

ASYMPTOTIC INTEGRATION OF A LINEAR FOURTH ORDER DIFFERENTIAL EQUATION OF POINCARÉ TYPE

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ABSTRACT. This article deals with the asymptotic behavior of fourth order differential equation where the coefficients are perturbations of linear constant coefficient equation. We introduce a change of variable and deduce that the new variable satisfies a third order differential equation of Riccati type. We assume three hypothesis. The first is the following: all roots of the characteristic polynomial associated to the fourth order linear equation has distinct real part. The other two hypothesis are related with the behavior of the perturbation functions. Under this general hypothesis we obtain four main results. The first two results are related with the application of fixed point theorem to prove that the Riccati equation has a unique solution. The next result concerns with the asymptotic behavior of the solutions of the Riccati equation. The fourth main theorem is introduced to establish the existence of a fundamental system of solutions and to precise formulas for the asymptotic behavior of the linear fourth order differential equation.

1. INTRODUCTION

In this paper we are interested in the following fourth order differential equation

$$y^{(4)} + \sum_{i=0}^3 [a_i + r_i(t)]y^{(i)} = 0, \quad a_i \in \mathbb{R} \quad \text{and} \quad r_i : \mathbb{R} \rightarrow \mathbb{R}. \quad (1.1)$$

This equation is a perturbation of the following constant coefficient equation:

$$y^{(4)} + \sum_{i=0}^3 a_i y^{(i)} = 0. \quad (1.2)$$

The classical analysis of (1.1) is mainly focus on two questions: the existence of a fundamental system of solutions and the characterization of the asymptotic behavior of its solutions. The first significative answers of both problems comes back to the seminal work of Poincaré [30] and has been investigated by several authors with long and rich history of results [4, 9, 17, 18]. However, although it is an old problem is a matter which does not lose its topicality and importance in the research community. For instance in the case of asymptotic behavior there are the following newer results [13, 14, 28, 32, 33]. In particular, in this contribution, we address the question of new explicit formulas for asymptotic behavior of nonoscillatory solutions for (1.1) by application of the scalar method introduced by Bellman in [2] (see also [3, 4]) and recently applied by Figueroa and Pinto [13, 14], Stepin [32, 33] and Pietruczuk [28].

Linear fourth-order differential equations appear in several areas of sciences and engineering as the more basic mathematical models. These simplified equations, arise from different linearization approaches used to give an ideal description of the physical phenomenon or in order to analyze (analytically solve or numerically simulate) the corresponding nonlinear governing equations. For instance, the one-dimensional of Euler-Bernoulli model in linear theory of elasticity [1, 34], the

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optimization of quadratic functionals in optimization theory [1], the mathematical model in viscoelastic flows [7, 22], and the biharmonic equations in radial coordinates in harmonic analysis [15, 21]. In particular, here we describe the last application. We recall that the biharmonic equation

$$\Delta^2 u(\mathbf{x}) = 0 \quad \text{in } \mathbb{R}^n, \quad \text{with } n \geq 5,$$

in radial coordinates with $r = \|\mathbf{x}\|$ and $\phi(r) = u(\mathbf{x})$, may be rewritten as follows

$$\phi^{(4)}(r) + \frac{2(n-1)}{r}\phi^{(3)}(r) + \frac{(n-1)(n-3)}{r^2}\phi^{(2)}(r) - \frac{(n-1)(n-3)}{r^3}\phi^{(1)}(r) = 0, \quad r \in]0, \infty[.$$

Now, by introducing the change of variable $v(t) = e^{-4t/(p-1)}\phi(e^t)$ for some $p > (n+4)(n-4)^{-1}$, the differential equation for ϕ can be transformed in the following equivalent equation

$$v^{(4)}(t) + K_3 v^{(3)}(t) + K_2 v^{(2)}(t) + K_1 v^{(1)}(t) + K_0 v(t) = 0, \quad t \in \mathbb{R}, \quad (1.3)$$

where

$$\begin{aligned} K_0 &= \frac{8}{(p-1)^4} \left[(n-2)(n-4)(p-1)^3 + 2(n^2 - 10n + 20)(p-1)^2 - 16(n-4)(p-1) + 32 \right], \\ K_1 &= -\frac{8}{(p-1)^3} \left[(n-2)(n-4)(p-1)^3 + 4(n^2 - 10n + 20)(p-1)^2 - 48(n-4)(p-1) + 128 \right], \\ K_2 &= \frac{1}{(p-1)^2} \left[(n^2 - 10n + 20)(p-1)^2 - 24(n-4)(p-1) + 96 \right], \\ K_3 &= \frac{2}{p-1} \left[(n-4)(p-1) - 8 \right], \end{aligned}$$

see [15] for further details. We note that the roots of the characteristic polynomial associated to the homogeneous equation are given by

$$\lambda_1 = 2\frac{p+1}{p-1} > \lambda_2 = \frac{4}{p-1} > 0 > \lambda_3 = \frac{4p}{p-1} - n > \lambda_4 = 2\frac{p+1}{p-1} - n. \quad (1.4)$$

Thus, the radial solutions of the biharmonic equation in an space of dimension $n \geq 5$ and with $p > (n+4)(n-4)^{-1}$ can be analysed by the linear fourth order differential equation (1.3) where the characteristic roots satisfy (1.4) which will be generalized by considering throughout of the paper the assumption (H_1) . See the list of assumptions given below at the end of the introduction.

Nowadays, there exist three big approaches to study the problem of asymptotic behavior of solutions for (1.1): the analytic theory, the nonanalytic theory and the scalar method. In a broad sense, we recall that the essence of the analytic theory consist in the assumption of some representation of the coefficients and of the solution, for instance power series representation (see [5] for details). Concerning to the nonanalytic theory, we know that the methods are procedures consisting of the two main steps: first a change of variable to transform (1.1) in a system of first order of Poincare type and then by the application of a diagonalization process to obtain the asymptotic formulas (for further details consult [9, 6, 11, 24]). Meanwhile, in the scalar method [4, 13, 14, 32, 33, 28, 2, 3] the asymptotic behavior of solutions for (1.1) is obtained by a change of variable which reduce (1.1) to a third order Riccati-type equation. Then, the results for (1.1) are derived by analyzing the asymptotic behavior of the Riccati equation. For instance in [3], Bellman present the analysis of the second order differential equation $u^{(2)} - (1 + f(t))u = 0$ by introducing the new variable $v = u^{(1)}/u$ which transform the linear perturbed equation in the following Riccati equation $v^{(1)} + v - (1 + f(t)) = 0$. Then, by assuming several conditions on the regularity and integrability of f , he obtains the formulas for characterization of the asymptotic behavior of u . For example in the case that $f(t) \rightarrow 0$ when $t \rightarrow \infty$, Bellman proves that there exists two linearly

independent solutions u_1 and u_2 , such that $(u_i^{(1)}/u_i)(t) \rightarrow (-1)^{i+1}$ when $t \rightarrow \infty$ and

$$\exp\left((-1)^{i+1}t - \int_{t_0}^t |f(\tau)|d\tau\right) \leq u_i(t) \leq \exp\left((-1)^{i+1}t + \int_{t_0}^t |f(\tau)|d\tau\right) \quad \text{for } i = 1, 2.$$

More details and a summarization of the results of the application of the scalar method to a special second order equation are given in [4].

Let us recall some classical results. The list of the results is non-exhaustive. Firstly, we recall that Poincaré, in [30] assumes two hypothesis:

- (P₁) λ is a simple characteristic root of (1.2) distinct of the real part of the any other characteristic root
- (P₂) the perturbation functions r_j are rational functions such that, for all $j = 0, \dots, 3$, $r_j(t) \rightarrow 0$ when $t \rightarrow \infty$.

Then, under (P₁)-(P₂), he deduce that $y(t)$, the solution of (1.1), has the following asymptotic behavior: $y^{(\ell)}(t)/y(t) \rightarrow (\lambda)^\ell$ for $\ell = 1, 2, 3, 4$ when $t \rightarrow \infty$. Afterwards, Perron [25] extends the results of Poincaré by assuming (P₁) and considering instead of (P₂) the hypothesis that the perturbation functions r_j are continuous functions such that, for all $j = 0, \dots, 3$, $r_j(t) \rightarrow 0$ when $t \rightarrow \infty$. Perhaps, other three important landmarks on the asymptotic behavior are the contributions of Levinson [23], Hartman-Wintner [19] and Harris and Lutz [16, 17]. In [23], Levinson analyze the non-autonomous system $\mathbf{x}'(t) = [\Lambda(t) + R(t)]\mathbf{x}(t)$ where Λ is a diagonal matrix and R is the perturbation matrix. Levinson assumes that the diagonal matrix satisfies a dichotomy condition and the perturbation function is continuous and belongs to $L^1([t_0, \infty[)$ and prove that a fundamental matrix X has the following asymptotic representation $X(t) = [I + o(1)] \exp\left(\int_{t_0}^t \Lambda(s)ds\right)$. Meanwhile, Hartman-Wintner assumes that the diagonal matrix satisfies a more strong condition than the Levinson dichotomy condition and the perturbation function is continuous and belongs to $L^p([t_0, \infty[)$ for some $p \in]1, 2]$ and prove that $X(t) = [I + o(1)] \exp\left(\int_{t_0}^t (\Lambda(s) + \text{diag}(R(s)))ds\right)$. In the seventies Harris and Lutz in [16] (see also [17, 12, 27, 26, 29]) find a change of variable to unify the results of Levinson and Hartman-Wintner. Other, important contributions are given for instance by [31, 10]. We comment that the application of Levinson and Hartman-Wintner results to (1.1) are not direct and should be done via the nonanalytic theory. Here, the main practical disadvantage is that, in most of the cases Λ and R are difficult to algebraic manipulation and the asymptotic formulas are only theoretical ones.

