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Study of a new class of nonlinear inextensible elastic bodies

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Abstract. In this paper, we study the consequences of the constraint of inextensibility with regard to a class of constitutive relations, where the strain is given as a function of the stress. Such constitutive equations belong to a wider class of implicit constitutive relations, which have been proposed recently in the literature.

Mathematics Subject Classification. 74A20.

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1. Introduction

Constraints such as rigidity, incompressibility and inextensibility refer to restrictions concerning the types of deformations a body can be subject to, see, for example, §30 of [47]. These kinds of restrictions have been considered within the context of classical linearized theory of elasticity by, for example, Poincare [27] and Love [22], and in the case of large elastic deformations by Green and Shield (see §3 of [18]), Rivlin [39], Rivlin and Ericksen and Rivlin (see §3 of [14]), and Adkins and Rivlin [1].

Constraints such as incompressibility introduce an indeterminacy in the specification of the stress and in fact make it possible to obtain solutions that would be otherwise impossible to obtain within the unconstrained class of bodies. For example, a static homogeneous deformation is not possible in a homogeneous compressible Cauchy elastic solid, if gravity is taken into consideration. However, such static solutions are possible within the context of incompressible homogeneous elastic bodies even if gravity is to be taken into account. This is due to the arbitrariness in the stress due to the constraint that one can manipulate. Similarly, while no inhomogeneous deformations are 'universal' within the class of isotropic homogeneous nonlinear Cauchy elastic bodies, as many as six classes of inhomogeneous deformations are possible in incompressible, isotropic nonlinear Cauchy elastic bodies (see Ericksen [13]). Rigidity is a constraint that allows for any distribution of stress in the body.

In the classical theory of nonlinear continuum mechanics, when considering the class of simple materials (see Noll [24]), the effects of the constraints have been incorporated by decomposing the stress in two parts, where one depends on the particular properties of the material, while the second part, called 'reaction stress', is assumed not to do work with any deformation compatible with the constraints, see §30 of [47], in particular pp. 70–71 therein. This approach (in particular the workless nature of the reaction stress) has been questioned by many investigators, see Antman and Marlow [2], Casey [9], and recently in particular by Rajagopal and Srinivasa[1] [30,31].

Recently, in a series of papers, Rajagopal and coworkers have studied a more general class of elastic bodies that cannot be classified as either Cauchy or Green elastic bodies [4,29,32–37]. The models belonging to the generalization can be used in describing the response of soft tissue [15], describing

[1] It would be fair to say that the assumption of the reaction stress being 'workless' is not well taken within the context of dissipative materials (see Rajagopal and Srinivasa [30,31]).

more accurately the phenomenon of fracture of solids [37], and leading to linearizations that provide a nonlinear relationship between the linearized strain and the stress as exhibited by many metallic alloys [42,45,48,49], the response of infrastructure materials such as concrete [16], and the implicit counterpart of models for fluids to describe noninvertible response exhibited by many colloids and suspensions [28]. This study concerns the development of constraints for the recent generalization of elasticity due to Rajagopal. In [8], such a study was carried out for constitutive relations that arise from the linearization of a subclass of the new constitutive relations, namely constitutive expressions for the linearized strain as a function (in general, nonlinear) of the stresses [4,5], within the context of incompressibility. In the present work, we continue the analysis of constraints for the same class of constitutive relations within the context of inextensibility.

The constraint of inextensibility has been used, for example, as a mathematical approximation for studying the behaviour of composites, where we have a matrix filled with one or more families of 'fibres' or 'cords', which have a much higher stiffness than the matrix, see, for example, §1 of [1], §7.1 of [17] (pp. 229 therein) and [44]. The constituents of the composites, namely the matrix and the fibres, are assumed to undergo no relative displacement (see, for example, §7.1 of [17]), and the body is assumed to be incapable of stretching in the direction of the fibres once the fibres are fully stretched. One could and does have composites wherein the fibres are allowed to stretch until the fibres break. In this study, we are considering the special class of composites that are incapable of stretching in the fibre direction.

In [1], we find one of the early studies considering such a constraint, where the concept of the tension in the direction of the fibre was introduced, and where the stress is decomposed into two parts, where one part is an indeterminate tension, and another part depends on the particular properties of the material (see Eqs. (2.5)–(2.8) and §7 of [1]). Such concepts have been generalized in subsequent works such as the book [17], where solutions for different boundary value problems have been presented considering large elastic deformations (see Chapter VII and §7.2–§7.17 therein); the article by Spencer [44], where in Sect. 2.1, the case of a linearized inextensible body (one family of fibres) is considered, while in Sect. 3.2, the case of an inextensible body undergoing large elastic deformations is studied; the papers [19,20], where an inextensible linearized elastic body is studied, within the context of isotropic bodies (Eq. 4.1 therein) when the density of fibres is low in comparison with the density of the composite as a whole, and within the purview of a transversely isotropic body (in the same direction as the inextensibility), when the density of fibres is large in comparison with the density of the composite; the paper by Pipkin [26], presents a detailed study of constraints (in a general sense) for linearized elastic bodies, where he introduces the notions of 'uniform' and 'nonuniform' constraints[2]; and finally we mention the papers [3,41] for some exact solutions and for some universal relations for the case of inextensible bodies within the context of large deformations, and [11,12,23,40] with regard to exact solutions in the case of inextensible elastic bodies within the context of linearized constitutive equations.

As mentioned earlier, in the present work, we study how the constraint of inextensibility can be incorporated, in the case of a nonlinear elastic transversely isotropic body, within the context of small displacement gradients for the new constitutive relation presented, for example, in [6,7,37]. The paper is arranged in the following manner: in Sect. 2, the basic equations of nonlinear elasticity are documented and the new classes of constitutive relations are presented. In Sect. 3, the inextensibility of the elastic body is considered, and the general constitutive representation for such a body is developed. In Sect. 3.1, the constraints of inextensibility and incompressibility are considered simultaneously. In Sect. 4, several simple boundary value problems are studied for the constrained bodies within the framework of this new class of constitutive equations. Finally, in Sect. 5, the case of inextensibility for a body within large elastic deformations is studied.

[2]A uniform constraint according to Pipkin is a constraint that does not depend explicitly on the position, while a nonuniform constraint does depend on the position.

