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L^p-THEORY FOR THE BOUSSINESQ SYSTEM

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SUMMARY OF THE THESIS TO OBTAIN THE DEGREE OF: Doctor en Ciencias de la Ingeniería, Mención Modelación Matemática
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L^p-THEORY FOR THE BOUSSINESQ SYSTEM

This thesis is dedicated to the study of the stationary Boussinesq system:

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} \quad \text{in } \Omega, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{0.1a}$$

$$-\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = h \quad \text{in } \Omega, \tag{0.1b}$$

where $\Omega \subset \mathbb{R}^3$ is an open bounded connected set; $\boldsymbol{u}$, π and θ are the velocity field, pressure and temperature of the fluid, respectively, and stand for the unknowns of the system; $\nu > 0$ is the kinematic viscosity of the fluid, $\kappa > 0$ is the thermal diffusivity of the fluid, $\boldsymbol{g}$ is the gravitational acceleration and h is a heat source applied to the fluid.

The aim of this thesis is the study of the L^p-theory for the stationary Boussinesq system in the context of two different types of boundary conditions for the velocity field. Indeed, in the first part of the thesis, we will consider a non-homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{u_b} \quad \text{on} \quad \Gamma, \tag{0.2}$$

where Γ denotes the boundary of the domain; meanwhile in the second part, the velocity field will be prescribed through a non-homogeneous Navier boundary condition

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\,\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{a}, \quad \text{on} \quad \Gamma, \tag{0.3}$$

where $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ is the strain tensor associated with the velocity field $\boldsymbol{u}$, $\boldsymbol{n}$ is the unit outward normal vector, $\boldsymbol{\tau}$ is the corresponding unit tangent vector, α and $\boldsymbol{a}$ are a friction scalar function and a tangential vector field defined both on the boundary, respectively. Further, the boundary condition for the temperature will be, in the first two parts of the thesis, a non-homogeneous Dirichlet boundary condition

$$\theta = \theta_b \quad \text{on} \quad \Gamma. \tag{0.4}$$

Then, firstly, we study the existence and uniqueness of the weak solution for the problem (0.1), (0.2) and (0.4) in the hilbertian case. Also, the existence of generalized solutions for $p \geq \frac{3}{2}$ and strong solutions for $1 < p < \infty$ is showed. Furthermore, the existence and uniqueness of the very weak solution is studied. It is worth to note that because a non-homogeneous Dirichlet boundary condition is considered for the velocity field, the fact that the boundary of the domain could be non-connected plays a fundamental role since it appears in an explicit way in the assumptions of some of the main results.

In the second part, we study the existence of weak solutions in the Hilbert case, as well as the existence of generalized solutions for $p > 2$ and strong solutions for $p \geq \frac{6}{5}$ for the problem (0.1), (0.3) and (0.4). Note that the assumption of a non-connected boundary, which was mentioned before, will not appear here due to the impermeability restriction on the boundary.

Finally, in the last part of this thesis, we study the L^p-theory for the Stokes equations with Navier boundary condition (0.3). Specifically, we deal with the $W^{1,p}$-regularity for $p \geq 2$ and the $W^{2,p}$-regularity for $p \geq \frac{6}{5}$.

RÉSUMÉ DE LA THÈSE POUR OBTENIR LE DEGRE DE: Doctor en Ciencias de la Ingeniería, Mención Modelación Matemática
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THÉORIE L^p POUR LE SYSTÈME DE BOUSSINESQ

Cette thèse est consacrée à l'étude du système de Boussinesq stationnaire:

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} \quad \text{dans } \Omega, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{dans } \Omega, \tag{0.1a}$$

$$-\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = h \quad \text{dans } \Omega, \tag{0.1b}$$

où $\Omega \subset \mathbb{R}^3$ est un ouvert, borné et connexe; les inconnues du système sont $\boldsymbol{u}$, π et θ: la vitesse, la pression et la température du fluide, respectivement; $\nu > 0$ est la viscosité cinématique du fluide, $\kappa > 0$ est la diffusivité thermique du fluide, $\boldsymbol{g}$ est l'accélération de la pesanteur et h est une source de chaleur appliquée au fluide.

L'objectif de cette thèse est l'étude de la théorie L^p pour le système de Boussinesq en considérant deux différents types de conditions aux limites du champ de vitesse. En effet, dans une premierè partie, nous considérons une condition de Dirichlet non homogène

$$\boldsymbol{u} = \boldsymbol{u_b} \quad \text{sur} \quad \Gamma, \tag{0.2}$$

où Γ désigne la frontière du domaine. Dans une deuxième partie, nous considéron une condition de Navier non homogène

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\,\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{a}, \quad \text{sur} \quad \Gamma, \tag{0.3}$$

où $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ est le tenseur de déformation associé au champ de vitesse $\boldsymbol{u}$, $\boldsymbol{n}$ est le vecteur normal unitaire extérieur, $\boldsymbol{\tau}$ est le correspondant vecteur tangent unitaire, α et $\boldsymbol{a}$ sont une fonction scalaire de friction et un champ de vecteur tangentiel donnés sur la frontière, respectivement. De plus, la condition aux limites pour la température sera, dans les deux premières parties, une condition aux limites de Dirichlet non homogène

$$\theta = \theta_b \quad \text{sur} \quad \Gamma. \tag{0.4}$$

Alors, premièrement, nous étudions l'existence et l'unicité d'une solution faible pour le problème (0.1), (0.2) et (0.4) dans le cas hilbertien. Également, l'existence de solutions généralisées pour $p \geq \frac{3}{2}$ et des solutions fortes pour $1 < p < \infty$ est démontrée. De plus, l'existence et l'unicité de la solution très faible sont étudiés. Il est intéressant de noter que puisque une condition de Dirichlet non homogène est considérée pour le champ de vitesse, le fait que la frontière du domaine pourrait être non-connexe joue un rôle fondamental puisque cela apparait de manière explicite dans les hypothèses des principaux résultats.

D'autre part, dans la deuxième partie, nous étudions l'existence de solutions faibles dans le cas hilbertien, ainsi que l'existence de solutions généralisées pour $p > 2$ et des solutions fortes pour $p \geq \frac{6}{5}$ pour le problème (0.1), (0.3) et (0.4). Notez que l'hypothèse d'une frontière non-connexe, mentionnée précédemment, ne figurait pas dans cette partie du travail en raison de la restriction d'imperméabilité de la frontière.

Enfin, dans la dernière partie de cette thèse, nous étudions la théorie L^p pour les équations de Stokes avec la condition de Navier (0.3). Plus précisément, nous examinons la régularité $W^{1,p}$ pour $p \geq 2$ et la régularité $W^{2,p}$ pour $p \geq \frac{6}{5}$.

RESUMEN DE LA TESIS PARA OPTAR EL GRADO DE: Doctor en Ciencias de la Ingeniería, Mención Modelación Matemática
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TEORÍA L^p PARA EL SISTEMA DE BOUSSINESQ

Esta tesis está dedicada al estudio del sistema de Boussinesq estacionario:

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} \quad \text{en } \Omega, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{en } \Omega, \tag{0.1a}$$

$$-\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = h \quad \text{en } \Omega, \tag{0.1b}$$

donde $\Omega \subset \mathbb{R}^3$ es un conjunto abierto, acotado y conexo; $\boldsymbol{u}$, π y θ representan el campo de velocidades, la presión y la temperatura del fluido, respectivamente, siendo éstas las incógnitas del sistema; $\nu > 0$ es la viscosidad cinemática del fluido, $\kappa > 0$ es la difusividad térmica del fluido, $\boldsymbol{g}$ es la aceleración de la gravedad y h es una fuente de calor aplicada al fluido.

El objetivo de esta tesis es el estudio de la teoría L^p para el sistema de Boussinesq estacionario considerando dos diferentes tipos de condiciones de frontera del campo de velocidades. En efecto, en una primera etapa, se considerará la condición de frontera de Dirichlet no homogéneo

$$\boldsymbol{u} = \boldsymbol{u_b} \quad \text{sobre} \quad \Gamma, \tag{0.2}$$

donde Γ denota la frontera del dominio; mientras que en una segunda etapa, el campo de velocidades tendrá prescrito la condición de frontera de Navier no homogéneo

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad 2\,[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\,\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{a}, \quad \text{sobre} \quad \Gamma, \tag{0.3}$$

donde $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ es el tensor de deformación asociado con el campo de velocidades $\boldsymbol{u}$, $\boldsymbol{n}$ es el vector normal unitario exterior, $\boldsymbol{\tau}$ es el correspondiente vector unitario tangente, α y $\boldsymbol{a}$ son una función de fricción y un campo vectorial tangencial definidas ambas sobre la frontera. Además, la condición de frontera para la temperatura será, en las dos primeras partes, la condición de frontera de Dirichlet no homogéneo

$$\theta = \theta_b \quad \text{sobre} \quad \Gamma. \tag{0.4}$$

Así, en primer lugar, estudiamos la existencia y unicidad de la solución débil para el problema (0.1), (0.2) y (0.4) en el caso hilbertiano. Además, la existencia de soluciones generalizadas para $p \geq \frac{3}{2}$ y soluciones fuertes para $1 < p < \infty$ es probada. También, se estudiará la existencia y unicidad de la solución muy débil. Vale la pena señalar que debido a que la condición de Dirichlet no homogénea es considerada para la velocidad, el hecho de que la frontera del dominio pueda ser no conexa juega un papel importante, ya que aparece de manera explícita en las hipótesis de algunos de los principales resultados.

Por otro lado, en la segunda etapa de la tesis, se estudiará la existencia de soluciones débiles en el caso de Hilbert, así como la existencia de soluciones generalizadas para $p > 2$ y soluciones fuertes para $p \geq \frac{6}{5}$ para el problema (0.1), (0.3) y (0.4). Tenga en cuenta que la suposición hecha anteriormente acerca de la no conexidad de la frontera no aparecerá aquí debido a la restricción de impermeabilidad en la frontera.

Finalmente, en la última parte de esta tesis, estudiamos la teoría L^p para las ecuaciones de Stokes con la condición de Navier (0.3). Más precisamente, nos ocuparemos de la regularidad $W^{1,p}$ para $p \geq 2$ y la regularidad $W^{2,p}$ para $p \geq \frac{6}{5}$.

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Paul

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Chapter 1

General Introduction

1.1 Preliminaries

This thesis is concerned with the study of the following **stationary Boussinesq system**:

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} \quad \text{in } \Omega, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$-\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = h \quad \text{in } \Omega, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^3$ is an open bounded connected set; $\boldsymbol{u}$, π and θ are the velocity field, pressure and temperature of the fluid, respectively, and stand for the unknowns of the system; $\nu > 0$ is the kinematic viscosity of the fluid, $\kappa > 0$ is the thermal diffusivity of the fluid, $\boldsymbol{g}$ is the gravitational acceleration and h is a heat source applied to the fluid.

The Boussinesq system (1.1) is a system of non linear partial differential equations which is formed by coupling the **Navier-Stokes equations** (1.1a) with the **convection-diffusion equation** (1.1b). Basically, this system describes the behaviour of a viscous incompressible fluid when is heated.

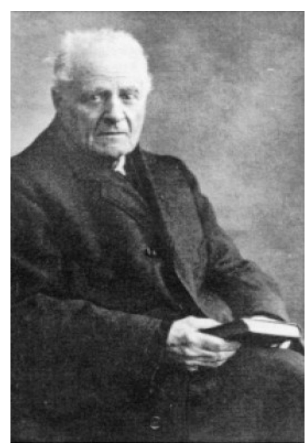

Figure 1.1: J. Boussinesq

This system was named after the French mathematician Joseph Valentin Boussinesq (Figure 1.1)[1] who realized that in some situations the variations of density in a fluid can be neglected in all the terms of the equations of motion of the fluid, except when they are multiplied by the acceleration of gravity. This assertion was published in his monograph entitled "Théorie analytique de la chaleur" in 1903, see [13][2].

It is a good idea to see, at least in a brief way, how to derive the Boussinesq equations. It will start by describing the general equations of the hydrodynamical flow of a viscous fluid

[1]Source: https://en.wikipedia.org/wiki/File:Joseph_Boussinesq.jpg. Visited on 07-15-2015.

[2]This system is also called the "Oberbeck–Boussinesq equations". According to Yaglom and Frisch [72], in Joseph's book [33], the author pointed out that the German physicist Anton Oberbeck used, still earlier, practically the same equations (and also the same modifications) in his papers [54] and [55]. According to Joseph, the prevalence of the term "Boussinesq equations" is due to Rayleigh [59] (which became an extremely popular work), who did not know Oberbeck's papers. De Boer [22] pointed out that the earlier result of Oberbeck [54] is nearly the same as the Boussinesq approximation, with the only difference that Oberbeck preserved density variations in the continuity equation.

in the three dimensional space with density ρ and velocity field $\boldsymbol{u}$. From the law of the *conservation of mass*, it follows the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0. \tag{1.2}$$

The following equations of motion (best known as the Navier-Stokes equations) result from the *conservation of the linear momentum*:

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{g} + \operatorname{div} \sigma, \tag{1.3}$$

where the differential operator $\frac{D(\cdot)}{Dt}$ stands for the material derivative

$$\left[\frac{D(\cdot)}{Dt} := \frac{\partial (\cdot)}{\partial t} + (\boldsymbol{u} \cdot \nabla)(\cdot) \right],$$

$\boldsymbol{g}$ is the gravitational acceleration and σ is the stress tensor given by

$$\sigma = -\pi I_3 + \mu \left(2\mathbb{D}(\boldsymbol{u}) - \frac{2}{3} \operatorname{div} \boldsymbol{u}\, I_3 \right) + \lambda \operatorname{div} \boldsymbol{u}\, I_3, \tag{1.4}$$

with π the pressure, μ the coefficient of dynamic viscosity of the fluid, λ the bulk viscosity (or second viscosity), I_3 the identity matrix of order 3 and $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$ the deformation tensor (or strain tensor) associated with the velocity field $\boldsymbol{u}$.

The equation of heat conduction, which is obtained thanks to the law of *conservation of energy*, is as follows:

$$\rho \frac{De}{Dt} = \operatorname{div}(k_{tc} \nabla \theta) - \pi \operatorname{div} \boldsymbol{u} + \Phi, \tag{1.5}$$

where e is the internal energy per unit mass of the fluid, k_{tc} is the thermal conductivity, θ is the temperature and Φ is the rate of viscous dissipation per unit volume of fluid defined by

$$\Phi = 2\mu \operatorname{Tr}(\mathbb{D}(\boldsymbol{u})^2) + \left(\lambda - \frac{2}{3}\mu \right) (\operatorname{div} \boldsymbol{u})^2, \tag{1.6}$$

where $\operatorname{Tr}(A)$ stands for the trace of the matrix A. It is known that for a fluid (gas or liquid) $e = c\,\theta$, where c is the specific heat of the fluid. It is noteworthy that the quantities ρ, μ, λ, k_{tc}, c and e are, in general, functions of the pressure π and the temperature θ.

Until this time we presented the general hydrodynamical equations for compressible heat conducting and diffusive flow of a viscous, nonhomogeneous fluid. However, as Boussinesq pointed out in [13], in many real situations, there are fluids for which the influence of pressure is unimportant and even the variation with temperature may be disregarded, except in so far as it modifies the action of gravity. For example, according to [33], in many real situations, there are fluids for which one needs to change the pressure by roughly five atmospheres to produce the same change of density as a temperature difference of 1°C. For example, it is easier to change the density of water inside of a container by heating than by squeezing. This implies that even vigorous motions of water will not introduce important buoyant forces other than those from temperature variations. Moreover, if the temperature varies little, therefore the density varies little too, except in the buoyancy which drives the motion (density gradients in a fluid means that gravitational potential energy can be converted into motion through the action of buoyant forces). Hence, the variation of density is neglected everywhere except in the buoyancy. In this way the quantities ρ, μ, λ, k_{tc} and c will depend just on the temperature θ.

With this in mind, for small temperature difference between the bottom and top of the layer of fluid, it follows that

$$\rho = \rho_0[1 - \alpha(\theta - \theta_0)], \tag{1.7}$$

where ρ_0 is the density of the fluid at the temperature θ_0 of the bottom of the layer and α is the constant coefficient of volumetric expansion. It is known from experiments that for a perfect gas, $\alpha \approx 3 \times 10^{-3} K^{-1}$ (K stands for degrees kelvin), and for a typical liquid $\alpha \approx 5 \times 10^{-4} K^{-1}$. If $\theta - \theta_0 \lesssim 10\ K$, then $\frac{\rho-\rho_0}{\rho_0} = \alpha(\theta - \theta_0) \ll 1$, but nevertheless the buoyancy $\boldsymbol{g}(\rho - \rho_0)$ is of the same order of magnitude as the inertia, acceleration or viscous stresses of the fluid and so is not negligible. For most real fluids, the variations of μ, k_{tc} and c with respect to the temperature θ is approximately less that α, so that, they will be considered as constants in the Boussinesq approximation. Realize that the coefficient of bulk viscosity λ is neglected because it only arises as a factor of div $\boldsymbol{u}$ which is of order α. Then, the Boussinesq approximation considers the thermodynamic variables as constants except the pressure and temperature, and except the density when is multiplied by $\boldsymbol{g}$.

The density fluctuations in the continuity equation (1.2) are of order α, so this approximation gives

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.8}$$

indicating that the fluid is incompressible. It follows from (1.4) that

$$\sigma = -\pi I_3 + 2\mu\mathbb{D}(\boldsymbol{u}). \tag{1.9}$$

Regarding $\rho = \rho_0$ in each term of the equations of motion, except in the buoyancy term which is given by (1.7), thanks to (1.8) and (1.9), the Navier-Stokes equations (1.3) become

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla\left(\frac{1}{\rho_0}\pi + \varphi\right) + \alpha(\theta - \theta_0)\boldsymbol{g} + \nu\Delta\boldsymbol{u}, \tag{1.10}$$

where $\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity, Δ is the Laplacian operator, and it is used the fact that the gravitational field is a conservative one, so there exists a scalar potential field φ such that $\boldsymbol{g} = \nabla\varphi$.

Now, as c and k_{tc} are constants, it is possible to take them outside the differentiation signs of the equation of heat conduction (1.5), and remembering that $\rho = \rho_0$ and by using (1.8), it follows that

$$\frac{D\theta}{Dt} = \kappa\Delta\theta + \frac{2\nu}{c}\Phi, \tag{1.11}$$

where $\kappa = \frac{k_{tc}}{c\rho_0}$ is the thermal diffusivity and

$$\Phi = \operatorname{Tr}(\mathbb{D}(\boldsymbol{u})^2).$$

Note that if U is a representative velocity scale of the flow, L a length scale and $\theta_0 - \theta_1$ a scale of temperature difference, then the ratio of the rate of production of heat by internal friction to the rate of transfer of heat is

$$\frac{\Phi}{\rho\frac{D(c\theta)}{Dt}} \approx \frac{\mu\frac{U^2}{L^2}}{\rho_0 c(\theta_0 - \theta_1)\frac{U}{L}} = \frac{\nu U}{c(\theta_0 - \theta_1)L},$$

where θ_1 stands for the temperature of the top of the layer of thickness L. From the experiments, it is known that $\frac{\nu}{c} \approx 10^{-8} Ks$ (s stands for seconds) for a typical gas and $\frac{\nu}{c} \approx 10^{-9} Ks$ for a typical liquid, which shows that the ratio is very small for both, gases and liquids. Therefore, under these situations, it is possible to neglect Φ. Finally, the heat equation (1.11) reduces to

$$\frac{D\theta}{Dt} = \kappa \Delta \theta. \tag{1.12}$$

Thus, the Boussinesq equations (1.10), (1.8) and (1.12) have been derived from the general hydrodynamical system. In summary, the Boussinesq approximation takes account the following simplified features which characterize the motion:

i. the motion is as if the fluid were incompressible, except that density changes are not ignored in the body-force terms of the momentum equations (the motion is driven by buoyancy);

ii. the density changes are induced by changes of temperature (and concentration), but not by pressure;

iii. the velocity gradients are sufficiently small so that the effect on the temperature of conversion of work to heat can be ignored;

iv. the dynamic viscosity μ, the thermal conductivity k_{tc} and the specific heat c are constants;

v. the equation of state $\rho = \rho(\theta)$ is given by (1.7).

There are many situations in which all the assumptions from above strongly characterize the flow. For instance, the emblematic natural convection phenomenon (see Figure 1.2)[3], known as Rayleigh – Bénard convection, satisfies all the conditions from (i) to (v). The mathematical explanation for this interesting physical phenomenon was given by the English physicist Lord Rayleigh (Figure 1.3)[4]. In fact, Rayleigh's paper [59] represents the starting point of many articles on thermal convection. According to Rayleigh, this phenomenon might have been first described by James Thomson in 1882[5], but the first quantitative experiments were made by the French physicist Henri Bénard in 1900 (Figure 1.4)[6]. In this way, Rayleigh wrote that Bénard worked with very thin layers of a liquid (several liquids were employed in the experiments), only about 1 mm deep, standing on a levelled metallic plate which was maintained at a uniform temperature.

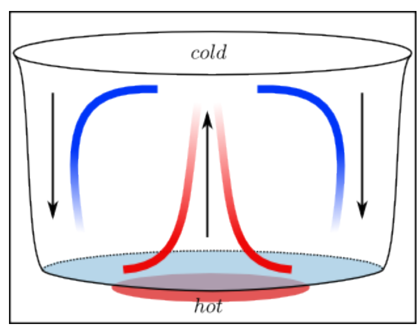

Figure 1.2: Pot filled with water. The bottom plate is heated, the top is cooled

The upper surface was usually free, and it was at a lower temperature because of its contact with the air. After a moment, a number of cells appeared in the liquid. Two phases

[3]Source: http://www.mis.mpg.de/applan/research/rayleigh.html. Visited on 07-15-2015.
[4]Source: http://www.potto.org/gasDynamics/node56.html. Visited on 07-15-2015.
[5]According to Wesfreid's paper [71], E.H. Weber had described polygonal structures in drop dissolutions in 1855. So, Weber could be the first one who described this type of geometries.
[6]Source: [71].

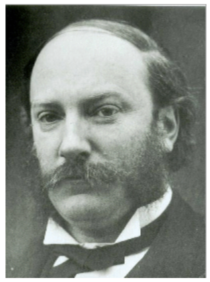

Figure 1.3: L. Rayleigh

Figure 1.4: H. Bénard

are distinguished, of different duration, the first being relatively very short. The limit of the first phase is described as the "semi-regular cellular regime"; in this state all the cells have already acquired surfaces nearly identical, their forms being nearly regular convex polygons of, in general, 4 to 7 sides (see Figure 1.5)[7].

The boundaries are vertical, and the circulation in each cell occurs with an ascension of the liquid in the middle of a cell and then the liquid descends at the common boundary between a cell and its neighbours. This phase is brief (1 or 2 seconds) for the less viscous liquids such as alcohol, benzine, etc., at ordinary temperatures. But in the case of very viscous liquids such as oils, if the flux of heat is small, the deformations are extremely slow and the first phase may last several minutes or more. The second phase has for its limit a permanent regime of regular hexagons. During this period the cells become equal, regular and align themselves.

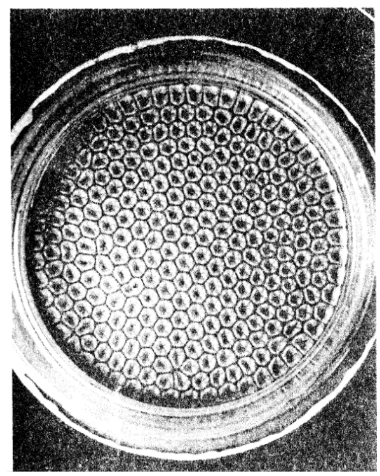

Figure 1.5: Bénard cells under an air surface

Encouraged by Bénard's experiments (see [71] to know about the scientific life of Henri Bénard), Rayleigh formulated the mathematical theory of convective instability of a layer of fluid between horizontal planes by using of the Boussinesq equations (see [25], [74], [15]). Then, he showed that instability would occur only when the temperature gradient was so large such that the dimensionless number $\frac{g\alpha\beta d^4}{\kappa\boldsymbol{u}}$ (nowadays called the *Rayleigh number*) exceeded a certain critical value. Here g is the value of the acceleration due to gravity, α the coefficient of thermal expansion of the fluid, β the magnitude of the vertical temperature gradient of the basic state of rest, d the depth of the layer of the fluid, κ its thermal diffusivity and ν its kinematic viscosity. Physically speaking, Rayleigh number measures the ratio of the destabilizing effect of buoyancy to the stabilizing effects of diffusion and dissipation.

It is worth to note that although Pearson [57] proved that most of the motions observed by Bénard were driven by the variation of surface tension with temperature and not by thermal instability of light fluid below heavy fluid, the convection in a horizontal layer of fluid heated from below is still called Bénard convection. However, Rayleigh's model is in accord with experiments on layers of fluid with rigid boundaries and on thicker layers with a free surface, because the variation of surface tension diminishes as the thickness of the layer increases.

In conclusion, the assumption (1.7) for the density requires that the maximum value of

[7]Source: [25]

$\alpha(\theta-\theta_0)$ is small compared to 1, that means, $\alpha(\theta-\theta_0)^* \ll 1$, where $(\theta-\theta_0)^*$ is the maximum value of $(\theta-\theta_0)$, see [31]. When this is true, it can be demonstrated that density variations due to changing temperature are negligible in both the continuity (1.2) and heat flow (1.5) equations, but dominant in the equation of motion (1.3). It is important to mention that the Boussinesq approximation is more appropriate for shallow layers of liquid (as in laboratory experiments) where hydrostatic compression is not important. However, for deep layers or for compressible fluids, equation (1.7) is not suitable. Thus, in addition to the requirement of small values of $\alpha(\theta-\theta_0)$, the Boussinesq approximation is only valid when the depth d of the convecting layer is small compared to the scale height H over which significant density variations occur, see [64], [24, p. 6] and [37, p. 135]. Further, more typically, assumptions (iv) and (v) do not hold. For example, the condition (v) is not fulfilled when treating convection near the critical point (4°C) at which the density of water has a relative maximum, see [33], because a nonlinear equation of state $\rho = \rho(\theta)$ arises. In very large scale systems (typical in geo-astrophysical applications) the variations of material properties cannot be neglected, then the condition (iv) is not fulfilled. Moreover, for flows in which the Mach number is sensibly different from zero (for example, in propagation of sound or shock waves), the fact that the velocity field is solenoidal and the independence of density on pressure are lost, and then conditions (i) and (ii) are not suitable for this kind of problems. Also, the Boussinesq approximation cannot be applied to high-speed gas flows where density variations induced by velocity divergence cannot be neglected. It is interesting to note that the Boussinesq equations are obtained as an asymptotic limit of the complete Navier-Stokes equations, see [46] for details.

Besides to model the thermal instability of fluids in hydrodynamics (formation of some patterns in the fluid when this is heated), there are other very important applications of the Boussinesq system such as in geophysics, modeling the convection in the earth's mantle (see [24] and Figure 1.6)[8], in magnetohydrodynamic flows (see [21]). It is commonly useful for analyzing oceanic and atmospheric flows (see [45], [61, Chapter 3], [53]). Also, some applications of the Boussinesq equations are the analysis of wave propagation in a density-stratified medium and turbulence in a stratified medium; for details of these applications review [37].

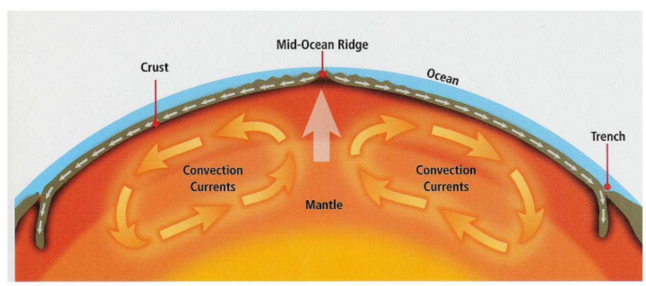

Figure 1.6: Convection currents in the earth's mantle move the tectonic plates and generate the earth's magnetic field

In the description of a physical phenomenon happening in a region of the space by means of differential equations, it is necessary to set boundary conditions in order to have proper solutions. Basically, in fluid mechanics boundary conditions can be regarded as coupling conditions between adjacent physical systems (for example, fluid-fluid or solid-fluid interactions). These interactions are fixed in order to satisfy some specific requirement or a specific situation of the phenomenon. These a priori known conditions are usually referred as "prescribed boundary conditions".

On a solid boundary or at the interface between two immiscible fluids, one boundary

[8]Source: http://mrrudgegeography.weebly.com/plate-tectonic-theory.html. Visited on 07-15-2015

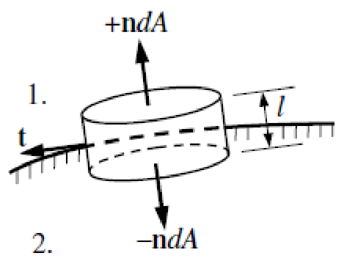

Figure 1.7: Interface between two mediums. Medium 1 is a fluid, and medium 2 is a solid or a second fluid that is immiscible with the first fluid

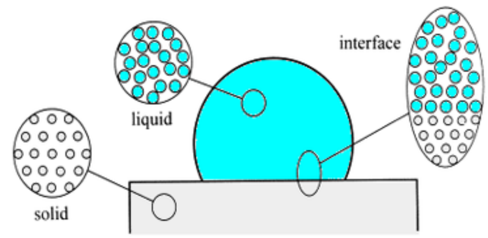

Figure 1.8: Liquid molecules tend to order near a solid-liquid interface

condition for the velocity may be deduced from the mass conservation law. Indeed, regarding a small cylindrical control volume through the interface which separates the two mediums, the velocity boundary condition can be determined by applying the equation of continuity in the small cylindrical control volume and then letting l (height of the cylindrical control volume) go to zero until the volume becomes the interface (see Figure 1.7)[9]. Then, considering this interface at rest, conservation of mass produces that $\rho_1 \boldsymbol{u}_1 \cdot \boldsymbol{n} = \rho_2 \boldsymbol{u}_2 \cdot \boldsymbol{n}$ at each point on the interface. Here ρ_i and $\boldsymbol{u}_i$ are the density and the velocity of the medium i, respectively; and $\boldsymbol{n}$ the normal unit vector. If medium 2 is a solid, then $\boldsymbol{u}_2 = \mathbf{0}$. If medium 1 and medium 2 are immiscible liquids, no mass flows across the boundary surface. Then, in either case, $\boldsymbol{u}_1 \cdot \boldsymbol{n} = 0$ on the boundary (that means, the boundary is impermeable).

One additional condition is needed to completely specify the problem and this is not consequence of any conservation law. This condition is the no-slip condition of a viscous fluid which is applicable at a solid boundary (see Figure 1.8)[10], and says that the tangential component of the velocity is null, i.e., $\boldsymbol{u}_1 \cdot \boldsymbol{\tau} = 0$ (or $\boldsymbol{u}_1 \times \boldsymbol{n} = 0$). Here $\boldsymbol{\tau}$ is a unit vector tangent to the boundary. Both conditions $\boldsymbol{u}_1 \cdot \boldsymbol{n} = 0$ and $\boldsymbol{u}_1 \cdot \boldsymbol{\tau} = 0$ (or $\boldsymbol{u}_1 \times \boldsymbol{n} = 0$) imply that $\boldsymbol{u}_1 = 0$ on the boundary. This condition is usually called *homogeneous Dirichlet boundary condition* or also, *no-slip condition*, and was suggested by the Irish mathematician and physicist George G. Stokes in 1843, see [65].

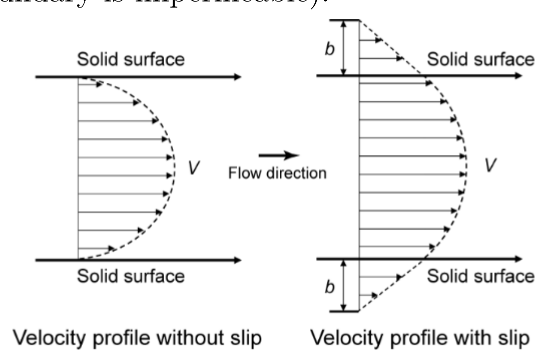

Figure 1.9: Velocity profiles of fluid flow without and with boundary slip. The degree of boundary slip at the solid–liquid interface is characterized by b (the slip length)

For many years, this boundary condition have been used in a lot of works, and sometimes it was chosen as a boundary condition just for routine. It was a difficult task to precise if the no-slip boundary condition is suitable for modeling certain phenomena. Nevertheless, in the following years, experimental evidence was in favor of the no-slip boundary condition for a

[9]Source: [37]
[10]Source: http://www.utwente.nl/tnw/pcf/education/masterprojects/ordering_of_molecules/. Visited 07-17-2015

large class of flows, and it became widely accepted for most liquid flows.

Relatively recent, it was showed in a rigourously way why a viscous fluid cannot slip on a wall covered by microscopic asperities (rugose wall), see [14], allowing the acceptance of the no-slip boundary condition. While this assumption has proved to be highly successful for a great variety of flow conditions, it has been found to be inadequate in certain situations such as in the mechanics of thin films, problems involving multiple interfaces, the flow of rarefied fluids, the flow of a liquid in a domain which has air as part of its boundary, the flow of a fluid in perforated domains, flow of blood through blood vessels (see [70]), the flow of a fluid regarding free boundary, etc. In this situation, the French engineering Claude Navier (see [51])[11], in 1823, proposed that the tangential velocity should be proportional to the tangential stress on the boundary, i.e., $2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}$, where $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$ is the deformation tensor (or linearized strain tensor) associated with the velocity field $\boldsymbol{u}$ and α is a friction function. The condition of impermeability of the boundary together with this last condition is known as the *Navier boundary condition* (see Figure 1.9)[12]. When α is a positive function, this condition is called a slip boundary condition with linear friction. Lately, the Navier boundary condition has raised its interest to the scientific community due to the interesting applications in modeling of physical phenomena such as in the examples mentioned before.