In this paper, we reorganize and reformulate the original scalar method of Bellman and then introduce new hypothesis in order to characterize the asymptotic behavior of the solutions for (1.1) by considering that the perturbation functions satisfy some restrictions in the L^p sense. Indeed, the scalar method is presented in three big steps (see section 2). First, we introduce a change of variable and deduce that the new variable is a solution of a Riccati-type equation. In a second step, in order to deduce the well posedness and the asymptotic behavior of the solution for the Riccati-type equation, we assume a general hypothesis about the linear part of (1.1) and the perturbation functions. Then, in a third step, we translate the results for the solution of the Riccati-type equation to the solution of (1.1). In this step, we deduce the existence of a fundamental system of solutions for (1.1) and conclude the process with the formulation and proof of the asymptotic integration formulas for the solutions of (1.1).

The main results of the paper are summarized in section 2. These results are obtained by considering the following hypothesis about the coefficients and perturbation functions of (1.1)

- (H₁) All roots of the characteristic polynomial $\lambda^4 + \sum_{i=0}^3 a_i \lambda^i$, associated to (1.2), has distinct real part or equivalently $\{\lambda_i, i = \overline{1, 4} : \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4\} \subset \mathbb{R}$ is the set of characteristic roots for (1.2).
- (H₂) For all $j = 0, \dots, 3$, the perturbation functions r_j are selected such that $\mathcal{L}(r_j)(t) \rightarrow 0$ when $t \rightarrow \infty$, where \mathcal{L} is the functional on $L^p([t_0, \infty[)$ defined as follows

$$\mathcal{L}(E)(t) = \int_{t_0}^{\infty} \left[|g(t, s)| + \left| \frac{\partial g}{\partial t}(t, s) \right| + \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| \right] |E(s)| ds. \quad (1.5)$$

Here $L^p([t_0, \infty[)$ is the space of measurable functions on $[t_0, \infty[$ for some $t_0 > 0$, such that are p -integrable in the sense of Lebesgue for $p \in [1, \infty[$ or essentially bounded for $p = \infty$.
(H₃) For all $j = 0, \dots, 3$, the perturbation functions r_j are belong to $\mathcal{F}_{\rho_i}([t_0, \infty[)$ defined by

$$\mathcal{F}_{\rho_i}([t_0, \infty[) = \left\{ E : [t_0, \infty[\rightarrow \mathbb{R} : \mathbb{F}_i(E)(t) \leq \rho_i, \quad t \geq t_0 \right\},$$

for each $i = 1, \dots, 4$, with $\rho_i \in]\mathbb{F}_i(1)(t), \infty[\subset \mathbb{R}^+$ a given (fix) number and the operators \mathbb{F}_i defined as follows

$$\begin{aligned} \mathbb{F}_1(E)(t) &= \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} |E(s)| ds, \\ \mathbb{F}_2(E)(t) &= \int_{t_0}^t e^{-(\lambda_1 - \lambda_2)(t-s)} |E(s)| ds + \int_t^\infty e^{-(\lambda_3 - \lambda_2)(t-s)} |E(s)| ds, \\ \mathbb{F}_3(E)(t) &= \int_{t_0}^t e^{-(\lambda_2 - \lambda_3)(t-s)} |E(s)| ds + \int_t^\infty e^{-(\lambda_4 - \lambda_3)(t-s)} |E(s)| ds, \\ \mathbb{F}_4(E)(t) &= \int_{t_0}^t e^{-(\lambda_3 - \lambda_4)(t-s)} |E(s)| ds. \end{aligned}$$

We should be comment that (H₁)-(H₂) are used to prove the existence of a fundamental system of solutions for (1.1) and (H₃) is needed in order to get the asymptotic behavior formulas for solutions of (1.1). The hypothesis (H₂) is new and is the natural generalization of the classical hypothesis introduced by Poincaré when the perturbation functions are integrable functions instead of rational functions. We note that a similar hypothesis to (H₂) was introduced by Figueroa and Pinto [13].

The paper is organized as follows. In section 2 we present the reformulated scalar method and the main results of this paper. Then, in section 3 we present the proofs of Theorems 2.1, 2.3 and 2.4.

2. REVISITED BELLMAN METHOD AND MAIN RESULTS

In this section we present the scalar method as a process of three steps. In each step we present the main results which proofs are deferred to section 3.

2.1. Step 1: Change of variable and reduction of the order. We introduce a little bit different change of variable to those originally proposed by Bellman. Here, in this paper, the new variable z is of the following type

$$z(t) = \frac{y^{(1)}(t)}{y(t)} - \mu \quad \text{or equivalently} \quad y(t) = \exp\left(\int_{t_0}^t (z(s) + \mu) ds\right), \quad (2.1)$$

where y is a solution of (1.1) and μ is an arbitrary root of the characteristic polynomial associated to (1.2). Then, by differentiation of $y(t)$ and by replacing the results of $y^\ell(t)$, $\ell = 0, \dots, 4$, in (1.1), we deduce that z is a solution of the following third order Riccati-type

$$\begin{aligned} z^{(3)} &+ [4\mu + a_3]z^{(2)} + [6\mu^2 + 3a_3\mu + a_2]z^{(1)} + [4\mu^3 + 3\mu^2a_3 + 2\mu a_2 + a_1]z \\ &+ r_3z^{(2)} + [3\mu r_3(t) + r_2(t)]z^{(1)} + [3\mu^2r_3(t) + 2\mu r_2(t) + r_1(t)]z + \mu^3r_3(t) \\ &+ \mu^2r_2(t) + \mu r(t) + r_0(t) + 4zz^{(2)} + [\mu + 3a_3 + 3r_3(t)]zz^{(1)} + 6z^2z^{(1)} \\ &+ 3[z^{(1)}]^2 + [6\mu^2 + 3\mu a_3 + a_2 + 3\mu r_3(t) + r_2(t)]z^2 + [4\mu + r_3(t)]z^3 + z^4 = 0. \end{aligned} \quad (2.2)$$

Now, if we define the operators Ψ^l and Ψ^n by the following relations

$$\begin{aligned} \Psi^l(\mu, h) &= h^{(3)} + [4\mu + a_3]h^{(2)} + [6\mu^2 + 3a_3\mu + a_2]h^{(1)} \\ &+ [4\mu^3 + 3\mu^2a_3 + 2\mu a_2 + a_1]h, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \Psi^n(\mu, h) &= r_3h^{(2)} + [3\mu r_3 + r_2]h^{(1)} + [3\mu^2r_3 + 2\mu r_2 + r_1]h + \mu^3r_3 \\ &+ \mu^2r_2 + \mu r_i + r_0 + 4hh^{(2)} + [\lambda\mu + 3a_3 + 3r_3]hh^{(1)} + 6h^2h^{(1)} \\ &+ 3[h^{(1)}]^2 + [6\mu^2 + 3\mu a_3 + a_2 + 3\mu r_3 + r_2]h^2 + [4\mu + r_3]h^3 + h^4, \end{aligned} \quad (2.4)$$

we note that (2.2) can be equivalently rewritten as follows

$$\Psi^l(\mu, z) + \Psi^n(\mu, z) = 0. \quad (2.5)$$

Note that Ψ^l and Ψ^n are linear and nonlinear operators, respectively. Thus, the analysis of original linear perturbed equation of fourth order (1.1) is translated to the analysis of a nonlinear third order equation (2.5). Moreover, we note that characteristic polynomials associated to (1.2) and to $\Psi^l(\lambda, z) = 0$ are related in the sense of the following Proposition.