2. Basic equations

2.1. Kinematics and the equation of equilibrium

Let $X \in \mathscr{B}$ denote a typical particle belonging to the abstract body $\mathscr{B}$. Let us denote by $\boldsymbol{\kappa}$ a one to one mapping, referred to as the placer, which maps the abstract body into a three-dimensional Euclidean space. By the motion of the body, we mean a one-parameter family of placers (the parameter being identified with time). Let $\boldsymbol{\kappa}_r$ denote a reference placer and $\boldsymbol{\kappa}_t$ the placer at time t, and let $\kappa_r(\mathscr{B})$ and $\kappa_t(\mathscr{B})$ denote the reference configuration and the configuration at time t, respectively. We can identify the motion with a one to one mapping $\boldsymbol{\chi}$ that assigns to each point $\mathbf{X} \in \kappa_r(\mathscr{B})$ a point $\mathbf{x} \in \kappa_t(\mathscr{B})$, that is

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t). \tag{1}$$

Let $\mathbf{u}$ denote the displacement of the body, that is

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \tag{2}$$

and $\mathbf{F}$ denote the gradient of the motion

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \tag{3}$$

where it is assumed that $J = \det \mathbf{F} > 0$. Then

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{F} - \mathbf{I}. \tag{4}$$

We define the Cauchy–Green tensors $\mathbf{B}$ and $\mathbf{C}$, and the Green-St. Venant strain $\mathbf{E}$ through

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}). \tag{5}$$

The linearized strain $\boldsymbol{\varepsilon}$ is defined through

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}\right). \tag{6}$$

In the present communication, we consider only quasi-static deformations; therefore, the Cauchy stress tensor must satisfy the equilibrium equation

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \mathbf{0}, \tag{7}$$

where ρ is the density of the body in the current configuration and $\mathbf{b}$ the body force.

The above definitions and equations are sufficient for our work, and more details about kinematics and the equation of motion can be found, for example, in [10,46].

2.2. Some new classes of constitutive relations

Based on the initial work of Rajagopal [29,32–34], several constitutive relations have been proposed by Rajagopal and his coworkers for elastic bodies, which cannot be interpreted as describing either Cauchy or Green elastic bodies [29,32–35]. One such class of constitutive relation is the implicit constitutive equation for isotropic elastic bodies of the form (see [35]) $\boldsymbol{\mathfrak{H}}(\mathbf{T}, \mathbf{B}) = \mathbf{0}$, where in general it is not possible to express either the Cauchy stress tensor $\mathbf{T}$ in terms of $\mathbf{B}$ or viceversa. As special cases of the above implicit relation, we have $\mathbf{T} = \boldsymbol{\mathfrak{g}}(\mathbf{B})$ and $\mathbf{B} = \boldsymbol{\mathfrak{K}}(\mathbf{T})$, where the first equation is the classical constitutive equation for Cauchy elastic bodies, while the second equation is a new class of constitutive relation.

When one assumes that $|\nabla \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, we have $\mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\varepsilon}$, and from $\mathbf{B} = \boldsymbol{\mathfrak{K}}(\mathbf{T})$, it is possible to show that the correct relation between the stresses and the linearized strain tensor $\boldsymbol{\varepsilon}$ should be of the form [25] $\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T})$. In the present work, we consider the class of linearized constitutive relations given by $\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T})$, and we furthermore assume that there exists a scalar function $\Pi = \Pi(\mathbf{T})$ such that

$$\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T}) = \frac{\partial \Pi}{\partial \mathbf{T}}. \tag{8}$$

2.3. Boundary value problems

As explained in detail in [8], when working with the constitutive equation (8), the procedure we use in order to solve boundary value problems is to study (8), (6) and (7) simultaneously:

$$\frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}) = \mathfrak{f}(\mathbf{T}), \tag{9}$$

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \mathbf{0}, \tag{10}$$

$$\mathbf{T}\mathbf{n} = \breve{\mathbf{t}} \quad \text{on} \quad \partial\kappa_t(\mathscr{B})_t, \quad \mathbf{u} = \breve{\mathbf{u}} \quad \text{on} \quad \partial\kappa_t(\mathscr{B})_u \tag{11}$$

where the boundary of the body in the current configuration is $\partial\kappa_t(\mathscr{B})$ and $\partial\kappa_t(\mathscr{B}) = \partial\kappa_t(\mathscr{B})_u \cup \partial\kappa_t(\mathscr{B})_t$, $\partial\kappa_t(\mathscr{B})_u \cap \partial\kappa_t(\mathscr{B})_t = \emptyset$, and $\breve{\mathbf{t}}$ and $\breve{\mathbf{u}}$ are the external traction and a known displacement field, respectively.

2.4. Constraints for the deformation in the classical theory of elasticity

Kinematical constraints of the deformation are defined as restrictions, which the field $\boldsymbol{\chi}$ must satisfy in all deformations the body is subjected to, for a given family of bodies, and are usually expressed mathematically as[3] (see, for example, §30 of [47]):

$$\lambda(\mathbf{C}) = 0. \tag{12}$$

The method of enforcing the constraint is usually based on the assumption that the stress tensor can be divided into two parts [47]

$$\mathbf{T} = \mathfrak{F}(\mathbf{F}) + \mathbf{T}_{\mathrm{N}}, \tag{13}$$

where the part $\mathfrak{F}(\mathbf{F})$ (usually referred as the extra stress) depends on the particular material being considered (the class of materials for which the constraint holds), and $\mathbf{T}_{\mathrm{N}}$ the part of the stress that enforces the constraint (usually referred to as the reaction stress) does not do work with any deformation compatible with the constraint (12) (see [31,47]). Such an assumption leads to the following expression for the stress tensor $\mathbf{T}_{\mathrm{N}}$

$$\mathbf{T}_{\mathrm{N}} = q\mathbf{F}\frac{\partial \lambda}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}}, \tag{14}$$

where q is a Lagrangian multiplier.

In the case of bodies that are inextensible in the direction $\mathbf{a}_0$ in the reference configuration, the expression for the function λ is (see [47]):

$$\lambda(\mathbf{C}) = \mathbf{a}_0 \cdot (\mathbf{C}\mathbf{a}_0) - 1. \tag{15}$$

As mentioned earlier, the constraint of inextensibility (15) has been used as a mathematical idealization of a certain class of composite materials, where we have a matrix reinforced by inextensible fibres (aligned in a preferred direction $\mathbf{a}_0$), inextensible in the sense that the fibres are much stiffer than the matrix [44]. In such a situation, as an approximation, it is assumed that the composite is inextensible in the direction

[3] It is possible, even for simple materials to have a constraint that is given in terms of the history of the deformation, as far as mathematics is concerned. However, none of the physically meaningful constraints such as rigidity, incompressibility or inextensibility require such an artifice.

$\mathbf{a}_0$. However, such a constraints would be valid only when there is tension in that particular direction,[4] in compression the ‘fibres’ may not present a significant resistance to deformation, and the matrix would be the only component of the composite resisting such compressive loads. In the case of modelling the behaviour of such composites, the following constraint would be more appropriate

$$\mathbf{a}_0 \cdot (\mathbf{C}\mathbf{a}_0) \leq 1. \tag{16}$$

If $|\nabla \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ the constraint (15) $\lambda(\mathbf{C}) = 0$ can be written as (see, for example, Eq. (17) of [44] and Eq. (4.2) of [26])

$$\mathbf{a} \cdot (\boldsymbol{\varepsilon}\mathbf{a}) = 0, \tag{17}$$

where $\mathbf{a}$ is the direction along which the body is inextensible in the current configuration, and where in general since $|\nabla \mathbf{u}| \sim O(\delta)$, as $\delta \ll 1$ it is not necessary to distinguish between $\mathbf{a}_0$ and $\mathbf{a}$. In the case of (16), the counterpart when $|\nabla \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ would be

$$\mathbf{a} \cdot (\boldsymbol{\varepsilon}\mathbf{a}) \leq 0. \tag{18}$$

3. An elastic body that is inextensible in a preferred direction

Let us assume that the elastic body of interest is subject to small displacement gradient, that in the case of Cauchy elasticity leads to the classical linearized elastic model. On considering small displacement gradients in the sense that $|\nabla \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, within the context of a transversely isotropic elastic solid with a preferred direction $\mathbf{a}$. Then, the scalar function Π defined in (8) depends on the following list of invariants[5] [43]:

$$I_1 = \mathrm{tr}\,\mathbf{T}, \quad I_2 = \frac{1}{2}\mathrm{tr}\,(\mathbf{T}^2), \quad I_3 = \frac{1}{3}\mathrm{tr}\,(\mathbf{T}^3), \quad I_4 = \mathbf{a} \cdot (\mathbf{T}\mathbf{a}), \quad I_5 = \mathbf{a} \cdot (\mathbf{T}^2\mathbf{a}). \tag{19}$$