1.2 Thesis Description and Main Results

The work carried out in this thesis covers the research done along three years under a joint supervised doctoral thesis between the *University of Chile* and the *Université de Pau et des Pays de l'Adour*.

This work will be focusing in the L^p-theory for the stationary Boussinesq system (1.1) regarding two different types of boundary conditions for the velocity field: in the first chapter, we consider the Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{u_b} \quad \text{on} \quad \Gamma, \tag{1.13}$$

where Γ denotes the boundary of the domain; meanwhile in the second one, the velocity field will have attached the Navier boundary condition

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{a}, \quad \text{on} \quad \Gamma. \tag{1.14}$$

Further, the boundary condition for the temperature will be, in both chapters, the Dirichlet boundary condition

$$\theta = \theta_b \quad \text{on} \quad \Gamma. \tag{1.15}$$

Next sections are dedicated to describe the main results of this work, leaving all the details for the subsequent chapters.

[11] After almost half of a century, Maxwell [44] derived this condition from the kinetic theory of gases

[12] Source: http://www.beilstein-journals.org/bjnano/single/articleFullText.htm?publicId=2190-4286-2-9. Visited 07-17-2015

1.2.1 Boussinesq system with Dirichlet boundary conditions for the velocity field

In **chapter 2**, the stationary Boussinesq system (1.1) is studied with the boundary conditions (1.13) and (1.15). The aim is to develop the L^p-theory for this problem, meaning with L^p-theory, the study of the existence of generalized solutions in $\boldsymbol{W}^{1,p}$, strong solutions in $\boldsymbol{W}^{2,p}$ and very weak solutions in $\boldsymbol{L}^p$.

This chapter has seven sections. In the **first section** we describe the problems under consideration and related literature. The **second section** is dedicated to summarize the main results of this chapter. The **third section** will be focusing on standardizing the notation to be used along the chapter and it will be given some useful statements which will play an important role in the proof of the main results. The **fourth section** will deal with the existence of weak solutions for (1.1)-(1.13)-(1.15) in the Hilbert case. This result, whose proof is based on applying the Leray-Schauder fixed point theorem, is established as follows.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with Lipschitz boundary Γ and let

$$\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega), \quad h \in H^{-1}(\Omega), \quad \boldsymbol{u_b} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \quad \theta_b \in H^{\frac{1}{2}}(\Gamma) \tag{1.16}$$

such that $\int_\Gamma \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = 0$. There exists $\delta_1 = \delta_1(\Omega) > 0$ such that if

$$\frac{1}{\nu} \sum_{i=1}^{m} \left| \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s \right| \leq \delta_1,$$

then problem (1.1)-(1.13)-(1.15) has at least one weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times H^1(\Omega)$. Further, if $\boldsymbol{u_b} = \boldsymbol{0}$ and $\theta_b = 0$, then the weak solution $(\boldsymbol{u}, \theta)$ satisfies the following estimates:

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C}{\nu\kappa} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|h\|_{H^{-1}(\Omega)},$$
$$\|\nabla \theta\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C}{\kappa} \|h\|_{H^{-1}(\Omega)},$$

with $C = C(\Omega) > 0$.

The **fifth section** is concerned with the L^p-regularity results for the Hilbertian weak solution. They are proved by using the regularity of the Poisson and the Stokes equations, and a suitable bootstrap argument. In this way, let $\Omega \subset \mathbb{R}^3$ be more regular than before (of class $\mathcal{C}^{1,1}$). It is supposed that

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega), \quad h \in W^{-1,p}(\Omega) \quad \text{and} \quad (\boldsymbol{u_b}, \theta_b) \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \times W^{1-\frac{1}{p},p}(\Gamma)$$

with $p > 2$, $r = \max\left\{\frac{3}{2}, \frac{3p}{3+p}\right\}$ if $p \neq 3$ and $r = \frac{3}{2} + \varepsilon$ if $p = 3$ for any fixed $0 < \varepsilon < \frac{1}{2}$. Then the weak solution in $\boldsymbol{H}^1(\Omega)$ for the Boussinesq system satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R} \times W^{1,p}(\Omega).$$

Moreover, if

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega), \quad h \in L^p(\Omega) \quad \text{and} \quad (\boldsymbol{u_b}, \theta_b) \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma) \times W^{2-\frac{1}{p},p}(\Gamma)$$

with $p \geq \frac{6}{5}$, $r = \max\left\{\frac{3}{2}, p\right\}$ if $p \neq \frac{3}{2}$ and $r = \frac{3}{2} + \varepsilon$ if $p = \frac{3}{2}$ for any fixed $0 < \varepsilon < \frac{1}{2}$. Then the weak solution in $\boldsymbol{H}^1(\Omega)$ for the Boussinesq system satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R} \times W^{2,p}(\Omega).$$

It is observed that the choice of the space for $\boldsymbol{g}$ is optimal to study $W^{1,p}(\Omega)$-regularity with $p > 2$ and $W^{2,p}(\Omega)$-regularity with $p \geq \frac{6}{5}$.

The **sixth section** deals with the existence and uniqueness of the very weak solution for the Boussinesq system (the definition is given in Section 2.6). So, if $\Omega \subset \mathbb{R}^3$ is of class $\mathcal{C}^{1,1}$, and letting $3 \leq p, r < \infty$,

$$\boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \quad \text{with} \quad < \boldsymbol{u_b} \cdot \boldsymbol{n}, 1 >_\Gamma = 0, \quad \boldsymbol{g} \in \boldsymbol{L}^q(\Omega),$$

$$h \in W^{-1,\frac{pr}{p+r}}(\Omega), \quad \theta_b \in W^{-\frac{1}{r},r}(\Gamma),$$

where $q = \max\left\{s, \frac{3}{2} + \varepsilon\right\}$ for any fixed $0 < \varepsilon < \frac{1}{2}$ and s given by

$$s > r' \quad \text{if} \quad p = 3, \quad \text{and} \quad s = \frac{3rp}{2rp + 3(r-p)} \quad \text{if} \quad p > 3 \quad \text{and} \quad \frac{1}{r} \leq \frac{1}{p} + \frac{2}{3},$$

where, in the case $p > 3$, s is chosen such that

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{(p')^{**}} = 1 \quad \text{with} \quad \frac{1}{(p')^{**}} = \frac{1}{p'} - \frac{2}{3}.$$

Then, there exists $\delta_2 = \delta_2(\Omega) > 0$ such that if

$$\frac{1}{\nu} \sum_{i=1}^{m} \left| \langle \boldsymbol{u_b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} \right| \leq \delta_2,$$

then there exists at least one very weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R} \times L^r(\Omega)$ of (1.1)-(1.13)-(1.15). Further, if $\boldsymbol{g} \in \boldsymbol{L}^t(\Omega)$, where $t = \max\{s, 2\}$, and if

$$\left(\frac{1}{\nu} + \frac{1}{\kappa}\right) \left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu} + \frac{1}{\kappa}\right) \|\boldsymbol{g}\|_{\boldsymbol{L}^t(\Omega)} \left(\frac{1}{\kappa} \|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)} \right) \right) \leq \delta_3$$

for some $\delta_3 = \delta_3(p, r, t, \Omega) > 0$, then this solution is unique.

This last result tells that it is possible to show the existence of very weak solutions for the Boussinesq system, if we just consider smallness of the fluxes of $\boldsymbol{u_b}$ through each connected component Γ_i of the boundary Γ, as in the case of Navier-Stokes equations, see Amrouche - Rodríguez [6] or in the linear Stokes case, see Conca [19].

Unlike the result showed in Kim [35], two important facts are taken into account. First a more general type of domain regarding a possible disconnected boundary is considered allowing in this way other kind of domains to analyze this system of equations. And second, the space where $\boldsymbol{g}$ lies is enlarged and hence the hypothesis $\boldsymbol{L}^\infty(\Omega)$ can be weakened and results for very weak solutions for the Boussinesq system can still be obtained.

Furthermore, it is possible to extend the regularity of the solution of (1.1)-(1.13)-(1.15) for $\frac{3}{2} \leq p < 2$ in $W^{1,p}(\Omega)$ and for $1 < p < \frac{6}{5}$ in $W^{2,p}(\Omega)$, by means of using some uniqueness and regularity results for the Oseen problem. Finally, **seventh section** is devoted to show some $\boldsymbol{H}^1(\Omega)$-estimates for the weak solution, and this serves as a tool to show the uniqueness of this solution.

1.2.2 Boussinesq system with Navier boundary conditions for the velocity field

Chapter 3 is concerned with the study of the stationary Boussinesq system (1.1) with the boundary conditions (1.14) and (1.15). The idea is to study the existence of weak solutions in the Hilbert case $\boldsymbol{H}^1(\Omega)$, generalized solutions in $\boldsymbol{W}^{1,p}(\Omega)$ and strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$.

Before starting the description of the structure of the next two chapters, we consider the following hypothesis for the friction function α which will be used along the next sections: there exists a real number α_* such that

$$\begin{cases} \alpha \geq \alpha_* \geq 0 & \text{with} \\ \alpha_* \geq 0 & \text{if } \Omega \text{ is not axisymmetric, or} \\ \alpha_* > 0 & \text{(or even, } \alpha(x) > 0 \text{ a.e. } x \in \Gamma\text{) otherwise.} \end{cases} \tag{H}$$

This chapter is composed by five sections. Similarly as in Chapter 1, **first, second and third sections** are concerned with the introduction of the Boussinesq problem, a review of some literature related to this problem, the main results of this chapter, the standardization of the notation to be used along this chapter and the presentation of some useful assertions which will play an important role in the proof of the main results. It is noteworthy that one of this important assertions is a *Korn-type inequality*. This type of inequality is very useful in problems where the deformation tensor appears. The **fourth section** is dedicated to the existence of weak solutions for (1.1)-(1.14)-(1.15) in the Hilbert case. Indeed, let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{2,1}$ and let

$$\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega),\ h \in H^{-1}(\Omega),\ \alpha \in L^{2+\varepsilon}(\Gamma) \text{ satisfying (H) for any } \varepsilon > 0 \text{ sufficiently small,}$$

$$\boldsymbol{a} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ such that } \boldsymbol{a}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,\ \theta_b \in H^{\frac{1}{2}}(\Gamma).$$

Then, problem (1.1)-(1.14)-(1.15) has a weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times H^1(\Omega)$. Further, if $\theta_b = 0$ on Γ, $\boldsymbol{u}$ and θ satisfy the following estimates:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{M_1}{\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right),$$

$$\|\nabla\theta\|_{L^2(\Omega)} \leq \frac{M_1}{\kappa}\|h\|_{H^{-1}(\Omega)},$$

with $M_1 = M_1(\Omega) > 0$ independent of α. Moreover, if there exists $\gamma = \gamma(\Omega) > 0$ such that

$$\nu\kappa \geq \gamma\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)},$$

then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{M_2}{\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\theta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right),$$

$$\|\theta\|_{H^1(\Omega)} \leq M_2\left[\left(1 + \frac{1}{\kappa}\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right)\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} + \frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)}\right],$$

with $M_2 = M_2(\Omega, \gamma) > 0$ independent of α. The proof of this result is based on applying the Leray-Schauder fixed point theorem.

Finally, in the **fifth section**, a study of the $\boldsymbol{W}^{1,p}$ and $\boldsymbol{W}^{2,p}$ regularity of the weak solutions for the problem (1.1)-(1.14)-(1.15) will be carried out. The proof is done by taking advantage of the regularity results for the Poisson equation with Dirichlet boundary condition and Stokes problem with Navier boundary condition. Indeed, supposing that

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega), \quad h \in W^{-1,p}(\Omega), \quad \alpha \in L^{t^*(p)}(\Gamma) \quad \text{satisfying} \quad (\mathrm{H})$$

$$\text{with } t^*(p) \text{ defined by } (4.14) \quad \text{and} \quad (\boldsymbol{a}, \theta_b) \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times W^{1-\frac{1}{p},p}(\Gamma)$$

with

$$p > 2, \quad r = \max\left\{\frac{3}{2}, \frac{3p}{3+p}\right\} \quad \text{if} \quad p \neq 3 \quad \text{and} \quad r = \frac{3}{2} + \varepsilon \quad \text{if} \quad p = 3$$

for any $\varepsilon > 0$ sufficiently small. Then the weak solution for (1.1)-(1.14)-(1.15) satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R} \times W^{1,p}(\Omega).$$

Moreover, if $\boldsymbol{g} \in \boldsymbol{L}^r(\Omega)$, $h \in L^p(\Omega)$,

$$\alpha \in H^{\frac{1}{2}}(\Gamma) \quad \text{if} \quad \frac{6}{5} \le p \le 2; \quad \alpha \in H^{\frac{1}{2}+\varepsilon}(\Gamma) \quad \text{if} \quad 2 < p < 3; \quad \alpha \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{if} \quad p \ge 3$$

satisfying (H) and

$$(\boldsymbol{a}, \theta_b) \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \times W^{2-\frac{1}{p},p}(\Gamma)$$

with

$$p \ge \frac{6}{5}, \quad r = \max\left\{\frac{3}{2}, p\right\} \quad \text{if} \quad p \neq \frac{3}{2} \quad \text{and} \quad r = \frac{3}{2} + \varepsilon \quad \text{if} \quad p = \frac{3}{2}$$

for any $\varepsilon > 0$ sufficiently small. Then the solution for (1.1)-(1.14)-(1.15) satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R} \times W^{2,p}(\Omega).$$

1.2.3 Stokes equations with Navier boundary condition

In **Chapter 4**, we deal with the study of the stationary Stokes equations

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g & \text{on } \Gamma, \\ 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain of class $\mathcal{C}^{1,1}$, Γ is the boundary of Ω, $\boldsymbol{u}$ and π are the velocity and pressure of the fluid, respectively, $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$ is the strain tensor associated with the velocity field $\boldsymbol{u}$, $\boldsymbol{n}$ is the unit outward normal vector, $\boldsymbol{\tau}$ is the corresponding unit tangent vector, $\boldsymbol{f}$ is an external force acting on the fluid, χ and g stand for the compressibility and permeability conditions, respectively, α is a friction scalar function and $\boldsymbol{h}$ is a tangential vector field on the boundary. In the case $\alpha > 0$, the Navier boundary condition is said to be a boundary condition with linear friction. In this chapter α will satisfy (H).

We are interested in the study of the existence of a unique weak solution $\boldsymbol{H}^1(\Omega)$, a unique generalized solution in $\boldsymbol{W}^{1,p}(\Omega)$ and a unique strong solution in $\boldsymbol{W}^{2,p}(\Omega)$.

In order to attain these results, we organize the chapter as follows: in the **first section** we introduce the problem to be considered. **Second section** provides a summary of the main results of this chapter. The notation to be used along this chapter and the presentation of some useful results will be treated in the **third section**. The **fourth section** is dedicated to prove the existence of a unique weak solution for the Stokes problem in $\boldsymbol{H}^1(\Omega)$. Indeed, let Ω be a bounded domain of class $\mathcal{C}^{1,1}$ and let us suppose $\chi = 0$ and $g = 0$. Also, let

$$\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega), \quad \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ such that } \boldsymbol{h}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \quad \alpha \in L^2(\Gamma)$$

with α verifying the hypothesis (H). Then the Stokes problem has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ which satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega, \alpha_*) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

where

$$C(\Omega, \alpha_*) = \begin{cases} C(\Omega) & \text{under the hypothesis } (H_1) \\ \frac{C(\Omega)}{\min\{1, \alpha_*\}} & \text{under the hypothesis } (H_2) \text{ with } \alpha \geq \alpha_* > 0 \\ C(\Omega) & \text{under the hypothesis } (H_2) \text{ with } \alpha(x) > 0 \text{ a.e. } x \in \Gamma. \end{cases}$$

We use the very useful Lax-Milgram theorem to prove the previous assertion.

Finally, in the **fifth section**, a study of the $\boldsymbol{W}^{1,p}(\Omega)$ for $p > 2$ and $\boldsymbol{W}^{2,p}(\Omega)$ for $p \geq \frac{6}{5}$ is carried out. In support of this, let us suppose $\chi = 0$, $g = 0$,

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega) \text{ with } r(p) \text{ is defined by (4.1)}, \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ such that } \boldsymbol{h}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,$$

and $\alpha \in L^{t^*(p)}(\Gamma)$ satisfying (H) with

$$t^*(p) = \begin{cases} 2 + \varepsilon & \text{if} \quad 2 < p \leq 3, \\ \frac{2}{3}\, p + \varepsilon & \text{if} \quad p > 3, \end{cases}$$

where $\varepsilon > 0$ is an arbitrary number sufficiently small. Then the Stokes problem has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Furthermore, let us suppose

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ such that } \boldsymbol{h}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,$$

and

$$\alpha \in H^{\frac{1}{2}}(\Gamma) \quad \text{if} \quad \frac{6}{5} \leq p \leq 2; \quad \alpha \in H^{\frac{1}{2}+\varepsilon}(\Gamma) \quad \text{if} \quad 2 < p < 3; \quad \alpha \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{if} \quad p \geq 3$$

where $\varepsilon > 0$ is an arbitrary number sufficiently small and α satisfies (H). Then the weak solution for the Stokes problem satisfies that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$.

The key of the proof of this result is to take advantage of the regularity results for the Stokes problem with Navier boundary condition when $\alpha = 0$.

Chapter 2

Boussinesq system with Dirichlet boundary conditions

Abstract

In this chapter we consider the stationary Boussinesq system with non-homogeneous Dirichlet boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ with a possibly disconnected boundary. Assuming that the fluxes of the velocity across each connected component of the boundary are sufficiently small, we prove the existence of weak, strong and very weak solutions of the stationary Boussinesq system in L^p-theory. As it is expected, we obtain the uniqueness of the solution by considering small data.

Keywords: Boussinesq system, natural convection, non-homogeneous Dirichlet boundary conditions, existence, uniqueness, L^p-regularity, weak solution, strong solution, very weak solution

2.1 Introduction

The work developed in this chapter is concerned with the existence, uniqueness and regularity of the solution for the stationary Boussinesq system with Dirichlet non-homogeneous boundary conditions. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{1,1}$. Consider the stationary Boussinesq system as follows:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ -\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u_b}\ ,\ \theta = \theta_b & \text{on } \Gamma, \end{cases} \tag{BS}$$

where Γ is the boundary of Ω which is not necessarily connected, i.e., it could be the disjoint union of its connected components Γ_i, with $i = 0, 1, \ldots, m$. In this context, Γ_0 will represent the exterior boundary which contains Ω and all the boundaries Γ_j, $j = 1, \ldots, m$. The unknowns are $\boldsymbol{u}$, π and θ which represent the velocity field, the pressure and the temperature of the fluid, respectively. The data are $\nu > 0$ the kinematic viscosity of the fluid, $\kappa > 0$ the thermal diffusivity of the fluid, $\boldsymbol{g}$ the gravitational acceleration, h a heat source applied on the fluid, $\boldsymbol{u_b}$ the velocity at the boundary and θ_b the temperature at the boundary. As we can see, this system of partial differential equations is formed by coupling the stationary Navier-Stokes equations and the stationary convection-diffusion equation for heat transfer.

Keep in mind the following facts concerning $\boldsymbol{g}$. Along this chapter, we will consider a non-zero gravitational acceleration $\boldsymbol{g}$. This assumption is not a strong or an unusual constraint over $\boldsymbol{g}$, on the contrary, in all the physical phenomena which are carried out into a gravitational field (like the Earth's gravitational field), gravity plays an important role in the development of such phenomena. In particular, in the Boussinesq approximation, the gravitational acceleration is the keystone of this formulation. Mathematically, if we consider $\boldsymbol{g} = \mathbf{0}$, then the Navier-Stokes equations and the convection-diffusion equation are decoupled, hence we loose the essential aspect of the Boussinesq system and we end by analyzing a different physical problem. Further, it is noteworthy that in practical cases, the gravitational acceleration $\boldsymbol{g}$, in fact, belongs to $\boldsymbol{L}^{\infty}(\Omega)$, but we consider important to relax this assumption for mathematical purposes. This means we can enlarge the space where $\boldsymbol{g}$ lies and we can still get solutions for our problem.

The Boussinesq system is considered as a good approximation to model the natural (or free) convection phenomenon. This physical phenomenon is a way of heat transfer which is carried out by the motion of the fluid without using external devices to produce that motion (forced convection), but only by the density differences resulting from temperature gradients within the fluid. We have an emblematic example of this phenomenon when we heat a pot of water. Indeed, when we start heating the pot, the water at the bottom of the pot begins to be heated firstly. This produces a reduction in the density of this part of the water and consequently, it rises to the surface. On the other hand, the water at the top of the pot is colder than the one at the bottom, consequently, its density is higher and hence, it descends to the bottom of the pot. This process is repeated again and again, generating circular currents in the water, known as convection currents, causing all the water moves inside the pot and with this, the water is completely heated.

The Boussinesq approximation was proposed by the French mathematician and physicist Joseph Boussinesq at the beginning of the twentieth century (1903) on his monograph [13]. From that time until our days, there have been many works related with this system, showing us the importance that it has had in theoretical and applied mathematics, physics, oceanography and many other sciences. Apart from Joseph Boussinesq's monograph, we have some references which address the mathematical deduction of the Boussinesq system, see, for example, [15, 10]. Also, there is an interesting work concerning the life of Joseph Boussinesq, the idea that led him to deduce this system and the applicability of his approximation in various physical problems, see [73].

What do we know about the solvability of the Boussinesq system when our domain has a non-connected boundary? Let us see some details. From the continuity equation we obtain the following necessary compatibility condition for the boundary data $\boldsymbol{u_b}$, in order to solve the Boussinesq system:

$$\int_{\Gamma} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = \sum_{i=0}^{m} \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = 0, \tag{2.1}$$

where $\boldsymbol{n}$ denotes the outward unit normal vector to Γ. It is important to mention that the existence of solutions for the Navier-Stokes equations and, consequently, for the Boussinesq system, considering merely the condition (2.1), is an open problem yet. In fact, during a long time the existence of weak solutions for the Navier-Stokes equations was proven under the following condition:

$$\int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = 0 \quad \text{for all} \quad i = 0, \dots, m, \tag{2.2}$$

see, for example, [40, 39, 20]. Clearly, condition (2.2) is stronger than (2.1) and, further, (2.2) does not allow the presence of sinks and sources along the boundary.

Afterwards, it was possible to prove existence of weak solutions for the Navier-Stokes equations weakening the condition (2.2) by assuming only smallness of these fluxes, see [12, 36, 27]. However, there are some special cases where the existence of weak solutions for the Navier-Stokes equations is known just considering the condition (2.1) without any information about the size of the fluxes across each connected component of the boundary. These special cases consider some symmetry hypothesis on the domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, and on the velocity boundary data $\boldsymbol{u_b}$, see [3, 49, 58, 26].

In connection with the stationary Boussinesq system, Morimoto [50] recently proved the existence of weak solutions considering the condition (2.2) and also proved the existence of weak solutions in the case of a symmetric planar domain and symmetric boundary data $\boldsymbol{u_b}$ considering just the condition (2.1).

There are other related works concerning the stationary Boussinesq system as for example some contributions done by Morimoto. Morimoto [47] studied the existence of weak solutions and their interior regularity (C^∞ regularity), in the case of a bounded domain Ω in $\mathbb{R}^3$ with C^2 connected boundary Γ which is divided in two disjoint subsets Γ_1 and Γ_2. The boundary conditions considered were Dirichlet homogeneous on the velocity and mixed on the temperature (Dirichlet non-homogeneous condition on Γ_1 and Neumann homogeneous condition on Γ_2). In a next work, assuming now that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with C^1 connected boundary Γ whose structure is the same as it was considered in [47], Morimoto extended his previous result on existence of weak solutions considering, this time, Neumann non-homogeneous condition on Γ_2. This time, he gave a result of uniqueness by imposing smallness condition on the norm of the solution, see [48]. In both articles, Morimoto took the gravitational acceleration $\boldsymbol{g}$ in $\boldsymbol{L}^\infty(\Omega)$ and did not consider a heat source in the convection-diffusion equation.

Assuming that Ω is a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, with Lipschitz connected boundary Γ, and considering the same structure of the boundary as in the articles of Morimoto [47, 48], Bernardi, Métivet and Pernaud-Thomas [10] proved existence and uniqueness of the weak solution for the Boussinesq system, considering Dirichlet non-homogeneous boundary condition for the velocity field, extending, in this manner, the results given by Morimoto. They considered, on the right hand side of the Navier-Stokes equations, a function which depends on the temperature and is continuously differentiable with bounded derivative and, also, they included a heat source in the convection-diffusion equation. It is interesting to note that for proving the existence of weak solutions, they used an important tool from nonlinear analysis: the topological degree theory. As in the work [48], they showed uniqueness if the norm of the solution is sufficiently small.

Let us see that there are other works about weak solutions for the stationary Boussinesq system which address other types of interesting problems. For instance, Gil' [28] studied the existence of weak solutions for a stationary Boussinesq system which appears in heat-mass transfer theory, in which, apart from consider the velocity and the temperature of the fluid, it takes account the concentration of material in a liquid. Kuraev [38] studied the existence of weak solutions considering a nonlinear boundary condition for the temperature in one part of the boundary. Also, we can find some works concerning the study of weak solutions for this system in exterior domains, see, for example, [56, 52].

As in the case of Stokes equations and Navier-Stokes equations, see [19, 6], concerning the work with very weak solutions for the Boussinesq system we refer two articles. Santos da

Rocha, Rojas-Medar M. A. and Rojas-Medar M. D. [60] studied the existence and uniqueness of the very weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^3(\Omega) \times W^{-1,3}(\Omega) \times L^2(\Omega)$ for the stationary Boussinesq system with Dirichlet non-homogeneous boundary conditions in $L^2(\Gamma)$ in a bounded domain of $\mathbb{R}^3$ with smooth enough connected boundary. Further, they considered the gravitational acceleration $\boldsymbol{g} \in \boldsymbol{L}^3(\Omega)$ and did not consider a heat source in the convection-diffusion equation. The proof of the existence was based on the Leray-Schauder fixed point theorem and the uniqueness was showed for sufficiently large viscosity. Recently, Kim [35] studied the existence and uniqueness of the very weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^q(\Omega) \times W^{-1,q}(\Omega) \times L^r(\Omega)$ of the stationary Boussinesq system with Dirichlet non-homogeneous boundary conditions in $\boldsymbol{W}^{-\frac{1}{q},q}(\Gamma) \times W^{-\frac{1}{r},r}(\Gamma)$ in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with connected boundary Γ of class C^2. He considered the gravitational acceleration $\boldsymbol{g} \in \boldsymbol{L}^\infty(\Omega)$ and did not consider a heat source in the convection-diffusion equation. This article is taken as a base for this work.

In our work we are focused in showing existence, uniqueness and L^p regularity of the weak solution for the stationary Boussinesq system with Dirichlet non-homogeneous boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ of class $\mathcal{C}^{1,1}$ which is not necessarily connected, and also, we consider the gravitational acceleration $\boldsymbol{g}$ in a weaker space than $\boldsymbol{L}^\infty(\Omega)$ and the presence of a heat source in the convection-diffusion equation. We prove existence of weak solutions in $H^1(\Omega)$ just considering smallness of the fluxes of $\boldsymbol{u_b}$ across each connected component Γ_i of the boundary Γ by applying a Leray-Schauder fixed point argument, and to prove uniqueness, we consider smallness condition on the norm of the data. In order to prove the regularity of the weak solution in $W^{1,p}(\Omega)$ with $p > 2$, and $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$, we use the regularity results for the Stokes and Poisson equations combining them with a bootstrap argument. For the regularity in $W^{1,p}(\Omega)$ and $W^{2,p}(\Omega)$ with $\frac{3}{2} \leq p < 2$ and $1 < p < \frac{6}{5}$, respectively, first we need to study the existence of very weak solutions for this system. With this result and using the existence and uniqueness of solutions for the Oseen problem, we can establish the desired regularity of the solution.

The work is organized as follows: in section 2, we describe the main results of this work. Section 3 is devoted to introduce some notations and to precise some useful results. In section 4, we study the existence of weak solutions of the stationary Boussinesq system. The regularity of the weak solution in $W^{1,p}(\Omega)$ for $p > 2$ and in $W^{2,p}(\Omega)$ for $p \geq \frac{6}{5}$ is dealt in section 5. Later, in section 6 we deal with the study of very weak solutions and then, we can prove the regularity of the solution in $W^{1,p}(\Omega)$ for $\frac{3}{2} \leq p < 2$ and in $W^{2,p}(\Omega)$ for $1 < p < \frac{6}{5}$. Finally, thanks to the study done in section 6, we can derive estimates for the weak solution in $H^1(\Omega)$ and consequently, we obtain the uniqueness of such solution. This is derived in section 7.

2.2 Main results

In this section, we summarize the main results of this chapter. The first theorem is concerned with the existence of weak solutions for the Boussinesq system. As you will realize, we just consider smallness of the fluxes of $\boldsymbol{u_b}$ across each connected component Γ_i of the boundary Γ to obtain the existence.

Theorem 2.2.1 (weak solutions of the Boussinesq system in $H^1(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ and let*

$$\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega), \quad h \in H^{-1}(\Omega), \quad \boldsymbol{u_b} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \quad \theta_b \in H^{\frac{1}{2}}(\Gamma)$$

such that $\int_\Gamma \boldsymbol{u_b} \cdot \boldsymbol{n}\, \mathrm{ds} = 0$. *There exists* $\delta_1 = \delta_1(\Omega) > 0$ *such that if*

$$\frac{1}{\nu} \sum_{i=1}^{m} \left| \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n}\, \mathrm{ds} \right| \leq \delta_1,$$

then the problem (BS) *has at least one weak solution* $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$. *Further, if* $\boldsymbol{u_b} = \boldsymbol{0}$ *and* $\theta_b = 0$, *then the weak solution* $(\boldsymbol{u}, \theta)$ *satisfies the following estimates:*

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C}{\nu\kappa} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|h\|_{H^{-1}(\Omega)},$$
$$\|\nabla \theta\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C}{\kappa} \|h\|_{H^{-1}(\Omega)},$$

with $C = C(\Omega) > 0$.

After the study of weak solutions in the case of Hilbert spaces, we are interested in study generalized and strong solutions in L^p-theory. In fact, the next two theorems deal with the L^p regularity of the weak solution of the Boussinesq system. In order to get these results, we use a classical bootstrap argument using regularity results of the Poisson and Stokes equations. Notice that for regularity in $W^{1,p}(\Omega)$, we begin by considering $p > 2$, and for regularity in $W^{2,p}(\Omega)$, we begin by considering $p \geq \frac{6}{5}$. For the cases $\frac{3}{2} \leq p < 2$ in $W^{1,p}(\Omega)$ and $1 < p < \frac{6}{5}$ in $W^{2,p}(\Omega)$, we will precise them later.

Theorem 2.2.2 (generalized solutions in $W^{1,p}(\Omega)$ with $p > 2$)**.** *Let*

$$h \in W^{-1,p}(\Omega) \quad \textit{and} \quad (\boldsymbol{u_b}, \theta_b) \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \times W^{1-\frac{1}{p},p}(\Gamma).$$

Let us suppose that

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega) \quad \textit{with} \quad r = \max\left\{\frac{3}{2}, \frac{3p}{3+p}\right\} \quad \textit{if} \quad p \neq 3, \quad \textit{and} \quad r = \frac{3}{2} + \varepsilon \quad \textit{if} \quad p = 3$$

for any fixed $0 < \varepsilon < \frac{1}{2}$. *Then the weak solution for the Boussinesq system given by Theorem* 2.2.1 *satisfies*

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega) \times W^{1,p}(\Omega).$$

Theorem 2.2.3 (strong solutions in $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$)**.** *Let*

$$h \in L^p(\Omega) \quad \textit{and} \quad (\boldsymbol{u_b}, \theta_b) \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma) \times W^{2-\frac{1}{p},p}(\Gamma).$$

Let us suppose that

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega) \quad \textit{with} \quad r = \max\left\{\frac{3}{2}, p\right\} \quad \textit{if} \quad p \neq \frac{3}{2}, \quad \textit{and} \quad r = \frac{3}{2} + \varepsilon \quad \textit{if} \quad p = \frac{3}{2}$$

for any fixed $0 < \varepsilon < \frac{1}{2}$. *Then the solution for the Boussinesq system given by Theorem* 2.2.1 *satisfies*

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega) \times W^{2,p}(\Omega).$$

The next theorem is concerned with very weak solutions of the Boussinesq system. As in the case of weak solutions, in order to show existence of very weak solutions, we just consider smallness of the fluxes of $\boldsymbol{u_b}$ through each connected component Γ_i of the boundary Γ. For proving uniqueness, we impose smallness condition on the norm of the data.

Theorem 2.2.4 (very weak solutions of the Boussinesq system). *Let $3 \le p, r < \infty$,*

$$\boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \quad \textit{with} \quad < \boldsymbol{u_b} \cdot \boldsymbol{n}, 1 >_{\Gamma} = 0, \quad \boldsymbol{g} \in \boldsymbol{L}^q(\Omega),$$

$$h \in W^{-1,\frac{pr}{p+r}}(\Omega), \quad \theta_b \in W^{-\frac{1}{r},r}(\Gamma),$$

where $q = \max\left\{s, \frac{3}{2}+\varepsilon\right\}$ for any fixed $0 < \varepsilon < \frac{1}{2}$ and s given by

$$s > r' \quad \textit{if} \quad p = 3, \quad \textit{and} \quad s = \frac{3rp}{2rp + 3(r-p)} \quad \textit{if} \quad p > 3 \quad \textit{and} \quad \frac{1}{r} \le \frac{1}{p} + \frac{2}{3}.$$

There exists $\delta_2 = \delta_2(\Omega) > 0$ such that if

$$\frac{1}{\nu}\sum_{i=1}^{m} \left|\langle \boldsymbol{u_b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}\right| \le \delta_2, \tag{2.3}$$

then there exists at least one very weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$ of (BS). Further, if $\boldsymbol{g} \in \boldsymbol{L}^t(\Omega)$, where $t = \max\{s, 2\}$, and if

$$\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^t(\Omega)}\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right) \le \delta_3$$

for some $\delta_3 = \delta_3(p, r, t, \Omega) > 0$, then this solution is unique.