Proposition 2.1. *Let us consider Ψ^l the operator defined in (2.3). If λ_i and λ_j are two distinct characteristic polynomials associated to (1.2), then $\lambda_j - \lambda_i$ is a root of the characteristic polynomial associated to the differential equation $\Psi^l(\lambda_i, z) = 0$.*

Proof. Considering $\lambda_i \neq \lambda_j$ satisfying the characteristic polynomial associated to (1.2), subtracting the equalities, dividing the result by $\lambda_j - \lambda_i$ and using the identities

$$\begin{aligned} \lambda_j^3 + \lambda_j^2\lambda_i + \lambda_j\lambda_i^2 + \lambda_i^3 &= (\lambda_j - \lambda_i)^3 + 4\lambda_i(\lambda_j - \lambda_i)^2 + 6\lambda_i^2(\lambda_j - \lambda_i) + 4\lambda_i^3 \\ a_3(\lambda_j^2 + \lambda_j\lambda_i + \lambda_i^2) &= a_3(\lambda_j - \lambda_i)^2 + 3a_3\lambda_i(\lambda_j - \lambda_i) + 3a_3\lambda_i^2 \\ a_2(\lambda_j - \lambda_i) &= a_2(\lambda_j - \lambda_i) + 2\lambda_i a_2, \end{aligned}$$

we deduce that $\Psi^l(\lambda_j - \lambda_i, z) = 0$. Thus, $\lambda_j - \lambda_i$ is a root of the characteristic polynomial associated to $\Psi^l(\lambda_i, z) = 0$ and the proof is concluded. \square

We note that the change of variable (2.1) can be applied by each characteristic root λ_i and the equation (2.5) should be satisfied with $\mu = \lambda_i$. Then, in order to distinguish that z is a solution of (2.5) with $\mu = \lambda_i$ we introduce the notation z_i . Hence, to conclude this step we precise the previous discussion in the following Lemma.

Lemma 2.1. *If hypothesis (H_1) is satisfied, then the fundamental system of solutions of (1.1) is given by*

$$y_i(t) = \exp\left(\int_{t_0}^t [\lambda_i + z_i(s)] ds\right), \quad \text{with } \{\mu_i, z_i\} \text{ solution of (2.5), } i \in \{1, 2, 3, 4\}. \quad (2.6)$$

2.2. Step 2: Well posedness and asymptotic behavior of the Riccati-type equation (2.2).

In this second step, we obtain three results. The first result is related to the conditions for the existence and uniqueness of a more general equation of that given in (2.2), see Theorem 2.1. Then, we introduce a second result concerning to the well posedness of (2.2), see Theorem 2.2. Finally, we present the result of asymptotic behavior for (2.2), see Theorem 2.3. Indeed, to be precise these three results are the following theorems:

Theorem 2.1. *Let us introduce the notation $C_0^2([t_0, \infty[)$ for the following space of functions*

$$C_0^2([t_0, \infty[) = \left\{ z \in C^2([t_0, \infty[, \mathbb{R}) \quad : \quad z, z^{(1)}, z^{(2)} \rightarrow 0 \text{ when } t \rightarrow \infty \right\}, \quad t_0 \in \mathbb{R},$$

and consider the equation

$$z^{(3)} + \sum_{i=0}^2 b_i z^{(i)} = \Omega(t) + F(t, z, z^{(1)}, z^{(2)}), \quad (b_0, b_1, b_2) \in \mathbb{R}^3, \quad (2.7)$$

where Ω and F are given functions such that the following restrictions

(\mathcal{R}_1) *There exists the functions $\hat{F}_1, \hat{F}_2, \Gamma : \mathbb{R}^4 \rightarrow \mathbb{R}$; $\Lambda_1, \Lambda_2 : \mathbb{R} \rightarrow \mathbb{R}^3$ and $\mathbf{C} \in \mathbb{R}^7$, such that*

$$\begin{aligned} F &= \hat{F}_1 + \hat{F}_2 + \Gamma, \\ \hat{F}_1(t, x_1, x_2, x_3) &= \Lambda_1(t) \cdot (x_1, x_2, x_3), \\ \hat{F}_2(t, x_1, x_2, x_3) &= \Lambda_2(t) \cdot (x_1 x_2, x_1^2, x_1^3), \\ \Gamma(t, x_1, x_2, x_3) &= \mathbf{C} \cdot (x_2^2, x_1 x_2, x_1 x_3, x_1^2, x_1^2 x_2, x_1^3, x_1^4), \end{aligned}$$

where “ \cdot ” denotes the canonical inner product in \mathbb{R}^n .

(\mathcal{R}_2) *The roots γ_i , $i = 1, 2, 3$, of the corresponding characteristic polynomial associated to the homogeneous part of (2.7) are real and simple.*

(\mathcal{R}_3) It is assumed that $\mathcal{L}(\Omega)(t) \rightarrow 0$, $\mathcal{L}(\|\Lambda_1\|_1)(t) \rightarrow 0$ and $\mathcal{L}(\|\Lambda_2\|_1)(t)$ is bounded, when $t \rightarrow \infty$. Here $\|\cdot\|_1$ denotes the norm of the sum in \mathbb{R}^n and \mathcal{L} is the operator defined on (1.5).

hold. Then, there exists a unique $z \in C_0^2([t_0, \infty[)$ solution of (2.7).

Theorem 2.2. Let us consider that the hypothesis (H_1) and (H_2) are satisfied. Then, for each $i = 1, \dots, 4$, the equation (2.5) has a unique solution $\{\mu_i, z_i\}$ with $z_i \in C_0^2([t_0, \infty[)$.

Theorem 2.3. Consider that the hypothesis (H_1), (H_2) and (H_3) are satisfied and for $i = 1, \dots, 4$, introduce the notation

$$A_i = \frac{1}{|\delta w_i|} \sum_{j=0}^2 \alpha_{j,i} \quad \text{and}$$

$$\sigma_i = 3|\lambda_i|^2 + 5|\lambda_i| + 3 + \left(19 + 7|\lambda_i| + |12\lambda_i + 3a_3| + |6\lambda_i^2 + 3\lambda_i a_3 + a_2|\right) \eta, \quad \eta \in]0, 1/2[,$$

with

$$\begin{aligned} \delta w_1 &= (\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2), & \delta w_2 &= (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_1), \\ \delta w_3 &= (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1), & \delta w_4 &= (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1), \\ \alpha_{j,1} &= |\lambda_4 - \lambda_3| |\lambda_2 - \lambda_1|^j + |\lambda_4 - \lambda_2| |\lambda_3 - \lambda_1|^j + |\lambda_3 - \lambda_2| |\lambda_4 - \lambda_1|^j \\ \alpha_{j,2} &= |\lambda_3 - \lambda_4| |\lambda_1 - \lambda_2|^j + |\lambda_1 - \lambda_3| |\lambda_4 - \lambda_2|^j + |\lambda_1 - \lambda_4| |\lambda_3 - \lambda_2|^j \\ \alpha_{j,3} &= |\lambda_2 - \lambda_1| |\lambda_4 - \lambda_3|^j + |\lambda_2 + \lambda_4 - 2\lambda_3| |\lambda_1 - \lambda_3|^j + |\lambda_1 + \lambda_4 - 2\lambda_3| |\lambda_2 - \lambda_3|^j \\ \alpha_{j,4} &= |\lambda_3 - \lambda_2| |\lambda_1 - \lambda_4|^j + |\lambda_3 - \lambda_1| |\lambda_2 - \lambda_4|^j + |\lambda_2 - \lambda_1| |\lambda_3 - \lambda_4|^j \end{aligned}$$

For each $i = 1, \dots, 4$, if $\rho_i \in]0, (A_i \sigma_i)^{-1}[$, then $\{\mu_i, z_i\}$, the solution of (2.5), has the following asymptotic behavior

$$z_i^{(\ell)}(t) = \begin{cases} O\left(\int_{t_0}^{\infty} e^{-\beta(t-s)} |p(\lambda_1, s)| ds\right), & i = 1, \quad \beta \in [\lambda_2 - \lambda_1, 0[, \\ O\left(\int_{t_0}^{\infty} e^{-\beta(t-s)} |p(\lambda_2, s)| ds\right), & i = 2, \quad \beta \in [\lambda_3 - \lambda_2, 0[, \\ O\left(\int_{t_0}^{\infty} e^{-\beta(t-s)} |p(\lambda_3, s)| ds\right), & i = 3, \quad \beta \in [\lambda_4 - \lambda_3, 0[, \\ O\left(\int_{t_0}^t e^{-\beta(t-s)} |p(\lambda_4, s)| ds\right), & i = 4, \quad \beta \in]0, \lambda_3 - \lambda_4], \end{cases} \quad (2.8)$$

where $p(\mu, s) = \mu r_3^3(s) + \mu r_2^2(s) + \mu r_1(s) + r_0(s)$.

2.3. Step 3: Existence of a fundamental system of solutions for (1.1) and its asymptotic behavior. Here we translate the results for the behavior of z (see Theorem 2.2) to the variable y via the relation (2.1).