Using $\Pi = \Pi(I_1, I_2, I_3, I_4, I_5)$ in (8) and the chain rule for the derivative, we obtain (see Eq. (4.8) of [4])

$$\boldsymbol{\varepsilon} = \Pi_1\mathbf{I} + \Pi_2\mathbf{T} + \Pi_3\mathbf{T}^2 + \Pi_4\mathbf{a} \otimes \mathbf{a} + \Pi_5[\mathbf{a} \otimes (\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a}) \otimes \mathbf{a}], \tag{20}$$

where $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, 3, 4, 5$.

Now, as the body cannot extend in the direction $\mathbf{a}$, replace (20) in (17), then we obtain the first-order partial differential equation

$$\Pi_1 + \Pi_2 I_4 + \Pi_3 I_5 + \Pi_4 + 2\Pi_5 I_4 = 0, \tag{21}$$

whose solution is found to be (see Chapter 1 of [21] for a general methodology to solve such first-order partial differential equations):

$$\Pi = \bar{\Pi}(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4), \tag{22}$$

where we have defined[6]

$$\bar{I}_1 = I_4 - I_1, \quad \bar{I}_2 = \frac{1}{2}I_1^2 + I_2 - I_1 I_4, \quad \bar{I}_3 = I_1^2 - 2I_1 I_4 + I_5, \tag{23}$$

$$\bar{I}_4 = -\frac{1}{3}I_1^3 + I_3 + I_1^2 I_4 - I_1 I_5. \tag{24}$$

[4] Compare this with the classical approach, where it is assumed that there is no stretching or contraction in the preferred direction $\mathbf{a}_0$, see, for example, pp. 34 of [40] and §7.1 of [17] (see (i) in pp. 229 therein).

[5] See also [4], where the notation $\mathbf{e}$ was used for the field $\mathbf{a}$, and [38].

[6] Please notice the error in Eq. (5.19) of [4].

Replacing (22) in (20), and taking into considering (23) and (24), after some manipulations, we obtain the general expression for the constitutive equation for a transversely isotropic body, which is inextensible in the direction of anisotropy:

$$\begin{aligned}\boldsymbol{\varepsilon} = {} & \frac{\partial \bar{\Pi}}{\partial \bar{I}_1}(\mathbf{a}\otimes\mathbf{a}-\mathbf{I}) + \frac{\partial \bar{\Pi}}{\partial \bar{I}_2}(-\bar{I}_1\mathbf{I}+\mathbf{T}-I_1\mathbf{a}\otimes\mathbf{a}) + \frac{\partial \bar{\Pi}}{\partial \bar{I}_3}[-2\bar{I}_1\mathbf{I}-2I_1\mathbf{a}\otimes\mathbf{a}+\mathbf{a}\otimes(\mathbf{Ta}) \\ & +(\mathbf{Ta})\otimes\mathbf{a}] + \frac{\partial \bar{\Pi}}{\partial \bar{I}_4}\{-\bar{I}_3\mathbf{I}+\mathbf{T}^2+I_1^2\mathbf{a}\otimes\mathbf{a}-I_1[\mathbf{a}\otimes(\mathbf{Ta})+(\mathbf{Ta})\otimes\mathbf{a}]\}. \end{aligned} \tag{25}$$

Regarding (18) by following the same procedure presented above, on assuming that $\Pi = \Pi(I_1, I_2, I_3, I_4, I_5)$, we obtain

$$\Pi = \begin{cases} \Pi(I_1, I_2, I_3, I_4, I_5) & \text{if} \quad \mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a}) < 0, \\ \bar{\Pi}(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4) & \text{if} \quad \mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a}) = 0. \end{cases} \tag{26}$$

3.1. An elastic body that is inextensible and incompressible

Let us study briefly a special case, where apart from considering the body to be inextensible in the direction $\mathbf{a}$, we also assume that is incompressible. Then, using (20) and on enforcing the constraint of incompressibility

$$\operatorname{tr}(\boldsymbol{\varepsilon}) = 0, \tag{27}$$

we obtain the first-order partial differential equation (see [4,8])

$$3\Pi_1 + \Pi_2 I_1 + 2\Pi_3 I_2 + \Pi_4 + 2\Pi_5 I_4 = 0. \tag{28}$$

Using the expression for Π from (22) that already satisfies the inextensibility constraint and using the chain rule for the partial derivatives, Eq. (28) becomes

$$-2\frac{\partial \bar{\Pi}}{\partial \bar{I}_1} - 3\frac{\partial \bar{\Pi}}{\partial \bar{I}_2}\bar{I}_1 - 4\frac{\partial \bar{\Pi}}{\partial \bar{I}_3}\bar{I}_1 + \frac{\partial \bar{\Pi}}{\partial \bar{I}_4}(2\bar{I}_2 - 3\bar{I}_3) = 0, \tag{29}$$

which is the equation that the inextensible body must satisfy in order to be incompressible as well. The solution of (29) is of the form (see [21])

$$\bar{\Pi} = \tilde{\Pi}(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3), \tag{30}$$

where we have defined (see (23) and (24))

$$\tilde{I}_1 = \bar{I}_2 - \frac{3}{4}\bar{I}_1^2, \quad \tilde{I}_2 = \bar{I}_3 - \bar{I}_1^2, \quad \tilde{I}_3 = \frac{1}{2}(\bar{I}_1^3 + 2\bar{I}_2\bar{I}_2 - 3\bar{I}_1\bar{I}_3 + 2\bar{I}_4), \tag{31}$$

and from (30) and (31), the expression for $\boldsymbol{\varepsilon}$ would be of the form:

$$\begin{aligned}\boldsymbol{\varepsilon} = {} & \frac{\partial \tilde{\Pi}}{\partial \tilde{I}_1}\left[\frac{1}{2}(I_1 - 3I_4)\mathbf{a}\otimes\mathbf{a} + \frac{1}{2}(I_4 - I_1)\mathbf{I} + \mathbf{T}\right] + \frac{\partial \tilde{\Pi}}{\partial \tilde{I}_2}[-2I_4\mathbf{a}\otimes\mathbf{a} + \mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] \\ & + \frac{\partial \tilde{\Pi}}{\partial \tilde{I}_3}\left\{\frac{1}{2}(8I_1I_4 - 5I_1^2 - 2I_2 + 3I_4^2 - 3I_5)\mathbf{a}\otimes\mathbf{a} + \frac{1}{2}(5I_1^2 + 2I_2 - 10I_1I_4 + 3I_4^2 + I_5)\mathbf{I}\right. \\ & \left. + \frac{1}{2}(I_1 - 3I_4)[\mathbf{a}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{a}] + \mathbf{T}^2\right\}. \end{aligned} \tag{32}$$

In this study, we restrict our attention to the model (25) for inextensible bodies and do not consider the additional constraint that it be incompressible as well.