The following two theorems deal with the regularity of the solution of the Boussinesq system in $W^{1,p}(\Omega)$ with $\frac{3}{2} \le p < 2$ and in $W^{2,p}(\Omega)$ with $1 < p < \frac{6}{5}$.

Theorem 2.2.5 (regularity $W^{1,p}(\Omega)$ with $\frac{3}{2} \le p < 2$). *Let us suppose that*

$$\boldsymbol{u_b} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \ \textit{satisfies (2.3)}, \ \theta_b \in W^{1-\frac{1}{p},p}(\Gamma), \ h \in W^{-1,p}(\Omega) \ \textit{and} \ \boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$$

for any fixed $0 < \varepsilon < \frac{1}{2}$. Then the very weak solution for the Boussinesq system given by Theorem 2.2.4 satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega) \times W^{1,p}(\Omega).$$

Theorem 2.2.6 (regularity $W^{2,p}(\Omega)$ with $1 < p < \frac{6}{5}$). *Let us suppose that*

$$\boldsymbol{u_b} \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma) \ \textit{satisfies (2.3)}, \ \theta_b \in W^{2-\frac{1}{p},p}(\Gamma), \ h \in L^p(\Omega) \ \textit{and} \ \boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$$

for any fixed $0 < \varepsilon < \frac{1}{2}$. Then the very weak solution for the Boussinesq system given by Theorem 2.2.4 satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega) \times W^{2,p}(\Omega).$$

Finally, the next result shows the estimates for the weak solution in $H^1(\Omega)$ and the uniqueness of such solution.

Theorem 2.2.7 (H^1-estimates for the weak solution and uniqueness)**.** *Let*

$$\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega), \quad h \in H^{-1}(\Omega), \quad \boldsymbol{u_b} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \quad \theta_b \in H^{\frac{1}{2}}(\Gamma)$$

such that $\int_\Gamma \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = 0$. *There exists* $\delta_4 = \delta_4(\Omega) > 0$ *such that if*

$$\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\Big)\right) \le \delta_4,$$

then the weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ *of* (BS) *satisfies the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C\left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\Big)\right),$$

$$\|\theta\|_{H^1(\Omega)} \le C\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right),$$

with $C = C(\Omega) > 0$. *Moreover, if the data satisfy that*

$$\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\Big)\right) \le \delta_5$$

for some $\delta_5 = \delta_5(\Omega) > 0$ *such that* $\delta_5 \le \delta_4$, *then the weak solution of* (BS) *is unique.*

2.3 Notations and some useful results

Throughout this work, we consider $\Omega \subset \mathbb{R}^3$ a bounded domain with boundary Γ of class $\mathcal{C}^{1,1}$. The term *domain* will be reserved for a nonempty open and connected set. In the case that Ω be another kind of set, we will point it out. Bold font for spaces means vector (or matrix) valued spaces, and their elements will be denoted with bold font also. We will denote by $\boldsymbol{n}$ the unit outward normal vector to Γ. Unless otherwise stated or unless the context otherwise requires, we will write with the same positive constant all the constants which depend on the same arguments in the estimations that will appear along this work.

We will denote as $\mathcal{D}(\Omega)$ the set of smooth functions with compact support in Ω. Let us define the following spaces:

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\Omega); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\},$$

$$\boldsymbol{H}^1_{0,\sigma}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}.$$

Recall that $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$, see [66, Theorem 1.6, p. 18]. Depending on the context, we use the following notation for the dual pairing:

$$\langle f, \varphi\rangle_\Omega := \langle f, \varphi\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \quad \text{or} \quad \langle f, \varphi\rangle_\Omega := \langle f, \varphi\rangle_{W^{-1,p}(\Omega), W^{1,p'}_0(\Omega)}.$$

Observe that this notation depends on which spaces the functions belong and keep in mind that it is also valid for vector valued functions.

Throughout this work, we use the following Sobolev inequality, see [39, Lemma 3, p. 10], and Poincaré inequality, which are valid for scalar and vector valued functions.

Lemma 2.3.1. (i) *Let $\Omega \subset \mathbb{R}^3$ be a domain. There exists a positive constant A_1, independent of Ω, such that for all $\varphi \in H_0^1(\Omega)$*

$$\|\varphi\|_{L^6(\Omega)} \leq A_1 \|\nabla\varphi\|_{\boldsymbol{L}^2(\Omega)}.$$

(ii) *Let $\Omega \subset \mathbb{R}^3$ be a domain which is bounded at least in one direction. There exists a positive constant A_2, depending on Ω, such that for all $\varphi \in H_0^1(\Omega)$*

$$\|\varphi\|_{H^1(\Omega)} \leq A_2 \|\nabla\varphi\|_{\boldsymbol{L}^2(\Omega)}.$$

Let B and b be the following trilinear forms:

$$B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}x \quad \text{for all} \quad (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \left(\boldsymbol{H}^1(\Omega)\right)^3,$$

$$b(\boldsymbol{u}, \theta, \tau) = \int_\Omega (\boldsymbol{u} \cdot \nabla\theta)\tau \, \mathrm{d}x \quad \text{for all} \quad \boldsymbol{u} \in \boldsymbol{H}^1(\Omega), (\theta, \tau) \in \left(H^1(\Omega)\right)^2.$$

Note, by using Hölder inequality, that

$$|B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)} \tag{2.4}$$

and

$$|b(\boldsymbol{u}, \theta, \tau)| \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \|\nabla\theta\|_{\boldsymbol{L}^2(\Omega)} \|\tau\|_{L^3(\Omega)}. \tag{2.5}$$

The following lemmas deal with some of the main properties of the trilinear forms B and b. Both lemmas are easily proven by applying (2.4), (2.5) and Sobolev embedding theorem to show the first claim and integration by parts to show the rest.

Lemma 2.3.2. *Let $\Omega \subset \mathbb{R}^3$ be an open set with Lipschitz boundary.*
(i) *The trilinear form B is continuous in $(\boldsymbol{H}^1(\Omega))^3$.*
(ii) *$B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -B(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$ for all $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in (\boldsymbol{H}^1(\Omega))^3$ with $\operatorname{div} \boldsymbol{u} = 0$ in Ω, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ or $\boldsymbol{v} = \boldsymbol{0}$ or $\boldsymbol{w} = \boldsymbol{0}$ on Γ.*
(iii) *$B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0$ for all $(\boldsymbol{u}, \boldsymbol{v}) \in (\boldsymbol{H}^1(\Omega))^2$ with $\operatorname{div} \boldsymbol{u} = 0$ in Ω, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ or $\boldsymbol{v} = \boldsymbol{0}$ on Γ.*

Lemma 2.3.3. *Let $\Omega \subset \mathbb{R}^3$ be an open set with Lipschitz boundary.*
(i) *The trilinear form b is continuous in $\boldsymbol{H}^1(\Omega) \times (H^1(\Omega))^2$.*
(ii) *$b(\boldsymbol{u}, \theta, \tau) = -b(\boldsymbol{u}, \tau, \theta)$ for all $(\boldsymbol{u}, \theta, \tau) \in \boldsymbol{H}^1(\Omega) \times (H^1(\Omega))^2$ with $\operatorname{div} \boldsymbol{u} = 0$ in Ω, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ or $\theta = 0$ or $\tau = 0$ on Γ.*
(iii) *$b(\boldsymbol{u}, \theta, \theta) = 0$ for all $(\boldsymbol{u}, \theta) \in \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ with $\operatorname{div} \boldsymbol{u} = 0$ in Ω, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ or $\theta = 0$ on Γ.*

When we address problems which involve Dirichlet non-homogeneous boundary conditions and we want to study the existence of weak solutions for such kind of problems, we usually use lift functions for these boundary conditions. In our case, we are interested in the existence of suitable lift functions for the Dirichlet non-homogeneous boundary conditions on the velocity field and the temperature.

The next lemma deals with the existence of a specific lift function for the boundary condition on the velocity, which satisfies a convenient estimate. The complete proof of this lemma is given in [27, Lemma IX.4.2, p. 610].

Lemma 2.3.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ and let $\boldsymbol{u_b} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ such that*

$$\int_\Gamma \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = 0.$$

Then, for all $\varepsilon > 0$ there exists $\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon} \in \boldsymbol{H}^1(\Omega)$ such that

$$\operatorname{div} \boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon} = 0, \quad \text{in } \Omega; \quad \boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon} = \boldsymbol{u_b}, \quad \text{on } \Gamma$$

and satisfies

$$|B(\boldsymbol{w}, \boldsymbol{u}_{\boldsymbol{b}}^{\boldsymbol{\varepsilon}}, \boldsymbol{w})| \le \left\{ \varepsilon + \sum_{i=0}^{m} K_i \left| \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s \right| \right\} \|\boldsymbol{\nabla w}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{2.6}$$

for all $\boldsymbol{w} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$, where each $K_i = K_i(\Omega) > 0$.

The following lemma is about the existence of a lift function for the boundary condition on the temperature that satisfies a suitable estimate.

Lemma 2.3.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary Γ and let $\theta_b \in H^{\frac{1}{2}}(\Gamma)$. Then, for all $\eta > 0$ there exists $\theta_b^\eta \in H^1(\Omega)$ such that $\theta_b^\eta = \theta_b$ on Γ and satisfies*

$$\|\theta_b^\eta\|_{L^3(\Omega)} \le \eta \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}. \tag{2.7}$$

Proof. To proof this lemma, we will use the idea given in [10, Lemma 2.8], but we introduce slight changes on it. Let us see some details. Since $\theta_b \in H^{\frac{1}{2}}(\Gamma)$, we know that there exists $\theta \in H^1(\Omega)$ such that $\theta = \theta_b$ on Γ and

$$\|\theta\|_{H^1(\Omega)} \le C_1 \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}$$

with $C_1 = C_1(\Omega) > 0$. Also, as Ω is bounded with Lipschitz boundary Γ, then the distance function d from $x \in \Omega$ up to the boundary Γ ($d(x) = \operatorname{dist}(x, \Gamma)$) belongs to $W^{1,\infty}(\Omega)$. Let $\varepsilon > 0$ be an arbitrary number. We can define the function $\chi_\varepsilon : \overline{\Omega} \longrightarrow [0,1]$ as

$$\chi_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad 0 \le d(x) \le \frac{\gamma(\varepsilon)}{2}, \\ 2\left(1 - \frac{1}{\gamma(\varepsilon)} d(x)\right) & \text{if} \quad \frac{\gamma(\varepsilon)}{2} \le d(x) \le \gamma(\varepsilon), \\ 0 & \text{if} \quad d(x) \ge \gamma(\varepsilon), \end{cases}$$

where $\gamma(\varepsilon) := \exp(-\frac{1}{\varepsilon})$. Clearly, $\chi_\varepsilon \in W^{1,\infty}(\Omega)$. Defining $\theta_b^\varepsilon = \chi_\varepsilon \theta$, we have that

$$\|\theta_b^\varepsilon\|_{L^3(\Omega)} \le C_2 \, |\Omega_1^\varepsilon|^{\frac{1}{6}} \, \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)},$$

where $C_2 = C_2(\Omega) > 0$ and $\Omega_1^\varepsilon = \left\{ x \in \overline{\Omega};\ 0 \le d(x) \le \gamma(\varepsilon) \right\}$. Since Ω_1^ε is an annular region, it follows that

$$|\Omega_1^\varepsilon| \le C_3 \gamma(\varepsilon)$$

with $C_3 = C_3(\Omega) > 0$. Further, due to the definition of $\gamma(\varepsilon)$, it is possible to deduce that for all $t > 0$

$$|\Omega_1^\varepsilon|^t \le C_4 \varepsilon$$

with $C_4 = C_4(t, \Omega) > 0$. In particular, $|\Omega_1^\varepsilon|^{\frac{1}{6}} \leq C_4\varepsilon$. Then

$$\|\theta_b^\varepsilon\|_{L^3(\Omega)} \leq C_5\varepsilon\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}$$

with $C_5 = C_5(\Omega) > 0$. Finally, for any $\eta > 0$, we can choose $\varepsilon = \frac{\eta}{C_5}$ and the estimate (2.7) is verified. $\square$

In part of our work, specifically, when we study the existence of very weak solutions for the Boussinesq system, we will use the following results concerning the existence and uniqueness of very weak solutions to the Poisson and Stokes equations.

First, let us consider the following Poisson problem:

$$\begin{cases} -\kappa\Delta\theta = f & \text{in } \Omega, \\ \theta = \theta_b & \text{on } \Gamma. \end{cases} \tag{P}$$

Proposition 2.3.6. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set of class $\mathcal{C}^{1,1}$. Let $f \in W^{-1,t}(\Omega)$ and $\theta_b \in W^{-\frac{1}{r},r}(\Gamma)$ with $t = 1 + \varepsilon$, for any fixed $0 < \varepsilon < 1$ if $1 < r \leq \frac{3}{2}$, and $t = \frac{3r}{r+3}$ if $r > \frac{3}{2}$. Then, the Poisson equation (P) has a unique very weak solution $\theta \in L^r(\Omega)$, which satisfies the estimate*

$$\|\theta\|_{L^r(\Omega)} \leq C\left(\frac{1}{\kappa}\|f\|_{W^{-1,t}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right),$$

where $C > 0$ is a constant depending only on ε, r and Ω when $1 < r \leq \frac{3}{2}$, and depending only on r and Ω when $r > \frac{3}{2}$.

Proof. Since the Poisson equation is linear, it is possible to split the problem in two parts:

$$\begin{cases} -\kappa\Delta\theta_1 = f & \text{in } \Omega, \\ \theta_1 = 0 & \text{on } \Gamma, \end{cases} \tag{P_1}$$

$$\begin{cases} -\kappa\Delta\theta_2 = 0 & \text{in } \Omega, \\ \theta_2 = \theta_b & \text{on } \Gamma. \end{cases} \tag{P_2}$$

Let us note that $\theta = \theta_1 + \theta_2$ is the unique very weak solution for (P). Then, let us focus in the solutions of (P_1) and (P_2).

First of all, thanks to [6, Theorem 7], (P_2) has a unique solution $\theta_2 \in L^r(\Omega)$ for all $1 < r < \infty$, which satisfies

$$\|\theta_2\|_{L^r(\Omega)} \leq C_2\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}$$

with $C_2 = C_2(r, \Omega) > 0$. Now, let us solve the problem (P_1).

Case $1 < r \leq \frac{3}{2}$: Since $f \in W^{-1,1+\varepsilon}(\Omega)$, by classical results of generalized solutions to the Poisson equation, we have that (P_1) has a unique solution $\theta_1 \in W_0^{1,1+\varepsilon}(\Omega) \hookrightarrow L^{\frac{3}{2}+\varepsilon'}(\Omega)$ for $\varepsilon' = \varepsilon'(\varepsilon) > 0$. Since $1 < r \leq \frac{3}{2}$, then $\theta_1 \in L^r(\Omega)$, and satisfies

$$\|\theta_1\|_{L^r(\Omega)} \leq \frac{C_1}{\kappa}\|f\|_{W^{-1,1+\varepsilon}(\Omega)}$$

with $C_1 = C_1(\varepsilon, r, \Omega) > 0$.

Case $r > \frac{3}{2}$: Since $f \in W^{-1,\frac{3r}{r+3}}(\Omega)$, by classical results of generalized solutions to the Poisson equation, we have that (P_1) has a unique solution $\theta_1 \in W_0^{1,\frac{3r}{r+3}}(\Omega) \hookrightarrow L^r(\Omega)$, and satisfies

$$\|\theta_1\|_{L^r(\Omega)} \leq \frac{C_1}{\kappa}\|f\|_{W^{-1,\frac{3r}{r+3}}(\Omega)}$$

with $C_1 = C_1(r, \Omega) > 0$. Finally, taking $C = \max\{C_1, C_2\}$, we have the desired estimate. $\square$

Let us consider the following Stokes problem:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u_b} & \text{on } \Gamma, \end{cases} \tag{S}$$

and let us introduce the following space:

$$\boldsymbol{X}_{r,p}(\Omega) = \left\{\boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,r}(\Omega);\ \operatorname{div} \boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\right\}, \quad 1 < r, p < \infty,$$

whose dual space can be characterized as follows:

$\boldsymbol{f} \in (\boldsymbol{X}_{r,p}(\Omega))'$ if and only if there exist $\mathbb{F}_0 = (f_{ij})_{1\le i,j\le 3}$ such that $\mathbb{F}_0 \in \boldsymbol{L}^{r'}(\Omega)$, $f_1 \in W^{-1,p'}(\Omega)$ and these satisfy that

$$\boldsymbol{f} = \nabla\cdot\mathbb{F}_0 + \nabla f_1.$$

Moreover,

$$\|\boldsymbol{f}\|_{(\boldsymbol{X}_{r',p'}(\Omega))'} = \max\left\{\|f_{ij}\|_{L^{r'}(\Omega)},\ 1\le i,j\le 3\,;\|f_1\|_{W^{-1,p'}(\Omega)}\right\}.$$

This characterization is proven in [6, Lemma 9]. Then, the following proposition is concerned with the existence and uniqueness of the very weak solution of problem (S), see [6, Theorem 11], for details of the proof.

Proposition 2.3.7. *For any $\boldsymbol{f} \in (\boldsymbol{X}_{r',p'}(\Omega))'$ and $\boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ which satisfies*

$$< \boldsymbol{u_b}\cdot\boldsymbol{n}, \boldsymbol{1} >_{W^{-\frac{1}{p},p}(\Gamma), W^{\frac{1}{p},p'}(\Gamma)} = 0 \quad \text{with} \quad \frac{1}{r} \le \frac{1}{p} + \frac{1}{3} \quad \text{and} \quad r \le p,$$

the Stokes problem (S) has a unique very weak solution $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$, which satisfies the estimate

$$\nu\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \le C\left(\|\boldsymbol{f}\|_{(\boldsymbol{X}_{r',p'}(\Omega))'} + \nu\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right)$$

with $C = C(p, r, \Omega) > 0$.

The following lemma will be used to estimate some terms which will appear later in our work. This lemma is easily proven by using regularization, Hölder inequality and Sobolev embedding arguments.

Lemma 2.3.8. *Let $u \in L^3(\Omega)$ and $v \in W^{1,p}(\Omega)$ with $1 \le p < 3$. Then $uv \in L^p(\Omega)$ and for all $\varepsilon > 0$ there exists $C_\varepsilon = C\left(\varepsilon, \|u\|_{L^3(\Omega)}\right) > 0$ such that*

$$\|uv\|_{L^p(\Omega)} \le \varepsilon\|v\|_{W^{1,p}(\Omega)} + C_\varepsilon\|v\|_{L^p(\Omega)}.$$

Proof. Using Hölder inequality and Sobolev embedding, we immediately have

$$\|uv\|_{L^p(\Omega)} \le C\|u\|_{L^3(\Omega)}\|v\|_{W^{1,p}(\Omega)}$$

with $C = C(p,\Omega) > 0$. Denoting by $\widetilde{u}$ the extension of u to $\mathbb{R}^3$ by zero outside of Ω, we have the following decomposition using the mollifier function ρ_ε:

$$u = \widetilde{u} * \rho_\varepsilon\big|_\Omega + \left(u - \widetilde{u} * \rho_\varepsilon\big|_\Omega\right).$$

Using this decomposition, Young inequality and again Hölder inequality and Sobolev embedding, we get that

$$\begin{aligned}\|uv\|_{L^p(\Omega)} &\leq \left\|\left(\widetilde{u} * \rho_\varepsilon\big|_\Omega\right) v\right\|_{L^p(\Omega)} + \left\|\left(u - \widetilde{u} * \rho_\varepsilon\big|_\Omega\right) v\right\|_{L^p(\Omega)} \\ &\leq C_\varepsilon \|v\|_{L^p(\Omega)} + \varepsilon \|v\|_{W^{1,p}(\Omega)}.\end{aligned}$$

□

Remark 2.3.9. As a classical method in the study of the Stokes and Navier-Stokes equations, the pressure π is obtained thanks to a variant of De Rham's theorem. Indeed, if $\boldsymbol{f} \in \boldsymbol{W}^{-1,p}(\Omega)$ for $1 < p < \infty$, satisfies that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \quad \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = 0,$$

then there exists $\pi \in L^p(\Omega)$ such that $\boldsymbol{f} = \nabla \pi$. For more details see [4, Theorem 2.8]. So, when we say that $(\boldsymbol{u},\theta) \in \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ is a weak solution of (*BS*), we mean that $(\boldsymbol{u},\pi,\theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ is a weak solution of (*BS*).

2.4 Weak solutions

In this section we are going to establish the existence of weak solutions for the Boussinesq system in the Hilbert space $H^1(\Omega)$. It is important to note that when we consider a domain whose boundary could be non-connected, in order to prove the existence of weak solutions we only need to assume smallness of the fluxes of the velocity across each connected component of the boundary.

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ and let*

$$\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega), \quad h \in H^{-1}(\Omega), \quad \boldsymbol{u_b} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \quad \theta_b \in H^{\frac{1}{2}}(\Gamma)$$

such that $\int_\Gamma \boldsymbol{u_b} \cdot \boldsymbol{n}\, \mathrm{d}s = 0$. There exists $\delta_1 = \delta_1(\Omega) > 0$ such that if

$$\frac{1}{\nu} \sum_{i=1}^{m} \left| \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n}\, \mathrm{d}s \right| \leq \delta_1, \tag{2.8}$$

then the problem (BS) has at least one weak solution $(\boldsymbol{u},\pi,\theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$. Further, if $\boldsymbol{u_b} = \boldsymbol{0}$ and $\theta_b = 0$ on Γ, then the weak solution $(\boldsymbol{u},\theta)$ satisfies the following estimates:

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C}{\nu\kappa} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|h\|_{H^{-1}(\Omega)}, \tag{2.9}$$

$$\|\nabla \theta\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C}{\kappa} \|h\|_{H^{-1}(\Omega)}, \tag{2.10}$$

with $C = C(\Omega) > 0$.

Proof. Let us define $\boldsymbol{H} := \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ as the Hilbert space equipped with the norm

$$\|(\boldsymbol{u}, \theta)\|_{\boldsymbol{H}} = \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\theta\|_{H^1(\Omega)}.$$

Let $(\boldsymbol{u}, \theta) \in \boldsymbol{H}$ given. Note that $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \mathrm{div}(\boldsymbol{u} \otimes \boldsymbol{u})$, $\boldsymbol{u} \cdot \nabla\theta = \mathrm{div}(\theta\boldsymbol{u})$ and thanks to the Sobolev embeddings, we have that $\boldsymbol{u} \otimes \boldsymbol{u}$ and $\theta\boldsymbol{u}$ are in $\boldsymbol{L}^3(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$. Then, by the existence and uniqueness of weak solutions to the Stokes and Poisson equations, there exists a unique $(\hat{\boldsymbol{u}}, \hat{\theta}, \hat{\pi}) \in \boldsymbol{H} \times L^2(\Omega)/\mathbb{R}$ weak solution of the following uncoupled system:

$$\begin{cases} -\nu\Delta\hat{\boldsymbol{u}} + \nabla\hat{\pi} = \hat{\theta}\boldsymbol{g} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} & \text{in } \Omega, \\ \mathrm{div}\ \hat{\boldsymbol{u}} = 0 & \text{in } \Omega, \\ -\kappa\Delta\hat{\theta} = h - \boldsymbol{u} \cdot \nabla\theta & \text{in } \Omega, \\ \hat{\boldsymbol{u}} = \boldsymbol{u_b}\ ,\ \hat{\theta} = \theta_b & \text{on } \Gamma. \end{cases} \tag{2.11}$$

Let $\mathcal{T} : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ be the operator such that $(\hat{\boldsymbol{u}}, \hat{\theta}) = \mathcal{T}(\boldsymbol{u}, \theta)$ is the unique weak solution to (2.11). Let us realize that a fixed point of the operator $\mathcal{T}$ is a weak solution of (*BS*). So, in order to apply the Leray-Schauder fixed point theorem, see [29, Theorem 11.3, p. 280], we must prove that $\mathcal{T}$ is a compact operator on $\boldsymbol{H}$ and

$$\exists C_1 > 0 \text{ such that } \|(\boldsymbol{u},\theta)\|_{\boldsymbol{H}} < C_1, \forall(\boldsymbol{u}, \theta) \in \boldsymbol{H} \text{ and } \forall\alpha \in [0, 1] \text{ such that } (\boldsymbol{u}, \theta) = \alpha\mathcal{T}(\boldsymbol{u}, \theta). \tag{2.12}$$

(i) *Let us prove that $\mathcal{T}$ is a compact operator.* Suppose $(\boldsymbol{u}, \theta) \in \boldsymbol{H}$, $(\boldsymbol{u}_n, \theta_n) \in \boldsymbol{H}$, $n \in \mathbb{N}$ and $(\boldsymbol{u}_n, \theta_n) \rightharpoonup (\boldsymbol{u}, \theta)$, in $\boldsymbol{H}$-weak. Let us define $(\hat{\boldsymbol{u}}_n, \hat{\theta}_n) := \mathcal{T}(\boldsymbol{u}_n, \theta_n)$ for all $n \in \mathbb{N}$. We obtain that $(\hat{\boldsymbol{u}}_n - \hat{\boldsymbol{u}}, \hat{\theta}_n - \hat{\theta})$ satisfies the following system:

$$\begin{cases} -\nu\Delta(\hat{\boldsymbol{u}}_n - \hat{\boldsymbol{u}}) + \nabla(\hat{\pi}_n - \hat{\pi}) = (\hat{\theta}_n - \hat{\theta})\boldsymbol{g} - [(\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] & \text{in } \Omega, \\ \mathrm{div}(\hat{\boldsymbol{u}}_n - \hat{\boldsymbol{u}}) = 0 & \text{in } \Omega, \\ -\kappa\Delta(\hat{\theta}_n - \hat{\theta}) = -(\boldsymbol{u}_n \cdot \nabla\theta_n - \boldsymbol{u} \cdot \nabla\theta) & \text{in } \Omega, \\ \hat{\boldsymbol{u}}_n - \hat{\boldsymbol{u}} = \boldsymbol{0}\ ,\ \hat{\theta}_n - \hat{\theta} = 0 & \text{on } \Gamma. \end{cases}$$

Note that $\hat{\boldsymbol{u}}_n - \hat{\boldsymbol{u}} \in \boldsymbol{H}_0^1(\Omega)$ and $\hat{\theta}_n - \hat{\theta} \in H_0^1(\Omega)$, then by applying usual estimates for weak solutions to the Poisson equation, we have

$$\begin{aligned} \kappa\|\hat{\theta}_n - \hat{\theta}\|_{H^1(\Omega)} &\leq C\|\boldsymbol{u}_n \cdot \nabla\theta_n - \boldsymbol{u} \cdot \nabla\theta\|_{H^{-1}(\Omega)} \\ &= C\|\mathrm{div}(\theta_n\boldsymbol{u}_n - \theta\boldsymbol{u})\|_{H^{-1}(\Omega)} \\ &\leq C\|\theta_n\boldsymbol{u}_n - \theta\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq C\left[\|(\theta_n - \theta)\boldsymbol{u}_n\|_{\boldsymbol{L}^2(\Omega)} + \|(\boldsymbol{u}_n - \boldsymbol{u})\theta\|_{\boldsymbol{L}^2(\Omega)}\right] \\ &\leq C\left[\|\theta_n - \theta\|_{L^3(\Omega)}\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\|\theta\|_{H^1(\Omega)}\right] \end{aligned}$$

with $C = C(\Omega) > 0$. Since $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$, in $\boldsymbol{H^1(\Omega)}$-weak and $\theta_n \rightharpoonup \theta$, in $H^1(\Omega)$-weak, therefore, $\boldsymbol{u}_n \to \boldsymbol{u}$, in $\boldsymbol{L^s(\Omega)}$ and $\theta_n \to \theta$, in $L^s(\Omega)$, for $1 \leq s < 6$. Then

$$\hat{\theta}_n \underset{n\to\infty}{\longrightarrow} \hat{\theta}, \text{ in } H^1(\Omega). \tag{2.13}$$

On the other hand, by using usual estimates for weak solutions to the Stokes equations, it follows that

$$
\begin{aligned}
\nu\|\hat{\boldsymbol{u}}_n-\hat{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)} &\le C\left[\|(\hat{\theta}_n-\hat{\theta})\boldsymbol{g}\|_{\boldsymbol{H}^{-1}(\Omega)}+\|(\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}\right]\\
&= C\left[\|(\hat{\theta}_n-\hat{\theta})\boldsymbol{g}\|_{\boldsymbol{H}^{-1}(\Omega)}+\|\mathrm{div}(\boldsymbol{u}_n\otimes\boldsymbol{u}_n-\boldsymbol{u}\otimes\boldsymbol{u})\|_{\boldsymbol{H}^{-1}(\Omega)}\right]\\
&\le C\left[\|(\hat{\theta}_n-\hat{\theta})\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}+\|\boldsymbol{u}_n\otimes\boldsymbol{u}_n-\boldsymbol{u}\otimes\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\right]\\
&\le C\left[\|\hat{\theta}_n-\hat{\theta}\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|(\boldsymbol{u}_n-\boldsymbol{u})\otimes\boldsymbol{u}_n\|_{\boldsymbol{L}^2(\Omega)}+\|(\boldsymbol{u}_n-\boldsymbol{u})\otimes\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\right]\\
&\le C\left[\|\hat{\theta}_n-\hat{\theta}\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{u}_n\|_{\boldsymbol{L}^6(\Omega)}+\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\right]\\
&\le C\left[\|\hat{\theta}_n-\hat{\theta}\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+(\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)})\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\right].
\end{aligned}
$$

Then

$$
\hat{\boldsymbol{u}}_n \underset{n\to\infty}{\longrightarrow} \hat{\boldsymbol{u}}, \text{ in } \boldsymbol{H}^1(\Omega). \tag{2.14}
$$

By (2.13) and (2.14), we deduce that $(\hat{\boldsymbol{u}}_n,\hat{\theta}_n)\to(\hat{\boldsymbol{u}},\hat{\theta})$, in $\boldsymbol{H}$. Therefore, $\mathcal{T}$ is a completely continuous operator, and since $\boldsymbol{H}$ is a reflexive space, then $\mathcal{T}$ is a compact operator in $\boldsymbol{H}$.

(ii) *Let us show the condition* (2.12). Let $(\boldsymbol{u},\theta)=\alpha\mathcal{T}(\boldsymbol{u},\theta)$ with $(\boldsymbol{u},\theta)\in\boldsymbol{H}$ and $\alpha\in[0,1]$. As $(\boldsymbol{u},\theta)=\alpha(\hat{\boldsymbol{u}},\hat{\theta})=(\alpha\hat{\boldsymbol{u}},\alpha\hat{\theta})$, then $(\hat{\boldsymbol{u}},\hat{\theta})=\mathcal{T}(\boldsymbol{u},\theta)=\mathcal{T}(\alpha\hat{\boldsymbol{u}},\alpha\hat{\theta})$ satisfies the following system:

$$
\left\{
\begin{aligned}
-\nu\Delta\hat{\boldsymbol{u}}+\nabla\hat{\pi}=\hat{\theta}\boldsymbol{g}-\alpha^2(\hat{\boldsymbol{u}}\cdot\nabla)\hat{\boldsymbol{u}} &\quad \text{in } \Omega,\\
\mathrm{div}\ \hat{\boldsymbol{u}}=0 &\quad \text{in } \Omega,\\
-\kappa\Delta\hat{\theta}=h-\alpha^2\hat{\boldsymbol{u}}\cdot\nabla\hat{\theta} &\quad \text{in } \Omega,\\
\hat{\boldsymbol{u}}=\boldsymbol{u_b}\ ,\ \hat{\theta}=\theta_b &\quad \text{on } \Gamma.
\end{aligned}
\right. \tag{2.15}
$$

We shall consider two cases depending on the values of the boundary data.

(a) *Case* $\boldsymbol{u_b}=\boldsymbol{0}$ *and* $\theta_b=0$. In this case, note that $\hat{\boldsymbol{u}}\in\boldsymbol{H}^1_{0,\sigma}(\Omega)$ and $\hat{\theta}\in H^1_0(\Omega)$, then multiplying by $\hat{\boldsymbol{u}}$ and by $\hat{\theta}$ the first and third equations of (2.15), respectively, and integrating by parts[1], we have

$$
\nu\int_\Omega|\nabla\hat{\boldsymbol{u}}|^2\,\mathrm{d}x=\int_\Omega\hat{\theta}\boldsymbol{g}\cdot\hat{\boldsymbol{u}}\,\mathrm{d}x \quad\text{and}\quad \kappa\int_\Omega|\nabla\hat{\theta}|^2\,\mathrm{d}x=\langle h,\hat{\theta}\rangle_\Omega.
$$

We have immediately that

$$
\|\nabla\hat{\theta}\|_{\boldsymbol{L}^2(\Omega)}\le\frac{D}{\kappa}\|h\|_{H^{-1}(\Omega)} \tag{2.16}
$$

with $D=D(\Omega)>0$, and

$$
\|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\le\frac{A_1^2}{\nu}\|\nabla\hat{\theta}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)},
$$

where A_1 is the constant given in Lemma 2.3.1. Then, by using (2.16), we have that

$$
\|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\le\frac{DA_1^2}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|h\|_{H^{-1}(\Omega)}. \tag{2.17}
$$

[1]Because of the low regularity of the functions ($\boldsymbol{H}^1(\Omega)$) we can not use integration by parts in a direct way. Actually, we consider the definition of derivatives in the sense of the distributions and then we use a density argument to obtain the desired integral equations for functions in $\boldsymbol{H}^1(\Omega)$.