Theorem 2.4. Let us assume that the hypothesis (H_1) and (H_2) are satisfied, denote by $W[y_1, \dots, y_4]$ the Wronskian of $\{y_1, \dots, y_4\}$, by

$$\pi_i = \prod_{k \in N_i} (\lambda_k - \lambda_i), \quad N_i = \{1, 2, 3, 4\} - \{i\}, \quad i = 1, \dots, 4,$$

by $p(\mu, s)$ the function defined in Theorem 2.3 and by F the functions defined in Theorem 2.1 with Λ_1, Λ_2 and \mathbf{C} given in (3.8). Then the equation (1.1) has a fundamental system of solutions given by (2.6). Moreover the following properties about the asymptotic behavior

$$\frac{y_i^{(\ell)}(t)}{y_i(t)} = (\lambda_i)^\ell, \quad \text{for } i \in \{1, 2, 3, 4\} \text{ and } \ell \in \{1, 2, 3, 4\}, \quad (2.9)$$

$$W[y_1, \dots, y_4] = (\lambda_4 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3), \quad (2.10)$$

are satisfied when $t \rightarrow \infty$. Furthermore, if (H_3) is satisfied, then

$$y_i(t) = e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t \left[p(\lambda_i, s) + F(s, z_i(s), z_i^{(1)}(s), z_i^{(2)}(s))\right] ds\right), \quad (2.11)$$

$$y_i^{(\ell)}(t) = \left(\lambda_i^{\ell-1} + o(1)\right) e^{\lambda_i(t-t_0)} \times \exp\left(\pi_i^{-1} \int_{t_0}^t \left[p(\lambda_i, s) + F(s, z_i(s), z_i^{(1)}(s), z_i^{(2)}(s))\right] ds\right), \quad \ell = \overline{2, 4}, \quad (2.12)$$

holds, when $t \rightarrow \infty$ with z_i given asymptotically by (2.8).

3. PROOF OF MAIN RESULTS

In this section we present the proofs of Theorems 2.1, 2.2, 2.3, and 2.4.

3.1. Proof of Theorem 2.1. Before of start the proof, we need define some notation about Green functions. First, let us consider the homogeneous equation associated to (2.7)

$$z^{(3)} + \sum_{i=0}^2 b_i z^{(i)} = 0, \quad (3.1)$$

and denote by γ_i , $i = 1, 2, 3$, the roots of the corresponding characteristic polynomial for (3.1). Then, the green function for (3.1) is defined by

$$g(t, s) = \frac{1}{\delta\gamma} \times \begin{cases} g_1(t, s), & (\operatorname{Re}\gamma_1, \operatorname{Re}\gamma_2, \operatorname{Re}\gamma_3) \in \mathbb{R}_{----}^3, \\ g_2(t, s), & (\operatorname{Re}\gamma_1, \operatorname{Re}\gamma_2, \operatorname{Re}\gamma_3) \in \mathbb{R}_{+---}^3, \\ g_3(t, s), & (\operatorname{Re}\gamma_1, \operatorname{Re}\gamma_2, \operatorname{Re}\gamma_3) \in \mathbb{R}_{++--}^3, \\ g_4(t, s), & (\operatorname{Re}\gamma_1, \operatorname{Re}\gamma_2, \operatorname{Re}\gamma_3) \in \mathbb{R}_{++++}^3, \end{cases} \quad (3.2)$$

where $\delta\gamma = (\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)$ and

$$\begin{aligned} g_1(t, s) &= \begin{cases} 0, & t \geq s, \\ (\gamma_3 - \gamma_2)e^{-\gamma_1(t-s)} + (\gamma_1 - \gamma_3)e^{-\gamma_2(t-s)} + (\gamma_2 - \gamma_1)e^{-\gamma_3(t-s)}, & t \leq s, \end{cases} \\ g_2(t, s) &= \begin{cases} (\gamma_1 - \gamma_2)e^{-\gamma_3(t-s)} - (\gamma_1 - \gamma_3)e^{-\gamma_2(t-s)}, & t \leq s, \\ (\gamma_2 - \gamma_3)e^{-\gamma_1(t-s)}, & t \geq s, \end{cases} \\ g_3(t, s) &= \begin{cases} (\gamma_2 - \gamma_1)e^{-\gamma_3(t-s)}, & t \leq s, \\ (\gamma_3 - \gamma_2)e^{-\gamma_1(t-s)} - (\gamma_3 - \gamma_1)e^{-\gamma_2(t-s)}, & t \geq s, \end{cases} \\ g_4(t, s) &= \begin{cases} (\gamma_3 - \gamma_2)e^{-\gamma_1(t-s)} + (\gamma_1 - \gamma_3)e^{-\gamma_2(t-s)} + (\gamma_2 - \gamma_1)e^{-\gamma_3(t-s)}, & t \geq s, \\ 0, & t \leq s. \end{cases} \end{aligned}$$

Further details on Green functions may be consultd in [4].

Now, we start the proof be noticing that, by the method of variation of parameters, the hypothesis (\mathcal{B}_2) , implies that the equation (2.7) is equivalent to the following integral equation

$$z(t) = \int_{t_0}^{\infty} g(t, s) \left[\Omega(s) + F\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right] ds, \quad (3.3)$$

where g is the Green function defined on (3.2). We recall that $C_0^2([t_0, \infty[)$ is a Banach space with the norm $\|z\|_0 = \sup_{t \geq t_0} \{|z(t)| + |z^{(1)}(t)| + |z^{(2)}(t)|\}$. Now, we define the operator T from $C_0^2([t_0, \infty[)$ to $C_0^2([t_0, \infty[)$ as follows

$$Tz(t) = \int_{t_0}^{\infty} g(t, s) \left[\Omega(s) + F\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right] ds. \quad (3.4)$$

Then, we note that (3.3) can be rewritten as the operator equation

$$Tz = z \quad \text{over} \quad D_{\eta} := \left\{ z \in C_0^2([t_0, \infty[) \quad : \quad \|z\|_0 \leq \eta \right\}, \quad (3.5)$$

where $\eta \in \mathbb{R}^+$ will be selected in order to apply the Banach fixed point theorem. Indeed, we have that

(a) T is well defined from $C_0^2([t_0, \infty[)$ to $C_0^2([t_0, \infty[)$. Let us consider an arbitrary $z \in C_0^2([t_0, \infty[)$. We note that

$$T^{(i)}z(t) = \int_{t_0}^{\infty} \frac{\partial^i g}{\partial t^i}(t, s) \left[\Omega(s) + F\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right] ds, \quad i = 1, 2.$$

Then, by the definition of g , we immediately deduce that $Tz, T^{(1)}z, T^{(2)}z \in C^2([t_0, \infty[, \mathbb{R})$. Furthermore, by the hypothesis (\mathcal{R}_1) , we can deduce the following estimate

$$\begin{aligned} |T^{(i)}z(t)| \leq & \int_{t_0}^{\infty} \left| \frac{\partial^i g}{\partial t^i}(t, s) \right| \left[|\Omega(s)| + \left| \hat{F}_1\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right| \right. \\ & \left. + \left| \hat{F}_2\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right| + \left| \Gamma\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right| \right] ds \end{aligned} \quad (3.6)$$

for each $i = 0, 1, 2$. Now, by application of the hypothesis (\mathcal{R}_3) , the properties of \hat{F}_1, \hat{F}_2 and Γ and the fact that $z \in C_0^2$, we have that the right hand side of (3.6) tends to 0 when $t \rightarrow \infty$. Then, $Tz, T^{(1)}z, T^{(2)}z \rightarrow 0$ when $t \rightarrow \infty$ or equivalently $Tz \in C_0^2$ for all $z \in C_0^2$.

(b) For all $\eta \in]0, 1[$, the set D_η is invariante under T . Let us consider $z \in D_\eta$. From (3.6), we can deduce the following estimate

$$\begin{aligned} \|Tz\|_0 \leq & \mathcal{L}(\Omega)(t) + \|z\|_0 \mathcal{L}\left(\|\Lambda_1\|_1\right)(t) + 2\left(\|z\|_0\right)^2 \mathcal{L}\left(\|\Lambda_2\|_1\right)(t) + \left(\|z\|_0\right)^3 \mathcal{L}\left(\|\Lambda_2\|_1\right)(t) \\ & + \left(\|z\|_0\right)^2 \left(\sum_{i=1}^4 |c_i| + (|c_5| + |c_6|)\|z\|_0 + |c_7| \left(\|z\|_0\right)^2 \right) \mathcal{L}(1)(t) \\ \leq & I_1 + I_2. \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} I_1 &= \mathcal{L}(\Omega)(t) \\ I_2 &= \|z\|_0 \left\{ \mathcal{L}\left(\|\Lambda_1\|_1\right)(t) + \left(2\mathcal{L}\left(\|\Lambda_2\|_1\right)(t) + \mathcal{L}\left(\|\mathbf{C}\|_1\right)(t) \right) \|z\|_0 \right. \\ & \quad \left. + \left(\mathcal{L}\left(\|\Lambda_2\|_1\right)(t) + \mathcal{L}\left(\|\mathbf{C}\|_1\right)(t) \right) \left(\|z\|_0\right)^2 + \mathcal{L}\left(\|\mathbf{C}\|_1\right)(t) \left(\|z\|_0\right)^3 \right\}. \end{aligned}$$

Now, by (\mathcal{R}_3) we deduce that $I_1 \rightarrow 0$ when $t \rightarrow \infty$. Similarly, by application of (\mathcal{R}_3) we can prove that the inequality

$$\begin{aligned} I_2 \leq & \eta^2 \left\{ \left(2\mathcal{L}\left(\|\Lambda_2\|_1\right)(t) + \mathcal{L}\left(\|\mathbf{C}\|_1\right)(t) \right) + \left(\mathcal{L}\left(\|\Lambda_2\|_1\right)(t) + \mathcal{L}\left(\|\mathbf{C}\|_1\right)(t) \right) \eta \right. \\ & \left. + \mathcal{L}\left(\|\mathbf{C}\|_1\right)(t) \eta^2 \right\} \\ \leq & \eta, \end{aligned}$$

holds when $t \rightarrow \infty$ in a right neighborhood of $\eta = 0$. Hence, by (3.7) and (\mathcal{R}_3) , we prove that $Tz \in D_\eta$ for all $z \in D_\eta$.