3.2. On the consequences of a stress tensor of the form $\mathbf{T} = q\mathbf{a} \otimes \mathbf{a}$

In the classical theory of elasticity, for inextensible bodies, from (14), we deduce that[7] $\mathbf{T}_{\mathrm{N}} = q\mathbf{a} \otimes \mathbf{a}$, and in this section, we study the effect of adding such a reaction stress tensor in (25). Consider the following decomposition of the stress tensor

$$\mathbf{T} = \mathbf{T}_{\mathrm{o}} + q\mathbf{a} \otimes \mathbf{a}, \tag{33}$$

where $q = q(\mathbf{x})$ is a scalar field and

$$\mathbf{a} \cdot (\mathbf{T}_{\mathrm{o}}\mathbf{a}) = 0. \tag{34}$$

Let us define

$$I_1^{\mathrm{o}} = \operatorname{tr}\mathbf{T}_{\mathrm{o}}, \quad I_2^{\mathrm{o}} = \frac{1}{2}\operatorname{tr}(\mathbf{T}_{\mathrm{o}}^2), \quad I_3^{\mathrm{o}} = \frac{1}{3}\operatorname{tr}(\mathbf{T}_{\mathrm{o}}^3), \quad I_4^{\mathrm{o}} = \mathbf{a} \cdot (\mathbf{T}_{\mathrm{o}}\mathbf{a}), \quad I_5^{\mathrm{o}} = \mathbf{a} \cdot (\mathbf{T}_{\mathrm{o}}^2\mathbf{a}), \tag{35}$$

where from (34), we have $I_4^{\mathrm{o}} = 0$. From (19) and (35), we can show that

$$I_1 = \operatorname{tr}\mathbf{T} = I_1^{\mathrm{o}} + q, \quad I_2 = \frac{1}{2}\operatorname{tr}(\mathbf{T}^2) = I_2^{\mathrm{o}} + qI_4^{\mathrm{o}} + \frac{q^2}{2} = I_2^{\mathrm{o}} + \frac{q^2}{2}, \tag{36}$$

$$I_3 = \frac{1}{3}\operatorname{tr}(\mathbf{T}^3) = I_3^{\mathrm{o}} + qI_5^{\mathrm{o}} + q^2 I_4^{\mathrm{o}} + \frac{q^3}{4} = I_3^{\mathrm{o}} + qI_5^{\mathrm{o}} + \frac{q^3}{4}, \tag{37}$$

$$I_4 = \mathbf{a} \cdot (\mathbf{T}\mathbf{a}) = I_4^{\mathrm{o}} + q = q, \quad I_5 = \mathbf{a} \cdot (\mathbf{T}^2\mathbf{a}) = I_5^{\mathrm{o}} + 2qI_4^{\mathrm{o}} + q^2 = I_5^{\mathrm{o}} + q^2. \tag{38}$$

Using these expressions in (23) and (24), it is easy to deduce that $\bar{I}_k = \bar{I}_k^{\mathrm{o}}$, $k = 1, 2, 3, 4$, where $\bar{I}_k^{\mathrm{o}}$ are the invariants (23) and (24) defined with respect to $\mathbf{T}_{\mathrm{o}}$. On the other hand, it follows from (33), (36)–(38) and (25) that

$$(I_1 - I_4)\mathbf{I} + \mathbf{T} - I_1\mathbf{a} \otimes \mathbf{a} = (I_1^{\mathrm{o}} - I_4^{\mathrm{o}})\mathbf{I} + \mathbf{T}_{\mathrm{o}} - I_1^{\mathrm{o}}\mathbf{a} \otimes \mathbf{a}, \tag{39}$$

$$\begin{aligned} &2(I_1 - I_4)\mathbf{I} - 2I_1\mathbf{a} \otimes \mathbf{a} + \mathbf{a} \otimes (\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a}) \otimes \mathbf{a} = 2(I_1^{\mathrm{o}} - I_4^{\mathrm{o}})\mathbf{I} - 2I_1^{\mathrm{o}}\mathbf{a} \otimes \mathbf{a} + \mathbf{a} \otimes (\mathbf{T}_{\mathrm{o}}\mathbf{a}) \\ &\quad + (\mathbf{T}_{\mathrm{o}}\mathbf{a}) \otimes \mathbf{a}, \end{aligned} \tag{40}$$

$$\begin{aligned} &(2I_1I_4 - I_1^2 - I_5)\mathbf{I} + \mathbf{T}^2 + I_1^2\mathbf{e} \otimes \mathbf{e} - I_1[\mathbf{a} \otimes (\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a}) \otimes \mathbf{a}] = (2I_1^{\mathrm{o}}I_4^{\mathrm{o}} - I_1^{\mathrm{o}2} - I_5^{\mathrm{o}})\mathbf{I} \\ &\quad + \mathbf{T}_{\mathrm{o}}^2 + I_1^{\mathrm{o}2}\mathbf{e} \otimes \mathbf{e} - I_1^{\mathrm{o}}[\mathbf{a} \otimes (\mathbf{T}_{\mathrm{o}}\mathbf{a}) + (\mathbf{T}_{\mathrm{o}}\mathbf{a}) \otimes \mathbf{a}]. \end{aligned} \tag{41}$$

Using all these results in (25), one can show the expression for the $\boldsymbol{\varepsilon}$ in (25) remains identical to that which is obtained by replacing $\mathbf{T}_{\mathrm{o}}$ by $\mathbf{T}_{\mathrm{o}} + q\mathbf{a} \otimes \mathbf{a}$. Therefore, we have

$$\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T}) = \mathfrak{f}(\mathbf{T}_{\mathrm{o}}). \tag{42}$$

3.3. Dimensionless expressions

With an aim towards a proper comparison of the results of our analysis and the results predicted by the linearized theory of elasticity, we proceed to nondimensionalize (20) and (25).