It follows from (2.17) and (2.16) that

$$\|(\boldsymbol{u},\theta)\|_{\boldsymbol{H}} = \alpha\|(\hat{\boldsymbol{u}},\hat{\theta})\|_{\boldsymbol{H}} \leq C_1,$$

where $C_1 = C_1\left(\Omega,\nu,\kappa,\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)},\|h\|_{H^{-1}(\Omega)}\right)$ is a positive constant independent of $(\boldsymbol{u},\theta)$ and α.

(b) *Case* $\boldsymbol{u_b} \neq \mathbf{0}$ *and* $\theta_b \neq 0$. Let us define $\hat{\boldsymbol{u}}_\varepsilon = \hat{\boldsymbol{u}} - \boldsymbol{u}_{\boldsymbol{b}}^\varepsilon$ and $\hat{\theta}_\eta = \hat{\theta} - \theta_b^\eta$, where $\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon$ and θ_b^η are the lift functions of the boundary conditions given by Lemma 2.3.4 and Lemma 2.3.5, respectively.

Using the definitions of $\hat{\boldsymbol{u}}_\varepsilon$ and $\hat{\theta}_\eta$ in (2.15), we get the following system:

$$\begin{cases} -\nu\Delta\hat{\boldsymbol{u}}_\varepsilon + \alpha^2(\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla)\hat{\boldsymbol{u}}_\varepsilon + \alpha^2(\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\cdot\nabla)\hat{\boldsymbol{u}}_\varepsilon + \nabla\hat{\pi} = \hat{\theta}_\eta\boldsymbol{g} - \alpha^2(\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla)\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon + \boldsymbol{F} & \text{in } \Omega,\\ \operatorname{div}\hat{\boldsymbol{u}}_\varepsilon = 0 & \text{in } \Omega,\\ -\kappa\Delta\hat{\theta}_\eta + \alpha^2\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla\hat{\theta}_\eta + \alpha^2\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\cdot\nabla\hat{\theta}_\eta = h - \alpha^2\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla\theta_b^\eta + G & \text{in } \Omega,\\ \hat{\boldsymbol{u}}_\varepsilon = \mathbf{0}\ ,\ \hat{\theta}_\eta = 0 & \text{on } \Gamma, \end{cases} \tag{2.18}$$

where

$$\boldsymbol{F} = \boldsymbol{F}(\varepsilon,\eta) = \nu\Delta\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon + \theta_b^\eta\boldsymbol{g} - \alpha^2(\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\cdot\nabla)\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon \in \boldsymbol{H}^{-1}(\Omega)$$

and

$$G = G(\varepsilon,\eta) = \kappa\Delta\theta_b^\eta - \alpha^2\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\cdot\nabla\theta_b^\eta \in H^{-1}(\Omega).$$

Noting that $\hat{\boldsymbol{u}}_\varepsilon \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ and $\hat{\theta}_\eta \in H^1_0(\Omega)$, we can choose them as test functions in the variational formulation of (2.18)[2], and therefore

$$\nu\int_\Omega |\nabla\hat{\boldsymbol{u}}_\varepsilon|^2\,\mathrm{d}x = \int_\Omega \hat{\theta}_\eta\boldsymbol{g}\cdot\hat{\boldsymbol{u}}_\varepsilon\,\mathrm{d}x - \ \alpha^2 B\left(\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\boldsymbol{u}_{\boldsymbol{b}}^{\boldsymbol{\varepsilon}},\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) + \langle\boldsymbol{F},\hat{\boldsymbol{u}}_\varepsilon\rangle_\Omega, \tag{2.19}$$

$$\kappa\int_\Omega |\nabla\hat{\theta}_\eta|^2\,\mathrm{d}x = \langle h,\hat{\theta}_\eta\rangle_\Omega + \ \alpha^2 b\left(\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\hat{\theta}_\eta,\theta_b^\eta\right) + \langle G,\hat{\theta}_\eta\rangle_\Omega. \tag{2.20}$$

Then, by using (2.5) and Lemma 2.3.1, we have from (2.20) that

$$\begin{aligned} \kappa\|\nabla\hat{\theta}_\eta\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \|h\|_{H^{-1}(\Omega)}\|\hat{\theta}_\eta\|_{H^1(\Omega)} + \|\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)}\|\theta_b^\eta\|_{L^3(\Omega)}\\ &\quad + \|G\|_{H^{-1}(\Omega)}\|\hat{\theta}_\eta\|_{H^1(\Omega)}\\ &\leq A_2\|h\|_{H^{-1}(\Omega)}\|\nabla\hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)} + A_1\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)}\|\theta_b^\eta\|_{L^3(\Omega)}\\ &\quad + A_2\|G\|_{H^{-1}(\Omega)}\|\nabla\hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)}\\ \kappa\|\nabla\hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)} &\leq C\left(\|h\|_{H^{-1}(\Omega)} + \|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\|\theta_b^\eta\|_{L^3(\Omega)} + \|G\|_{H^{-1}(\Omega)}\right) \end{aligned} \tag{2.21}$$

with $C = C(\Omega) > 0$, and from (2.19) we have

$$\begin{aligned} \nu\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \|\hat{\theta}_\eta\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\|\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^6(\Omega)} + |B\left(\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\boldsymbol{u}_{\boldsymbol{b}}^{\boldsymbol{\varepsilon}},\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right)| + \|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)}\|\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}\\ &\leq A_1\|\hat{\theta}_\eta\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} + |B\left(\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\boldsymbol{u}_{\boldsymbol{b}}^{\boldsymbol{\varepsilon}},\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right)|\\ &\quad + A_2\|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\\ &\leq A_1^2\|\nabla\hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} + |B\left(\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\boldsymbol{u}_{\boldsymbol{b}}^{\boldsymbol{\varepsilon}},\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right)|\\ &\quad + A_2\|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

[2]In a similar way as the previous case, we simply use the definition of derivatives in the sense of the distributions combined with a density argument to obtain the desired integral equations.

Choosing $\boldsymbol{w} = \hat{\boldsymbol{u}}_\varepsilon$ in (2.6) and defining $K = K(\Omega) := \max\limits_{0\le i\le m} K_i$, we obtain

$$\begin{aligned}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|^2_{\boldsymbol{L}^2(\Omega)} \le \frac{A_1^2}{\nu}\|\nabla\hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\\ + \frac{1}{\nu}\left\{\varepsilon + K\sum_{i=0}^{m}\left|\int_{\Gamma_i}\boldsymbol{u_b}\cdot\boldsymbol{n}\,\mathrm{d}s\right|\right\}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{A_2}{\nu}\|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)},\end{aligned}$$

and by the hypothesis (2.8), we get

$$\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \le \frac{A_1^2}{\nu}\|\nabla\hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \left(\frac{\varepsilon}{\nu} + 2K\delta_1\right)\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} + \frac{A_2}{\nu}\|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)}. \quad (2.22)$$

Let us find estimates for $\boldsymbol{F}$ and G. Let $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$, it follows that

$$\begin{aligned}|\langle\boldsymbol{F},\boldsymbol{v}\rangle_\Omega| &\le |\langle\nu\Delta\boldsymbol{u}^\varepsilon_{\boldsymbol{b}},\boldsymbol{v}\rangle_\Omega| + \left|\int_\Omega\theta_b^\eta\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}x\right| + \left|B\left(\boldsymbol{u}^{\boldsymbol{\varepsilon}}_{\boldsymbol{b}},\boldsymbol{u}^{\boldsymbol{\varepsilon}}_{\boldsymbol{b}},\boldsymbol{v}\right)\right|\\ &\le \nu\left|\int_\Omega\nabla\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}:\nabla\boldsymbol{v}\,\mathrm{d}x\right| + \|\theta_b^\eta\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^6(\Omega)} + \|\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\Omega)}\\ &\le \nu\|\nabla\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + A_1\|\theta_b^\eta\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\\ &\quad + A_2C_2C_3\|\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|^2_{\boldsymbol{H}^1(\Omega)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)},\end{aligned}$$

where $C_2>0$ is the constant involved in the embedding $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$ and $C_3>0$ is the constant involved in the embedding $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$. Then, we have that

$$\begin{aligned}\|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)} &= \sup_{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}=1}|\langle\boldsymbol{F},\boldsymbol{v}\rangle_\Omega|\\ &\le \nu\|\nabla\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|_{\boldsymbol{L}^2(\Omega)} + A_1\|\theta_b^\eta\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + A_2C_2C_3\|\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|^2_{\boldsymbol{H}^1(\Omega)}\\ &\le D_1,\end{aligned}$$

where

$$D_1 = D_1(\varepsilon,\eta) := \widetilde{C}\left(\nu\|\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta_b^\eta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right)$$

with $\widetilde{C}=\widetilde{C}(\Omega)>0$, is a constant independent of $(\boldsymbol{u},\theta)$ and α.

In a similar way, it is showed that

$$\|G\|_{H^{-1}(\Omega)} \le D_2,$$

where $D_2 = D_2(\varepsilon,\eta) := \widetilde{C}\left(\kappa\|\theta_b^\eta\|_{H^1(\Omega)} + \|\boldsymbol{u}^\varepsilon_{\boldsymbol{b}}\|_{\boldsymbol{H}^1(\Omega)}\|\theta_b^\eta\|_{L^3(\Omega)}\right)$ is a constant independent of $(\boldsymbol{u},\theta)$ and α.

Using (2.7), (2.21) and the estimates for $\boldsymbol{F}$ and G in (2.22), it follows that

$$\begin{aligned}
\|\nabla \hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} &\leq \frac{A_1^2 C}{\nu\kappa}\left(\|h\|_{H^{-1}(\Omega)} + \|\nabla \hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\|\theta_b^\eta\|_{L^3(\Omega)} + D_2\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \\
&\quad + \left(\frac{\varepsilon}{\nu} + 2K\delta_1\right)\|\nabla \hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} + \frac{A_2}{\nu}D_1 \\
&\leq \frac{1}{\nu}\left(A_2 D_1 + \frac{A_1^2 C}{\kappa}D_2\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \frac{A_1^2 C}{\kappa}\|h\|_{H^{-1}(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right) \\
&\quad + \left(\frac{A_1^2 C}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_b^\eta\|_{L^3(\Omega)} + \frac{\varepsilon}{\nu} + 2K\delta_1\right)\|\nabla \hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq \frac{1}{\nu}\left(A_2 D_1 + \frac{A_1^2 C}{\kappa}D_2\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \frac{A_1^2 C}{\kappa}\|h\|_{H^{-1}(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right) \\
&\quad + \left(\frac{A_1^2 C\eta}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} + \frac{\varepsilon}{\nu} + 2K\delta_1\right)\|\nabla \hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Taking $\eta = \frac{\nu\kappa}{8A_1^2 C\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}}$, $\varepsilon = \frac{\nu}{8}$ and $\delta_1 = \frac{1}{8K}$, we have

$$\|\nabla \hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_4}{\nu}\left(D_1 + \frac{1}{\kappa}D_2\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right) \tag{2.23}$$

with $C_4 = C_4(\Omega) = 2\max\{A_1^2 C, A_2\}$. Now, by using (2.7), the value of η and (2.23), we deduce from (2.21) that

$$\|\nabla \hat{\theta}_\eta\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_5}{\kappa}\left(\|h\|_{H^{-1}(\Omega)} + D_2 + \frac{\kappa}{\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}}D_1\right) \tag{2.24}$$

with $C_5 = C_5(\Omega) > 0$. It follows from (2.23) and (2.24) that

$$\begin{aligned}
\|(\boldsymbol{u}, \theta)\|_{\boldsymbol{H}} &= \alpha\|(\hat{\boldsymbol{u}}, \hat{\theta})\|_{\boldsymbol{H}} \\
&\leq \|\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)} + \|\hat{\theta}_\eta\|_{H^1(\Omega)} + \|\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\|_{\boldsymbol{H}^1(\Omega)} + \|\theta_b^\eta\|_{H^1(\Omega)} \\
&\leq C_1,
\end{aligned}$$

where $C_1 = C_1\left(\Omega, \nu, \kappa, \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}, \|h\|_{H^{-1}(\Omega)}, \|\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\|_{\boldsymbol{H}^1(\Omega)}, \|\theta_b^\eta\|_{H^1(\Omega)}\right)$ is a positive constant independent of $(\boldsymbol{u}, \theta)$ and α.

Finally, by Leray-Schauder fixed point theorem and Remark 2.3.9, there exists at least one $(\boldsymbol{u}, \theta, \pi) \in \boldsymbol{H} \times L^2(\Omega)$ such that (*BS*) is satisfied.

(iii) *Proof of estimates* (2.9) *and* (2.10). From the Leray-Schauder fixed point theorem, we have that $\hat{\boldsymbol{u}} = \boldsymbol{u}$ and $\hat{\theta} = \theta$, and hence if $\boldsymbol{u_b} = \boldsymbol{0}$ and $\theta_b = 0$, we have directly the desired estimates from (2.17) and (2.16).

Therefore, the theorem is totally proven. □

Remark 2.4.2. (i) If the boundary is connected ($m = 0$), then the condition (2.8) is always satisfied because, by hypothesis, the flux of $\boldsymbol{u_b}$ through the boundary Γ is zero. Then, in this case, we have at least one solution for (*BS*) just considering the compatibility condition for $\boldsymbol{u_b}$, i.e., $\int_\Gamma \boldsymbol{u_b} \cdot \boldsymbol{n}\, \mathrm{d}s = 0$.

(ii) We know that $\Gamma = \bigcup_{i=0}^{m} \Gamma_i$ with $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$. Note, however, that it is sufficient to have smallness condition for $\sum_{i=1}^{m} \left| \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s \right|$ instead of $\sum_{i=0}^{m} \left| \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s \right|$. This is because as $\int_{\Gamma} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = 0$, then

$$\int_{\Gamma_0} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = -\sum_{i=1}^{m} \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s.$$

So, with the condition (2.8), we have immediately that

$$\frac{1}{\nu} \sum_{i=0}^{m} \left| \int_{\Gamma_i} \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s \right| \leq 2\delta_1.$$

(iii) The estimates for $\boldsymbol{u}$ and θ when the boundary data are non-zero, we will prove them later by using estimates of the very weak solutions of the Boussinesq system.

2.5 Regularity of the weak solution

We can rewrite the Boussinesq system (*BS*) in order to have the structure of the Stokes and Poisson equations as follows

$$\begin{cases} -\nu \Delta \boldsymbol{u} + \nabla \pi = \theta \boldsymbol{g} - (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u_b} & \text{on } \Gamma \\ -\kappa \Delta \theta = h - \boldsymbol{u} \cdot \nabla \theta & \text{in } \Omega, \\ \theta = \theta_b & \text{on } \Gamma, \end{cases}$$

and then we are going to take advantage of well-known regularity results for the Stokes and Poisson equations. Realize that if we want to show the regularity $W^{1,p}(\Omega)$ and $W^{2,p}(\Omega)$ of the solution of the Boussinesq system, then we need more regularity for the domain Ω. This is the reason why we consider Ω of class $\mathcal{C}^{1,1}$.

Theorem 2.5.1 (regularity $W^{1,p}(\Omega)$ with $p > 2$)**.** *Let us suppose that*

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega), \quad h \in W^{-1,p}(\Omega) \quad \textit{and} \quad (\boldsymbol{u_b}, \theta_b) \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \times W^{1-\frac{1}{p},p}(\Gamma)$$

with

$$p > 2, \quad r = \max\left\{\frac{3}{2}, \frac{3p}{3+p}\right\} \quad \textit{if} \quad p \neq 3 \quad \textit{and} \quad r = \frac{3}{2} + \varepsilon \quad \textit{if} \quad p = 3$$

for any fixed $0 < \varepsilon < \frac{1}{2}$. Then the weak solution for the Boussinesq system given by Theorem 2.4.1 satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega) \times W^{1,p}(\Omega).$$

Proof. Let $(\boldsymbol{u}, \theta) \in \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ be a weak solution for the Boussinesq system. We know that $H^1(\Omega) \hookrightarrow L^6(\Omega)$. Then $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{W}^{-1,3}(\Omega)$. Also, $\operatorname{div}(\theta \boldsymbol{u}) \in W^{-1,3}(\Omega)$ because $\theta \in L^6(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{L}^6(\Omega)$. We must note that $W^{-1,3}(\Omega) \hookrightarrow W^{-1,p}(\Omega)$ if $p \leq 3$. Then, we have three cases:

(i) *Case* $2 < p < 3$: As $h \in W^{-1,p}(\Omega)$, by the regularity of the Poisson equation we have $\theta \in W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. Since $2 < p < 3$, we have that $\frac{6}{5} < \frac{3p}{3+p} < \frac{3}{2}$, and then $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$. Hence, $\theta\boldsymbol{g} \in \boldsymbol{L}^s(\Omega)$ with $\frac{1}{s} = \frac{1}{p} + \frac{1}{3}$, but $\boldsymbol{L}^s(\Omega) \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$. Consequently, thanks to the regularity of the Stokes equations, we have $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$.

(ii) *Case* $p = 3$: In view of $\operatorname{div}(\theta\boldsymbol{u}) \in W^{-1,3}(\Omega)$ and $h \in W^{-1,3}(\Omega)$, by regularity of the Poisson equation, we have that $\theta \in W^{1,3}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q < \infty$. Then, since $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$, we have $\theta\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{W}^{-1,3}(\Omega)$ and thanks to the regularity of the Stokes equations we get $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$ and $\pi \in L^3(\Omega)$.

(iii) *Case* $p > 3$: From the previous case, we have that $(\boldsymbol{u}, \theta) \in \boldsymbol{W}^{1,3}(\Omega) \times W^{1,3}(\Omega)$. Therefore, $(\boldsymbol{u}, \theta) \in \boldsymbol{L}^t(\Omega) \times L^t(\Omega)$, for any $t \ge 1$, and then $\boldsymbol{u} \otimes \boldsymbol{u} \in \boldsymbol{L}^t(\Omega)$, $\theta\boldsymbol{u} \in \boldsymbol{L}^t(\Omega)$ for any $t \ge 1$. Consequently, $\operatorname{div}(\theta\boldsymbol{u}) \in W^{-1,p}(\Omega)$ and as $h \in W^{-1,p}(\Omega)$, by regularity of the Poisson equation, we have $\theta \in W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. Further, as $p > 3$, $\frac{3p}{3+p} > \frac{3}{2}$, and $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega)$, consequently $\theta\boldsymbol{g} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega) \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$. We have that $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \in \boldsymbol{W}^{-1,p}(\Omega)$, and by regularity of the Stokes equations, we get that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$. $\square$

Remark 2.5.2. We must note that the choice of the space, where $\boldsymbol{g}$ lies, is optimal in order to study the regularity $W^{1,p}(\Omega)$ with $p > 2$. Indeed, for the case $2 < p < 3$, we know that $\theta \in W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$ and then $\theta\boldsymbol{g} \in \boldsymbol{L}^s(\Omega)$ with $\frac{1}{s} = \frac{1}{p^*} + \frac{1}{r}$, where r is the number that we need to determine. If we want to use the regularity of the Stokes equations, we need to have $\theta\boldsymbol{g} \in \boldsymbol{W}^{-1,p}(\Omega)$, what means that $\boldsymbol{L}^s(\Omega) \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$. This is equivalent to say that $\boldsymbol{W}_0^{1,p'}(\Omega) \hookrightarrow \boldsymbol{L}^{s'}(\Omega)$, where $\frac{1}{s'} = \frac{1}{p'} - \frac{1}{3}$. We have that

$$\frac{1}{r} = \frac{1}{s} - \frac{1}{p} + \frac{1}{3} = -\frac{1}{s'} + \frac{1}{p'} + \frac{1}{3} = \frac{2}{3}$$

and hence $r = \frac{3}{2} = \max\left\{\frac{3}{2}, \frac{3p}{3+p}\right\}$ because $2 < p < 3$.

The other cases for p are analyzed in a similar way, and this proves that r is optimal to obtain the regularity $W^{1,p}(\Omega)$ with $p > 2$.

Theorem 2.5.3 (regularity $W^{2,p}(\Omega)$ with $p \ge \frac{6}{5}$)**.** *Let us suppose that*

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega), \quad h \in L^p(\Omega) \quad \textit{and} \quad (\boldsymbol{u_b}, \theta_b) \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma) \times W^{2-\frac{1}{p},p}(\Gamma)$$

with

$$p \ge \frac{6}{5}, \quad r = \max\left\{\frac{3}{2}, p\right\} \quad \textit{if} \quad p \ne \frac{3}{2} \quad \textit{and} \quad r = \frac{3}{2} + \varepsilon \quad \textit{if} \quad p = \frac{3}{2}$$

for any fixed $0 < \varepsilon < \frac{1}{2}$*. Then the solution for the Boussinesq system given by Theorem 2.4.1 satisfies*

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega) \times W^{2,p}(\Omega).$$

Proof. Let $(\boldsymbol{u}, \theta) \in \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ be a weak solution for the Boussinesq system. By the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{u} \cdot \nabla\theta \in L^{\frac{3}{2}}(\Omega)$. Then, we have three cases:

(i) *Case* $\frac{6}{5} \leq p < \frac{3}{2}$: As $h \in L^p(\Omega)$, by the regularity of the Poisson equation we have $\theta \in W^{2,p}(\Omega) \hookrightarrow L^{p^{**}}(\Omega)$ with $\frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{3}$. In view of $p < \frac{3}{2}$, we have that $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$. Hence, $\theta\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ and $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$, consequently, thanks to the regularity of the Stokes equations, we have $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)$.

(ii) *Case* $p = \frac{3}{2}$: Since $\boldsymbol{u}\cdot\nabla\theta \in L^{\frac{3}{2}}(\Omega)$ and $h \in L^{\frac{3}{2}}(\Omega)$, by the regularity of the Poisson equation, we have $\theta \in W^{2,\frac{3}{2}}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$. Since $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$, then $\theta\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, and finally, by the regularity of the Stokes equations $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$ and $\pi \in W^{1,\frac{3}{2}}(\Omega)$.

(iii) *Case* $p > \frac{3}{2}$: From the previous case, we have that $(\boldsymbol{u},\theta) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{2,\frac{3}{2}}(\Omega)$. Note that $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^r(\Omega)$ for all $r \geq 1$. Further, as $\theta \in W^{2,\frac{3}{2}}(\Omega) \hookrightarrow W^{1,3}(\Omega)$, this implies that $\nabla\theta \in \boldsymbol{L}^3(\Omega)$, which results that $\boldsymbol{u}\cdot\nabla\theta \in L^s(\Omega)$ for all $1 \leq s < 3$.

(a) If $\frac{3}{2} < p < 3$, we have $h - \boldsymbol{u}\cdot\nabla\theta \in L^p(\Omega)$, and thanks to the regularity of the Poisson equation $\theta \in W^{2,p}(\Omega)$.
In the same way, since $p > \frac{3}{2}$, then $W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, and as $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, then $\theta\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$. Since $\nabla\boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$, by Hölder inequality we have $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^s(\Omega)$ for all $1 \leq s < 3$. It follows that $\theta\boldsymbol{g} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, and by regularity of the Stokes equations $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)$.

(b) Suppose now that $3 \leq p < \infty$. From the above result, we have that $(\boldsymbol{u},\theta) \in \boldsymbol{W}^{2,3-\delta}(\Omega) \times W^{2,3-\delta}(\Omega)$ for any $0 < \delta < \frac{3}{2}$. Then, we get that $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$ and $\nabla\theta \in \boldsymbol{W}^{1,3-\delta}(\Omega) \hookrightarrow \boldsymbol{L}^t(\Omega)$, for any $3 < t < \infty$. This implies that $\boldsymbol{u}\cdot\nabla\theta \in \boldsymbol{L}^p(\Omega)$ and as $h \in L^p(\Omega)$, thanks to the regularity of the Poisson equation we have that $\theta \in W^{2,p}(\Omega)$. Finally, $\theta\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ and, by the same process as was made before for $\nabla\theta$, we conclude that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, and by regularity of the Stokes equations $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)$. □

Remark 2.5.4. (i) As in Remark 2.5.2, the choice of the space for $\boldsymbol{g}$ is optimal in order to study the regularity $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$. Indeed, for the case $\frac{6}{5} \leq p < \frac{3}{2}$, we already know that $\theta \in W^{2,p}(\Omega) \hookrightarrow L^{p^{**}}(\Omega)$ with $\frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{3}$ and hence $\theta\boldsymbol{g} \in \boldsymbol{L}^s(\Omega)$ with $\frac{1}{s} = \frac{1}{p^{**}} + \frac{1}{r}$, where r is the number to determine. In order to use the regularity of the Stokes equations, we need to have $\theta\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, and this happens if s is at least equal to p. Then,

$$\frac{1}{r} = \frac{1}{s} - \frac{1}{p^{**}} = \frac{1}{p} - \frac{1}{p} + \frac{2}{3}$$

and hence $r = \frac{3}{2} = \max\left\{\frac{3}{2}, p\right\}$ because $\frac{6}{5} \leq p < \frac{3}{2}$.

In the same way, we can analyze the other cases of p, and therefore we conclude that r is optimal to obtain the regularity $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$.
(ii) With the help of the existence of very weak solutions of (*BS*), we will try later the regularity of the solution in $W^{1,p}(\Omega)$ with $\frac{3}{2} \leq p < 2$ and the regularity $W^{2,p}(\Omega)$ with $1 < p < \frac{6}{5}$.

2.6 Very weak solutions

In the article [35], Kim carried out a study of very weak solutions of the stationary Boussinesq system, considering the boundary of the domain connected, the gravitational acceleration $\boldsymbol{g}$ belonging to $\boldsymbol{L}^\infty(\Omega)$ and did not consider a heat source in the convection-diffusion equation.

In our work, we are interested in finding $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$ very weak solution of (BS) over a domain in $\mathbb{R}^3$ which does not necessarily have a connected boundary, and we also consider the gravitational acceleration $\boldsymbol{g}$ belongs to a weaker space and the presence of a heat source in the convection-diffusion equation. With these considerations, it is still possible to get the existence and uniqueness of the very weak solution of (BS), as we will see it later.

Let us define the following space:

$$\boldsymbol{W}^{1,p}_{0,\sigma}(\Omega) := \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}_0(\Omega); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}.$$

Remind that $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $\boldsymbol{W}^{1,p}_{0,\sigma}(\Omega)$, see [4, Theorem 2.9]. We also use the following notations:

$$\langle \boldsymbol{u_b}, \boldsymbol{\chi}\rangle_\Gamma := \langle \boldsymbol{u_b}, \boldsymbol{\chi}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)},$$

$$\langle \theta_b, \psi\rangle_\Gamma := \langle \theta_b, \psi\rangle_{W^{-\frac{1}{r},r}(\Gamma),W^{\frac{1}{r},r'}(\Gamma)}.$$

Assuming that

$$\boldsymbol{g} \in \boldsymbol{L}^s(\Omega), \quad h \in W^{-1,\frac{pr}{p+r}}(\Omega), \quad \theta_b \in W^{-\frac{1}{r},r}(\Gamma),$$

$$\boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \quad \text{such that} \quad < \boldsymbol{u_b} \cdot \boldsymbol{n}, 1 >_\Gamma = 0,$$

where

$$3 \le p < \infty, \;\; p' < r < \infty$$

and

$$s > r' \quad \text{if} \quad p = 3, \quad \text{and} \quad s = \frac{3rp}{2rp + 3(r-p)} \quad \text{if} \quad p > 3 \quad \text{and} \quad \frac{1}{r} \le \frac{1}{p} + \frac{2}{3}, \tag{2.25}$$

where, in this last case ($p > 3$), s is chosen such that

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{(p')^{**}} = 1 \quad \text{with} \quad \frac{1}{(p')^{**}} = \frac{1}{p'} - \frac{2}{3}; \tag{2.26}$$

we call a very weak solution of (BS) the pair $(\boldsymbol{u}, \theta) \in \boldsymbol{L}^p(\Omega) \times L^r(\Omega)$ such that satisfies the following equalities:

$$\int_\Omega \boldsymbol{u} \cdot (-\nu \Delta \boldsymbol{\chi} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\chi})\, \mathrm{d}x = \int_\Omega \theta \boldsymbol{g} \cdot \boldsymbol{\chi}\, \mathrm{d}x - \nu \left\langle \boldsymbol{u_b}, \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma \quad \forall \boldsymbol{\chi} \in \boldsymbol{W}^{2,p'}(\Omega) \cap \boldsymbol{W}^{1,p'}_{0,\sigma}(\Omega), \tag{2.27}$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla \varphi\, \mathrm{d}x = \langle \boldsymbol{u_b} \cdot \boldsymbol{n}, \varphi\rangle_\Gamma \quad \forall \varphi \in W^{1,p'}(\Omega),$$

$$\int_\Omega \theta \cdot (-\kappa \Delta \psi - \boldsymbol{u} \cdot \nabla \psi)\, \mathrm{d}x = \langle h, \psi\rangle_\Omega - \kappa \left\langle \theta_b, \frac{\partial \psi}{\partial \boldsymbol{n}} \right\rangle_\Gamma \quad \forall \psi \in W^{2,r'}(\Omega) \cap W^{1,r'}_0(\Omega).$$

Remark 2.6.1. In the case when $p > 3$, the restriction $\frac{1}{r} \le \frac{1}{p} + \frac{2}{3}$ is given to avoid that s takes negative values. Note also that $s \to \infty$ when $p \to \infty$; justifying why the data $\boldsymbol{g}$ is supposed to lie in $\boldsymbol{L}^\infty(\Omega)$ in the work [35]. But as we will see later, this hypothesis is too strong and is not required to prove existence, uniqueness and regularity of the solution to the Boussinesq system.

Remark 2.6.2. In order to see that the definition of very weak solution of the Boussinesq system is well-defined, we need to show that the following integrals have sense:

$$I_1 := \int_\Omega \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{\chi}\,\mathrm{d}x,\quad I_2 := \int_\Omega \theta\boldsymbol{g}\cdot\boldsymbol{\chi}\,\mathrm{d}x,\quad I_3 := \int_\Omega \theta(\boldsymbol{u}\cdot\nabla\psi)\,\mathrm{d}x.$$

Indeed, we have two cases:

(i) If $p=3$, then we have the following embeddings:

$$\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)\hookrightarrow \boldsymbol{L}^3(\Omega)\quad\text{and}\quad \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)\hookrightarrow \boldsymbol{L}^q(\Omega)\quad\text{for all}\quad 1\le q<\infty.$$

It follows that integrals I_1 and I_2 are finite. On the other hand, the embedding $\boldsymbol{W}^{1,r'}(\Omega)\hookrightarrow \boldsymbol{L}^{(r')^*}(\Omega)$ with $\frac{1}{(r')^*}=\frac{1}{r'}-\frac{1}{3}$ since $r>\frac{3}{2}$ by hypothesis, guarantees that I_3 is finite.

(ii) If $p>3$, we have that $\boldsymbol{W}^{1,p'}(\Omega)\hookrightarrow \boldsymbol{L}^{(p')^*}(\Omega)$ with $\frac{1}{(p')^*}=\frac{1}{p'}-\frac{1}{3}$, and then I_1 is finite. Thanks to (2.26), it is immediately that I_2 is finite. Finally, since $p>3$ then $r'<p$, and we must consider the three cases $r'<3$, $r'=3$ and $r'>3$ in order to show that I_3 is finite. When $r'<3$, $\boldsymbol{W}^{1,r'}(\Omega)\hookrightarrow \boldsymbol{L}^{(r')^*}(\Omega)$ with $\frac{1}{(r')^*}=\frac{1}{r'}-\frac{1}{3}$, and together with $p>3$ ensure that I_3 is finite. When $r'=3$, $\boldsymbol{W}^{1,r'}(\Omega)\hookrightarrow \boldsymbol{L}^q(\Omega)$ for all $1\le q<\infty$, and this tell us that I_3 is finite. And in the last case when $r'>3$, then $\boldsymbol{W}^{1,r'}(\Omega)\hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $p'<r<\infty$ guarantees that I_3 is finite.

We must also note that $\langle h,\psi\rangle_\Omega$ is well-defined. In fact, because $p\ge 3$ we have that $W^{2,r'}(\Omega)\hookrightarrow W^{1,\frac{pr'}{p-r'}}(\Omega)$ with $\frac{pr'}{p-r'}$ the conjugate exponent of $\frac{pr}{p+r}$. Consequently, it is sensible to define a very weak solution as we did it above.

Remark 2.6.3. From (2.27), we have that for all $\boldsymbol{\chi}\in \boldsymbol{W}_0^{2,p'}(\Omega)$ with $\operatorname{div}\boldsymbol{\chi}=0$:

$$\langle -\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\theta\boldsymbol{g},\boldsymbol{\chi}\rangle_{\boldsymbol{W}^{-2,p}(\Omega),\boldsymbol{W}_0^{2,p'}(\Omega)}=0,$$

and using a variant of De Rham's theorem, see [4, Lemma 2.7], it follows that there exists a unique $\pi\in W^{-1,p}(\Omega)/\mathbb{R}$ such that

$$-\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla\pi=\theta\boldsymbol{g},\ \text{ in }\ \boldsymbol{W}^{-2,p}(\Omega).$$

Hence, when we say that $(\boldsymbol{u},\theta)\in \boldsymbol{L}^p(\Omega)\times L^r(\Omega)$ is a very weak solution of (*BS*), we mean that $(\boldsymbol{u},\pi,\theta)\in \boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)\times L^r(\Omega)$ is a very weak solution of (*BS*).

Before to establish the existence of very weak solutions of the Boussinesq system, we need a result about very weak solutions of the convection-diffusion equation for heat transfer which will be useful later.