(c) T is a contraction for $\eta \in]0, 1/2[$. Let $z_1, z_2 \in D_\eta$, by the hypothesis (\mathcal{R}_1) and algebraic rearrangements, we follow that

$$\begin{aligned} \|Tz_1 - Tz_2\|_0 \leq & \|z_1 - z_2\|_0 \sum_{i=0}^2 \int_{t_0}^{\infty} \left| \frac{\partial^i g}{\partial t^i}(t, s) \right| \|\Lambda_1(s)\|_1 ds \\ & + \|z_1 - z_2\|_0 \max\{2\eta, 3\eta^2\} \sum_{i=0}^2 \int_{t_0}^{\infty} \left| \frac{\partial^i g}{\partial t^i}(t, s) \right| \|\Lambda_2(s)\|_1 ds \end{aligned}$$

$$\begin{aligned}
& + \|z_1 - z_2\|_0 \max \left\{ 2\eta, 3\eta^2, 4\eta^3 \right\} \sum_{i=0}^2 \int_{t_0}^{\infty} \left| \frac{\partial^i g}{\partial t^i}(t, s) \right| \|C\|_1 ds \\
\leq & \|z_1 - z_2\|_0 \left\{ \mathcal{L}(\|\Lambda_1\|_1) + \max \left\{ 2\eta, 3\eta^2 \right\} \mathcal{L}(\|\Lambda_2\|_1) \right. \\
& \left. + \max \left\{ 2\eta, 3\eta^2, 4\eta^3 \right\} \mathcal{L}(\|C\|_1) \right\}.
\end{aligned}$$

Then, by application of (\mathcal{R}_3) , we deduce that T is a contraction, since, for an arbitrary $\eta \in]0, 1/2[$, we have that $\max \left\{ 2\eta, 3\eta^2 \right\} = \max \left\{ 2\eta, 3\eta^2, 4\eta^3 \right\} = 2\eta < 1$. Hence, from (a)-(c) and application of Banach fixed point theorem, we deduce that there exists a unique $z \in D_\eta \subset C_0^2([t_0, \infty[)$ solution of (3.5).

3.2. Proof of Theorem 2.2. The proof of the Theorem 2.2 follows by application Theorem 2.1. Indeed, in the next lines we verify the hypothesis (\mathcal{R}_1) - (\mathcal{R}_3) . First, the hypothesis (\mathcal{R}_1) is satisfied since (2.5) can be rewritten as (2.7). More precisely, if λ_i denotes an arbitrary characteristic root of (1.2), we have that the constant coefficients $b_j, j = 0, 1, 2$, in (2.7) are defined by

$$b_0 = 4\lambda_i^3 + 3\lambda_i^2 a_3 + 2\lambda_i a_2 + a_1, \quad b_1 = 6\lambda_i^2 + 3\lambda_i a_3 + a_2, \quad b_2 = 4\lambda_i + a_3, \quad (3.8a)$$

the functions $\Omega : \mathbb{R} \rightarrow \mathbb{R}$ and $\Lambda_1, \Lambda_2 : \mathbb{R} \rightarrow \mathbb{R}^3$ and the constant $C \in \mathbb{R}^7$ defining the function F are given by

$$\Omega(t) = -(\lambda_i^3 r_3(t) + \lambda_i^2 r_2(t) + \lambda_i r_1(t) + r_0(t)), \quad \Lambda_1(t) = (b(t), f(t), h(t)), \quad (3.8b)$$

$$\Lambda_2(t) = (p(t), f(t), h(t)), \quad C = -\left(3, 12\lambda_i + 3a_3, 6\lambda_i^2 + 3\lambda_i a_3 + a_2, 4, 6, 4\lambda_i, 1\right), \quad (3.8c)$$

with

$$b(t) = -(3\lambda_i^2 r_3(t) + 2\lambda_i r_2(t) + r_1(t)), \quad f(t) = -(3\lambda_i r_3(t) + r_2(t)), \quad p(t) = 3h(t) = -3r_3(t). \quad (3.8d)$$

In second place, by application of Proposition 2.1, we deduce that the hypothesis (\mathcal{R}_2) is satisfied. Meanwhile, we note that (H_2) implies (\mathcal{R}_3) . Thus, we deduce that conclusion of the Theorem 2.2 is valid.

3.3. Proof of Theorem 2.3. First we present some useful bounds concerning to the Green functions g_i defined on (3.2). In the case of g_1 and g_4 , for $i \in \mathbb{N} \cup \{0\}$, we have the following bound

$$\left| \frac{\partial^i g_\ell}{\partial t^i}(t, s) \right| \leq \left(|\gamma_3 - \gamma_2| |\gamma_1|^i + |\gamma_1 - \gamma_3| |\gamma_2|^i + |\gamma_2 - \gamma_1| |\gamma_3|^i \right) e^{-\alpha_\ell(t-s)}, \quad \ell = \{1, 4\}, \quad (3.9)$$

with $\alpha_1 = \max \left\{ \gamma_1, \gamma_2, \gamma_3 \right\}$ and $\alpha_4 = \min \left\{ \gamma_1, \gamma_2, \gamma_3 \right\}$. Similarly for g_2 and g_3 , for $i \in \mathbb{N} \cup \{0\}$, we have that

$$\left| \frac{\partial^i g_2}{\partial t^i}(t, s) \right| \leq \begin{cases} \left(|\gamma_2 - \gamma_1| |\gamma_3|^i + |\gamma_3 - \gamma_1| |\gamma_2|^i \right) e^{-\max\{\gamma_2, \gamma_3\}(t-s)}, & t \leq s, \\ |\gamma_3 - \gamma_2| |\gamma_1|^i e^{-\gamma_1(t-s)}, & t \geq s, \end{cases} \quad (3.10)$$

$$\left| \frac{\partial^i g_3}{\partial t^i}(t, s) \right| \leq \begin{cases} |\gamma_2 - \gamma_1| |\gamma_3|^i e^{-\gamma_3(t-s)}, & t \leq s, \\ \left(|\gamma_3 - \gamma_2| |\gamma_1|^i + |\gamma_3 - \gamma_1| |\gamma_2|^i \right) e^{-\min\{\gamma_1, \gamma_2\}(t-s)}, & t \geq s. \end{cases} \quad (3.11)$$

The proof of the bounds (3.9)-(3.11) are straightforward by application of the algebraic properties of the exponential function.

Proof of (2.8) with $\ell = 1$. Let us denote by T the operator defined in (3.4) and by z_1 the solution of the equation (2.5) associated with the characteristic root λ_1 of (1.2). Now, on D_η with $\eta \in]0, 1/2[$, we define the sequence $\omega_{n+1} = T\omega_n$ with $\omega_0 = 0$, we have that $\omega_n \rightarrow z_1$ when $n \rightarrow \infty$. This fact is a consequence of the contraction property of T . We note that the hypothesis (H_1) and Proposition (2.1) implies that all roots of the corresponding characteristic polynomial for (3.1) with b_i defined on (3.8a) are negative, since, $0 > \gamma_1 = \lambda_2 - \lambda_1 > \gamma_2 = \lambda_3 - \lambda_1 > \gamma_3 = \lambda_4 - \lambda_1$. Moreover,

by (3.8b) we note that the following identity $\Omega(s) = p(\lambda_1, s)$ holds. Then, the Green function g defined on (3.2) is given by $(\delta\gamma)^{-1}g_1$. Naturally the operator T can be rewritten equivalently as follows

$$Tz(t) = \frac{1}{\delta\gamma} \int_t^\infty g_1(t, s) \left[p(\lambda_1, s) + F\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right] ds, \quad \text{for } t \geq t_0, \quad (3.12)$$

since $g_1(t, s) = 0$ for $s \in [t_0, t]$. Thus, the proof of (2.8) with $\ell = 1$ is reduced to prove that

$$\exists \Phi_n \in \mathbb{R}_+ : \sum_{j=0}^2 |\omega_n^{(j)}(t)| \leq \Phi_n \int_t^\infty e^{-\beta(t-\tau)} |p(\lambda_1, \tau)| d\tau, \quad \forall t \geq t_0, \quad (3.13)$$

$$\exists \Phi \in \mathbb{R}_+ : \Phi_n \rightarrow \Phi \text{ when } n \rightarrow \infty. \quad (3.14)$$