Let us define the dimensionless stress tensor $\hat{\mathbf{T}} = \frac{1}{\sigma_o}\mathbf{T}$, where σ_o is a characteristic value for the stress. We then obtain, from (19), dimensionless invariants (see [8]):

$$\hat{I}_1 = \operatorname{tr}\hat{\mathbf{T}}, \quad \hat{I}_2 = \frac{1}{2}\operatorname{tr}(\hat{\mathbf{T}}^2), \quad \hat{I}_3 = \frac{1}{3}\operatorname{tr}(\hat{\mathbf{T}}^3), \quad \hat{I}_4 = \mathbf{a} \cdot (\hat{\mathbf{T}}\mathbf{a}) = \frac{I_4}{\sigma_o}, \quad \hat{I}_5 = \mathbf{a} \cdot (\hat{\mathbf{T}}^2\mathbf{a}) = \frac{I_5}{\sigma_o^2}. \tag{43}$$

Let us define the dimensionless function [8] $\hat{\Pi}$ as $\hat{\Pi} = \frac{\Pi}{\sigma_o}$. Using the chain rule for the derivatives, we have $\frac{\partial \Pi}{\partial I_1} = \frac{\partial \Pi}{\partial \hat{I}_1}\frac{1}{\sigma_o}$, $\frac{\partial \Pi}{\partial I_2} = \frac{\partial \Pi}{\partial \hat{I}_2}\frac{1}{\sigma_o^2}$, $\frac{\partial \Pi}{\partial I_3} = \frac{\partial \Pi}{\partial \hat{I}_3}\frac{1}{\sigma_o^3}$, $\frac{\partial \Pi}{\partial I_4} = \frac{1}{\sigma_o}\frac{\partial \Pi}{\partial \hat{I}_4}$ and $\frac{\partial \Pi}{\partial I_5} = \frac{1}{\sigma_o^2}\frac{\partial \Pi}{\partial \hat{I}_5}$. Using these expressions in

[7] See, for example, Eq. 24 of [44], Eq. 8 of [40] and Eq. 3.7 of [26].

(8), it is easy to show that

$$\boldsymbol{\varepsilon}=\hat{\Pi}_1\mathbf{I}+\hat{\Pi}_2\hat{\mathbf{T}}+\hat{\Pi}_3\hat{\mathbf{T}}^2+\hat{\Pi}_4\mathbf{a}\otimes\mathbf{a}+\hat{\Pi}_5[\mathbf{a}\otimes(\hat{\mathbf{T}}\mathbf{a})+(\hat{\mathbf{T}}\mathbf{a})\otimes\mathbf{a}], \tag{44}$$

where $\hat{\Pi}_i=\frac{\partial\hat{\Pi}}{\partial\hat{I}_i}$, $i=1,2,3,4,5$. Replacing in (17), we obtain the solution $\hat{\Pi}=\check{\Pi}(\check{I}_1,\check{I}_2,\check{I}_3,\check{I}_4)$, where

$$\check{I}_1=\hat{I}_4-\hat{I}_1,\quad \check{I}_2=\frac{1}{2}(\hat{I}_1^2+2\hat{I}_2-2\hat{I}_1\hat{I}_4),\quad \check{I}_3=\hat{I}_1^2-2\hat{I}_1\hat{I}_4+\hat{I}_5,$$
$$\check{I}_4=\frac{1}{3}(3\hat{I}_3-\hat{I}_1^3+3\hat{I}_1^2\hat{I}_4-3\hat{I}_1\hat{I}_5), \tag{45}$$

and so from (44), we finally obtain

$$\begin{aligned}\boldsymbol{\varepsilon}=&\frac{\partial\check{\Pi}}{\partial\check{I}_1}(\mathbf{a}\otimes\mathbf{a}-\mathbf{I})+\frac{\partial\check{\Pi}}{\partial\check{I}_2}[(\hat{I}_1-\hat{I}_4)\mathbf{I}+\hat{\mathbf{T}}-\hat{I}_1\mathbf{a}\otimes\mathbf{a}]+\frac{\partial\check{\Pi}}{\partial\check{I}_3}[2(\hat{I}_1-\hat{I}_4)\mathbf{I}-2\hat{I}_1\mathbf{a}\otimes\mathbf{a}\\&+\mathbf{a}\otimes(\hat{\mathbf{T}}\mathbf{a})+(\hat{\mathbf{T}}\mathbf{a})\otimes\mathbf{a}]+\frac{\partial\check{\Pi}}{\partial\check{I}_4}\{(2\hat{I}_1\hat{I}_4-\hat{I}_1^2-\hat{I}_5)\mathbf{I}+\hat{\mathbf{T}}^2+\hat{I}_1^2\mathbf{a}\otimes\mathbf{a}\\&-\hat{I}_1[\mathbf{a}\otimes(\hat{\mathbf{T}}\mathbf{a})+(\hat{\mathbf{T}}\mathbf{a})\otimes\mathbf{a}]\},\end{aligned} \tag{46}$$

which is the dimensionless counterpart of (25).

3.4. The linearized constitutive equation

Let us carry out a further linearization by assuming that $|\hat{\mathbf{T}}|\sim O(\delta)$, $\delta\ll 1$, which is equivalent to saying $|\mathbf{T}|\ll\sigma_o$. Such a linearization leads to the classical linearized elastic model. We can express the functions $\frac{\partial\check{\Pi}}{\partial\check{I}_i}(\hat{\mathbf{T}})$ as a truncated Taylor series (assuming that $\check{\Pi}$ is sufficiently smooth) around $\hat{\mathbf{T}}=\mathbf{0}$ in index notation as:

$$\frac{\partial\check{\Pi}}{\partial\check{I}_i}(\hat{\mathbf{T}})\approx\frac{\partial\check{\Pi}}{\partial\check{I}_i}(\hat{\mathbf{0}})+\left(\frac{\partial^2\check{\Pi}}{\partial\check{I}_i\partial\check{I}_j}\frac{\partial\check{I}_j}{\partial\hat{T}_{mn}}\right)_{\hat{\mathbf{T}}=\mathbf{0}}\hat{T}_{mn},\quad i,j=1,2,3,4. \tag{47}$$

Using this expansion in (46) after some manipulations (neglecting terms of order $O(\delta^r)$, $r\geq 2$), which for brevity are not presented here, we obtain the linearized equation (compare with Eq. 24 of [44])

$$\boldsymbol{\varepsilon}=[\alpha_1+(\beta+\alpha_2+2\alpha_3)(\hat{I}_1-\hat{I}_4)]\mathbf{I}+\alpha_2\hat{\mathbf{T}}-(\alpha_2+2\alpha_3)\mathbf{a}\otimes\mathbf{a}+\alpha_3[\mathbf{a}\otimes(\hat{\mathbf{T}}\mathbf{a})+(\hat{\mathbf{T}}\mathbf{a})\otimes\mathbf{a}], \tag{48}$$

where we have defined

$$\alpha_l=\frac{\partial\check{\Pi}}{\partial\check{I}_l}(\mathbf{0}),\quad \beta=\frac{\partial^2\check{\Pi}}{\partial\check{I}_1^2}(\mathbf{0}),\quad l=1,2,3. \tag{49}$$

4. Simple boundary value problems

In this section, we first study two simple boundary value problems, where we assume the distributions of stresses and strains are homogeneous; thereafter, we consider the results presented in Sect. 3.2, and we study problems concerning the inhomogeneous distributions of stresses and strains, and we present partial solutions of some boundary value problem using the decomposition of the stress (33), i.e. adding $q\mathbf{a}\otimes\mathbf{a}$ to the stress field.