Proposition 2.6.4. *Let $3\le p<\infty$ and $p'<r<\infty$. Let us consider the following closed ball:*

$$\boldsymbol{B}=\left\{\boldsymbol{u}\in \boldsymbol{L}^p(\Omega);\frac{1}{\kappa}\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\le\rho_0\right\},$$

for some $\rho_0=\rho_0(p,r,\Omega)>0$. Then, for all $h\in W^{-1,\frac{pr}{p+r}}(\Omega)$, $\theta_b\in W^{-\frac{1}{r},r}(\Gamma)$ and $\boldsymbol{u}\in\boldsymbol{B}$, there exists a unique very weak solution $\theta=\mathcal{P}\boldsymbol{u}\in L^r(\Omega)$ of the following linear problem:

$$\begin{cases} -\kappa\Delta\theta+\boldsymbol{u}\cdot\nabla\theta=h & \text{in }\Omega,\\ \qquad\qquad\quad \theta=\theta_b & \text{on }\Gamma.\end{cases}\tag{CD}$$

Further, for any $\boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{B}$, we have the following estimates:

$$\|\mathcal{P}\boldsymbol{u}\|_{L^r(\Omega)} \leq C_0 \left(\frac{1}{\kappa} \|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)} \right), \tag{2.28}$$

$$\|\mathcal{P}\boldsymbol{u}_1 - \mathcal{P}\boldsymbol{u}_2\|_{L^r(\Omega)} \leq \frac{C_0}{\kappa} \left(\frac{1}{\kappa} \|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)} \right) \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)} \tag{2.29}$$

with $C_0 = C_0(p, r, \Omega) > 0$.

Proof. The proof is fairly similar to the given in [35, Lemma 3.1], but in our case $h \neq 0$.

Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ be fixed and $\theta \in L^r(\Omega)$. As $\boldsymbol{u} \cdot \nabla\theta = \operatorname{div}(\theta\boldsymbol{u}) \in W^{-1,\frac{pr}{p+r}}(\Omega)$, $h \in W^{-1,\frac{pr}{p+r}}(\Omega)$ and taking into account that $\frac{pr}{p+r} \geq \frac{3r}{3+r}$ since $p \geq 3$, by Proposition 2.3.6 there exists a unique very weak solution $\hat{\theta} = \mathcal{L}\theta \in L^r(\Omega)$ of

$$\begin{cases} -\kappa\Delta\hat{\theta} = h - \boldsymbol{u} \cdot \nabla\theta & \text{in } \Omega, \\ \theta = \theta_b & \text{on } \Gamma, \end{cases}$$

and satisfies the following estimate:

$$\|\mathcal{L}\theta\|_{L^r(\Omega)} \leq C \left(\frac{1}{\kappa} \|h - \boldsymbol{u} \cdot \nabla\theta\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)} \right)$$

for all $\theta \in L^r(\Omega)$ with $C = C(p, r, \Omega) > 0$. Then,

$$\|\mathcal{L}\theta\|_{L^r(\Omega)} \leq \frac{C}{\kappa} \|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \left(\frac{C_1}{\kappa} \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \right) \|\theta\|_{L^r(\Omega)} + C\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)} \tag{2.30}$$

with $C_1 = C_1(p, r, \Omega) > 0$. Furthermore, for any $\theta_1, \theta_2 \in L^r(\Omega)$ we have that

$$\begin{aligned} \|\mathcal{L}\theta_1 - \mathcal{L}\theta_2\|_{L^r(\Omega)} &\leq \frac{C}{\kappa} \|\boldsymbol{u} \cdot \nabla(\theta_1 - \theta_2)\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} \\ &= \frac{C}{\kappa} \|\operatorname{div}((\theta_1 - \theta_2)\boldsymbol{u})\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} \\ &\leq \frac{C_1}{\kappa} \|(\theta_1 - \theta_2)\boldsymbol{u}\|_{\boldsymbol{L}^{\frac{pr}{p+r}}(\Omega)}, \end{aligned}$$

and then

$$\|\mathcal{L}\theta_1 - \mathcal{L}\theta_2\|_{L^r(\Omega)} \leq \left(\frac{C_1}{\kappa} \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \right) \|\theta_1 - \theta_2\|_{L^r(\Omega)}. \tag{2.31}$$

Suppose that $\boldsymbol{u} \in \boldsymbol{B}$ with $\rho_0 = \frac{1}{2C_1}$. Then, by (2.31), we have that $\mathcal{L}$ is a contraction mapping on the Banach space $L^r(\Omega)$. Then, by the Banach fixed point theorem, $\mathcal{L}$ has a unique fixed point $\theta \in L^r(\Omega)$ and by (2.30) we have that

$$\|\theta\|_{L^r(\Omega)} \leq \frac{2C}{\kappa} \|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + 2C\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}.$$

Until here, we have that for any $\boldsymbol{u} \in \boldsymbol{B}$, there exists a unique $\theta = \mathcal{P}\boldsymbol{u} \in L^r(\Omega)$ solution of (*CD*) such that

$$\|\mathcal{P}\boldsymbol{u}\|_{L^r(\Omega)} \leq \frac{2C}{\kappa} \|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + 2C\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}.$$

Suppose now $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{B}$ and let us define $\theta_1 = \mathcal{P}\boldsymbol{u}_1$ and $\theta_2 = \mathcal{P}\boldsymbol{u}_2$. Then,

$$
\begin{aligned}
\|\theta_1 - \theta_2\|_{L^r(\Omega)} &\le \frac{C}{\kappa}\|\boldsymbol{u}_1\cdot\nabla\theta_1 - \boldsymbol{u}_2\cdot\nabla\theta_2\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} \\
&= \frac{C}{\kappa}\|\mathrm{div}(\theta_1\boldsymbol{u}_1 - \theta_2\boldsymbol{u}_2)\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} \\
&\le \frac{C_1}{\kappa}\|\theta_1\boldsymbol{u}_1 - \theta_2\boldsymbol{u}_2\|_{\boldsymbol{L}^{\frac{pr}{p+r}}(\Omega)} \\
&\le \frac{C_1}{\kappa}\left(\|\theta_1\|_{L^r(\Omega)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)} + \|\theta_1-\theta_2\|_{L^r(\Omega)}\|\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}\right) \\
&\le \frac{C_1}{\kappa}\left(\frac{2C}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + 2C\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)} \\
&\quad + \frac{1}{2}\|\theta_1 - \theta_2\|_{L^r(\Omega)},
\end{aligned}
$$

and so,

$$
\|\mathcal{P}\boldsymbol{u}_1 - \mathcal{P}\boldsymbol{u}_2\|_{L^r(\Omega)} \le \frac{4CC_1}{\kappa}\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)},
$$

and taking $C_0 = \max\{2C, 4CC_1\}$, the proof is finished. $\square$

The next proposition is devoted to show the existence of at least one very weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$ of (BS) under smallness considerations of the data.

Proposition 2.6.5. *For $3 \le p < \infty$ and $p' < r < \infty$, there exists $\rho_1 = \rho_1(p, r, \Omega) > 0$ such that if*

$$
\boldsymbol{g} \in \boldsymbol{L}^s(\Omega), \quad h \in W^{-1,\frac{pr}{p+r}}(\Omega), \quad \theta_b \in W^{-\frac{1}{r},r}(\Gamma),
$$
$$
\boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \quad \textit{such that} \quad <\boldsymbol{u_b}\cdot\boldsymbol{n}, 1>_\Gamma = 0,
$$

where s satisfies (2.25) and

$$
\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right) \le \rho_1, \tag{2.32}
$$

then there exists at least one very weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$ of (BS) satisfying the following estimates:

$$
\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le C\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right), \tag{2.33}
$$

$$
\|\theta\|_{L^r(\Omega)} \le C\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right), \tag{2.34}
$$

with $C = C(p, r, \Omega) > 0$.

Proof. The proof is similar to the given in [35, Theorem 1.5], but we consider $h \ne 0$ and $\boldsymbol{g}$ lies in a weaker space than $\boldsymbol{L}^\infty(\Omega)$.

Let ρ_0, C_0, $\boldsymbol{B}$ and $\mathcal{P}$ be the constants, closed ball and solution operator established in Proposition 2.6.4. Given $\boldsymbol{u} \in \boldsymbol{B}$, there exists a unique very weak solution $\bar{\boldsymbol{u}} = \mathcal{T}\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ of the following Stokes problem:

$$\begin{cases} -\nu\Delta\bar{\boldsymbol{u}} + \nabla\bar{\pi} = \boldsymbol{g}\mathcal{P}\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div}\bar{\boldsymbol{u}} = 0 & \text{in } \Omega, \\ \bar{\boldsymbol{u}} = \boldsymbol{u_b} & \text{on } \Gamma. \end{cases} \tag{2.35}$$

Indeed, since the Stokes equations are linear, it is possible to split the problem in two parts:

$$\begin{cases} -\nu\Delta\bar{\boldsymbol{u}}_1 + \nabla\bar{\pi}_1 = \boldsymbol{g}\mathcal{P}\boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div}\bar{\boldsymbol{u}}_1 = 0 & \text{in } \Omega, \\ \bar{\boldsymbol{u}}_1 = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{S_1}$$

$$\begin{cases} -\nu\Delta\bar{\boldsymbol{u}}_2 + \nabla\bar{\pi}_2 = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div}\bar{\boldsymbol{u}}_2 = 0 & \text{in } \Omega, \\ \bar{\boldsymbol{u}}_2 = \boldsymbol{u_b} & \text{on } \Gamma. \end{cases} \tag{S_2}$$

Let us note that $\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}_1 + \bar{\boldsymbol{u}}_2$ and $\bar{\pi} = \bar{\pi}_1 + \bar{\pi}_2$ is the unique very weak solution for (2.35). So, let us focus in the solutions of (S_1) and (S_2). Then, since s satisfies (2.25), we must consider two cases:

(i) If $p = 3$ and $r > \frac{3}{2}$, then $s > r'$, and so $\boldsymbol{g}\mathcal{P}\boldsymbol{u} \in \boldsymbol{L}^{\frac{rs}{r+s}}(\Omega)$ with $\frac{rs}{r+s} > 1$, which implies that (S_1) has a unique solution $\bar{\boldsymbol{u}}_1 \in \boldsymbol{W}^{2,\frac{rs}{r+s}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. As $\boldsymbol{u}\otimes\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, then $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) \in (\boldsymbol{X}_{3,\frac{3}{2}}(\Omega))'$ and using Proposition 2.3.7, we have that (S_2) has a unique solution $\bar{\boldsymbol{u}}_2 \in \boldsymbol{L}^3(\Omega)$.

(ii) If $p > 3$ and $r > p'$, then $s = \frac{3rp}{2rp+3(r-p)}$, and so $\boldsymbol{g}\mathcal{P}\boldsymbol{u} \in \boldsymbol{L}^{\frac{3p}{3+2p}}(\Omega)$ which implies that (S_1) has a unique solution $\bar{\boldsymbol{u}}_1 \in \boldsymbol{W}^{2,\frac{3p}{3+2p}}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$. In view of $\boldsymbol{u}\otimes\boldsymbol{u} \in \boldsymbol{L}^{\frac{p}{2}}(\Omega)$, then $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) \in (\boldsymbol{X}_{\frac{p}{p-2},p'}(\Omega))'$ and using Proposition 2.3.7 again, we have that (S_2) has a unique solution $\bar{\boldsymbol{u}}_2 \in \boldsymbol{L}^p(\Omega)$. Note that the hypothesis about the exponents in Proposition 2.3.7 are satisfied because $\frac{2}{p} \leq \frac{1}{p} + \frac{1}{3}$ and $\frac{p}{2} \leq p$.

Thanks to estimates for the Stokes problem, this solution satisfies that

$$\begin{aligned} \|\mathcal{T}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &\leq C_1\left(\frac{1}{\nu}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^p(\Omega)} + \frac{1}{\nu}\|\boldsymbol{g}\mathcal{P}\boldsymbol{u}\|_{\boldsymbol{L}^{\frac{rs}{r+s}}(\Omega)} + \|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) \\ &\leq C_1\left(\frac{1}{\nu}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^p(\Omega)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|\mathcal{P}\boldsymbol{u}\|_{L^r(\Omega)} + \|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) \end{aligned}$$

with $C_1 = C_1(p,r,\Omega) > 0$, and by (2.28), we have

$$\begin{aligned} \|\mathcal{T}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_2\Bigg(&\frac{1}{\nu}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^p(\Omega)} + \frac{1}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)} \\ &+\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\Bigg) \end{aligned} \tag{2.36}$$

for some $C_2 = C_2(p,r,\Omega) > 1$.

On the other hand, taking account that $(\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u}_1 - (\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u}_2 = \operatorname{div}(\boldsymbol{u}_1\otimes\boldsymbol{u}_1 - \boldsymbol{u}_2\otimes\boldsymbol{u}_2)$, we have for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{B}$

$$\|\mathcal{T}\boldsymbol{u}_1 - \mathcal{T}\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)} \leq C_1\left[\frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|\mathcal{P}\boldsymbol{u}_1 - \mathcal{P}\boldsymbol{u}_2\|_{L^r(\Omega)} + \frac{1}{\nu}\|\boldsymbol{u}_1\otimes\boldsymbol{u}_1 - \boldsymbol{u}_2\otimes\boldsymbol{u}_2\|_{\boldsymbol{L}^{\frac{p}{2}}(\Omega)}\right],$$

and by using (2.29), we have

$$\|\mathcal{T}\boldsymbol{u}_1-\mathcal{T}\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}\le C_2\Bigg[\frac{1}{\nu\kappa^2}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)}+\frac{1}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}$$
$$+\frac{1}{\nu}\left(\|\boldsymbol{u}_1\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}\right)\Bigg]\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}$$
$$\le C_2\Bigg[\frac{1}{\kappa}\left(\frac{1}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)}+\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}+\frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)$$
$$+\frac{1}{\nu}\left(\|\boldsymbol{u}_1\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}\right)\Bigg]\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}$$

and then

$$\|\mathcal{T}\boldsymbol{u}_1-\mathcal{T}\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}\le C_2\left[\frac{1}{\nu}\left(\|\boldsymbol{u}_1\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}\right)+\frac{1}{\kappa}\mathcal{R}(\boldsymbol{g},h,\boldsymbol{u_b},\theta_b)\right]\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}, \tag{2.37}$$

where

$$\mathcal{R}(\boldsymbol{g},h,\boldsymbol{u_b},\theta_b):=\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}+\frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\Big(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)}+\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\Big).$$

Let us define

$$\boldsymbol{B_1}=\left\{\boldsymbol{w}\in\boldsymbol{L}^p(\Omega):\|\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}\le\frac{3}{2}C_2\mathcal{R}(\boldsymbol{g},h,\boldsymbol{u_b},\theta_b)\right\}.$$

Consequently, if $\boldsymbol{u}\in\boldsymbol{B_1}$ and using (2.32), we have that

$$\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\le\frac{3}{2}C_2\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\mathcal{R}(\boldsymbol{g},h,\boldsymbol{u_b},\theta_b)\le\frac{3}{2}C_2\rho_1,$$

and choosing ρ_1 such that $\rho_1\le\frac{2\rho_0}{3C_2}$, it follows that $\boldsymbol{u}\in\boldsymbol{B}$. Then, returning to (2.36) and using this last inequality, we have for each $\boldsymbol{u}\in\boldsymbol{B_1}$

$$\|\mathcal{T}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\le\left(\frac{C_2}{\nu}\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\right)\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}+C_2\Bigg[\frac{1}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)}$$
$$+\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}+\frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\Bigg]$$
$$\le C_2\mathcal{R}(\boldsymbol{g},h,\boldsymbol{u_b},\theta_b)+\frac{3}{2}C_2^2\rho_1\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$$
$$\le\left(\frac{9}{4}C_2^3\rho_1+C_2\right)\mathcal{R}(\boldsymbol{g},h,\boldsymbol{u_b},\theta_b).$$

Taking $\rho_1=\min\left\{\frac{2\rho_0}{3C_2},\frac{2}{9C_2^2}\right\}$, we deduce that

$$\|\mathcal{T}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\le\frac{3}{2}C_2\mathcal{R}(\boldsymbol{g},h,\boldsymbol{u_b},\theta_b);$$

hence $\mathcal{T}$ is an operator defined from the complete metric space $\boldsymbol{B_1}$ into $\boldsymbol{B_1}$. Further, using (2.37) and again (2.32) and remembering that $C_2 > 1$, we have that for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{B_1}$

$$\begin{aligned} \|\mathcal{T}\boldsymbol{u}_1 - \mathcal{T}\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)} &\leq C_2(1+3C_2)\rho_1\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}, \\ &\leq \frac{8}{9}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Then, thanks to the Banach fixed point theorem, $\mathcal{T}$ has a unique fixed point $\boldsymbol{u} \in \boldsymbol{B_1}$. Defining $\theta = \mathcal{P}\boldsymbol{u}$ and using Remark 2.6.3, we have that $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$ is a very weak solution of (BS). The estimates for $\boldsymbol{u}$ and θ follow from the definition of $\boldsymbol{B_1}$ and the estimate (2.28), respectively. □

Remark 2.6.6. (i) For the proof of Proposition 2.6.5, we do not use the fact that the boundary Γ could be non-connected. This means that this proof is valid for connected or non-connected boundary.
(ii) In the proofs of Proposition 2.6.4 and Proposition 2.6.5, we use the existence of very weak solutions of the Poisson equation (Proposition 2.3.6) and the Stokes equations (Proposition 2.3.7), and note that the spaces, where the right hand sides lie in these propositions, are different from the ones used in [35, Theorem 2.2 and Theorem 2.1].

The following two auxiliary results (Proposition 2.6.7 and Proposition 2.6.8) will help us to show Theorem 2.6.10. The first auxiliary result is concerning weak solutions of a perturbed Boussinesq system and the second one is about the existence and uniqueness of strong solutions to a linear problem. Proposition 2.6.7 will allow us to show the existence of a very weak solution of the Boussinesq system and Proposition 2.6.8 will help us to show the uniqueness of such solution.

Proposition 2.6.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ and let $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, $\boldsymbol{u}_a \in \boldsymbol{L}^3(\Omega)$, $\theta_a \in L^3(\Omega)$ given vectors and scalar fields, where $\operatorname{div} \boldsymbol{u}_a = 0$ in Ω. There exists $\rho_2 = \rho_2(\Omega) > 0$ such that if*

$$\frac{1}{\nu}\left(\sum_{i=1}^{m}\left|\int_{\Gamma_i} \boldsymbol{u_b}\cdot\boldsymbol{n}\,\mathrm{d}s\right| + \|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_a\|_{L^3(\Omega)}\right) \leq \rho_2, \tag{2.38}$$

where $\boldsymbol{u_b} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ satisfies $\int_\Gamma \boldsymbol{u_b}\cdot\boldsymbol{n}\,\mathrm{d}s = 0$, then for each $h \in H^{-1}(\Omega)$ and $\theta_b \in H^{\frac{1}{2}}(\Gamma)$, there exists at least one weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ of the following problem:

$$\left\{\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}_a\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_a + \nabla\pi = \theta\boldsymbol{g} &\quad \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 &\quad \text{in } \Omega, \\ -\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta + \boldsymbol{u}\cdot\nabla\theta_a + \boldsymbol{u}_a\cdot\nabla\theta = h &\quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u_b}\ ,\ \theta = \theta_b &\quad \text{on } \Gamma. \end{aligned}\right. \tag{BS_P}$$

Proof. This result can be proven by adapting Kim's proof to our case, see [35, Proposition 4.1].

Let us define $\boldsymbol{H} := \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ as the Hilbert space equipped with the usual norm. Let $(\boldsymbol{u}, \theta) \in \boldsymbol{H}$ given. Since $\boldsymbol{u}_a \in \boldsymbol{L}^3(\Omega)$ and $\theta_a \in L^3(\Omega)$, we have that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}_a\cdot\nabla)\boldsymbol{u} +$

$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_a = \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}+\boldsymbol{u}_a\otimes\boldsymbol{u}+\boldsymbol{u}\otimes\boldsymbol{u}_a)\in \boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{u}\cdot\nabla\theta+\boldsymbol{u}\cdot\nabla\theta_a+\boldsymbol{u}_a\cdot\nabla\theta = \operatorname{div}(\theta\boldsymbol{u}+\theta_a\boldsymbol{u}+\theta\boldsymbol{u}_a)\in H^{-1}(\Omega)$. Then, there exists a unique $(\hat{\boldsymbol{u}},\hat{\theta},\hat{\pi})\in \boldsymbol{H}\times L^2(\Omega)/\mathbb{R}$ weak solution of the following system:

$$\begin{cases} -\nu\Delta\hat{\boldsymbol{u}}+\nabla\hat{\pi} = \hat{\theta}\boldsymbol{g}-[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+(\boldsymbol{u}_a\cdot\nabla)\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_a] & \text{in } \Omega,\\ \operatorname{div}\hat{\boldsymbol{u}}=0 & \text{in } \Omega,\\ -\kappa\Delta\hat{\theta} = h-(\boldsymbol{u}\cdot\nabla\theta+\boldsymbol{u}\cdot\nabla\theta_a+\boldsymbol{u}_a\cdot\nabla\theta) & \text{in } \Omega,\\ \hat{\boldsymbol{u}}=\boldsymbol{u_b}\ ,\ \hat{\theta}=\theta_b & \text{on } \Gamma. \end{cases} \tag{2.39}$$

Let $\mathcal{S}:\boldsymbol{H}\longrightarrow\boldsymbol{H}$ be the operator such that $(\hat{\boldsymbol{u}},\hat{\theta})=\mathcal{S}(\boldsymbol{u},\theta)$ is the unique weak solution to (2.39). We are interested in finding a fixed point of the operator $\mathcal{S}$. In such a way, we are going to use the Leray-Schauder fixed point theorem.

(i) *Let us show that the operator $\mathcal{S}$ is compact.* Suppose $(\boldsymbol{u},\theta)\in\boldsymbol{H}$, $(\boldsymbol{u}_n,\theta_n)\in\boldsymbol{H}$, $n\in\mathbb{N}$ and $(\boldsymbol{u}_n,\theta_n)\rightharpoonup(\boldsymbol{u},\theta)$, in $\boldsymbol{H}$-weak. Defining $(\hat{\boldsymbol{u}}_n,\hat{\theta}_n):=\mathcal{S}(\boldsymbol{u}_n,\theta_n)$ for all $n\in\mathbb{N}$, by applying Lemma 2.3.8 for all $\varepsilon>0$ and by the identities $\boldsymbol{v}\cdot\nabla\varphi=\operatorname{div}(\varphi\boldsymbol{v})$ and $(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}=\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w})$, we obtain that $(\hat{\boldsymbol{u}}_n-\hat{\boldsymbol{u}},\hat{\theta}_n-\hat{\theta})$ satisfies the following inequalities:

$$\begin{aligned} \kappa\|\hat{\theta}_n-\hat{\theta}\|_{H^1(\Omega)} &\le C_1\left(\|\theta_n\boldsymbol{u}_n-\theta\boldsymbol{u}+\theta_a(\boldsymbol{u}_n-\boldsymbol{u})+(\theta_n-\theta)\boldsymbol{u}_a\|_{\boldsymbol{L}^2(\Omega)}\right)\\ &\le C_1\big(\|\theta_n-\theta\|_{L^3(\Omega)}\|\boldsymbol{u}_n\|_{\boldsymbol{L}^6(\Omega)}+\|\theta\|_{L^6(\Omega)}\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}+\varepsilon\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\\ &\quad+\alpha_\varepsilon\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}+\varepsilon\|\theta_n-\theta\|_{H^1(\Omega)}+\beta_\varepsilon\|\theta_n-\theta\|_{L^2(\Omega)}\big)\\ &\le C_1\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)}\|\theta_n-\theta\|_{L^3(\Omega)}+C_1\|\theta\|_{H^1(\Omega)}\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}+\varepsilon\|(\boldsymbol{u}_n-\boldsymbol{u},\theta_n-\theta)\|_{\boldsymbol{H}}\\ &\quad+C_\varepsilon\|(\boldsymbol{u}_n-\boldsymbol{u},\theta_n-\theta)\|_{\boldsymbol{L}^2(\Omega)\times L^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} \nu\|\hat{\boldsymbol{u}}_n-\hat{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)} &\le C_2\Big(\|(\hat{\theta}_n-\hat{\theta})\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}+\|\boldsymbol{u}_n\otimes\boldsymbol{u}_n-\boldsymbol{u}\otimes\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\\ &\quad+\|\boldsymbol{u}_a\otimes(\boldsymbol{u}_n-\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}+\|(\boldsymbol{u}_n-\boldsymbol{u})\otimes\boldsymbol{u}_a\|_{\boldsymbol{L}^2(\Omega)}\Big)\\ &\le C_2\Big(\|\hat{\theta}_n-\hat{\theta}\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\\ &\quad+\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}+2\varepsilon\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}+2D_\varepsilon\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\Big)\\ &\le C_2\|\hat{\theta}_n-\hat{\theta}\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+C_2(\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)})\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\\ &\quad+2\varepsilon\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}+2D_\varepsilon\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \end{aligned}$$

with $C_\varepsilon=C_\varepsilon\left(\varepsilon,\Omega,\|\theta_a\|_{L^3(\Omega)},\|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)}\right)>0$, $D_\varepsilon=D_\varepsilon\left(\varepsilon,\Omega,\|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)}\right)>0$, $C_1=C_1(\Omega)>0$ and $C_2=C_2(\Omega)>0$.

Since $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$, in $\boldsymbol{H^1(\Omega)}$-weak and $\theta_n\rightharpoonup\theta$, in $H^1(\Omega)$-weak, therefore, $\boldsymbol{u}_n\to\boldsymbol{u}$, in $\boldsymbol{L}^t(\Omega)$ and $\theta_n\to\theta$, in $L^t(\Omega)$, for $1\le t<6$. Also, $\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)}$ and $\|\theta_n\|_{H^1(\Omega)}$ are bounded sequences, so, letting $n\to\infty$, we have that

$$\begin{aligned} \kappa\limsup_{n\to\infty}\|\hat{\theta}_n-\hat{\theta}\|_{H^1(\Omega)} &\le C\varepsilon+C_\varepsilon\limsup_{n\to\infty}\|(\boldsymbol{u}_n-\boldsymbol{u},\theta_n-\theta)\|_{\boldsymbol{L}^2(\Omega)\times L^2(\Omega)}\\ &=C\varepsilon \end{aligned}$$

for all $\varepsilon>0$. This implies that

$$\hat{\theta}_n\underset{n\to\infty}{\longrightarrow}\hat{\theta},\ \text{in } H^1(\Omega). \tag{2.40}$$

In the same way, we have that

$$\begin{aligned}\nu \limsup_{n\to\infty} \|\hat{\boldsymbol{u}}_n - \hat{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)} &\leq 2C\varepsilon + 2D_\varepsilon \limsup_{n\to\infty} \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \\ &= 2C\varepsilon\end{aligned}$$

for all $\varepsilon > 0$, and then

$$\hat{\boldsymbol{u}}_n \underset{n\to\infty}{\longrightarrow} \hat{\boldsymbol{u}}, \text{ in } \boldsymbol{H}^1(\Omega). \tag{2.41}$$

By (2.40) and (2.41), we deduce that $(\hat{\boldsymbol{u}}_n, \hat{\theta}_n) \to (\hat{\boldsymbol{u}}, \hat{\theta})$, in $\boldsymbol{H}$ and therefore, $\mathcal{S}$ is a compact operator in $\boldsymbol{H}$.

(ii) *Let us show that all the fixed points of the operator $\alpha\mathcal{S}$ are bounded by the same constant C for all $\alpha \in [0,1]$.* Let $(\boldsymbol{u}, \theta) = \alpha\mathcal{S}(\boldsymbol{u}, \theta)$ with $(\boldsymbol{u}, \theta) \in \boldsymbol{H}$ and $\alpha \in [0,1]$. Then $(\hat{\boldsymbol{u}}, \hat{\theta}) = \mathcal{S}(\boldsymbol{u}, \theta)$ satisfies the following system:

$$\left\{\begin{aligned} -\nu\Delta\hat{\boldsymbol{u}} + \nabla\hat{\pi} = \hat{\theta}\boldsymbol{g} - \alpha\left[(\alpha\hat{\boldsymbol{u}}\cdot\nabla)\hat{\boldsymbol{u}} + (\boldsymbol{u}_a\cdot\nabla)\hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{u}_a\right] \quad &\text{in } \Omega, \\ \operatorname{div} \hat{\boldsymbol{u}} = 0 \quad &\text{in } \Omega, \\ -\kappa\Delta\hat{\theta} = h - \alpha\left[\alpha\hat{\boldsymbol{u}}\cdot\nabla\hat{\theta} + \hat{\boldsymbol{u}}\cdot\nabla\theta_a + \boldsymbol{u}_a\cdot\nabla\hat{\theta}\right] \quad &\text{in } \Omega, \\ \hat{\boldsymbol{u}} = \boldsymbol{u_b}\ ,\ \hat{\theta} = \theta_b \quad &\text{on } \Gamma. \end{aligned}\right. \tag{2.42}$$

We shall consider two cases depending on the values of the boundary data.

(a) *Case $\boldsymbol{u_b} = \boldsymbol{0}$ and $\theta_b = 0$.* In this case, note that $\hat{\boldsymbol{u}} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ and $\hat{\theta} \in H^1_0(\Omega)$. Then multiplying by $\hat{\boldsymbol{u}}$ and by $\hat{\theta}$ the first and third equations of (2.42), respectively, and integrating by parts, we have

$$\nu \int_\Omega |\nabla\hat{\boldsymbol{u}}|^2 \,\mathrm{d}x = \int_\Omega \hat{\theta}\boldsymbol{g}\cdot\hat{\boldsymbol{u}} \,\mathrm{d}x + \alpha B\left(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}, \boldsymbol{u_a}\right),$$

$$\kappa \int_\Omega |\nabla\hat{\theta}|^2 \,\mathrm{d}x = \langle h, \hat{\theta}\rangle_\Omega + \alpha\, b\left(\hat{\boldsymbol{u}}, \hat{\theta}, \theta_a\right).$$

We have immediately that

$$\|\nabla\hat{\theta}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{D}{\kappa}\left(\|h\|_{H^{-1}(\Omega)} + \|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\|\theta_a\|_{L^3(\Omega)}\right), \tag{2.43}$$

$$\|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{D}{\nu}\left(\|\nabla\hat{\theta}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)}\right),$$

with $D = D(\Omega) > 0$. Then, by using (2.43) and (2.38), we have that

$$\begin{aligned}\|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} &\leq \frac{D}{\nu}\Bigg[\frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|h\|_{H^{-1}(\Omega)} \\ &\qquad + \left(\|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_a\|_{L^3(\Omega)}\right)\|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\Bigg] \\ &\leq D\left(\frac{1}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|h\|_{H^{-1}(\Omega)} + \rho_2\|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\right).\end{aligned}$$

Taking $\rho_2 = \frac{1}{2D}$, it follows that

$$\|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{D}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|h\|_{H^{-1}(\Omega)}. \tag{2.44}$$

By applying (2.44), from (2.43) we obtain

$$\|\nabla\hat{\theta}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{D}{\kappa}\left(1+\frac{1}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_a\|_{L^3(\Omega)}\right)\|h\|_{H^{-1}(\Omega)}.$$

Finally,

$$\|(\boldsymbol{u},\theta)\|_{\boldsymbol{H}} = \alpha\|(\hat{\boldsymbol{u}},\hat{\theta})\|_{\boldsymbol{H}} \leq D\left(\|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\hat{\theta}\|_{\boldsymbol{L}^2(\Omega)}\right) \leq C,$$

where $C = C\left(\Omega,\nu,\kappa,\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)},\|h\|_{H^{-1}(\Omega)},\|\theta_a\|_{L^3(\Omega)}\right)$ is a positive constant independent of $(\boldsymbol{u},\theta)$ and α.