Indeed, we prove (3.13) by induction on n and deduce that (3.14) is a consequence of the construction of the sequence $\{\Phi_n\}$. Firstly, we prove (3.13). Note that for $n = 1$ the estimate (3.13) is satisfied with $\Phi_1 = A_1$. It can be proved immediatly by the definition of the operator T given on (3.12), the property $F(s, 0, 0, 0) = 0$, the estimate (3.9) and the hypothesis that $\beta \in [\lambda_2 - \lambda_1, 0[$, since

$$\begin{aligned} \sum_{j=0}^2 |\omega_1^{(j)}(t)| &= \sum_{j=0}^2 |T^{(j)}\omega_0(t)| = \frac{1}{|\delta\gamma|} \sum_{j=0}^2 \int_t^\infty \left| \frac{\partial^j g_1}{\partial t^j}(t, s) \right| |p(\lambda_1, s)| ds \\ &\leq \frac{1}{|\delta\gamma|} \sum_{j=0}^2 \left(|\gamma_3 - \gamma_2||\gamma_1|^j + |\gamma_1 - \gamma_3||\gamma_2|^j + |\gamma_2 - \gamma_1||\gamma_3|^j \right) \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-\tau)} |p(\lambda_1, \tau)| d\tau \\ &= A_1 \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-\tau)} |p(\lambda_1, \tau)| d\tau \leq A_1 \int_t^\infty e^{-\beta(t-\tau)} |p(\lambda_1, \tau)| d\tau. \end{aligned}$$

Now, assuming that (3.13) is valid for $n = k$, we prove that (3.13) is also valid for $n = k + 1$. However, before of prove the estimate (3.13) for $n = k + 1$, we note that by the hypothesis (H₃) (i.e. the perturbations are belong to $\mathcal{F}_1([t_0, \infty[))$, the notation (3.8) and the fact that $\max\{\eta, \eta^2, \eta^3\} = \eta$ for $\eta \in]0, 1/2[$, we deduce the following estimates

$$\begin{aligned} &\left| p(\lambda_1, s) + F\left(s, \omega_k(s), \omega_k^{(1)}(s), \omega_k^{(2)}(s)\right) \right| \\ &\leq |a(s)| + |b(s)||\omega_k(s)| + |f(s)||\omega_k^{(1)}(s)| + |h(s)||\omega_k^{(2)}(s)| \\ &\quad + |p(s)||\omega_k^{(1)}(s)||\omega_k(s)| + |f(s)||\omega_k(s)|^2 + |h(s)||\omega_k(s)|^3 \\ &\quad + |C_1||\omega_k^{(1)}(s)|^2 + |C_2||\omega_k(s)||\omega_k^{(1)}(s)| + |C_3||\omega_k(s)|^2 + |C_4||\omega_k(s)||\omega_k^{(2)}(s)| \\ &\quad + |C_5||\omega_k(s)|^2|\omega_k^{(2)}(s)| + |C_6||\omega_k(s)|^3 + |C_7||\omega_k(s)|^4 \\ &\leq |p(\lambda_1, s)| + \left[|b(s)| + |f(s)| + |h(s)| + |p(s)|\eta + |f(s)|\eta + |h(s)|\eta^2 \right. \\ &\quad \left. + \left(\sum_{i=1}^4 |C_i| \right) \eta + \left(\sum_{i=5}^6 |C_i| \right) \eta^2 \right] \max \left\{ |\omega_k(s)|, |\omega_k^{(1)}(s)|, |\omega_k^{(2)}(s)| \right\} \\ &\leq |p(\lambda_1, s)| + \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \left\| \left(\omega_k, \omega_k^{(1)}, \omega_k^{(2)} \right)(s) \right\|_1, \end{aligned} \quad (3.15a)$$

$$\int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} |b(s)| ds \leq (3|\lambda_1|^2 + 2|\lambda_1| + 1)\rho_1, \quad (3.15b)$$

$$\int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} |f(s)| ds \leq (3|\lambda_1| + 1)\rho_1, \quad (3.15c)$$

$$\int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} |p(s)| ds \leq 3\rho_1, \quad \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} |h(s)| ds \leq \rho_1. \quad (3.15d)$$

Using (3.12), the notation (3.8), the inductive hypothesis, the inequality (3.9) and the estimates (3.15) we have that

$$\begin{aligned}
\sum_{j=0}^2 |\omega_{k+1}^{(j)}(t)| &= \sum_{j=0}^2 |T^{(j)}\omega_k(t)| \\
&= \frac{1}{|\delta\gamma|} \sum_{j=0}^2 \left| \int_t^\infty \frac{\partial^j g_1}{\partial t^j}(t, s) \left[p(\lambda_1, s) + F\left(s, \omega_k(s), \omega_k^{(1)}(s), \omega_k^{(2)}(s)\right) \right] ds \right| \\
&\leq \frac{1}{|\delta\gamma|} \sum_{j=0}^2 \int_t^\infty \left| \frac{\partial^j g_1}{\partial t^j}(t, s) \right| \left| p(\lambda_1, s) + F\left(s, \omega_k(s), \omega_k^{(1)}(s), \omega_k^{(2)}(s)\right) \right| ds \\
&\leq A_1 \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} \left\{ |p(\lambda_1, s)| + \left[\|(b, f, h)(s)\|_1 \right. \right. \\
&\quad \left. \left. + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \left\| \left(\omega_k, \omega_k^{(1)}, \omega_k^{(2)} \right)(s) \right\|_1 \right\} ds \\
&\leq A_1 \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} \left\{ |p(\lambda_1, s)| + \left[\|(b, f, h)(s)\|_1 \right. \right. \\
&\quad \left. \left. + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \Phi_k \int_s^\infty e^{-\beta(s-\tau)} |p(\lambda_1, \tau)| d\tau \right\} ds \\
&\leq A_1 \left\{ 1 + \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \Phi_k ds \right\} \\
&\quad \times \left\{ \int_t^\infty e^{-\beta(t-\tau)} |p(\lambda_1, \tau)| d\tau \right\} \\
&\leq A_1 \left(1 + \Phi_k \rho_1 \sigma_1 \right) \int_t^\infty e^{-\beta(t-\tau)} |p(\lambda_1, \tau)| d\tau,
\end{aligned}$$

Then, by the induction process, (3.13) is satisfied with $\Phi_n = A_1(1 + \Phi_{n-1} \rho_1 \sigma_1)$. Now, using recursively the definition of $\Phi_{n-2}, \dots, \Phi_2$, we can rewrite Φ_n as the sum of terms of a geometric progression where the common ratio is given by $\rho_1 A_1 \sigma_1$. Then, the existence of Φ satisfying (3.14) follows by the hypothesis that $\rho_1 A_1 \sigma_1 \in]0, 1[$ for $\eta \in]0, 1/2[$. More precisely, we have that

$$\lim_{n \rightarrow \infty} \Phi_n = A_1 \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \left(\rho_1 A_1 \sigma_1 \right)^i = A_1 \lim_{n \rightarrow \infty} \frac{[(\rho_1 A_1 \sigma_1)^n - 1]}{\rho_1 A_1 \sigma_1 - 1} = \frac{A_1}{1 - \rho_1 A_1 \sigma_1} = \Phi > 0.$$

Hence, (3.13)-(3.14) are valid and the proof of (2.8) with $\ell = 1$ is concluded by passing to the limit the sequence $\{\Phi_n\}$ when $n \rightarrow \infty$ and in the topology of $C_0^2([t_0, \infty])$.