4.1. The biaxial extension of a thin plate

In this first problem, let us consider the plate defined through $-L_1/2 \leq x_1 \leq L_1/2$, $-L_2/2 \leq x_2 \leq L_2/2$, $0 \leq x_3 \leq h$, where $h \ll L_1$, $h \ll L_2$. Let us assume that the plate is under a homogeneous stress distribution of the form

$$\mathbf{T} = \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2 \tag{50}$$

and let us consider the case where the preferred direction $\mathbf{a}$ is given as

$$\mathbf{a} = \cos\xi \mathbf{e}_1 + \sin\xi \mathbf{e}_2, \quad 0 \leq \xi \leq \frac{\pi}{2}. \tag{51}$$

On substituting (50) and (51) into (19), we obtain

$$I_1 = \sigma_1 + \sigma_2, \quad I_2 = \frac{1}{2}(\sigma_1^2 + \sigma_2^2), \quad I_3 = \frac{1}{3}(\sigma_1^3 + \sigma_2^3), \tag{52}$$

$$I_4 = \sigma_1 \cos^2\xi + \sigma_2 \sin^2\xi, \quad I_5 = \sigma_1^2 \cos^2\xi + \sigma_2^2 \sin^2\xi, \tag{53}$$

therefore, in terms of the invariants defined in (23), (24), we have

$$\bar{I}_1 = \sigma_1(\cos^2\xi - 1) + \sigma_2(\sin^2\xi - 1), \tag{54}$$

$$\bar{I}_2 = \bar{I}_3 = \sigma_1^2(1 - \cos^2\xi) + \sigma_2^2(1 - \sin^2\xi), \quad \bar{I}_4 = 0, \tag{55}$$

$$\tag{56}$$

and finally from (25), the nonzero components of the strain tensor are:

$$\varepsilon_{11} = \frac{\partial\bar{\Pi}}{\partial\bar{I}_1}(\cos^2\xi - 1) + 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_2}\sigma_1 \sin^2\xi + 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_3}\sigma_1 \sin^2\xi, \tag{57}$$

$$\varepsilon_{22} = \frac{\partial\bar{\Pi}}{\partial\bar{I}_1}(\sin^2\xi - 1) + 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_2}\sigma_1 \cos^2\xi + 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_3}\sigma_1 \cos^2\xi, \tag{58}$$

$$\begin{aligned}\varepsilon_{33} = &-\frac{\partial\bar{\Pi}}{\partial\bar{I}_1} + \frac{\partial\bar{\Pi}}{\partial\bar{I}_2}[\sigma_1(1 - \cos^2\xi) + \sigma_2(1 - \sin^2\xi)] + \frac{\partial\bar{\Pi}}{\partial\bar{I}_3}[\sigma_1 + \sigma_2 \\ &+ (\sigma_2 - \sigma_1)\cos(2\xi)] + \frac{\partial\bar{\Pi}}{\partial\bar{I}_4}[(\sigma_1^2 - \sigma_2^2)\cos(2\xi) - \sigma_1^2 - \sigma_2^2],\end{aligned} \tag{59}$$

$$\varepsilon_{12} = \left[\frac{\partial\bar{\Pi}}{\partial\bar{I}_1} - \frac{\partial\bar{\Pi}}{\partial\bar{I}_2}(\sigma_1 + \sigma_2) - \frac{\partial\bar{\Pi}}{\partial\bar{I}_3}(\sigma_1 + \sigma_2)\right]\cos\xi \sin\xi. \tag{60}$$

Since the stress field is constant, it satisfies the equilibrium equation (7) (without body forces) automatically, while from $(6)_3$ and (57)–(60), we obtain unique expressions for the components of the displacement field (up to a rigid body motion).

4.2. A slab under a state of simple shear stress

Let us consider the slab $-L_i/2 \leq x_i \leq L_i/2$, $i = 1, 2, 3$ under the effect of the shear stress distribution

$$\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{61}$$

and let us assume the same expression for $\mathbf{a}$ as in (51). Using (61) and (51) in (19), we obtain

$$I_1 = I_3 = 0, \quad I_2 = I_5 = \tau^2, \quad I_4 = 2\tau \sin\xi \cos\xi. \tag{62}$$

Using these expressions for the invariants in (23), (24), we have

$$\bar{I}_1 = 2\tau \sin\xi \cos\xi, \quad \bar{I}_2 = \bar{I}_3 = \tau^2, \quad \bar{I}_4 = 0. \tag{63}$$

Using (22) in (25) and repeating the above calculations for the nonzero components of the strains, we obtain:

$$\varepsilon_{11} = \frac{\partial\bar{\Pi}}{\partial\bar{I}_1}(\cos^2\xi - 1) - 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_2}\tau\sin\xi\cos\xi - 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_3}\tau\sin\xi\cos\xi, \tag{64}$$

$$\varepsilon_{22} = \frac{\partial\bar{\Pi}}{\partial\bar{I}_1}(\sin^2\xi - 1) - 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_2}\tau\sin\xi\cos\xi - 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_3}\tau\sin\xi\cos\xi, \tag{65}$$

$$\varepsilon_{33} = -\frac{\partial\bar{\Pi}}{\partial\bar{I}_1} - 2\frac{\partial\bar{\Pi}}{\partial\bar{I}_2}\tau\sin\xi\cos\xi - \frac{\partial\bar{\Pi}}{\partial\bar{I}_4}\tau^2, \tag{66}$$

$$\varepsilon_{12} = \frac{\partial\bar{\Pi}}{\partial\bar{I}_1}\sin\xi\cos\xi + \frac{\partial\bar{\Pi}}{\partial\bar{I}_2}\tau + \frac{\partial\bar{\Pi}}{\partial\bar{I}_3}\tau. \tag{67}$$

Again since $\mathbf{T}$ from (61) is constant, the equilibrium equations (without body forces) (7) are satisfied and from $(6)_3$ and (64)–(67), we can obtain unique expressions (up to a rigid body motion) for the displacement field.

4.3. A simple example wherein the reaction stress is of the form $q\mathbf{a}\otimes\mathbf{a}$

In this problem, we use the notation x, y, z for x_i, $i = 1, 2, 3$, respectively. Let us work with a three-dimensional body defined through $a \leq x \leq b$, where in the planes $y - z$ the body may have an arbitrary geometry (smooth enough such that our calculations are valid). Let us assume this body is under the effect of the following stress field

$$\mathbf{T} = \mathbf{T}_{\mathrm{o}}(x) + q\mathbf{e}_1 \otimes \mathbf{e}_1, \tag{68}$$

where we have assumed that $\mathbf{a} = \mathbf{e}_1$ (we need to remember that $\mathbf{a}\cdot(\mathbf{T}_{\mathrm{o}}\mathbf{a}) = 0$), then:

$$\begin{aligned}\mathbf{T}_{\mathrm{o}}(x) = {} & \sigma_2(x)\mathbf{e}_2\otimes\mathbf{e}_2 + \sigma_3(x)\mathbf{e}_3\otimes\mathbf{e}_3 + \tau_{12}(x)(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1) \\ & + \tau_{13}(x)(\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1) + \tau_{23}(x)(\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2).\end{aligned} \tag{69}$$

Using (68) and (69) in (7) and neglecting body forces in the equilibrium equations, we obtain

$$\frac{\partial q}{\partial x} = 0, \quad \frac{\mathrm{d}\tau_{12}}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}\tau_{13}}{\mathrm{d}x} = 0. \tag{70}$$

Let us assume that q is of the form

$$q = \zeta(x) + \vartheta(y, z), \tag{71}$$

then by virtue of (70), we have the solution

$$\zeta(x) = c_1, \quad \tau_{12} = c_2, \quad \tau_{13} = c_3, \tag{72}$$

where c_i, $i = 1, 2, 3$ are constants.