(b) *Case* $\boldsymbol{u_b}\neq\boldsymbol{0}$ *and* $\theta_b\neq 0$. Defining $\hat{\boldsymbol{u}}_\varepsilon = \hat{\boldsymbol{u}} - \boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon}$ and $\hat{\theta}_\eta = \hat{\theta} - \theta_b^\eta$, where $\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon}$ and θ_b^η are the lift functions of the boundary conditions given by Lemma 2.3.4 and Lemma 2.3.5, respectively, and using them in (2.42), we get

$$\left\{\begin{array}{rl}
-\nu\Delta\hat{\boldsymbol{u}}_\varepsilon + \alpha^2(\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla)\hat{\boldsymbol{u}}_\varepsilon + \alpha^2(\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla)\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon} + \alpha^2(\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon}\cdot\nabla)\hat{\boldsymbol{u}}_\varepsilon & \\
+\alpha(\boldsymbol{u}_a\cdot\nabla)\hat{\boldsymbol{u}}_\varepsilon + \alpha(\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla)\boldsymbol{u}_a + \nabla\hat{\pi} = \hat{\theta}_\eta\boldsymbol{g} + \boldsymbol{F} & \text{in } \Omega,\\
\operatorname{div}\hat{\boldsymbol{u}}_\varepsilon = 0 & \text{in } \Omega,\\
-\kappa\Delta\hat{\theta}_\eta + \alpha^2\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla\hat{\theta}_\eta + \alpha^2\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon}\cdot\nabla\hat{\theta}_\eta + \alpha\boldsymbol{u}_a\cdot\nabla\hat{\theta}_\eta = h & \\
-\alpha^2\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla\theta_b^\eta - \alpha\hat{\boldsymbol{u}}_\varepsilon\cdot\nabla\theta_a + G & \text{in } \Omega,\\
\hat{\boldsymbol{u}}_\varepsilon = \boldsymbol{0}\ ,\ \hat{\theta}_\eta = 0 & \text{on } \Gamma,
\end{array}\right. \tag{2.45}$$

where

$$\boldsymbol{F} = \boldsymbol{F}(\varepsilon,\eta) = \nu\Delta\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon} + \theta_b^\eta\boldsymbol{g} - \alpha^2(\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon}\cdot\nabla)\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon} - \alpha(\boldsymbol{u}_a\cdot\nabla)\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon} - \alpha(\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon}\cdot\nabla)\boldsymbol{u}_a \in \boldsymbol{H}^{-1}(\Omega)$$

and

$$G = G(\varepsilon,\eta) = \kappa\Delta\theta_b^\eta - \alpha^2\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon}\cdot\nabla\theta_b^\eta - \alpha\boldsymbol{u}_{\boldsymbol{b}}^{\varepsilon}\cdot\nabla\theta_a - \alpha\boldsymbol{u}_a\cdot\nabla\theta_b^\eta \in H^{-1}(\Omega).$$

Using $\hat{\boldsymbol{u}}_\varepsilon \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ and $\hat{\theta}_\eta \in H^1_0(\Omega)$ as test functions of the variational formulation of (2.45) and since $\operatorname{div}\boldsymbol{u}_a = 0$ in Ω, we have

$$\nu\int_\Omega |\nabla\hat{\boldsymbol{u}}_\varepsilon|^2\,\mathrm{d}x = \int_\Omega \hat{\theta}_\eta\boldsymbol{g}\cdot\hat{\boldsymbol{u}}_\varepsilon\,\mathrm{d}x - \alpha^2 B\left(\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\boldsymbol{u}_{\boldsymbol{b}}^{\boldsymbol{\varepsilon}},\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}\right) + \alpha B\left(\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\boldsymbol{u}_{\boldsymbol{a}}\right) + \langle\boldsymbol{F},\hat{\boldsymbol{u}}_\varepsilon\rangle_\Omega, \tag{2.46}$$

$$\kappa\int_\Omega |\nabla\hat{\theta}_\eta|^2\,\mathrm{d}x = \langle h,\hat{\theta}_\eta\rangle_\Omega + \langle G,\hat{\theta}_\eta\rangle_\Omega + \alpha\, b\left(\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}},\hat{\theta}_\eta,\alpha\theta_b^\eta + \theta_a\right). \tag{2.47}$$

Then, from (2.47), and using (2.5), Lemma 2.3.1 and $0\leq\alpha\leq 1$, it follows that

$$\kappa\|\nabla\hat{\theta}_\eta\|_{L^2(\Omega)} \leq C_1\left(\|h\|_{H^{-1}(\Omega)} + \|G\|_{H^{-1}(\Omega)} + \left(\|\theta_b^\eta\|_{L^3(\Omega)} + \|\theta_a\|_{L^3(\Omega)}\right)\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\right) \tag{2.48}$$

with $C_1 = C_1(\Omega) > 0$. From (2.46), (2.48), (2.6), Lemma 2.3.1 and defining $K = K(\Omega) :=$

$\max_{0\leq i\leq m} K_i$, it follows that

$$\begin{aligned}
\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} &\leq \frac{C_2}{\nu}\Big[\|\nabla\hat{\theta}_\eta\|_{L^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)} + \Big\{\varepsilon + \|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)} \\
&\quad + \sum_{i=0}^{m} K_i\left|\int_{\Gamma_i}\boldsymbol{u_b}\cdot\boldsymbol{n}\,\mathrm{d}s\right|\Big\}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\Big] \\
&\leq \frac{C_2}{\nu}\Big[\frac{C_1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big(\|h\|_{H^{-1}(\Omega)} + \|G\|_{H^{-1}(\Omega)}\Big) + \|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)} + \Big\{\varepsilon + \|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)} \\
&\quad + K\sum_{i=0}^{m}\left|\int_{\Gamma_i}\boldsymbol{u_b}\cdot\boldsymbol{n}\,\mathrm{d}s\right| + \frac{C_1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big(\|\theta_b^\eta\|_{L^3(\Omega)} + \|\theta_a\|_{L^3(\Omega)}\Big)\Big\}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\Big] \\
&\leq \frac{C_2C_3}{\nu}\Big[\frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big(\|h\|_{H^{-1}(\Omega)} + \|G\|_{H^{-1}(\Omega)}\Big) + \|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)} + \Big\{\varepsilon + \|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)} \\
&\quad + \sum_{i=0}^{m}\left|\int_{\Gamma_i}\boldsymbol{u_b}\cdot\boldsymbol{n}\,\mathrm{d}s\right| + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big(\|\theta_b^\eta\|_{L^3(\Omega)} + \|\theta_a\|_{L^3(\Omega)}\Big)\Big\}\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\Big]
\end{aligned}$$

with $C_2 = C_2(\Omega) > 0$ and $C_3 = C_3(\Omega) = \max\{1, C_1, K\}$.

With a simple calculation, it is possible to see that $\|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)} \leq d_1(\varepsilon,\eta)$ and $\|G\|_{H^{-1}(\Omega)} \leq d_2(\varepsilon,\eta)$, where $d_1(\varepsilon,\eta)$ and $d_2(\varepsilon,\eta)$ are two positive constants independent of $(\boldsymbol{u},\theta)$ and α, defined by

$$d_1(\varepsilon,\eta) = C_4\left(\nu\|\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\|_{\boldsymbol{H}^1(\Omega)} + \|\theta_b^\eta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\|_{\boldsymbol{H}^1(\Omega)}\right),$$

$$\begin{aligned}
d_2(\varepsilon,\eta) = C_4\big(&\kappa\|\theta_b^\eta\|_{H^1(\Omega)} + \|\theta_b^\eta\|_{H^1(\Omega)}\|\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\|_{\boldsymbol{H}^1(\Omega)} + \|\theta_a\|_{L^3(\Omega)}\|\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\|_{\boldsymbol{H}^1(\Omega)} \\
&+\|\theta_b^\eta\|_{H^1(\Omega)}\|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)}\big),
\end{aligned}$$

with $C_4 = C_4(\Omega) > 0$. By using (2.7) and (2.38), we get

$$\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{A(\varepsilon,\eta)}{\nu}C_2C_3 + C_2C_3\left(\frac{\varepsilon}{\nu} + 2\rho_2 + \frac{\eta}{\nu\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right)\|\nabla\hat{\boldsymbol{u}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}$$

with $A(\varepsilon,\eta) = \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big(\|h\|_{H^{-1}(\Omega)} + d_2(\varepsilon,\eta)\Big) + d_1(\varepsilon,\eta)$. Taking $\eta = \frac{\nu\kappa}{8C_2C_3\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}}$, $\varepsilon = \frac{\nu}{8C_2C_3}$, $\rho_2 = \frac{1}{8C_2C_3}$, it follows that

$$\|(\boldsymbol{u},\theta)\|_{\boldsymbol{H}} \leq \|(\hat{\boldsymbol{u}},\hat{\theta})\|_{\boldsymbol{H}} \leq C,$$

where

$$C = C\left(\Omega,\nu,\kappa,\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)},\|h\|_{H^{-1}(\Omega)},\|\theta_a\|_{L^3(\Omega)},\|\boldsymbol{u}_a\|_{\boldsymbol{L}^3(\Omega)},\|\theta_b^\eta\|_{H^1(\Omega)},\|\boldsymbol{u}_{\boldsymbol{b}}^\varepsilon\|_{\boldsymbol{H}^1(\Omega)}\right)$$

is a positive constant independent of $(\boldsymbol{u},\theta)$ and α.

Then, by the Leray-Schauder fixed point theorem together with the De Rham's theorem, there exists at least one $(\boldsymbol{u},\theta,\pi) \in \boldsymbol{H}\times L^2(\Omega)$ such that (BS_P) is satisfied. □

Proposition 2.6.8. *Let $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{L}^3(\Omega)$, $\theta_2 \in L^3(\Omega)$ given vectors and scalar fields where* $\operatorname{div} \boldsymbol{u}_1 = 0$ *in Ω. There exists $\rho_3 = \rho_3(\Omega) > 0$ such that if*

$$\frac{1}{\nu}\left(\|\boldsymbol{u}_2\|_{\boldsymbol{L}^3(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\theta_2\|_{L^3(\Omega)}\right) \leq \rho_3, \tag{2.49}$$

then for every $\boldsymbol{f}_1 \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $f_2 \in L^{\frac{3}{2}}(\Omega)$, there exists a unique strong solution

$$(\boldsymbol{y}, \pi, z) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)/\mathbb{R} \times W^{2,\frac{3}{2}}(\Omega)$$

of the following linear problem:

$$\left\{\begin{array}{rl} -\nu\Delta\boldsymbol{y} - (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{y} - (\nabla\boldsymbol{y})^T\boldsymbol{u}_2 + \nabla\pi - \theta_2\nabla z = \boldsymbol{f}_1 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{y} = 0 & \text{in } \Omega, \\ -\kappa\Delta z - \boldsymbol{u}_1\cdot\nabla z - \boldsymbol{g}\cdot\boldsymbol{y} = f_2 & \text{in } \Omega, \\ \boldsymbol{y} = \boldsymbol{0}\ ,\ z = 0 & \text{on } \Gamma. \end{array}\right. \tag{2.50}$$

Proof. This result can be proven in a fairly similar way as in the proof given in [35, Lemma 5.1]. We want to emphasize that the main changes in the proof are concentrated in the terms related with the gravitational acceleration $\boldsymbol{g}$.

Let $\boldsymbol{W} = \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{2,\frac{3}{2}}(\Omega)$ be the reflexive Banach space equipped with the usual norm. Let $(\boldsymbol{y}, z) \in \boldsymbol{W}$. Since $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$, we have that $(\boldsymbol{u}_1\cdot\nabla)\boldsymbol{y}$, $(\nabla\boldsymbol{y})^T\boldsymbol{u}_2$, $\theta_2\nabla z$ are in $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$, and $\boldsymbol{u}_1\cdot\nabla z \in L^{\frac{3}{2}}(\Omega)$, $\boldsymbol{g}\cdot\boldsymbol{y} \in L^m(\Omega)$ for any $1 \leq m < 2$. Hence, it follows that there exists a unique $(\bar{\boldsymbol{y}}, \bar{\pi}, \bar{z}) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)/\mathbb{R} \times W^{2,\frac{3}{2}}(\Omega)$ solution of the Stokes-Poisson system

$$\left\{\begin{array}{rl} -\nu\Delta\bar{\boldsymbol{y}} + \nabla\bar{\pi} = \boldsymbol{f}_1 + (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{y} + (\nabla\boldsymbol{y})^T\boldsymbol{u}_2 + \theta_2\nabla z & \text{in } \Omega, \\ \operatorname{div} \bar{\boldsymbol{y}} = 0 & \text{in } \Omega, \\ -\kappa\Delta\bar{z} = f_2 + \boldsymbol{u}_1\cdot\nabla z + \boldsymbol{g}\cdot\boldsymbol{y} & \text{in } \Omega, \\ \bar{\boldsymbol{y}} = \boldsymbol{0}\ ,\ \bar{z} = 0 & \text{on } \Gamma. \end{array}\right. \tag{SP}$$

Let us define $\mathcal{S}$ the solution operator of (SP), i.e., for each $(\boldsymbol{y}, z) \in \boldsymbol{W}$, $(\bar{\boldsymbol{y}}, \bar{z}) = \mathcal{S}(\boldsymbol{y}, z) \in \boldsymbol{W}$. Let us note that a fixed point of $\mathcal{S}$ is a solution of (2.50). So, we are going to apply the Leray-Schauder fixed point theorem to find a fixed point of $\mathcal{S}$.

(i) *Let us prove that $\mathcal{S}$ is a compact operator.* Suppose $(\boldsymbol{y}, z), (\boldsymbol{y}_n, z_n) \in \boldsymbol{W}$, $n \in \mathbb{N}$ and $(\boldsymbol{y}_n, z_n) \rightharpoonup (\boldsymbol{y}, z)$, in $\boldsymbol{W}-$weak. Let us define $(\bar{\boldsymbol{y}}_n, \bar{z}_n) := \mathcal{S}(\boldsymbol{y}_n, z_n)$ for all $n \in \mathbb{N}$. Then, we obtain that $(\bar{\boldsymbol{y}}_n - \bar{\boldsymbol{y}}, \bar{z}_n - \bar{z})$ satisfies the following system:

$$\left\{\begin{array}{rl} -\nu\Delta(\bar{\boldsymbol{y}}_n - \bar{\boldsymbol{y}}) + \nabla(\bar{\pi}_n - \bar{\pi}) = (\boldsymbol{u}_1\cdot\nabla)(\boldsymbol{y}_n - \boldsymbol{y}) + (\nabla(\boldsymbol{y}_n - \boldsymbol{y}))^T\boldsymbol{u}_2 & \\ +\theta_2\nabla(z_n - z) & \text{in } \Omega, \\ \operatorname{div}(\bar{\boldsymbol{y}}_n - \bar{\boldsymbol{y}}) = 0 & \text{in } \Omega, \\ -\kappa\Delta(\bar{z}_n - \bar{z}) = \boldsymbol{u}_1\cdot\nabla(z_n - z) + \boldsymbol{g}\cdot(\boldsymbol{y}_n - \boldsymbol{y}) & \text{in } \Omega, \\ \bar{\boldsymbol{y}}_n - \bar{\boldsymbol{y}} = \boldsymbol{0}\ ,\ \bar{z}_n - \bar{z} = 0 & \text{on } \Gamma. \end{array}\right.$$

By using usual estimates for strong solutions to the Stokes equations and Lemma 2.3.8, it follows that

$$\begin{aligned} \nu\|\bar{\boldsymbol{y}}_n - \bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)} &\leq C\|(\boldsymbol{u}_1\cdot\nabla)(\boldsymbol{y}_n - \boldsymbol{y}) + (\nabla(\boldsymbol{y}_n - \boldsymbol{y}))^T\boldsymbol{u}_2 + \theta_2\nabla(z_n - z)\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \\ &\leq \varepsilon\|(\boldsymbol{y}_n - \boldsymbol{y}, z_n - z)\|_{\boldsymbol{W}} + C_\varepsilon\|(\boldsymbol{y}_n - \boldsymbol{y}, z_n - z)\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)\times W^{1,\frac{3}{2}}(\Omega)} \end{aligned}$$

for any $\varepsilon > 0$ with $C_\varepsilon = C\left(\varepsilon, \Omega, \|\boldsymbol{u}_1\|_{\boldsymbol{L}^3(\Omega)}, \|\boldsymbol{u}_2\|_{\boldsymbol{L}^3(\Omega)}, \|\theta_2\|_{L^3(\Omega)}\right) > 0$. As $W^{2,\frac{3}{2}}(\Omega)$ is embedded compactly in $W^{1,\frac{3}{2}}(\Omega)$ and $\|(\boldsymbol{y}_n - \boldsymbol{y}, z_n - z)\|_{\boldsymbol{W}}$ is bounded, we let $n \to \infty$ and it follows that

$$\begin{aligned}\nu \limsup_{n\to\infty} \|\bar{\boldsymbol{y}}_n - \bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)} &\leq C\varepsilon + C_\varepsilon \limsup_{n\to\infty} \|(\boldsymbol{y}_n - \boldsymbol{y}, z_n - z)\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)\times W^{1,\frac{3}{2}}(\Omega)} \\ &= C\varepsilon\end{aligned}$$

for all $\varepsilon > 0$. Then, we deduce that

$$\bar{\boldsymbol{y}}_n \underset{n\to\infty}{\longrightarrow} \bar{\boldsymbol{y}}, \text{ in } \boldsymbol{W}^{2,\frac{3}{2}}(\Omega).$$

On the other hand, by applying usual estimates for strong solutions to the Poisson equation and again Lemma 2.3.8, we have

$$\begin{aligned}\kappa\|\bar{z}_n - \bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} &\leq C\|\boldsymbol{u}_1 \cdot \nabla(z_n - z) + \boldsymbol{g}\cdot(\boldsymbol{y}_n - \boldsymbol{y})\|_{L^{\frac{3}{2}}(\Omega)} \\ &\leq \varepsilon\|z_n - z\|_{W^{2,\frac{3}{2}}(\Omega)} + D_\varepsilon\|z_n - z\|_{W^{1,\frac{3}{2}}(\Omega)} + C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{y}_n - \boldsymbol{y}\|_{\boldsymbol{L}^6(\Omega)}\end{aligned}$$

for any $\varepsilon > 0$ with $D_\varepsilon = D\left(\varepsilon, \Omega, \|\boldsymbol{u}_1\|_{\boldsymbol{L}^3(\Omega)}\right) > 0$. Since $W^{2,\frac{3}{2}}(\Omega)$ is embedded compactly in the spaces $W^{1,\frac{3}{2}}(\Omega)$ and $L^6(\Omega)$, and $\|z_n - z\|_{W^{2,\frac{3}{2}}(\Omega)}$ is bounded, we let $n \to \infty$ and then

$$\begin{aligned}\kappa \limsup_{n\to\infty} \|\bar{z}_n - \bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} &\leq C\varepsilon + D_\varepsilon \limsup_{n\to\infty} \|z_n - z\|_{W^{1,\frac{3}{2}}(\Omega)} \\ &\quad + C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \limsup_{n\to\infty} \|\boldsymbol{y}_n - \boldsymbol{y}\|_{\boldsymbol{L}^6(\Omega)} \\ &= C\varepsilon\end{aligned}$$

for all $\varepsilon > 0$. Then, we deduce that

$$\bar{z}_n \underset{n\to\infty}{\longrightarrow} \bar{z}, \text{ in } W^{2,\frac{3}{2}}(\Omega).$$

Therefore, $\mathcal{S}$ is a compact operator in $\boldsymbol{W}$.

(ii) *Let us show that all the fixed points of $\alpha\mathcal{S}$ with $\alpha \in [0,1]$, are bounded by a constant independent of $(\boldsymbol{y}, z)$ and α.* Let $(\boldsymbol{y}, z) = \alpha\mathcal{S}(\boldsymbol{y}, z)$ with $(\boldsymbol{y}, z) \in \boldsymbol{W}$. Then $(\bar{\boldsymbol{y}}, \bar{z}) = \mathcal{S}(\alpha\bar{\boldsymbol{y}}, \alpha\bar{z})$ satisfies the following system:

$$\left\{\begin{aligned} -\nu\Delta\bar{\boldsymbol{y}} + \nabla\bar{\pi} = \boldsymbol{f}_1 + \alpha(\boldsymbol{u}_1\cdot\nabla)\bar{\boldsymbol{y}} + \alpha(\nabla\bar{\boldsymbol{y}})^T\boldsymbol{u}_2 + \alpha\theta_2\nabla\bar{z} &\quad \text{in } \Omega, \\ \operatorname{div} \bar{\boldsymbol{y}} = 0 &\quad \text{in } \Omega, \\ -\kappa\Delta\bar{z} = f_2 + \alpha\boldsymbol{u}_1\cdot\nabla\bar{z} + \alpha\boldsymbol{g}\cdot\bar{\boldsymbol{y}} &\quad \text{in } \Omega, \\ \bar{\boldsymbol{y}} = \boldsymbol{0}\ ,\ \bar{z} = 0 &\quad \text{on } \Gamma. \end{aligned}\right. \tag{2.51}$$

Thanks to strong estimates for the Stokes and Poisson equations, and Lemma 2.3.8, we can get the following estimates:

$$\begin{aligned}\nu\|\bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)} &\leq C\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \varepsilon_1\|\bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)} + \varepsilon_2\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} \\ &\quad + C_1\left(\|\bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} + \|\bar{z}\|_{W^{1,\frac{3}{2}}(\Omega)}\right),\end{aligned} \tag{2.52}$$

$$\kappa\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} \leq C\|f_2\|_{L^{\frac{3}{2}}(\Omega)} + \varepsilon_2\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} + C_2\|\bar{z}\|_{W^{1,\frac{3}{2}}(\Omega)} + C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\bar{\boldsymbol{y}}\|_{\boldsymbol{L}^6(\Omega)} \tag{2.53}$$

for all $\varepsilon_1, \varepsilon_2 > 0$, and some positive constants $C = C(\Omega)$, $C_2 = C_2\left(\Omega, \varepsilon_2, \|\boldsymbol{u}_1\|_{\boldsymbol{L}^3(\Omega)}\right)$ and $C_1 = C_1\left(\Omega, \varepsilon_1, \varepsilon_2, \|\boldsymbol{u}_1\|_{\boldsymbol{L}^3(\Omega)}, \|\boldsymbol{u}_2\|_{\boldsymbol{L}^3(\Omega)}, \|\theta_2\|_{L^3(\Omega)}\right)$.

Thanks to a well-known compactness lemma, see [43, Lemma 5.1, p. 58], we have that

$$\|\bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} \leq \eta_1\|\bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)} + C_3\|\bar{\boldsymbol{y}}\|_{\boldsymbol{L}^2(\Omega)},$$

$$\|\bar{z}\|_{W^{1,\frac{3}{2}}(\Omega)} \leq \eta_2\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} + C_4\|\bar{z}\|_{L^2(\Omega)}$$

for all $\eta_1, \eta_2 > 0$, where $C_3 = C_3(\eta_1)$ and $C_4 = C_4(\eta_2)$ are positive constants. Taking $\eta_1 = \frac{\varepsilon_1}{C_1}$ and $\eta_2 = \frac{\varepsilon_2}{\max\{C_1, C_2\}}$ in the last inequalities, and returning to (2.52) and (2.53), it follows that

$$\begin{aligned}\nu\|\bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)} \leq{}& C\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + 2\varepsilon_1\|\bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)} + 2\varepsilon_2\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} + C_1C_3\|\bar{\boldsymbol{y}}\|_{\boldsymbol{L}^2(\Omega)} \\ &+ C_1C_4\|\bar{z}\|_{L^2(\Omega)},\end{aligned}$$

$$\kappa\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} \leq C\|f_2\|_{L^{\frac{3}{2}}(\Omega)} + 2\varepsilon_2\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} + C_2C_4\|\bar{z}\|_{L^2(\Omega)} + C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\bar{\boldsymbol{y}}\|_{\boldsymbol{L}^6(\Omega)}. \tag{2.54}$$

Since $H^1(\Omega) \hookrightarrow L^6(\Omega)$, and taking $\varepsilon_1 = \frac{\nu}{4}$ and $\varepsilon_2 = \frac{\kappa}{4}$, we have that

$$\frac{\nu}{2}\|\bar{\boldsymbol{y}}\|_{\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)} \leq C\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \frac{\kappa}{2}\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} + C_1C_3\|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)} + C_1C_4\|\bar{z}\|_{H^1(\Omega)}, \tag{2.55}$$

$$\frac{\kappa}{2}\|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} \leq C\|f_2\|_{L^{\frac{3}{2}}(\Omega)} + C_2C_4\|\bar{z}\|_{H^1(\Omega)} + C_5\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)} \tag{2.56}$$

with $C_5 = C_5(\Omega) > 0$. From (2.55) and (2.56), it follows that

$$\|(\bar{\boldsymbol{y}}, \bar{z})\|_{\boldsymbol{W}} \leq D\left(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|f_2\|_{L^{\frac{3}{2}}(\Omega)} + \|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)} + \|\bar{z}\|_{H^1(\Omega)}\right) \tag{2.57}$$

with $D = D\left(\Omega, \nu, \kappa, \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}, \|\boldsymbol{u}_1\|_{\boldsymbol{L}^3(\Omega)}, \|\boldsymbol{u}_2\|_{\boldsymbol{L}^3(\Omega)}, \|\theta_2\|_{L^3(\Omega)}\right) > 0$.

Then, we need to find some estimates for $\bar{\boldsymbol{y}}$ and $\bar{z}$ in the space $H^1(\Omega)$. Since

$$\bar{\boldsymbol{y}} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \cap \boldsymbol{W}^{1,\frac{3}{2}}_{0,\sigma}(\Omega) \hookrightarrow \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega),$$

$$\bar{z} \in W^{2,\frac{3}{2}}(\Omega) \cap W^{1,\frac{3}{2}}_0(\Omega) \hookrightarrow W^{1,2}_0(\Omega),$$

we can take $(\bar{\boldsymbol{y}}, \bar{z})$ as the test functions of the weak formulation of (2.51), and then because $\operatorname{div}\boldsymbol{u}_1 = 0$, we have

$$\nu\int_\Omega |\nabla\bar{\boldsymbol{y}}|^2\,\mathrm{d}x = \int_\Omega (\boldsymbol{f}_1 + \alpha(\nabla\bar{\boldsymbol{y}})^T\boldsymbol{u}_2 + \alpha\theta_2\nabla\bar{z})\cdot\bar{\boldsymbol{y}}\,\mathrm{d}x, \tag{2.58}$$

$$\kappa\int_\Omega |\nabla\bar{z}|^2\,\mathrm{d}x = \int_\Omega (f_2 + \alpha\boldsymbol{g}\cdot\bar{\boldsymbol{y}})\bar{z}\,\mathrm{d}x. \tag{2.59}$$

From (2.59), we have that

$$\|\bar{z}\|_{H^1(\Omega)} \leq \frac{D_1}{\kappa}\left(\|f_2\|_{L^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)}\right) \tag{2.60}$$

with $D_1 = D_1(\Omega) > 0$, and from (2.58), we have that

$$\|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{D_2}{\nu}\left(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}_2\|_{\boldsymbol{L}^3(\Omega)} + \|\theta_2\|_{L^3(\Omega)}\|\bar{z}\|_{H^1(\Omega)}\right) \tag{2.61}$$

with $D_2 = D_2(\Omega) > 0$. Using (2.60) in (2.61), we get

$$\begin{aligned}\|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{D_3}{\nu}&\left(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \frac{1}{\kappa}\|\theta_2\|_{L^3(\Omega)}\|f_2\|_{L^{\frac{3}{2}}(\Omega)}\right) + \\ &\frac{D_3}{\nu}\left(\|\boldsymbol{u}_2\|_{\boldsymbol{L}^3(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\theta_2\|_{L^3(\Omega)}\right)\|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)}\end{aligned}$$

with $D_3 = D_3(\Omega) > 0$. Taking $\rho_3 = \frac{1}{2D_3}$ in (2.49), we have that

$$\|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)} \leq 2\frac{D_3}{\nu}\left(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \frac{1}{\kappa}\|\theta_2\|_{L^3(\Omega)}\|f_2\|_{L^{\frac{3}{2}}(\Omega)}\right),$$

and

$$\|\bar{z}\|_{H^1(\Omega)} \leq \frac{D_1}{\kappa}\left(\|f_2\|_{L^{\frac{3}{2}}(\Omega)} + 2\frac{D_3}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\left(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \frac{1}{\kappa}\|\theta_2\|_{L^3(\Omega)}\|f_2\|_{L^{\frac{3}{2}}(\Omega)}\right)\right).$$

Thus,

$$\|\bar{\boldsymbol{y}}\|_{\boldsymbol{H}^1(\Omega)} + \|\bar{z}\|_{H^1(\Omega)} \leq D_4\left(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|f_2\|_{L^{\frac{3}{2}}(\Omega)}\right)$$

with $D_4 = D_4\left(\Omega, \nu, \kappa, \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}, \|\theta_2\|_{L^3(\Omega)}\right) > 0$. Returning to (2.57), we have that

$$\|\mathcal{S}(\boldsymbol{y}, z)\|_{\boldsymbol{W}} = \|(\bar{\boldsymbol{y}}, \bar{z})\|_{\boldsymbol{W}} \leq C\left(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|f_2\|_{L^{\frac{3}{2}}(\Omega)}\right), \tag{2.62}$$

and consequently,

$$\begin{aligned}\|(\boldsymbol{y}, z)\|_{\boldsymbol{W}} &= \alpha\|(\bar{\boldsymbol{y}}, \bar{z})\|_{\boldsymbol{W}}, \\ &\leq C\left(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|f_2\|_{L^{\frac{3}{2}}(\Omega)}\right),\end{aligned}$$

where $C = C\left(\Omega, \nu, \kappa, \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}, \|\boldsymbol{u}_1\|_{\boldsymbol{L}^3(\Omega)}, \|\boldsymbol{u}_2\|_{\boldsymbol{L}^3(\Omega)}, \|\theta_2\|_{L^3(\Omega)}\right) > 0$ is a constant independent of $(\boldsymbol{y}, z)$ and α.

Then, by applying the Leray-Schauder fixed point theorem and the De Rham's theorem, we have that there exists a solution $(\boldsymbol{y}, \pi, z) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)/\mathbb{R} \times W^{2,\frac{3}{2}}(\Omega)$ of (2.50). The uniqueness of the solution follows directly from the estimate (2.62), because the solution of the linear problem (2.50) depends continuously on the data $\boldsymbol{f}_1$ and f_2. $\square$

Remark 2.6.9. Why do we use $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ instead of either $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ or $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$ for any fixed $0 < \varepsilon < \frac{1}{2}$ as we did in some results before?

Firstly, $\boldsymbol{g}$ cannot belong to $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$ because for obtaining $z \in W^{2,\frac{3}{2}}(\Omega)$, from the system (2.50) we observe that $\boldsymbol{g} \cdot \boldsymbol{y}$ must lie in $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$ in order to apply the regularity of the Poisson equation, but this is not possible because $\boldsymbol{y} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$ which is not embedded in $\boldsymbol{L}^\infty(\Omega)$. Then, this option is discarded.

Finally, the case $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$ for any fixed $0 < \varepsilon < \frac{1}{2}$ works very well in the proof until we arrive to an estimate similar to (2.54). Indeed, we would have

$$\kappa \|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} \leq C\|f_2\|_{L^{\frac{3}{2}}(\Omega)} + 2\varepsilon_2 \|\bar{z}\|_{W^{2,\frac{3}{2}}(\Omega)} + C_2C_4\|\bar{z}\|_{L^2(\Omega)} + C\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)} \|\bar{\boldsymbol{y}}\|_{\boldsymbol{L}^{q(\varepsilon)}(\Omega)}$$

with $q(\varepsilon) = \frac{3(3+2\varepsilon)}{4\varepsilon}$. Notice that for our purposes we must find an estimate for $\bar{\boldsymbol{y}}$ in $\boldsymbol{H}^1(\Omega)$, but the value of $q(\varepsilon) > 6$ for all $0 < \varepsilon < \frac{1}{2}$. Then, it is not possible to pass from the norm in $\boldsymbol{L}^{q(\varepsilon)}(\Omega)$ to the norm in $\boldsymbol{H}^1(\Omega)$. So, this option is also discarded.

Then, the weakest space that $\boldsymbol{g}$ could belong for our purposes is $\boldsymbol{L}^2(\Omega)$.

The next theorem tell us that it is possible to show the existence of very weak solutions for the Boussinesq system, if we just consider smallness of the fluxes of $\boldsymbol{u_b}$ through each connected component Γ_i of the boundary Γ. However, to show this, it is necessary to consider other hypothesis for r and $\boldsymbol{g}$, as we will point out later.

Theorem 2.6.10. *Let $3 \leq p, r < \infty$,*

$$\boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \quad \textit{with} \quad < \boldsymbol{u_b} \cdot \boldsymbol{n}, 1 >_\Gamma = 0, \quad \boldsymbol{g} \in \boldsymbol{L}^q(\Omega),$$

$$h \in W^{-1,\frac{pr}{p+r}}(\Omega), \quad \theta_b \in W^{-\frac{1}{r},r}(\Gamma),$$

where $q = \max\left\{s, \frac{3}{2}+\varepsilon\right\}$ for any fixed $0 < \varepsilon < \frac{1}{2}$ and s given by condition (2.25). There exists $\delta_2 = \delta_2(\Omega) > 0$ such that if

$$\frac{1}{\nu}\sum_{i=1}^m \left|\langle \boldsymbol{u_b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}\right| \leq \delta_2, \tag{2.63}$$

then there exists at least one very weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$ of (BS). Further, if $\boldsymbol{g} \in \boldsymbol{L}^t(\Omega)$, where $t = \max\{s, 2\}$, and if

$$\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^t(\Omega)}\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right) \leq \delta_3 \tag{2.64}$$

for some $\delta_3 = \delta_3(p, r, t, \Omega) > 0$, then this solution is unique.

Proof. Existence. Since $\boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ with $< \boldsymbol{u_b} \cdot \boldsymbol{n}, 1 >_\Gamma = 0$, then for any $\delta > 0$ there exists $\boldsymbol{u}_{\boldsymbol{b}}^\delta \in \boldsymbol{C}^\infty(\Gamma)$ such that $\int_\Gamma \boldsymbol{u}_{\boldsymbol{b}}^\delta \cdot \boldsymbol{n}\, \mathrm{d}s = 0$,

$$\left\|\boldsymbol{u}_{\boldsymbol{b}}^\delta - \boldsymbol{u_b}\right\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq \delta, \tag{2.65}$$

and

$$\sum_{i=0}^m \left|\int_{\Gamma_i} \boldsymbol{u}_{\boldsymbol{b}}^\delta \cdot \boldsymbol{n}\, \mathrm{d}s\right| \leq 2\sum_{i=0}^m \left|\langle \boldsymbol{u_b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}\right|, \tag{2.66}$$

see [6, Lemma 15]. Also, as $h \in W^{-1,\frac{pr}{p+r}}(\Omega)$ and $\theta_b \in W^{-\frac{1}{r},r}(\Gamma)$, it is possible to show that for all $\delta > 0$ there exist $h^\delta \in \mathcal{D}(\Omega)$ and $\theta_b^\delta \in C^\infty(\Gamma)$ such that

$$\left\| h^\delta - h \right\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} \leq \delta, \tag{2.67}$$

and

$$\left\| \theta_b^\delta - \theta_b \right\|_{W^{-\frac{1}{r},r}(\Gamma)} \leq \delta. \tag{2.68}$$

Let us define

$$\boldsymbol{v_b^\delta} := \boldsymbol{u_b} - \boldsymbol{u_b^\delta} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma), \quad f^\delta := h - h^\delta \in W^{-1,\frac{pr}{p+r}}(\Omega), \quad \sigma_b^\delta := \theta_b - \theta_b^\delta \in W^{-\frac{1}{r},r}(\Gamma).$$

Note that $< \boldsymbol{v_b^\delta} \cdot \boldsymbol{n}, 1 >_\Gamma = 0$. Let us set

$$\gamma := \left(\frac{1}{\nu} + \frac{1}{\kappa} \right) \left[1 + \left(\frac{1}{\nu} + \frac{1}{\kappa} \right) \|\boldsymbol{g}\|_{\boldsymbol{L}^q(\Omega)} \left(\frac{1}{\kappa} + 1 \right) \right]. \tag{2.69}$$

Realize that in order to study the existence of very weak solutions of (BS) considering only the smallness condition (2.63), we can split the problem in two parts. Indeed, firstly we want to find $(\boldsymbol{u}_1^\delta, \pi_1^\delta, \theta_1^\delta)$ solution of the problem:

$$\begin{cases} -\nu \Delta \boldsymbol{u}_1^\delta + (\boldsymbol{u}_1^\delta \cdot \nabla)\boldsymbol{u}_1^\delta + \nabla \pi_1^\delta = \theta_1^\delta \boldsymbol{g} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_1^\delta = 0 & \text{in } \Omega, \\ -\kappa \Delta \theta_1^\delta + \boldsymbol{u}_1^\delta \cdot \nabla \theta_1^\delta = f^\delta & \text{in } \Omega, \\ \boldsymbol{u}_1^\delta = \boldsymbol{v_b^\delta} \ , \ \theta_1^\delta = \sigma_b^\delta & \text{on } \Gamma, \end{cases} \tag{BS_1}$$

and then to find $(\boldsymbol{u}_2^\delta, \pi_2^\delta, \theta_2^\delta)$ solution of the problem:

$$\begin{cases} -\nu \Delta \boldsymbol{u}_2^\delta + (\boldsymbol{u}_2^\delta \cdot \nabla)\boldsymbol{u}_2^\delta + (\boldsymbol{u}_1^\delta \cdot \nabla)\boldsymbol{u}_2^\delta + (\boldsymbol{u}_2^\delta \cdot \nabla)\boldsymbol{u}_1^\delta + \nabla \pi_2^\delta = \theta_2^\delta \boldsymbol{g} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_2^\delta = 0 & \text{in } \Omega, \\ -\kappa \Delta \theta_2^\delta + \boldsymbol{u}_2^\delta \cdot \nabla \theta_2^\delta + \boldsymbol{u}_2^\delta \cdot \nabla \theta_1^\delta + \boldsymbol{u}_1^\delta \cdot \nabla \theta_2^\delta = h - f^\delta & \text{in } \Omega, \\ \boldsymbol{u}_2^\delta = \boldsymbol{u_b} - \boldsymbol{v_b^\delta} \ , \ \theta_2^\delta = \theta_b - \sigma_b^\delta & \text{on } \Gamma. \end{cases} \tag{BS_2}$$

Note that $(\boldsymbol{u}, \pi, \theta) = (\boldsymbol{u}_1^\delta + \boldsymbol{u}_2^\delta, \pi_1^\delta + \pi_2^\delta, \theta_1^\delta + \theta_2^\delta)$ is solution of the original problem. Let us focus in solving each problem.

