varphi_s, \tag{10}$$

where C_1 is an integration constant, which is determined by $\theta(x \to \pm\infty) = 0$. At dominant order, θ can be approximated to $\theta(x) \simeq 2\delta e^{-2\sqrt{\sigma}x} + C_1\partial_x\varphi_s$.

Analogous to the above procedure, we can determine the correction to the phase front final position, using the amended phase solution, indeed

$$\frac{2}{C}\int[\partial_\zeta\theta(\partial_\zeta\varphi_s)^2 - \gamma\theta\cos(2\varphi_s)\partial_\zeta\varphi_s]\,d\zeta = A + Be^{-2\sqrt{\sigma}x_f}. \tag{11}$$

Introducing the ansatz $x_f \equiv x_f^* + \Delta x_f^*$ in the above equation, we obtain

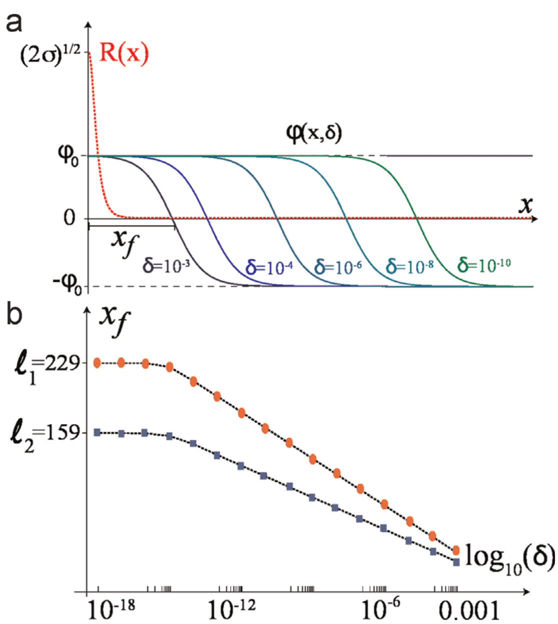

Fig. 4. Size of the phase shielding structure of amplitude Eq. (3) as a function of the nonlinear dissipation. (a) Phase shielding soliton with different nonlinear dissipation intensities obtained with $\Delta_s = 0.1$, $\nu_s = -0.1$ and $\gamma=0.011$. (b) x_f versus nonlinear dissipation δ by $\Delta_s = 0.1$, $\nu_s = -0.1$ ($\nu_s = -0.08$) and $\gamma=0.011$ ($\gamma=0.015$). l_i stands for typical size of phase shielding soliton without nonlinear dissipation.

$$\Delta x_f^* = \frac{1}{2\sqrt{\sigma}B}(\Delta A e^{2\sqrt{\sigma}x_f^*} + \Delta B) \tag{12}$$

with $\Delta A \equiv 2C_1\langle(-\partial_{\zeta\zeta}\varphi_s\partial_\zeta\varphi_s + \gamma\cos(2\varphi_s)\partial_\zeta\varphi_s)|\partial_\zeta\varphi_s\rangle/C$ and $\Delta B \equiv 4\delta\langle(2\sqrt{\sigma}\partial_\zeta\varphi_s + \gamma\cos(2\varphi_s))e^{-2\sqrt{\sigma}\zeta}|\partial_\zeta\varphi_s\rangle/C$.

It is important to note that although $\theta(x)$ is a small correction function to the phase shape, it strongly affects the phase front final position. The parameters which control the position of the phase front (A, B) are multiplied by an exponential factor that depends on the undisturbed phase front position [cf. the first term on the right side of Eq. (12)]. This causes the high sensitivity of the phase structure to a modification of the nonlinear dissipation. Fig. 4 shows the large modification of the phase structure, and in particular its size, when the nonlinear dissipation intensity is slightly modified. Note that for small values of δ, the size of phase structure approaches to a saturation value [$l \equiv x_f^*(\delta = 0)$]. Hence, this limit is well described by the PDNLS approximation. A similar sensitivity effect on the phase equilibrium position has been reported for traveling phase shielding solitons propagating in inhomogeneous medium [25].

The origin of these dramatic changes in the phase structure is that the nonlinear dissipation term in Eq. (4a) does not allow homogeneous phase equilibria. Therefore, the system cannot exhibit localized structure with homogeneous phase or propagative phase fronts. Then, the PSS in dissipation nonlinear systems are characterized by displaying a relaxation dynamics in the phase structure.

In two dimensions, the phase structure has a large radius of curvature, compared to the size of the localized structure modulus [11]. Therefore, one can neglect the effects of curvature and obtain similar results to those found in one dimension [11].

4. Conclusion

We have shown that the degenerated optical parametric oscillators exhibit transversal localized structures with shield-like phase structure in one and two spatial dimensions. Such phase structure is characterized by a pronounced bell shape and a phase front position closer to the module structure, in contrast with the usual PDNLS phase shielding soliton. Starting from DOPO describing equations, we have derived an adequate amplitude equation which allows us to describe correctly these modified phase shielding solitons. The phase structure is controlled by the interplay between the detuning, the external pump, and the losses of the cavity.

In the bistability regime of optical parametric oscillators, localized states or domain walls are observed, characterized by exhibiting damped spatial oscillatory tails [26,27]. These tails are responsible for the stability of these solutions. In the monostability region, these solutions with damped spatial oscillatory tails disappear. In contrast, phase shielding solitons are only observed.

PDNLS model, Eq. (3), owns a rich spatiotemporal behavior. However, as we have shown a small perturbation of this model changes radically the steady states. Therefore, a careful analysis of higher order terms effects is necessary. The consequence of the phase structure in the dynamics of dissipative solitons has not yet been revealed, but opens the question of its possible implications in the interaction and undiscovered behaviors of these localized states.

Acknowledgments

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