Let us assume that the stress tensor field (68) produces the following displacement field[8]

$$u_i(x, y, z) = \upsilon_i(x) + g_i y + h_i z, \quad i = 1, 2, 3, \tag{73}$$

where g_i, h_i, $i = 1, 2, 3$ are constants. Using (73) in $(6)_3$ and appealing to (42), we obtain the relations

$$\frac{\mathrm{d}\upsilon_1}{\mathrm{d}x} = \mathfrak{f}_{11}(\mathbf{T}_{\mathrm{o}}(x)), \quad g_2 = \mathfrak{f}_{22}(\mathbf{T}_{\mathrm{o}}(x)), \quad h_3 = \mathfrak{f}_{33}(\mathbf{T}_{\mathrm{o}}(x)), \tag{74}$$

$$g_1 + \frac{\mathrm{d}\upsilon_2}{\mathrm{d}x} = 2\mathfrak{f}_{12}(\mathbf{T}_{\mathrm{o}}(x)), \quad h_1 + \frac{\mathrm{d}\upsilon_3}{\mathrm{d}x} = 2\mathfrak{f}_{13}(\mathbf{T}_{\mathrm{o}}(x)), \quad h_2 + g_3 = 2\mathfrak{f}_{23}(\mathbf{T}_{\mathrm{o}}(x)), \tag{75}$$

[8] It is possible that other solutions for the displacement field are possible, considering (68), (69) and (25), but in this communication we do not discuss about such possible nonuniqueness for the displacement field.

where the components $\mathfrak{f}_{ij}$ of $\mathfrak{f}$ are obtained from (25), which for the sake of brevity are not presented explicitly here.

Solving for υ_i, $i = 1, 2, 3$ from $(74)_1$ and $(75)_{1,2}$ and (73), we obtain

$$u_1 = \int_a^x \mathfrak{f}_{11}(\mathbf{T}_\mathrm{o}(\eta))\,\mathrm{d}\eta + \upsilon_{1_o} + g_1 y + h_1 z, \tag{76}$$

$$u_2 = 2\int_a^x \mathfrak{f}_{12}(\mathbf{T}_\mathrm{o}(\eta))\,\mathrm{d}\eta - g_1 x + \upsilon_{2_o} + g_2 y + h_2 z, \tag{77}$$

$$u_3 = 2\int_a^x \mathfrak{f}_{13}(\mathbf{T}_\mathrm{o}(\eta))\,\mathrm{d}\eta - h_2 x + \upsilon_{3_o} + g_3 y + h_3 z, \tag{78}$$

where υ_{i_o}, $i = 1, 2, 3$ are constants.

Regarding boundary conditions, let us assume that at $x = a$ the displacement is zero, i.e. $\mathbf{u}(a, y, z) = \mathbf{0}$, therefore, from (76) to (78), we conclude that such a condition is satisfied if $\upsilon_{i_o} = 0$, $g_i = 0$, $h_i = 0$, $i = 1, 2, 3$. With regard to the boundary at $x = b$, let us assume that we apply an external traction there, i.e. $\mathbf{Tn} = \breve{\mathbf{t}}$, and considering (68), (69) and since $\mathbf{n} = \mathbf{e}_1$ we have

$$\breve{\mathbf{t}} = (c_1 + \vartheta(y, z))\mathbf{e}_1 + c_1\mathbf{e}_2 + c_3\mathbf{e}_3. \tag{79}$$

This is the external load that we need in order to produce the displacement (76), (78) such that the boundary value problem is solvable under the assumptions that have been made for the forms for the stress and displacement fields. As a consequence of (71), we have $\vartheta(y, z)$ and thus $\sigma_2(x)$, $\sigma_3(x)$ and $\tau_{23}(x)$ can be found by solving the, in general nonlinear, algebraic equations $(74)_{2,3}$ and $(75)_3$.

In finding the above solution for the boundary value problem, we have used a semi-inverse approach. As the governing equations in general are nonlinear, it is possible that there might be other solutions than of the form sought in the semi-inverse approach.

4.4. Another example wherein the reaction stress is of the form $q\mathbf{a} \otimes \mathbf{a}$

We repeat the methodology presented in the previous section, within the context of a cylindrical body. Consider the body defined in cylindrical coordinates by $a \le r \le b$. In the $\theta - z$ plane, the geometry of the body can be arbitrary. Let us assume that this body is under the influence of the stress tensor field

$$\mathbf{T} = \mathbf{T}_\mathrm{o} + q\mathbf{e}_r \otimes \mathbf{e}_r, \tag{80}$$

where $\mathbf{a} = \mathbf{e}_r$ and

$$\begin{aligned}\mathbf{T}_\mathrm{o} = {} & \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r)\mathbf{e}_z \otimes \mathbf{e}_z + \tau_{r\theta}(r)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) \\ & + \tau_{rz}(r)(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + \tau_{\theta z}(r)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta),\end{aligned} \tag{81}$$

i.e. $\mathbf{e}_r \cdot (\mathbf{T}_\mathrm{o}\mathbf{e}_r) = 0$. Replacing (80), (81) in the equilibrium equations (7) (neglecting body forces), we obtain

$$\frac{\partial q}{\partial r} + \frac{1}{r}(q - \sigma_\theta) = 0, \quad \frac{\mathrm{d}\tau_{r\theta}}{\mathrm{d}r} + \frac{2\tau_{r\theta}}{r} = 0, \quad \frac{\mathrm{d}\tau_{rz}}{\mathrm{d}r} + \frac{\tau_{rz}}{r} = 0. \tag{82}$$

Let us assume that $q = q(r)$, then from $(82)_1$ we find, for example, q as:

$$q(r) = \frac{1}{r}\int_a^r \sigma_\theta(\eta)\,\mathrm{d}\eta - \frac{c_r}{r}, \tag{83}$$

while from $(82)_{2,3}$, we obtain

$$\tau_{r\theta}(r) = \frac{c_\theta}{r^2}, \quad \tau_{rz}(r) = \frac{c_z}{r}, \tag{84}$$

where c_r, c_θ, c_z are constants.