(i) *Solution for* (BS_1): For solving (BS_1), we are going to apply Proposition 2.6.5. Since $q = \max\left\{s, \frac{3}{2} + \varepsilon\right\}$, from (2.65), (2.67), (2.68) and (2.69), it follows that

$$\begin{aligned} &\left(\frac{1}{\nu} + \frac{1}{\kappa} \right) \left[\left\| \boldsymbol{v_b^\delta} \right\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \frac{1}{\nu} \|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)} \left(\frac{1}{\kappa} \left\| f^\delta \right\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \left\| \sigma_b^\delta \right\|_{W^{-\frac{1}{r},r}(\Gamma)} \right) \right] \\ &\qquad \leq \delta \left(\frac{1}{\nu} + \frac{1}{\kappa} \right) \left[1 + \frac{1}{\nu} \|\boldsymbol{g}\|_{\boldsymbol{L}^q(\Omega)} \left(\frac{1}{\kappa} + 1 \right) \right] \\ &\qquad \leq \widetilde{C} \delta \left(\frac{1}{\nu} + \frac{1}{\kappa} \right) \left[1 + \frac{1}{\nu} \|\boldsymbol{g}\|_{\boldsymbol{L}^q(\Omega)} \left(\frac{1}{\kappa} + 1 \right) \right] \\ &\qquad \leq \widetilde{C} \delta \gamma, \end{aligned}$$

where $\widetilde{C} = \widetilde{C}(\Omega, s, \varepsilon) > 0$ is the constant related with the continuous embedding $L^q(\Omega) \hookrightarrow L^s(\Omega)$. Since $3 \leq p, r < \infty$ and taking $\delta \leq \frac{\rho_1}{\widetilde{C}\gamma}$, where ρ_1 is the number defined in Proposition

2.6.5, we have that $\boldsymbol{v}_{\boldsymbol{b}}^{\delta}$, σ_b^{δ}, f^{δ} and $\boldsymbol{g}$ satisfy the hypothesis of Proposition 2.6.5, hence there exists $(\boldsymbol{u}_1^{\delta}, \pi_1^{\delta}, \theta_1^{\delta}) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R} \times L^r(\Omega)$ very weak solution of (BS_1), and by (2.33) and (2.34), satisfies the inequality:

$$\|\boldsymbol{u}_1^{\delta}\|_{\boldsymbol{L}^p(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^q(\Omega)}\|\theta_1^{\delta}\|_{L^r(\Omega)} \leq C\left(\|\boldsymbol{v}_{\boldsymbol{b}}^{\delta}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^q(\Omega)}\right)\right.$$
$$\left.\times\left(\frac{1}{\kappa}\|f^{\delta}\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\sigma_b^{\delta}\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right)$$

with $C = C(p, r, \Omega) > 0$. As $3 \leq p, r < \infty$ and $q = \max\left\{s, \frac{3}{2} + \varepsilon\right\}$, in particular, we have that

$$\|\boldsymbol{u}_1^{\delta}\|_{\boldsymbol{L}^3(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_1^{\delta}\|_{L^3(\Omega)} \leq C_1\left(\|\boldsymbol{v}_{\boldsymbol{b}}^{\delta}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^q(\Omega)}\right.$$
$$\left.\times\left(\frac{1}{\kappa}\|f^{\delta}\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\sigma_b^{\delta}\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right) \tag{2.70}$$

with $C_1 = C_1(p, r, s, \varepsilon, \Omega) > 0$.

(ii) *Solution for* (BS_2): For solving (BS_2), we are going to use Proposition 2.6.7. Using successively (2.66), (2.70), (2.63), (2.69) and considering in particular that $h^{\delta} = h - f^{\delta} \in H^{-1}(\Omega)$, $\boldsymbol{u}_{\boldsymbol{b}}^{\delta} = \boldsymbol{u}_{\boldsymbol{b}} - \boldsymbol{v}_b^{\delta} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and $\theta_b^{\delta} = \theta_b - \sigma_b^{\delta} \in H^{\frac{1}{2}}(\Gamma)$, it follows that

$$\begin{aligned}
&\frac{1}{\nu}\sum_{i=1}^{m}\left|\int_{\Gamma_i}(\boldsymbol{u}_{\boldsymbol{b}} - \boldsymbol{v}_b^{\delta})\cdot\boldsymbol{n}\,\mathrm{d}s\right| + \frac{1}{\nu}\left(\|\boldsymbol{u}_1^{\delta}\|_{\boldsymbol{L}^3(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_1^{\delta}\|_{L^3(\Omega)}\right) \leq \\
&\leq \frac{2}{\nu}\sum_{i=1}^{m}\left|\langle\boldsymbol{u}_{\boldsymbol{b}}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i}\right| + \frac{C_1}{\nu}\left(\|\boldsymbol{v}_{\boldsymbol{b}}^{\delta}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^q(\Omega)}\right.\\
&\quad\left.\times\left(\frac{1}{\kappa}\|f^{\delta}\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\sigma_b^{\delta}\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right)\\
&\leq 2\delta_2 + C_1\delta\gamma.
\end{aligned}$$

Taking $\delta_2 = \frac{\rho_2}{4}$ and $\delta = \min\left\{\frac{\rho_1}{\widetilde{C}\gamma}, \frac{\rho_2}{2C_1\gamma}\right\}$, where ρ_2 is the number defined in Proposition 2.6.7, we have that there exists $(\boldsymbol{u}_2^{\delta}, \pi_2^{\delta}, \theta_2^{\delta}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ weak solution of (BS_2).

If p and r belong to the closed interval $[3, 6]$, it is clear that

$$(\boldsymbol{u}, \pi, \theta) = (\boldsymbol{u}_1^{\delta} + \boldsymbol{u}_2^{\delta}, \pi_1^{\delta} + \pi_2^{\delta}, \theta_1^{\delta} + \theta_2^{\delta}) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$$

is a very weak solution of (BS). Now, we must show the existence of very weak solutions for all $3 \leq p, r < \infty$.

Notice that we can consider more regularity for the following functions appearing in (BS_2): $h - f^{\delta} \in W^{-1,3}(\Omega)$, $\boldsymbol{u}_{\boldsymbol{b}} - \boldsymbol{v}_{\boldsymbol{b}}^{\delta} \in \boldsymbol{W}^{\frac{2}{3},3}(\Gamma)$ and $\theta_b - \sigma_b^{\delta} \in W^{\frac{2}{3},3}(\Gamma)$. Then by applying regularity results of the Poisson and Stokes equations, we can deduce that $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$ and $\theta \in L^r(\Omega)$ for all $p \geq 3$ and $r \geq 3$. Indeed, by the previous step, we have that $(\boldsymbol{u}_2^{\delta}, \theta_2^{\delta}) \in \boldsymbol{L}^6(\Omega) \times L^6(\Omega)$ and $(\boldsymbol{u}_1^{\delta}, \theta_1^{\delta}) \in \boldsymbol{L}^p(\Omega) \times L^r(\Omega)$ for all $3 \leq p, r < \infty$, in particular $(\boldsymbol{u}_1^{\delta}, \theta_1^{\delta}) \in \boldsymbol{L}^6(\Omega) \times L^6(\Omega)$. Then, since

$$\boldsymbol{u}_2^{\delta}\cdot\nabla\theta_2^{\delta} + \boldsymbol{u}_2^{\delta}\cdot\nabla\theta_1^{\delta} + \boldsymbol{u}_1^{\delta}\cdot\nabla\theta_2^{\delta} = \operatorname{div}(\theta_2^{\delta}\boldsymbol{u}_2^{\delta} + \theta_1^{\delta}\boldsymbol{u}_2^{\delta} + \theta_2^{\delta}\boldsymbol{u}_1^{\delta}) \in W^{-1,3}(\Omega) \tag{2.71}$$

and $h - f^\delta \in W^{-1,3}(\Omega)$, by applying regularity results of the Poisson equation in (BS_2), we have that $\theta_2^\delta \in W^{1,3}(\Omega) \hookrightarrow L^t(\Omega)$ for all $1 \le t < \infty$. Hence, $\theta_2^\delta \boldsymbol{g} \in \boldsymbol{L}^m(\Omega)$ with $1 \le m < q$, but since $q = \max\left\{s, \frac{3}{2} + \varepsilon\right\}$, we have that $q \ge \frac{3}{2} + \varepsilon$, for any fixed $0 < \varepsilon < \frac{1}{2}$, and this implies that $\theta_2^\delta \boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{W}^{-1,3}(\Omega)$. Further, in a similar way like in (2.71), we have that $(\boldsymbol{u}_2^\delta \cdot \nabla)\boldsymbol{u}_2^\delta + (\boldsymbol{u}_1^\delta \cdot \nabla)\boldsymbol{u}_2^\delta + (\boldsymbol{u}_2^\delta \cdot \nabla)\boldsymbol{u}_1^\delta \in W^{-1,3}(\Omega)$ and by applying regularity results of the Stokes equations in (BS_2), it follows that $\boldsymbol{u}_2^\delta \in \boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^t(\Omega)$ for all $1 \le t < \infty$ and $\pi_2^\delta \in L^3(\Omega) \hookrightarrow W^{-1,k}(\Omega)$ for all $3 \le k < \infty$.

Finally,

$$(\boldsymbol{u}, \pi, \theta) = (\boldsymbol{u}_1^\delta + \boldsymbol{u}_2^\delta, \pi_1^\delta + \pi_2^\delta, \theta_1^\delta + \theta_2^\delta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$$

is a very weak solution of (BS) for all $p \ge 3$ and $r \ge 3$.

Uniqueness. Let $(\boldsymbol{u}_1, \pi_1, \theta_1) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$ be a very weak solution of (BS). Since $t = \max\{s, 2\}$ and by the hypothesis (2.64), we have that

$$\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right)$$
$$\le C_2\delta_3,$$

where $C_2 = C_2(s, \Omega) > 0$ is the constant related with the continuous embedding $L^t(\Omega) \hookrightarrow L^s(\Omega)$. If $\delta_3 \le \frac{\rho_1}{C_2}$, where ρ_1 is the number defined in Proposition 2.6.5, then by Proposition 2.6.5, there exists $(\boldsymbol{u}_2, \pi_2, \theta_2) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega)$ very weak solution of (BS) such that satisfies the following estimate:

$$\|\boldsymbol{u}_2\|_{\boldsymbol{L}^p(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^t(\Omega)}\|\theta_2\|_{L^r(\Omega)} \le C\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^t(\Omega)}\right)\right.$$
$$\left.\times\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right) \tag{2.72}$$

for $C = C(p, r, \Omega) > 0$. Hence, let us define $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\theta = \theta_1 - \theta_2$ and $\pi = \pi_1 - \pi_2$. We can see that $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega) \times L^r(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega) \times W^{-1,3}(\Omega) \times L^3(\Omega)$ is a very weak solution to the following system:

$$\begin{cases} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_2 + \nabla\pi = \theta\boldsymbol{g} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ -\kappa\Delta\theta + \boldsymbol{u}_1 \cdot \nabla\theta + \boldsymbol{u} \cdot \nabla\theta_2 = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0}\ ,\ \theta = 0 & \text{on } \Gamma. \end{cases}$$

This means that $(\boldsymbol{u}, \theta) \in \boldsymbol{L}^3(\Omega) \times L^3(\Omega)$ satisfies

$$\int_\Omega \boldsymbol{u}\cdot(-\nu\Delta\boldsymbol{\chi} - (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{\chi} - (\nabla\boldsymbol{\chi})^T\boldsymbol{u}_2 + \nabla\varphi - \theta_2\nabla\psi)\,\mathrm{d}x + \int_\Omega \theta(-\kappa\Delta\psi - \boldsymbol{u}_1\cdot\nabla\psi - \boldsymbol{g}\cdot\boldsymbol{\chi})\,\mathrm{d}x = 0 \tag{2.73}$$

for all $\boldsymbol{\chi} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \cap \boldsymbol{W}_{0,\sigma}^{1,\frac{3}{2}}(\Omega)$, $\varphi \in W^{1,\frac{3}{2}}(\Omega)$, $\psi \in W^{2,\frac{3}{2}}(\Omega) \cap W_0^{1,\frac{3}{2}}(\Omega)$, where we use the identity $(\boldsymbol{u} \cdot \nabla)\boldsymbol{\chi} \cdot \boldsymbol{u}_2 = \boldsymbol{u} \cdot [(\nabla\boldsymbol{\chi})^T\boldsymbol{u}_2]$.

From Proposition 2.6.8, we know that for every $\boldsymbol{f}_1 \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $f_2 \in L^{\frac{3}{2}}(\Omega)$, there exists a unique strong solution

$$(\boldsymbol{y}, \pi, z) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)/\mathbb{R} \times W^{2,\frac{3}{2}}(\Omega)$$

of (2.50) provided we have (2.49). Let us verify that (2.49) holds. Indeed, since $3 \leq p, r < \infty$ and $t = \max\{s, 2\}$, from (2.72), we have that

$$\frac{1}{\nu}\left(\|\boldsymbol{u}_2\|_{\boldsymbol{L}^3(\Omega)} + \frac{1}{\kappa}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\theta_2\|_{L^3(\Omega)}\right) \leq \frac{C_3}{\nu}\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^t(\Omega)}\right.$$
$$\left.\times\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right)$$

with $C_3 = C_3(p, r, t, \Omega) > 0$. Then, taking $\delta_3 = \min\left\{\frac{\rho_1}{C_2}, \frac{\rho_3}{C_3}\right\}$ with ρ_3 as in Proposition 2.6.8, we get immediately (2.49). Therefore, if we take $(\boldsymbol{\chi}, \varphi, \psi) = (\boldsymbol{y}, \pi, z)$, it follows from (2.73) that

$$\int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{u}\,\mathrm{d}x + \int_\Omega f_2\,\theta\,\mathrm{d}x = 0$$

for all $\boldsymbol{f}_1 \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $f_2 \in L^{\frac{3}{2}}(\Omega)$. Hence, $\boldsymbol{u} = 0$ and $\theta = 0$, and the uniqueness is proven. ☐

Remark 2.6.11. (i) Note that when the boundary Γ is connected, the condition (2.63) is always satisfied because, by hypothesis, the flux of $\boldsymbol{u_b}$ across the boundary is equal to zero.
(ii) When $h = 0$, $\boldsymbol{u_b} = \boldsymbol{0}$ and $\theta_b = 0$, the sufficient condition for uniqueness (2.64) is always satisfied. Then, in this case the unique very weak solution is the trivial one, $\boldsymbol{u} = \boldsymbol{0}$, $\pi = c$ and $\theta = 0$, where c is any real number.
(iii) Notice that if we only want to show the existence of very weak solutions of (*BS*) for $3 \leq p, r \leq 6$, it is sufficient to consider $\boldsymbol{g} \in \boldsymbol{L}^q(\Omega)$ with $q = \max\{s, \frac{3}{2}\}$. But if we want to extend the existence of very weak solutions for $p, r \geq 6$, we need to assume a little more regularity for $\boldsymbol{g}$, i.e., $\boldsymbol{g} \in \boldsymbol{L}^q(\Omega)$ with $q = \max\{s, \frac{3}{2} + \varepsilon\}$ for any fixed $0 < \varepsilon < \frac{1}{2}$.

The next two theorems tell us that it is possible to extend the regularity of the solution of (*BS*) for $\frac{3}{2} \leq p < 2$ in $W^{1,p}(\Omega)$ and for $1 < p < \frac{6}{5}$ in $W^{2,p}(\Omega)$.

Theorem 2.6.12 (regularity $W^{1,p}(\Omega)$ with $\frac{3}{2} \leq p < 2$)**.** *Let us suppose that*

$$\boldsymbol{u_b} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \textit{ satisfies } (2.63),\ \theta_b \in W^{1-\frac{1}{p},p}(\Gamma),\ \ h \in W^{-1,p}(\Omega) \textit{ and } \boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$$

for any fixed $0 < \varepsilon < \frac{1}{2}$. Then the solution for the Boussinesq system given by Theorem 2.6.10 satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega) \times W^{1,p}(\Omega).$$

Proof. We observe that $W^{-1,p}(\Omega) \hookrightarrow W^{-1,\frac{3}{2}}(\Omega)$ and $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{3},3}(\Gamma)$. Then, all the hypothesis of Theorem 2.6.10 are verified, i.e.,

$$h \in W^{-1,\frac{3}{2}}(\Omega),\ \boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{3},3}(\Gamma) \quad \text{satisfies } (2.63),\ \theta_b \in W^{-\frac{1}{3},3}(\Gamma) \quad \text{and} \quad q = \frac{3}{2} + \varepsilon,$$

therefore there exists $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{L}^3(\Omega) \times W^{-1,3}(\Omega) \times L^3(\Omega)$ very weak solution of (*BS*).

We have that $\boldsymbol{u} \cdot \nabla\theta = \operatorname{div}(\theta\boldsymbol{u}) \in W^{-1,\frac{3}{2}}(\Omega)$ and if $h \in W^{-1,\frac{3}{2}}(\Omega)$, by using regularity of the Poisson equation, we deduce that $\theta \in W^{1,\frac{3}{2}}(\Omega)$. Further, $\theta\boldsymbol{g} \in \boldsymbol{L}^t(\Omega)$, where $\frac{1}{t} = \frac{1}{3} + \frac{1}{\frac{3}{2}+\varepsilon}$, then it is easy to see that $\theta \boldsymbol{g} \in \boldsymbol{W}^{-1,\frac{3}{2}+\varepsilon}(\Omega)$. Since $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u}) \in \boldsymbol{W}^{-1,\frac{3}{2}}(\Omega)$, by applying regularity of the Stokes equations, we conclude that $\boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega)$. At this point, we have showed the theorem for $p = \frac{3}{2}$.

Suppose now that $\frac{3}{2} < p < 2$. We will use a recurrence argument to show the regularity in this case. Indeed, from the previous step, we know that

$$\theta \boldsymbol{g} \in \boldsymbol{W}^{-1,p_0(\varepsilon)}(\Omega) \quad \text{with} \quad p_0(\varepsilon) = \frac{3}{2} + \varepsilon.$$

As $\boldsymbol{u} \in \boldsymbol{L}^3(\Omega)$ with div $\boldsymbol{u} = 0$, by using the existence of generalized solutions of the Oseen problem, see [6, Theorem 15], we know that there exists $(\boldsymbol{w}, \hat{\pi}) \in \boldsymbol{W}^{1,p_0(\varepsilon)}(\Omega) \times L^{p_0(\varepsilon)}(\Omega)$ solution of the problem

$$\begin{cases} -\nu\Delta\boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} + \nabla\hat{\pi} = \theta\boldsymbol{g} & \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} = \boldsymbol{u_b} & \text{on } \Gamma. \end{cases} \tag{O_1}$$

Defining $\boldsymbol{y} := \boldsymbol{u} - \boldsymbol{w} \in \boldsymbol{L}^{p_0(\varepsilon)}(\Omega)$ and $\chi := \pi - \hat{\pi} \in W^{-1,p_0(\varepsilon)}(\Omega)$, we have that $(\boldsymbol{y}, \chi) \in \boldsymbol{L}^{p_0(\varepsilon)}(\Omega) \times W^{-1,p_0(\varepsilon)}(\Omega)$ is a very weak solution of the problem

$$\begin{cases} -\nu\Delta\boldsymbol{y} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{y} + \nabla\chi = \boldsymbol{0} & \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{y} = 0 & \text{in } \Omega, \\ \boldsymbol{y} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{O_2}$$

Then, by using the uniqueness of very weak solutions of the Oseen problem, see [6, Theorem 17], we deduce that $\boldsymbol{y} = \boldsymbol{0}$ and $\nabla\chi = \boldsymbol{0}$, and hence $\boldsymbol{u} \in \boldsymbol{W}^{1,p_0(\varepsilon)}(\Omega)$ and $\pi \in L^{p_0(\varepsilon)}(\Omega)$[3]. Since $\theta \in L^3(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{W}^{1,p_0(\varepsilon)}(\Omega) \hookrightarrow \boldsymbol{L}^{p_0^*}(\Omega)$ with $\frac{1}{p_0^*} = \frac{1}{p_0(\varepsilon)} - \frac{1}{3}$, we observe that $\theta\boldsymbol{u} \in \boldsymbol{L}^{r_0(\varepsilon)}(\Omega)$ and consequently

$$\mathrm{div}(\theta\boldsymbol{u}) \in W^{-1,r_0(\varepsilon)}(\Omega) \quad \text{with} \quad r_0(\varepsilon) = \frac{3}{2} + \varepsilon.$$

Then, by applying the regularity of the Poisson equation, it follows that $\theta \in W^{1,r_0(\varepsilon)}(\Omega)$.

Now, as $W^{1,r_0(\varepsilon)}(\Omega) \hookrightarrow L^{r_0^*}(\Omega)$ with $\frac{1}{r_0^*} = \frac{1}{r_0(\varepsilon)} - \frac{1}{3}$, then it is possible to see, by using the Sobolev embedding theorem, that

$$\theta\boldsymbol{g} \in \boldsymbol{W}^{-1,p_1(\varepsilon)}(\Omega) \quad \text{with} \quad \frac{1}{p_1(\varepsilon)} = \frac{2}{\frac{3}{2}+\varepsilon} - \frac{2}{3}.$$

Note that $p_1(\varepsilon) > p_0(\varepsilon)$ because $0 < \varepsilon < \frac{1}{2}$ and therefore, by applying the Oseen argument, we conclude that $\boldsymbol{u} \in \boldsymbol{W}^{1,p_1(\varepsilon)}(\Omega)$ and $\pi \in L^{p_1(\varepsilon)}(\Omega)$. Further, since $\theta \in L^{r_0^*}(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{W}^{1,p_1(\varepsilon)}(\Omega) \hookrightarrow \boldsymbol{L}^{p_1^*}(\Omega)$ with $\frac{1}{p_1^*} = \frac{1}{p_1(\varepsilon)} - \frac{1}{3}$, it follows that $\theta\boldsymbol{u} \in \boldsymbol{L}^{r_1(\varepsilon)}(\Omega)$ and consequently

$$\mathrm{div}(\theta\boldsymbol{u}) \in W^{-1,r_1(\varepsilon)}(\Omega) \quad \text{with} \quad \frac{1}{r_1(\varepsilon)} = \frac{3}{\frac{3}{2}+\varepsilon} - \frac{4}{3}.$$

Therefore, by applying the regularity of the Poisson equation, it follows that $\theta \in W^{1,r_1(\varepsilon)}(\Omega)$ which is more regular than before because $r_1(\varepsilon) > r_0(\varepsilon)$.

[3]This process of applying successively the existence of generalized solutions and the uniqueness of very weak solutions for the Oseen problem in order to conclude that $\boldsymbol{u}$ and π are more regular, we use it several times. So, we will refer to this process as the Oseen argument.

By using, consecutively, the Sobolev embedding theorem, the Oseen argument and the Poisson regularity, we can deduce that $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p_k(\varepsilon)}(\Omega) \times L^{p_k(\varepsilon)}(\Omega) \times W^{1,r_k(\varepsilon)}(\Omega)$, where

$$\frac{1}{p_k(\varepsilon)} = \frac{2}{3} - \alpha_\varepsilon 2^k, \qquad \frac{1}{r_k(\varepsilon)} = \frac{2}{3} - \alpha_\varepsilon (2^{k+1} - 1), \qquad \alpha_\varepsilon = \frac{2}{3} - \frac{1}{\frac{3}{2} + \varepsilon} > 0 \tag{2.74}$$

with $k \in \mathbb{N}$. Further, it is possible to show that $(p_k)_k$ and $(r_k)_k$ are strictly increasing sequences of positive numbers for all $0 \leq k < \beta_\varepsilon$ with $\beta_\varepsilon = \frac{\ln(\frac{1}{2} + \frac{1}{3\alpha_\varepsilon})}{\ln 2} > 0$.

To finish the proof, we must show that there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $k_0 < \beta_\varepsilon$, and p_{k_0} and r_{k_0} reach the value of 2. Indeed, by using the definitions of the sequences $(p_k)_k$ and $(r_k)_k$, we have that

$$k_0 = \left\lfloor \frac{\ln\left(\frac{1}{6\alpha_\varepsilon}\right)}{\ln 2} \right\rfloor + 1, \tag{2.75}$$

where $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a, is the desired natural number which allow us to conclude that $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega) \times W^{1,p}(\Omega)$. $\square$

Theorem 2.6.13 (regularity $W^{2,p}(\Omega)$ with $1 < p < \frac{6}{5}$)**.** *Let us suppose that*

$$\boldsymbol{u_b} \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma) \textit{ satisfies } (2.63), \ \theta_b \in W^{2-\frac{1}{p},p}(\Gamma), \ \ h \in L^p(\Omega) \textit{ and } \boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$$

for any fixed $0 < \varepsilon < \frac{1}{2}$. Then the solution for the Boussinesq system given by Theorem 2.6.10 satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega) \times W^{2,p}(\Omega).$$

Proof. We observe that $L^p(\Omega) \hookrightarrow W^{-1,\frac{3}{2}}(\Omega)$ and $\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma)$. Then, by Theorem 2.6.12 there exists $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \times L^{\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$ weak solution of (*BS*).

As in Theorem 2.6.12, we will use a recurrence argument to show the regularity for $1 < p < \frac{6}{5}$. Since $\theta \in L^3(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$, we have that

$$\theta \boldsymbol{g} \in \boldsymbol{L}^{p_0(\varepsilon)}(\Omega) \quad \text{with} \quad \frac{1}{p_0(\varepsilon)} = \frac{1}{\frac{3}{2} + \varepsilon} + \frac{1}{3}.$$

Therefore, as $\boldsymbol{u} \in \boldsymbol{L}^3(\Omega)$ with $\operatorname{div} \boldsymbol{u} = 0$, we can use the existence of strong solutions to the Oseen problem, see [6, Theorem 16], and consequently there exists $(\boldsymbol{w}, \hat{\pi}) \in \boldsymbol{W}^{2,p_0(\varepsilon)}(\Omega) \times W^{1,p_0(\varepsilon)}(\Omega)$ solution of (O_1).

Defining $\boldsymbol{y} := \boldsymbol{u} - \boldsymbol{w} \in \boldsymbol{W}^{1,p_0(\varepsilon)}(\Omega)$ and $\chi := \pi - \hat{\pi} \in L^{p_0(\varepsilon)}(\Omega)$, we have that $(\boldsymbol{y}, \chi) \in \boldsymbol{W}^{1,p_0(\varepsilon)}(\Omega) \times L^{p_0(\varepsilon)}(\Omega)$ is a weak solution of (O_2). Then, thanks to the uniqueness of generalized solutions of the Oseen problem, see [6, Theorem 15], we deduce that $\boldsymbol{y} = \boldsymbol{0}$ and $\nabla \chi = \boldsymbol{0}$, what means that $\boldsymbol{u} \in \boldsymbol{W}^{2,p_0(\varepsilon)}(\Omega)$ and $\pi \in W^{1,p_0(\varepsilon)}(\Omega)$[4]. Since $\nabla \theta \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{W}^{2,p_0(\varepsilon)}(\Omega) \hookrightarrow \boldsymbol{L}^{p_0^{**}}(\Omega)$ with $\frac{1}{p_0^{**}} = \frac{1}{p_0(\varepsilon)} - \frac{2}{3}$, we have that

$$\boldsymbol{u} \cdot \nabla \theta \in L^{r_0(\varepsilon)}(\Omega) \quad \text{with} \quad \frac{1}{r_0(\varepsilon)} = \frac{1}{\frac{3}{2} + \varepsilon} + \frac{1}{3}.$$

[4]As in the Theorem 2.6.12, the process of applying successively the existence of strong solutions and the uniqueness of generalized solutions for the Oseen problem in order to conclude that $\boldsymbol{u}$ and π are more regular, we will call it again the Oseen argument for simplicity.

Then, by applying the regularity of the Poisson equation, it follows that $\theta \in W^{2,r_0(\varepsilon)}(\Omega)$.

Now, since $W^{2,r_0(\varepsilon)}(\Omega) \hookrightarrow L^{r_0^{**}}(\Omega)$ with $\frac{1}{r_0^{**}} = \frac{1}{r_0(\varepsilon)} - \frac{2}{3}$, we have that

$$\theta \boldsymbol{g} \in \boldsymbol{L}^{p_1(\varepsilon)}(\Omega) \quad \text{with} \quad \frac{1}{p_1(\varepsilon)} = \frac{2}{\frac{3}{2}+\varepsilon} - \frac{1}{3}.$$

As $0 < \varepsilon < \frac{1}{2}$, note that $p_1(\varepsilon) > p_0(\varepsilon)$ and by applying the Oseen argument, it follows that $\boldsymbol{u} \in \boldsymbol{W}^{2,p_1(\varepsilon)}(\Omega)$ and $\pi \in W^{1,p_1(\varepsilon)}(\Omega)$. Furthermore, as $\nabla\theta \in L^{r_0^*}(\Omega)$ with $\frac{1}{r_0^*} = \frac{1}{r_0(\varepsilon)} - \frac{1}{3}$ and $\boldsymbol{u} \in \boldsymbol{W}^{2,p_1(\varepsilon)}(\Omega) \hookrightarrow \boldsymbol{L}^{p_1^{**}}(\Omega)$ with $\frac{1}{p_1^{**}} = \frac{1}{p_1(\varepsilon)} - \frac{2}{3}$, it follows that

$$\boldsymbol{u} \cdot \nabla\theta \in L^{r_1(\varepsilon)}(\Omega) \quad \text{with} \quad \frac{1}{r_1(\varepsilon)} = \frac{3}{\frac{3}{2}+\varepsilon} - 1.$$

Therefore, by applying the regularity of the Poisson equation, it follows that $\theta \in W^{2,r_1(\varepsilon)}(\Omega)$ which is more regular than before because $r_1(\varepsilon) > r_0(\varepsilon)$.

By applying the Sobolev embedding theorem, the Oseen argument and the Poisson regularity, we can deduce that $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p_m(\varepsilon)}(\Omega) \times W^{1,p_m(\varepsilon)}(\Omega) \times W^{2,r_m(\varepsilon)}(\Omega)$, where

$$\frac{1}{p_m(\varepsilon)} = 1 - \alpha_\varepsilon 2^m, \qquad \frac{1}{r_m(\varepsilon)} = 1 - \alpha_\varepsilon (2^{m+1} - 1)$$

with α_ε the number given in (2.74) and $m \in \mathbb{N}$. Furthermore, we can show that $(p_m)_m$ and $(r_m)_m$ are strictly increasing sequences of positive numbers for all $m < \gamma_\varepsilon$ with $\gamma_\varepsilon = \frac{\ln(\frac{1}{2}+\frac{1}{2\alpha_\varepsilon})}{\ln 2} > 0$.

Finally, we must show that there exists $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that $m_0 < \gamma_\varepsilon$, and p_{m_0} and r_{m_0} reach the value of $\frac{6}{5}$. Indeed, by using the definitions of the sequences $(p_m)_m$ and $(r_m)_m$, we deduce that $m_0 = k_0$ with k_0 the natural number given in (2.75). Therefore, we conclude that $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega) \times W^{2,p}(\Omega)$. □

Remark 2.6.14. The two previous results are valid if $p \geq 2$ for the regularity $W^{1,p}(\Omega)$ and if $p \geq \frac{6}{5}$ for the regularity $W^{2,p}(\Omega)$, by considering $\boldsymbol{g}$ as in Theorem 2.5.1 and Theorem 2.5.3, respectively.

2.7 Estimates and uniqueness of the weak solution

This section deals with the estimates for weak solutions in $H^1(\Omega)$ of the Boussinesq system and with the uniqueness of this solution. We address these issues at the end of our work because when we want to find H^1-estimates of the solutions, we realize that it is very helpful to use the estimates for very weak solutions given in Proposition 2.6.5 to obtain suitable estimates for the velocity and the temperature of the fluid which agree with the estimates for the solutions to the Navier-Stokes and the convection-diffusion equations. Further, in order to prove uniqueness of the weak solution, we will make use of the estimates obtained for the weak solution and will assume that the data are sufficiently small.