Proof of (2.8) with $\ell = 2$. Let us denote by z_2 the solution of the equation (2.5) associated with the characteristic root λ_2 of (1.2). Similarly to the case $\ell = 1$ we define the sequence $\omega_{n+1} = T\omega_n$ with $\omega_0 = 0$ and, by the contraction property of T , we can deduce that $\omega_n \rightarrow z_2$ when $n \rightarrow \infty$. In this case, we note that $\Omega(s) = p(\lambda_2, s)$. Moreover, by Proposition (2.1) we have that $\gamma_1 = \lambda_1 - \lambda_2 > 0 > \gamma_2 = \lambda_3 - \lambda_2 > \gamma_3 = \lambda_4 - \lambda_2$. Then the Green function g defined on (3.2) is given by $(\delta\gamma)^{-1}g_2$. Thereby, the operator T can be rewrited equivalently as follows

$$\begin{aligned}
Tz(t) &= \frac{1}{\delta\gamma} \int_{t_0}^\infty g_2(t, s) \left[p(\lambda_2, s) + F\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right] ds \\
&= \frac{1}{\delta\gamma} \left\{ \int_{t_0}^t (\gamma_2 - \gamma_3) e^{-\gamma_1(t-s)} \left[p(\lambda_2, s) + F\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right] ds \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_t^\infty \left[(\gamma_2 - \gamma_1)e^{-\gamma_3(t-s)} - (\gamma_3 - \gamma_1)e^{-\gamma_2(t-s)} \right] \\
& \quad \times \left[p(\lambda_2, s) + F\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right] ds \Big\}. \tag{3.16}
\end{aligned}$$

Then, the proof of (2.8) with $\ell = 2$ is reduced to prove

$$\exists \Phi_n \in \mathbb{R}_+ \quad : \quad \sum_{j=0}^2 |\omega_n^{(j)}(t)| \leq \Phi_n \int_{t_0}^\infty e^{-\beta(t-\tau)} |p(\lambda_2, \tau)| d\tau, \quad \forall t \geq t_0, \tag{3.17}$$

$$\exists \Phi \in \mathbb{R}_+ \quad : \quad \Phi_n \rightarrow \Phi \text{ when } n \rightarrow \infty. \tag{3.18}$$

In the induction step for $n = 1$ the estimate (3.17) is satisfied with $\Phi_1 = A_2$, since by the definition of the operator T given on (3.16), the property $F(s, 0, 0, 0) = 0$, the estimates of type (3.10) and the fact that $\beta \in [\lambda_3 - \lambda_2, 0[\subset [\lambda_3 - \lambda_2, \lambda_1 - \lambda_2]$, we deduce the following bound

$$\begin{aligned}
\sum_{j=0}^2 |\omega_1^{(j)}(t)| &= \sum_{j=0}^2 |T^{(j)}\omega_0(t)| \\
&\leq \frac{1}{|\delta\gamma|} \left(\sum_{j=0}^2 |\gamma_2 - \gamma_3| |\gamma_1|^j \right) \int_{t_0}^t e^{-(\lambda_1 - \lambda_2)(t-s)} |p(\lambda_2, s)| ds \\
&\quad + \frac{1}{|\delta\gamma|} \left(\sum_{j=0}^2 |\gamma_2 - \gamma_1| |\gamma_3|^j + |\gamma_3 - \gamma_1| |\gamma_2|^j \right) \int_t^\infty e^{-(\lambda_3 - \lambda_2)(t-s)} |p(\lambda_2, s)| ds \\
&\leq A_2 \left\{ \int_{t_0}^t e^{-\beta(t-s)} |p(\lambda_2, \tau)| ds + \int_t^\infty e^{-\beta(t-s)} |p(\lambda_2, \tau)| ds \right\} \\
&= A_2 \int_{t_0}^\infty e^{-\beta(t-\tau)} |p(\lambda_2, \tau)| d\tau.
\end{aligned}$$

Noticing that a similar inequalities to (3.15), with λ_2 instead of λ_1 and integration on $[t_0, \infty[$ instead of $[t, \infty[$, we deduce that

$$\begin{aligned}
J_1 &:= \int_{t_0}^t e^{-(\lambda_1 - \lambda_2)(t-s)} |p(\lambda_2, s)| ds + \int_t^\infty e^{-(\lambda_3 - \lambda_2)(t-s)} |p(\lambda_2, s)| ds \\
&\leq \int_{t_0}^t e^{-\beta(t-s)} |p(\lambda_2, \tau)| ds + \int_t^\infty e^{-\beta(t-s)} |p(\lambda_2, \tau)| ds = \int_{t_0}^\infty e^{-\beta(t-\tau)} |p(\lambda_2, \tau)| d\tau, \\
J_2 &:= \int_{t_0}^t e^{-(\lambda_1 - \lambda_2)(t-s)} \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \\
&\quad \times \left\| \left(\omega_k, \omega_k^{(1)}, \omega_k^{(2)} \right)(s) \right\|_1 ds + \int_t^\infty e^{-(\lambda_1 - \lambda_2)(t-s)} \left[\|(b, f, h)(s)\|_1 \right. \\
&\quad \left. + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \times \left\| \left(\omega_k, \omega_k^{(1)}, \omega_k^{(2)} \right)(s) \right\|_1 ds \\
&\leq \int_{t_0}^t e^{-\beta(t-s)} \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \left\| \left(\omega_k, \omega_k^{(1)}, \omega_k^{(2)} \right)(s) \right\|_1 ds \\
&\quad + \int_t^\infty e^{-\beta(t-s)} \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \left\| \left(\omega_k, \omega_k^{(1)}, \omega_k^{(2)} \right)(s) \right\|_1 ds \\
&= \int_{t_0}^\infty e^{-\beta(t-s)} \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \left\| \left(\omega_k, \omega_k^{(1)}, \omega_k^{(2)} \right)(s) \right\|_1 ds
\end{aligned}$$

$$\begin{aligned}
&\leq \Phi_k \int_{t_0}^{\infty} e^{-\beta(t-s)} \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \int_{t_0}^{\infty} e^{-\beta(s-\tau)} |p(\lambda_2, \tau)| d\tau \Big\} ds \\
&= \Phi_k \int_{t_0}^{\infty} \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \int_{t_0}^{\infty} e^{-\beta(t-\tau)} |p(\lambda_2, \tau)| d\tau \Big\} ds \\
&\leq \Phi_k \rho_2 \sigma_2 \int_{t_0}^{\infty} e^{-\beta(t-\tau)} |p(\lambda_2, \tau)| d\tau.
\end{aligned}$$

Then, the general induction step can be proved as follows

$$\begin{aligned}
\sum_{j=0}^2 |\omega_{k+1}^{(j)}(t)| &= \sum_{j=0}^2 |T^{(j)} \omega_k(t)| \\
&\leq \frac{1}{|\delta\gamma|} \left(\sum_{j=0}^2 |\gamma_2 - \gamma_3| |\gamma_1|^j \right) \int_{t_0}^t e^{-(\lambda_1 - \lambda_2)(t-s)} \left| p(\lambda_2, s) + F(s, z(s), z^{(1)}(s), z^{(2)}(s)) \right| ds \\
&\quad + \frac{1}{|\delta\gamma|} \left(\sum_{j=0}^2 |\gamma_1 + \gamma_2| |\gamma_3|^j + |\gamma_1 + \gamma_3| |\gamma_2|^j \right) \\
&\quad \times \int_t^{\infty} e^{-(\lambda_3 - \lambda_2)(t-s)} \left| p(\lambda_2, s) + F(s, z(s), z^{(1)}(s), z^{(2)}(s)) \right| ds \\
&\leq A_2 \left[\int_{t_0}^t e^{-(\lambda_1 - \lambda_2)(t-s)} \left\{ |p(\lambda_2, s)| + \left[\|(b, f, h)(s)\|_1 + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \right. \right. \\
&\quad \times \left. \left\| (\omega_k, \omega_k^{(1)}, \omega_k^{(2)})(s) \right\|_1 \right\} ds + \int_t^{\infty} e^{-(\lambda_3 - \lambda_2)(t-s)} \left\{ |p(\lambda_2, s)| + \left[\|(b, f, h)(s)\|_1 \right. \right. \\
&\quad \left. \left. + \left(\|(p, f, h)(s)\|_1 + \|\mathbf{C}\|_1 \right) \eta \right] \left\| (\omega_k, \omega_k^{(1)}, \omega_k^{(2)})(s) \right\|_1 \right\} ds \Big] \\
&= A_2 [J_1 + J_2] \\
&\leq A_2 (1 + \Phi_k \rho_2 \sigma_2) \int_{t_0}^{\infty} e^{-\beta(t-\tau)} |p(\lambda_2, \tau)| d\tau.
\end{aligned}$$

Hence the thesis of the inductive steps holds with $\Phi_n = A_2(1 + \Phi_{n-1} \rho_2 \sigma_2)$. Now, proceeding in analogous way to the case $\ell = 1$, we find that (3.18) is satisfied with $\Phi = A_2/(1 - \rho_2 \sigma_2 A_2) > 0$. Thus, the sequence $\{\Phi_n\}$ is convergent and z_2 (the limit of ω_n in the topology of $C_0^2([t_0, \infty))$) satisfies (2.8).

Proof of (2.8) with $\ell = 3$. The proof of the case $\ell = 3$ is similar to case $\ell = 2$.