Let us assume the above stress field produces the following displacement field in the body:

$$u_r = \upsilon_r(r) + q_r z + k, \quad u_\theta = \upsilon_\theta(r), \quad u_z = \upsilon_z(r) + q_z z, \tag{85}$$

where q_r, q_z, k are constants. Using this displacement field in $(6)_3$ and recalling (42), we obtain

$$\frac{\mathrm{d}\upsilon_r}{\mathrm{d}r} = \mathfrak{f}_{rr}(\mathbf{T}_{\mathrm{o}}(r)), \quad \frac{\upsilon_r}{r} = \mathfrak{f}_{\theta\theta}(\mathbf{T}_{\mathrm{o}}(r)), \quad q_z = \mathfrak{f}_{zz}(\mathbf{T}_{\mathrm{o}}(r)), \tag{86}$$

$$\frac{\mathrm{d}\upsilon_\theta}{\mathrm{d}r} - \frac{\upsilon_\theta}{r} = 2\mathfrak{f}_{r\theta}(\mathbf{T}_{\mathrm{o}}(r)), \quad q_r + \frac{\mathrm{d}\upsilon_z}{\mathrm{d}r} = 2\mathfrak{f}_{rz}(\mathbf{T}_{\mathrm{o}}(r)), \quad 0 = \mathfrak{f}_{\theta z}(\mathbf{T}_{\mathrm{o}}(r)). \tag{87}$$

From $(86)_{1,2}$, we can find a unique solution for $\upsilon_r(r)$ if the following (in general nonlinear) differential equation, which arises from compatibility considerations, is satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r\mathfrak{f}_{\theta\theta}(\mathbf{T}_{\mathrm{o}}(r))\right] = \mathfrak{f}_{rr}(\mathbf{T}_{\mathrm{o}}(r)). \tag{88}$$

If (88) is satisfied, from $(86)_2$, we have the solution

$$\upsilon_r(r) = r\mathfrak{f}_{\theta\theta}(\mathbf{T}_{\mathrm{o}}(r)). \tag{89}$$

From $(87)_{1,2}$, (89) and (85), we obtain the solutions

$$u_r = r\mathfrak{f}_{\theta\theta}(\mathbf{T}_{\mathrm{o}}(r)) + q_r z + k, \tag{90}$$

$$u_\theta = 2r\int_a^r \frac{1}{\eta}\mathfrak{f}_{r\theta}(\mathbf{T}_{\mathrm{o}}(\eta))\,\mathrm{d}\eta + r\upsilon_{\theta_o}, \tag{91}$$

$$u_z = 2\int_a^r \mathfrak{f}_{rz}(\mathbf{T}_{\mathrm{o}}(\eta))\,\mathrm{d}\eta - q_r r + q_z z, \tag{92}$$

where υ_{θ_o} is a constant.

With regard to the boundary conditions, let us assume that on the boundary $r = a$ we have the condition $\mathbf{u} = \mathbf{0}$, then from (90)–(92) that condition is met if $k = -a\mathfrak{f}_{\theta\theta}(\mathbf{T}_{\mathrm{o}}(a))$ and $q_r = q_z = \upsilon_{\theta_o} = 0$. Regarding the boundary at $r = b$ we assume that there is an external traction $\breve{\mathbf{t}}$ applied there, and using (80), (81) from $\mathbf{Tn} = \breve{\mathbf{t}}$ we have

$$\breve{\mathbf{t}} = \left[\frac{1}{b}\int_a^b \sigma_\theta(\eta)\,\mathrm{d}\eta - \frac{c_r}{b}\right]\mathbf{e}_r + \frac{c_\theta}{b^2}\mathbf{e}_\theta + \frac{c_z}{b}\mathbf{e}_z, \tag{93}$$

which is the necessary external traction for the solution (90)–(92) to be valid.

In this problem, $\sigma_\theta(r)$, $\sigma_z(r)$ and $\tau_{\theta z}(r)$ can be found by solving the (in general nonlinear) ordinary differential equation (88) and the nonlinear algebraic equations $(86)_3$ and $(87)_3$. In the case of the previous boundary value problem, we are using a semi-inverse approach solutions other than those of the form sought are possible.

5. Case of large elastic deformations

In this last section, let us consider briefly the case when $|\nabla \mathbf{u}|$ is large, where in this case ∇ is the gradient defined with respect to the reference configuration. The results in this case are very similar to the case of small gradient of the displacement field presented in the previous sections. The constraint for a body that is inextensible in the direction $\mathbf{a}_0$ is given in terms of the tensor $\mathbf{C}$ (see (12) and (15)), and that equation can be rewritten in terms of $\mathbf{E}$ (see $(5)_3$) as

$$\mathbf{a}_0 \cdot (\mathbf{E}\mathbf{a}_0) = 0. \tag{94}$$

Let us consider an implicit constitutive relation of the form (see [33,34]) $\mathfrak{F}(\mathbf{S}, \mathbf{E}) = \mathbf{0}$ (where $\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathrm{T}}$ is the second Piola-Kirchhoff stress tensor), and its subclass $\mathbf{E} = \mathfrak{g}(\mathbf{S})$. Furthermore, let us assume again that there exists a scalar function $\Omega = \Omega(\mathbf{S})$ such that

$$\mathbf{E} = \mathfrak{g}(\mathbf{S}) = \frac{\partial \Omega}{\partial \mathbf{S}}. \tag{95}$$

In the case $\Omega = \Omega(\mathbf{S})$ is a transversely isotropic function the direction of anisotropy being $\mathbf{a}_0$ (reference configuration), then $\Omega = \Omega(J_k)$, $k = 1, 2, 3, 4, 5$, where J_k are given as in (19) but interchanging $\mathbf{T}$ by $\mathbf{S}$ and $\mathbf{a}$ by $\mathbf{a}_0$, i.e. $J_1 = \mathrm{tr}\,\mathbf{S}$, $J_2 = \frac{1}{2}\mathrm{tr}\,(\mathbf{S}^2)$, $J_3 = \frac{1}{3}\mathrm{tr}\,(\mathbf{S}^3)$, $J_4 = \mathbf{a}_0 \cdot (\mathbf{S}\mathbf{a}_0)$ and $J_5 = \mathbf{a}_0 \cdot (\mathbf{S}^2\mathbf{a}_0)$. In this case, we have

$$\mathbf{E} = \Omega_1 \mathbf{I} + \Omega_2 \mathbf{S} + \Omega_3 \mathbf{S}^2 + \Omega_4 \mathbf{a}_0 \otimes \mathbf{a}_0 + \Omega_5[\mathbf{a}_0 \otimes (\mathbf{S}\mathbf{a}_0) + (\mathbf{S}\mathbf{a}_0) \otimes \mathbf{a}_0], \tag{96}$$

where $\Omega_k = \frac{\partial \Omega}{\partial J_k}$, $k = 1, 2, 3, 4, 5$. Replacing (96) in (94) and following the same procedure as presented for the case of small gradient of the displacement field in Sect. 3, it is possible to show that the particular expression for Ω for the body to be inextensible in the direction $\mathbf{a}_0$ must be of the form

$$\Omega = \bar{\Omega}(\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{J}_4), \tag{97}$$

where $\bar{J}_1 = J_4 - J_1$, $\bar{J}_2 = \frac{1}{2}J_1^2 + J_2 - J_1 J_4$, $\bar{J}_3 = J_1^2 - 2J_1 J_4 + J_5$ and $\bar{J}_4 = -\frac{1}{3}J_1^3 + J_3 + J_1^2 J_4 - J_1 J_5$.

The constraint $\mathbf{a}_0 \cdot (\mathbf{E}\mathbf{a}_0) \le 0$ can be treated in the same way as in Sect. 3.

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