Proof of (2.8) with $\ell = 4$. Similarly to the preceding cases, let us denote by z_4 the solution of the equation (2.5) associated with the characteristic root λ_4 of (1.2). We start by defining the sequence $\omega_{n+1} = T\omega_n$ with $\omega_0 = 0$ and note that by the contraction property of T , we can deduce that $\omega_n \rightarrow z_4$ when $n \rightarrow \infty$. Now, in this case, we have that $\Omega(s) = p(\lambda_4, s)$, $\gamma_1 = \lambda_1 - \lambda_4 > \gamma_2 = \lambda_2 - \lambda_4 > \gamma_3 = \lambda_3 - \lambda_4 > 0$ (see Proposition (2.1)) and $g = (\delta\gamma)^{-1} g_4$ (see (3.2)). Then, we can deduce that the operator T is given by

$$\begin{aligned}
Tz(t) &= \frac{1}{\delta\gamma} \int_{t_0}^{\infty} g_4(t, s) \left[p(\lambda_2, s) + F(s, z(s), z^{(1)}(s), z^{(2)}(s)) \right] ds \\
&= \frac{1}{\delta\gamma} \left\{ \int_{t_0}^t \left[(\gamma_3 - \gamma_2) e^{-\gamma_1(t-s)} + (\gamma_1 - \gamma_3) e^{-\gamma_2(t-s)} + (\gamma_2 - \gamma_1) e^{-\gamma_3(t-s)} \right] \right.
\end{aligned}$$

$$\times \left[p(\lambda_2, s) + F\left(s, z(s), z^{(1)}(s), z^{(2)}(s)\right) \right] ds \Big\}, \quad \text{for } t \geq t_0. \quad (3.19)$$

Thus, to proof (2.8) with $\ell = 4$ is enough prove the following facts

$$\exists \Phi_n \in \mathbb{R}_+ : \sum_{j=0}^2 |\omega_n^{(j)}(t)| \leq \Phi_n \int_{t_0}^t e^{-\beta(t-\tau)} |p(\lambda_4, \tau)| d\tau, \quad \forall t \geq t_0, \quad (3.20)$$

$$\exists \Phi \in \mathbb{R}_+ : \Phi_n \rightarrow \Phi \text{ when } n \rightarrow \infty. \quad (3.21)$$

Now, by (3.19), (3.9) and the property $F(s, 0, 0, 0) = 0$, we note the induction step for $n = 1$ the estimate (3.20) is satisfied with $\Phi_1 = A_4$ and $\beta \in]0, \lambda_3 - \lambda_4]$. Indeed, we can deduce the following estimate

$$\begin{aligned} \sum_{j=0}^2 |\omega_1^{(j)}(t)| &= \sum_{j=0}^2 |T^{(j)} \omega_0(t)| \\ &\leq \frac{1}{|\delta\gamma|} \sum_{j=0}^2 |\gamma_3 - \gamma_2| |\gamma_1|^j + |\gamma_1 - \gamma_3| |\gamma_2|^j + |\gamma_2 - \gamma_1| |\gamma_3|^j \int_{t_0}^t e^{-(\lambda_3 - \lambda_4)(t-s)} |p(\lambda_4, s)| ds \\ &\leq A_4 \int_{t_0}^t e^{-\beta(t-\tau)} |p(\lambda_4, \tau)| d\tau. \end{aligned}$$

Then, we can prove that the general induction step holds with $\Phi_n = A_4(1 + \Phi_{n-1} \rho_4 \sigma_4)$, since proceeding as in the case $\ell = 1$, we can deduce the following estimate

$$\sum_{j=0}^2 |\omega_{k+1}^{(j)}(t)| = \sum_{j=0}^2 |T^{(j)} \omega_k(t)| \leq A_4 \left(1 + \Phi_k \rho_4 \sigma_4\right) \int_{t_0}^t e^{-\beta(t-\tau)} |p(\lambda_4, \tau)| d\tau.$$

Hence (3.21) is satisfied with $\Phi = A_4/(1 - \rho_4 \sigma_4 A_4) > 0$. Thus, the sequence $\{\Phi_n\}$ is convergent and z_4 (the limit of ω_n in the topology of $C_0^2([t_0, \infty])$) satisfies (2.8).

3.4. Proof of Theorem 2.4. By Lemma 2.1, we have that the fundamental system of solutions for (1.1) is given by (2.6). Moreover, by (2.6) we deduce the identities

$$\frac{y_i^{(1)}(t)}{y_i(t)} = [\lambda_i + z_i(t)] \quad (3.22)$$

$$\frac{y_i^{(2)}(t)}{y_i(t)} = [\lambda_i + z_i(t)]^2 + z_i^{(1)}(t), \quad (3.23)$$

$$\frac{y_i^{(3)}(t)}{y_i(t)} = [\lambda_i + z_i(t)]^3 + 3[\lambda_i + z_i(t)]z_i^{(1)}(t) + z_i^{(2)}(t), \quad (3.24)$$

$$\begin{aligned} \frac{y_i^{(4)}(t)}{y_i(t)} &= [\lambda_i + z_i(t)]^4 + 6[\lambda_i + z_i(t)]^2 z_i^{(1)}(t) + 3[z_i^{(1)}(t)]^2 \\ &\quad + 4[\lambda_i + z_i(t)]z_i^{(2)}(t) + z_i^{(3)}(t), \end{aligned} \quad (3.25)$$

Now, using the facts that $z_i \in C_0^2([t_0, \infty])$ and $\{\mu_i, z_i\}$ is a solution of (2.5), we deduce the proof of (2.9). Now, by the definition of the $W[y_1, \dots, y_n]$, some algebraic rearrangements and (2.9), we deduce (2.10).

The proof of (2.11) follows by the identity

$$\begin{aligned} \int_{t_0}^t e^{-a\tau} \int_{\tau}^{\infty} e^{-as} H(s) ds d\tau &= -\frac{1}{a} \left[\int_t^{\infty} e^{-a(t-s)} H(s) ds - \int_{t_0}^{\infty} e^{-a(t_0-s)} H(s) ds \right] \\ &\quad + \frac{1}{a} \int_{t_0}^t H(\tau) d\tau \end{aligned} \quad (3.26)$$

and by (3.4)-(3.5). Now we develop the proof for $i = 1$. Indeed, by (2.6) we have that

$$y_1(t) = \exp\left(\int_{t_0}^t (\lambda_1 + z_1(\tau))d\tau\right) = e^{\lambda_1(t-t_0)} \exp\left(\int_{t_0}^t z_1(\tau)d\tau\right). \quad (3.27)$$

By (3.4)-(3.5) and (3.26), we have that

$$\begin{aligned} \int_{t_0}^t z_1(\tau)d\tau &= \frac{1}{\delta\gamma} \int_{t_0}^t \int_{t_0}^{\infty} g_1(\tau, s) \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) d\tau \\ &= \frac{1}{\delta\gamma} \int_{t_0}^t \int_{\tau}^{\infty} \left[(\gamma_3 - \gamma_2)e^{-\gamma_1(\tau-s)} + (\gamma_1 - \gamma_3)e^{-\gamma_2(\tau-s)} + (\gamma_2 - \gamma_1)e^{-\gamma_3(\tau-s)} \right] \\ &\quad \times \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) d\tau \\ &= \frac{1}{\delta\gamma} \left[\frac{\gamma_3 - \gamma_1}{\gamma_1} + \frac{\gamma_1 - \gamma_3}{\gamma_2} + \frac{\gamma_2 - \gamma_1}{\gamma_3} \right] \int_{t_0}^t \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) ds \\ &\quad + \frac{1}{\delta\gamma} \left[\frac{\gamma_3 - \gamma_1}{\gamma_1} \left\{ \int_t^{\infty} e^{-\gamma_1(t-s)} \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) ds \right. \right. \\ &\quad \left. \left. - \int_{t_0}^{\infty} e^{-\gamma_1(t_0-s)} \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) ds \right\} \right] \\ &\quad + \frac{1}{\delta\gamma} \left[\frac{\gamma_1 - \gamma_3}{\gamma_2} \left\{ \int_t^{\infty} e^{-\gamma_2(t-s)} \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) ds \right. \right. \\ &\quad \left. \left. - \int_{t_0}^{\infty} e^{-\gamma_2(t_0-s)} \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) ds \right\} \right] \\ &\quad + \frac{1}{\delta\gamma} \left[\frac{\gamma_2 - \gamma_1}{\gamma_3} \left\{ \int_t^{\infty} e^{-\gamma_3(t-s)} \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) ds \right. \right. \\ &\quad \left. \left. - \int_{t_0}^{\infty} e^{-\gamma_3(t_0-s)} \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) ds \right\} \right] \\ &= \frac{1}{\gamma_1\gamma_2\gamma_3} \int_{t_0}^t \left(p(\lambda_1, s) + F(s, z_1(s), z_1^{(1)}(s), z_1^{(2)}(s)) \right) ds + o(1) \end{aligned}$$

Then, (2.11) is valid for $i = 1$, since $\gamma_1\gamma_2\gamma_3 = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1) = \pi_1$. The proof of (2.11) for $i = 2, 3, 4$ is analogous. Now the proof of (2.12) follows by (2.11) and (3.22)-(3.25).

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