Theorem 2.7.1. *Let*

$$\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega), \quad h \in H^{-1}(\Omega), \quad \boldsymbol{u_b} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma), \quad \theta_b \in H^{\frac{1}{2}}(\Gamma)$$

such that $\int_\Gamma \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = 0$. *There exists* $\delta_4 = \delta_4(\Omega) > 0$ *such that if*

$$\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right)\right) \le \delta_4, \tag{2.76}$$

then the weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ *of* (*BS*) *satisfies the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C\left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right)\right), \tag{2.77}$$

$$\|\theta\|_{H^1(\Omega)} \le C\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right), \tag{2.78}$$

with $C = C(\Omega) > 0$.

Proof. Since $H^{\frac{1}{2}}(\Gamma) \hookrightarrow W^{-\frac{1}{6},6}(\Gamma)$, we deduce that $\boldsymbol{u_b} \in \boldsymbol{W}^{-\frac{1}{6},6}(\Gamma)$ and $\theta_b \in W^{-\frac{1}{6},6}(\Gamma)$, and from (2.76) we have

$$\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{6},6}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{6},6}(\Gamma)}\right)\right) \le C_1\delta_4$$

with $C_1 = C_1(\Omega) > 0$. Taking $\delta_4 = \frac{\rho_1}{C_1}$, where $\rho_1 = \rho_1(\Omega) > 0$ is the constant given in Proposition 2.6.5 and noting that the exponent s given by (2.25) is equal to $\frac{3}{2}$, it follows, thanks precisely to Proposition 2.6.5, that there exists a very weak solution $(\boldsymbol{u}, \theta) \in \boldsymbol{L}^6(\Omega) \times L^6(\Omega)$ of (*BS*) satisfying the following estimates:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \le C_2\left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{6},6}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{6},6}(\Gamma)}\right)\right),$$

$$\|\theta\|_{L^6(\Omega)} \le C_2\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{6},6}(\Gamma)}\right),$$

with $C_2 = C_2(\Omega) > 0$, and a fortiori

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \le C_3\left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right)\right), \tag{2.79}$$

$$\|\theta\|_{L^6(\Omega)} \le C_3\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right), \tag{2.80}$$

with $C_3 = C_3(\Omega) > 0$.

Moreover, since $\boldsymbol{u} \in \boldsymbol{L}^6(\Omega)$ and $\theta \in L^6(\Omega)$, we have that $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \in \boldsymbol{W}^{-1,3}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$ and $\theta\boldsymbol{g} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$, then by applying regularity of the Stokes equations, we deduce that $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. Also, we have that $\boldsymbol{u} \cdot \nabla\theta = \operatorname{div}(\theta\boldsymbol{u}) \in W^{-1,3}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$, and by applying regularity of the Poisson equation, it follows that $\theta \in H^1(\Omega)$.

As $\boldsymbol{u_b} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ such that $\int_\Gamma \boldsymbol{u_b} \cdot \boldsymbol{n} \, \mathrm{d}s = 0$, there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{v} = \boldsymbol{u_b} \quad \text{on} \quad \Gamma$$

and

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C_4 \|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \tag{2.81}$$

with $C_4 = C_4(\Omega) > 0$. In the same way, since $\theta_b \in H^{\frac{1}{2}}(\Gamma)$ there exists $\xi \in H^1(\Omega)$ such that $\xi = \theta_b$ on Γ and

$$\|\xi\|_{H^1(\Omega)} \leq C_5 \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} \tag{2.82}$$

with $C_5 = C_5(\Omega) > 0$. Defining $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ and $\hat{\theta} = \theta - \xi \in H^1_0(\Omega)$, and replacing them in (*BS*), we obtain the following system

$$\left\{ \begin{aligned} -\nu\Delta\hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}} \cdot \nabla)\hat{\boldsymbol{u}} + (\boldsymbol{v} \cdot \nabla)\hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}} \cdot \nabla)\boldsymbol{v} + \nabla\pi = (\hat{\theta} + \xi)\boldsymbol{g} + \nu\Delta\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} & \quad \text{in } \Omega, \\ \operatorname{div} \hat{\boldsymbol{u}} = 0 & \quad \text{in } \Omega, \\ -\kappa\Delta\hat{\theta} + \hat{\boldsymbol{u}} \cdot \nabla\hat{\theta} + \boldsymbol{v} \cdot \nabla\hat{\theta} = h + \kappa\Delta\xi - (\hat{\boldsymbol{u}} + \boldsymbol{v}) \cdot \nabla\xi & \quad \text{in } \Omega, \\ \hat{\boldsymbol{u}} = \boldsymbol{0} \ , \ \hat{\theta} = 0 & \quad \text{on } \Gamma. \end{aligned} \right. \tag{2.83}$$

We can choose $\hat{\boldsymbol{u}}$ and $\hat{\theta}$ as test functions in the variational formulation of (2.83), and it follows that

$$\nu \int_\Omega |\nabla\hat{\boldsymbol{u}}|^2 \, \mathrm{d}x = \int_\Omega (\hat{\theta} + \xi)\boldsymbol{g} \cdot \hat{\boldsymbol{u}} \, \mathrm{d}x - B\left(\hat{\boldsymbol{u}}, \boldsymbol{v}, \hat{\boldsymbol{u}}\right) - \nu \int_\Omega \nabla\boldsymbol{v} \cdot \nabla\hat{\boldsymbol{u}} \, \mathrm{d}x - B\left(\boldsymbol{v}, \boldsymbol{v}, \hat{\boldsymbol{u}}\right), \tag{2.84}$$

$$\kappa \int_\Omega |\nabla\hat{\theta}|^2 \, \mathrm{d}x = \langle h, \hat{\theta}\rangle_\Omega - \kappa \int_\Omega \nabla\xi \cdot \nabla\hat{\theta} \, \mathrm{d}x + b\left(\hat{\boldsymbol{u}} + \boldsymbol{v}, \hat{\theta}, \xi\right). \tag{2.85}$$

By using Lemma 2.3.1, Hölder inequality and taking into account that we want to take advantage of the estimates (2.79) and (2.80), from (2.84) we have

$$\begin{aligned} \|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_6}{\nu} \Big[& \|\theta\|_{L^6(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \Big(\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{L}^6(\Omega)} \Big) \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \\ & + \nu \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} \Big] \end{aligned}$$

with $C_6 = C_6(\Omega) > 0$. From (2.79), we have that

$$\begin{aligned} \left(\frac{1}{\nu} + \frac{1}{\kappa}\right) \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \leq C_3 \left(\frac{1}{\nu} + \frac{1}{\kappa}\right) \Big(& \|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \\ & + \frac{1}{\nu} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \left(\frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} \right) \Big), \end{aligned}$$

and then using (2.76), we obtain

$$\left(\frac{1}{\nu} + \frac{1}{\kappa}\right) \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \leq C_3 \delta_4. \tag{2.86}$$

By applying (2.80), (2.81) and (2.86), it follows that

$$\begin{aligned} \|\nabla\hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_8}{\nu} \Big[& \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \left(\frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} \right) \\ & + \Big(\nu + \|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \Big) \|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \nu \|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \|\boldsymbol{u_b}\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \Big], \end{aligned}$$

where $C_8 = C_6 \max\{C_3, C_4, C_4^2, C_4^2 C_7, C_3 C_4 \delta_4\}$ and $C_7 > 0$ is the constant of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. From (2.76), we have that $\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \leq \nu\delta_4$, then

$$\|\boldsymbol{u_b}\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \leq \nu\delta_4 \|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)},$$

and we obtain immediately that

$$\|\nabla \hat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \leq C_9 \left[\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \left(\frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} \right) \right],$$

where $C_9 = 2C_8(1+\delta_4)$. Finally, as $\boldsymbol{u} = \hat{\boldsymbol{u}} + \boldsymbol{v}$, it follows that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C_{10} \left[\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \left(\frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} \right) \right],$$

where $C_{10} = A_2 C_9 + C_4$ and A_2 is the constant given in Lemma 2.3.1.

On the other hand, from (2.85) it follows that

$$\|\nabla \hat{\theta}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_{11}}{\kappa} \left(\|h\|_{H^{-1}(\Omega)} + \kappa \|\xi\|_{H^1(\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \|\xi\|_{H^1(\Omega)} \right)$$

with $C_{11} = C_{11}(\Omega) > 0$. By using (2.86) and (2.82), we have

$$\|\nabla \hat{\theta}\|_{\boldsymbol{L}^2(\Omega)} \leq C_{12} \left(\frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} \right),$$

where $C_{12} = C_{11} \max\{1, C_5(1 + C_3\delta_4)\}$. Finally, as $\theta = \hat{\theta} + \xi$, it follows that

$$\|\theta\|_{H^1(\Omega)} \leq C_{13} \left(\frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} \right),$$

where $C_{13} = A_2 C_{12} + C_5$. By taking $C = \max\{C_{10}, C_{13}\}$, the theorem is proven. $\square$

Remark 2.7.2. (i) It is noteworthy that the H^1-estimates of the weak solution is obtained if we consider smallness of the data, because in that case it is possible to use the estimates given in Proposition 2.6.5.
(ii) When $\boldsymbol{u_b} = \boldsymbol{0}$ and $\theta_b = 0$, the estimates (2.77) and (2.78) coincide with the estimates (2.9) and (2.10), respectively.
(iii) Thanks to Theorem 2.7.1 is possible to derive the corresponding estimate for the pressure $\pi \in L^2(\Omega)$. Indeed, from the equations $\nabla\pi = \nu\Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \theta\boldsymbol{g}$ we have

$$\begin{aligned} \|\dot{\pi}\|_{L^2(\Omega)/\mathbb{R}} &\leq C_1 \|\nabla\pi\|_{H^{-1}(\Omega)} \\ &\leq C_1 \left(\nu \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|_{H^1(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \right) \end{aligned}$$

with $C_1 > 0$ depends on Ω. By using (2.76) and (2.77), we have that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \nu C \delta_4, \tag{2.87}$$

where C and δ are the numbers given in Theorem 2.7.1. Finally, by applying (2.87), (2.77) and (2.78), it follows that

$$
\begin{aligned}
\|\dot{\pi}\|_{L^2(\Omega)/\mathbb{R}} &\leq C_1 \left(\nu\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \nu C\delta_4\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\theta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right) \\
&\leq C_2\nu \left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right)\right)
\end{aligned}
$$

with $C_2 = \max\{C_1, CC_1\delta_4\}$.

In a similar way, thanks to Proposition 2.6.5 we have that

$$
\|\dot{\pi}\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C_3\nu \left(\|\boldsymbol{u_b}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \frac{1}{\nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^s(\Omega)}\left(\frac{1}{\kappa}\|h\|_{W^{-1,\frac{pr}{p+r}}(\Omega)} + \|\theta_b\|_{W^{-\frac{1}{r},r}(\Gamma)}\right)\right)
$$

with $C_3 = C_3(\Omega, p, r) > 0$.

With the help of Theorem 2.7.1, we can derive the uniqueness of the weak solution in $H^1(\Omega)$ of the Boussinesq system.

Theorem 2.7.3. *If the data satisfy that*

$$
\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right)\right) \leq \delta_5 \tag{2.88}
$$

for some $\delta_5 = \delta_5(\Omega) > 0$, then the weak solution of (BS) is unique.

Proof. Let $(\boldsymbol{u}_1, \theta_1)$, $(\boldsymbol{u}_2, \theta_2)$ be two solutions in $\boldsymbol{H}^1(\Omega)\times H^1(\Omega)$ of (BS). Then, we obtain the following system:

$$
\left\{
\begin{aligned}
-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_1 + (\boldsymbol{u}_2\cdot\nabla)\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} &\quad \text{in } \Omega,\\
\operatorname{div}\boldsymbol{u} = 0 &\quad \text{in } \Omega,\\
-\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta_1 + \boldsymbol{u}_2\cdot\nabla\theta = 0 &\quad \text{in } \Omega,\\
\boldsymbol{u} = 0\ ,\ \theta = 0 &\quad \text{on } \Gamma,
\end{aligned}
\right. \tag{2.89}
$$

where $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2 \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$, $\theta = \theta_1 - \theta_2 \in H^1_0(\Omega)$ and $\pi = \pi_1 - \pi_2 \in L^2(\Omega)/\mathbb{R}$. Multiplying by $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ the first equation and by $\psi \in H^1_0(\Omega)$ the third equation of (2.89), and integrating by parts, we have:

$$
\nu\left(\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega)} + B\left(\boldsymbol{u}, \boldsymbol{u_1}, \boldsymbol{\varphi}\right) + B\left(\boldsymbol{u_2}, \boldsymbol{u}, \boldsymbol{\varphi}\right) = \left(\theta\boldsymbol{g}, \boldsymbol{\varphi}\right)_{\boldsymbol{L}^2(\Omega)}, \quad \forall\boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\sigma}(\Omega),
$$

$$
\kappa\left(\nabla\theta, \nabla\psi\right)_{\boldsymbol{L}^2(\Omega)} + b\left(\boldsymbol{u}, \theta_1, \psi\right) + b\left(\boldsymbol{u_2}, \theta, \psi\right) = 0, \quad \forall\psi \in H^1_0(\Omega).
$$

Taking $\boldsymbol{\varphi} = \boldsymbol{u}$ and $\psi = \theta$, it follows that

$$
\nu\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + B\left(\boldsymbol{u}, \boldsymbol{u_1}, \boldsymbol{u}\right) + B\left(\boldsymbol{u_2}, \boldsymbol{u}, \boldsymbol{u}\right) = \left(\theta\boldsymbol{g}, \boldsymbol{u}\right)_{\boldsymbol{L}^2(\Omega)},
$$

$$
\kappa\|\nabla\theta\|^2_{\boldsymbol{L}^2(\Omega)} + b\left(\boldsymbol{u}, \theta_1, \theta\right) + b\left(\boldsymbol{u_2}, \theta, \theta\right) = 0.
$$

By applying Lemmas 2.3.2 and 2.3.3, it follows that

$$
\nu\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} = \left(\theta\boldsymbol{g}, \boldsymbol{u}\right)_{\boldsymbol{L}^2(\Omega)} + B\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u_1}\right),
$$

$$\kappa \|\nabla \theta\|^2_{\boldsymbol{L}^2(\Omega)} = b\,(\boldsymbol{u}, \theta, \theta_1).$$

Therefore, by using (2.4), (2.5) and Sobolev embedding theorem, we have

$$\nu \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq C_1(\|\nabla \theta\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{u}_1\|_{\boldsymbol{H}^1(\Omega)} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}), \tag{2.90}$$

$$\kappa \|\nabla \theta\|_{\boldsymbol{L}^2(\Omega)} \leq C_2 \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\theta_1\|_{H^1(\Omega)}, \tag{2.91}$$

where C_1 and C_2 are positive constants depending on Ω. Replacing (2.91) into (2.90), we get

$$\begin{aligned}
\nu \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} &\leq C_1 \left(\frac{C_2}{\kappa} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\theta_1\|_{H^1(\Omega)} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}_1\|_{\boldsymbol{H}^1(\Omega)} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \right) \\
&\leq C_1 \left(\frac{C_2}{\kappa} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\theta_1\|_{H^1(\Omega)} + \|\boldsymbol{u}_1\|_{\boldsymbol{H}^1(\Omega)} \right) \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq C_3 \left(\frac{1}{\kappa} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\theta_1\|_{H^1(\Omega)} + \|\boldsymbol{u}_1\|_{\boldsymbol{H}^1(\Omega)} \right) \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}
\end{aligned}$$

with $C_3 = C_1 \max\{1, C_2\}$. Considering $\delta_5 \leq \delta_4$, Theorem 2.7.1 guarantees that the pair $(\boldsymbol{u}_1, \theta_1)$ satisfies the inequalities (2.77) and (2.78), and it follows that

$$\begin{aligned}
\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{CC_3}{\nu} &\left(\|\boldsymbol{u_b}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \left(\frac{1}{\nu} + \frac{1}{\kappa} \right) \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \left(\frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} + \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} \right) \right) \\
&\times \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Choosing $\delta_5 = \min\{\delta_4, \frac{1}{2CC_3}\}$, we can use the assumption (2.88) to conclude that $\boldsymbol{u} = 0$ and, a fortiori, $\theta = 0$; therefore, the solution is unique. □

Remark 2.7.4. In the case that $h = 0$, $\boldsymbol{u_b} = \boldsymbol{0}$ and $\theta_b = 0$, the condition (2.88) is always satisfied. Then $\theta = 0$, $\boldsymbol{u} = \boldsymbol{0}$ and $\pi = c$, where c is any real number. So, the unique weak solution is the trivial one.

Chapter 3

Boussinesq system with Navier boundary conditions

Abstract

In this chapter we are concerned with the stationary Boussinesq system with non-homogeneous Navier boundary conditions for the velocity field in a bounded domain $\Omega \subset \mathbb{R}^3$ of class $\mathcal{C}^{2,1}$. We prove the existence of generalized and strong solutions of the stationary Boussinesq system in L^p-theory.

Keywords: Boussinesq system, non-homogeneous Navier boundary conditions, existence, regularity, generalized solution, strong solution

3.1 Introduction

The work developed in this chapter is concerned with the existence and regularity of the solution for the stationary Boussinesq system with Navier boundary condition for the velocity and Dirichlet boundary condition for the temperature. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{2,1}$. Consider the following stationary Boussinesq system:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ -\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = h & \text{in } \Omega, \end{cases} \tag{BS}$$

attached with the following boundary conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \text{(3.1a)} \qquad 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{a} \qquad \text{(3.1b)}$$

$$\theta = \theta_b, \tag{3.2}$$

where Γ is the boundary of Ω, $\boldsymbol{u}$, π and θ are the velocity, pressure and temperature of the fluid, respectively, $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ is the deformation tensor (or linearized strain tensor) associated with the velocity field $\boldsymbol{u}$, $\boldsymbol{n}$ is the unit outward normal vector, $\boldsymbol{\tau}$ is the corresponding unit tangent vector, $\nu > 0$ is the kinematic viscosity of the fluid, $\kappa > 0$ is the thermal diffusivity of the fluid, $\boldsymbol{g}$ is the gravitational acceleration and h is a heat source applied on the fluid. Furthermore, we prescribe the following scalar and vector fields on the boundary: α is a friction scalar function, $\boldsymbol{a}$ is a tangential vector field and θ_b is the temperature. In the case $\alpha > 0$, the boundary condition (3.1) is called Navier boundary condition with linear friction.

Along this chapter, we will consider a non-zero gravitational acceleration $\boldsymbol{g}$. This assumption is done to avoid the decoupling of the Navier-Stokes equations and the convection-diffusion equation. Further, it is noteworthy that in fact $\boldsymbol{g}$ belongs to $\boldsymbol{L}^{\infty}(\Omega)$, but we consider important to relax this assumption because, in mathematical sense, we can enlarge the space in which $\boldsymbol{g}$ lies and we realize that it is still possible to get the solution for the Boussinesq system.

Many of the problems that are studied in Fluid Mechanics consider no-slip boundary condition, that is, the famous Dirichlet boundary condition. This condition is suitable when we study phenomena in which the boundary of the domain is solid, for example a wall. In this case, the velocity of the fluid at the wall is equal to the velocity of the wall. In most situations, the walls are not moving, so the velocity of the fluid is zero. In the case when the wall is in motion (for example drag flows), the velocity of the fluid is equal to the wall's velocity. An interesting explanation about the adherence of a fluid on a wall with microscopic asperities is found in [14]. There are other very interesting problems in which this boundary condition is not appropriate or is difficult to apply, for example, when a fluid forms part of the boundary of the domain (case of perforated boundary in which air or other fluid is in contact with the fluid contained inside the domain), or when we want to model flows with free boundary, see [63], or when the fluid is in presence of a boundary layer, see [14, Remark 6]. In these cases we have to use the Navier boundary condition, so-called slip boundary condition. This boundary condition was proposed by Claude Navier in 1823, see [51]. It is worth to say that (3.1a) means that the boundary is impermeable, and (3.1b) means that the friction forces on the boundary are proportional to the tangential component of the velocity.

In the literature we can find many works concerning the study of Stokes, Navier-Stokes and related systems attached with the Navier boundary condition. Among them we can cite the article written by Solonnikov and Ščadilov in 1973, see [63], who were the first to study the existence of weak solutions and strong solutions in the Hilbert case for the stationary Stokes problem. In [34] was shown the existence, uniqueness and regularity of the solution for the evolutionary Navier-Stokes equations by expressing the Navier boundary conditions in a suitable way depending on the vorticity of the velocity field. Previously, it was studied the Stokes and Navier-Stokes equations with three types of boundary conditions, i.e., the velocity is given on a portion of the boundary, the pressure and the tangential component of the velocity is given on a second portion of the boundary, and the normal component of the velocity and the tangential component of the vorticity is given on the rest of the boundary, see [8]. A study of the L^p-theory of the Stokes equations and the Hilbert theory for Navier-Stokes equations is found in [5]. In the field of numerical analysis, we can find the articles [41] and [42] which studied a numerical approximation of the solution to the Navier-Stokes equations by using Multilevel Methods. Also, in [67] and [68] we can find a numerical approximation of the solution to the Navier-Stokes equations but using Finite Element Method. All the articles mentioned above consider Navier boundary condition with $\alpha = 0$. On the other hand, in the case $\alpha > 0$, we can cite [9] in which we can find a study of weak and strong solutions for a generalized Stokes system with non-homogeneous Navier boundary condition in the Hilbert case, [18] and [17] deal with the homogenization theory to the Stokes and Navier-Stokes equations and [32] shows a Finite Elements approximation for the Navier-Stokes equations.

It is noteworthy that in the literature, to the best of my knowledge, L^p-theory for the Boussinesq system with Navier boundary condition has never been studied. And precisely, in our work we are focused in studying the existence of weak solutions in the Hilbertian case, and later, the L^p regularity of these weak solutions in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ. Further, we regard a general acceleration of gravity instead of the usual one (constant acceleration of gravity $\boldsymbol{g}$) because we are concerned in showing existence of solutions in a larger space for the gravitational acceleration $\boldsymbol{g}$.

We prove existence of weak solutions in $H^1(\Omega)$ for any data in suitable spaces by applying a Leray-Schauder fixed point argument. In order to prove the regularity of the weak solution in $W^{1,p}(\Omega)$ with $p > 2$, and $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$, we use a bootstrap argument by using the regularity results for the Stokes equations with Navier boundary condition (with $\alpha = 0$) and Poisson equation with Dirichlet boundary condition.

This chapter is organized as follows: in section 2, we describe our main results. Section 3 is devoted to introduce some notations and to precise some results which will be useful for the proofs of the main results. In section 4, we study the existence of weak solutions for the problem (BS)-(3.1)-(3.2). Finally, the regularity of the weak solution in $W^{1,p}(\Omega)$ for $p > 2$ and in $W^{2,p}(\Omega)$ for $p \geq \frac{6}{5}$ is dealt in section 5.

3.2 Main results

This section is devoted to present the main results of this chapter. The first theorem is concerned with the existence of weak solutions for the Boussinesq system with Navier boundary conditions for any data in suitable spaces.

Theorem 3.2.1 (weak solutions of the Boussinesq system in $H^1(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{2,1}$ and let*

$$\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega),\ h \in H^{-1}(\Omega),\ \alpha \in L^{2+\varepsilon}(\Gamma) \text{ satisfying (H) for any } \varepsilon > 0 \text{ sufficiently small,}$$

$$\boldsymbol{a} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ such that } \boldsymbol{a}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,\ \theta_b \in H^{\frac{1}{2}}(\Gamma).$$

Then, problem (BS)-(3.1)-(3.2) has a weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times H^1(\Omega)$. Further, if $\theta_b = 0$ on Γ, $\boldsymbol{u}$ and θ satisfy the following estimates:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{M_1}{\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right),$$

$$\|\nabla\theta\|_{L^2(\Omega)} \leq \frac{M_1}{\kappa}\|h\|_{H^{-1}(\Omega)},$$

with $M_1 = M_1(\Omega) > 0$ independent of α. Moreover, if there exists $\gamma = \gamma(\Omega) > 0$ such that

$$\nu\kappa \geq \gamma\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)},$$

then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{M_2}{\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\theta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right),$$

$$\|\theta\|_{H^1(\Omega)} \leq M_2\left[\left(1 + \frac{1}{\kappa}\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right)\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} + \frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)}\right],$$

with $M_2 = M_2(\Omega, \gamma) > 0$ independent of α.

Next two theorems deal with the L^p regularity of the weak solution of the Boussinesq system with Navier boundary conditions. In order to get these results, we use a classical bootstrap argument by using regularity results of the Poisson equation with Dirichlet boundary condition and Stokes equations with Navier boundary condition with $\alpha > 0$. We consider $p > 2$ for regularity in $W^{1,p}(\Omega)$ and $p \geq \frac{6}{5}$ for regularity in $W^{2,p}(\Omega)$.

Theorem 3.2.2 (generalized solutions in $W^{1,p}(\Omega)$ with $p > 2$)**.** *Let us suppose that*

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega), \quad h \in W^{-1,p}(\Omega), \quad \alpha \in L^{t^*(p)}(\Gamma) \quad \textit{satisfying} \quad \text{(H)}$$

$$\textit{with } t^*(p) \textit{ is defined by } (4.14) \textit{ and } \quad (\boldsymbol{a}, \theta_b) \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times W^{1-\frac{1}{p},p}(\Gamma)$$

with

$$p > 2, \quad r = \max\left\{\frac{3}{2}, \frac{3p}{3+p}\right\} \quad \textit{if} \quad p \neq 3 \quad \textit{and} \quad r = \frac{3}{2} + \varepsilon \quad \textit{if} \quad p = 3$$

for any $\varepsilon > 0$ sufficiently small. Then the weak solution for (BS)-(3.1)-(3.2) *given by Theorem 3.2.1 satisfies*

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R} \times W^{1,p}(\Omega).$$

Theorem 3.2.3 (strong solutions in $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$)**.** *Let us suppose that $\boldsymbol{g} \in \boldsymbol{L}^r(\Omega)$, $h \in L^p(\Omega)$,*

$$\alpha \in H^{\frac{1}{2}}(\Gamma) \quad \textit{if} \quad \frac{6}{5} \leq p \leq 2; \quad \alpha \in H^{\frac{1}{2}+\varepsilon}(\Gamma) \quad \textit{if} \quad 2 < p < 3; \quad \alpha \in W^{1-\frac{1}{p},p}(\Gamma) \quad \textit{if} \quad p \geq 3$$

satisfying (H) *and*

$$(\boldsymbol{a}, \theta_b) \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \times W^{2-\frac{1}{p},p}(\Gamma)$$

with

$$p \geq \frac{6}{5}, \quad r = \max\left\{\frac{3}{2}, p\right\} \quad \textit{if} \quad p \neq \frac{3}{2} \quad \textit{and} \quad r = \frac{3}{2} + \varepsilon \quad \textit{if} \quad p = \frac{3}{2}$$

for any $\varepsilon > 0$ sufficiently small. Then the solution for (BS)-(3.1)-(3.2) *given by Theorem 3.2.1 satisfies*

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R} \times W^{2,p}(\Omega).$$

3.3 Notations and some useful results

Throughout this work, we consider $\Omega \subset \mathbb{R}^3$ a bounded domain with boundary Γ of class $\mathcal{C}^{2,1}$. We use *domain* to stand for a nonempty open and connected set. Later, we will use the term *axisymmetric* to stand for a nonempty set which is generated by rotation around an axis. In the case that Ω is another kind of set, we will point it out. Bold font for spaces means vector (or matrix) valued spaces, and their elements will be denoted with bold font also. We will denote by $\boldsymbol{n}$ and $\boldsymbol{\tau}$ the unit outward normal vector and the unit tangent vector on Γ, respectively. Unless otherwise stated or unless the context otherwise requires, we will write with the same positive constant all the constants which depend on the same arguments in the estimations that will appear along this work.

If $1 < p < \infty$, then p' will denote the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If $\boldsymbol{u}$ is a vector field in $\mathbb{R}^3$, i.e., $\boldsymbol{u} = (u_i)$ with $i \in \{1, 2, 3\}$, $\nabla \boldsymbol{u}$ stands for a second order tensor in $\mathbb{R}^{3\times 3}$ defined by

$$(\nabla \boldsymbol{u})_{ij} = \frac{\partial u_j}{\partial x_i}$$

with $i, j \in \{1, 2, 3\}$. Further, if $\mathbb{T}$ is a second order tensor in $\mathbb{R}^{3\times 3}$, i.e., $\mathbb{T} = (t_{ij})$ with $i, j \in \{1, 2, 3\}$, div $\mathbb{T}$ stands for a vector field in $\mathbb{R}^3$ defined by

$$(\text{div } \mathbb{T})_i = \sum_{k=1}^{3} \frac{\partial t_{ki}}{\partial x_k}$$

with $i \in \{1, 2, 3\}$. With this notation, the Laplace operator of a vector field $\boldsymbol{u}$ can be written in divergence form as

$$\Delta \boldsymbol{u} = \text{div}(\nabla \boldsymbol{u}).$$

Nevertheless, in mathematical models for which the strain tensor takes part as a boundary condition (e.g., as a surface traction in elasticity theory, as a slip condition in fluid mechanics, etc), the previous representation of the Laplace operator is not suitable when we are trying to find the variational formulation of the problem in order to well-define a weak solution. In these cases, we are very encouraged to use the following identity:

$$\Delta \boldsymbol{u} = 2\text{div } \mathbb{D}(\boldsymbol{u}) - \nabla(\text{div } \boldsymbol{u}).$$

Let us define the operator $\boldsymbol{E}$ from $\boldsymbol{W}^{1,p}(\Omega)$ into $(\boldsymbol{W}^{1,p}(\Omega))^{3\times 3}$ as $\boldsymbol{E}(\boldsymbol{v}) = \mathbb{D}(\boldsymbol{v})$. Then, the kernel of the operator $\boldsymbol{E}$ ($\ker \boldsymbol{E}$) is the space of rigid motions in $\mathbb{R}^3$, i.e.,

$$\ker \boldsymbol{E} = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \exists\, \boldsymbol{b} \in \mathbb{R}^3 \text{ and } \exists\, \boldsymbol{c} \in \mathbb{R}^3 \text{ such that } \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x} + \boldsymbol{c}, \text{ a.e. } \boldsymbol{x} \in \Omega\}. \tag{3.3}$$

The proof follows easily from [16, Theorem 6.15-2,p. 406] by adapting it to the $\boldsymbol{L}^p$ case.

We will denote by $\mathcal{D}(\Omega)$ the set of smooth functions (infinitely differentiable functions) with compact support in Ω and by $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ the subspace of $\boldsymbol{\mathcal{D}}(\Omega)$ formed by divergence-free vector functions in Ω. Further, we will work with the closed subspace $\boldsymbol{V}^p_{\sigma,T}(\Omega)$ of $\boldsymbol{W}^{1,p}(\Omega)$ formed by divergence-free vector functions with null normal trace, that is,

$$\boldsymbol{V}^p_{\sigma,T}(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \text{div } \boldsymbol{v} = 0 \ \text{ in } \Omega \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma\right\},$$

for $1 < p < \infty$.

In order to study the $\boldsymbol{L}^p$-regularity of the solutions for the problem (BS)-(3.1)-(3.2), we need definitions of some Banach spaces and linked results. Let us define for all $1 < p < \infty$ the following space:

$$\boldsymbol{H}^p(\text{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \text{div } \boldsymbol{v} \in L^p(\Omega)\}$$

which is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\text{div},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\text{div } \boldsymbol{v}\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$

Also, we define $\boldsymbol{H}^p_0(\text{div}, \Omega)$ as the closure of $\boldsymbol{\mathcal{D}}(\Omega)$ in the norm of $\boldsymbol{H}^p(\text{div}, \Omega)$, that is, $\boldsymbol{H}^p_0(\text{div}, \Omega) = \overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\|\cdot\|_{\boldsymbol{H}^p(\text{div},\Omega)}}$. As in the Hilbertian case (see [30, Theorem 2.4, p. 27

and Theorem 2.6, p. 29]), it is possible to show that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathrm{div},\Omega)$ (see [62, Proposition 1.0.2, p. 17]) and $\boldsymbol{H}^p_0(\mathrm{div},\Omega)$ is characterized by

$$\boldsymbol{H}^p_0(\mathrm{div},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

Furthermore, we have the following characterization for the dual space of $\boldsymbol{H}^p_0(\mathrm{div},\Omega)$

$$(\boldsymbol{H}^p_0(\mathrm{div},\Omega))' = \{\boldsymbol{\psi} + \nabla\varphi;\ \boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega) \text{ and } \varphi \in L^{p'}(\Omega)\},$$

see [62, Proposition 1.0.4, p. 20].

The following results are concerned with the existence of solutions in $\boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{W}^{2,p}(\Omega)$, for $1 < p < \infty$, of the following Stokes problem:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\ \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g\ ,\ 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{SNb}$$

These results will play an important role in the study of the $\boldsymbol{L}^p$-regularity of the solutions to (BS)-(3.1)-(3.2), see [5, Corollary 3.8, Theorem 3.9, Theorem 4.1] for their proofs.

Proposition 3.3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{2,1}$. For $1 < p < \infty$, let $\boldsymbol{f} \in (\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega))'$, $\chi \in L^p(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ satisfying the following compatibility conditions:*

$$\boldsymbol{h}\cdot\boldsymbol{n} = 0, \text{ on } \Gamma, \tag{3.4}$$

$$\int_\Omega \chi \,\mathrm{d}x = \int_\Gamma g \,\mathrm{d}s, \tag{3.5}$$

$$\langle \boldsymbol{f}, \boldsymbol{\beta}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_\Gamma = 0, \tag{3.6}$$

where $\boldsymbol{\beta} := \boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{b}\times\boldsymbol{x}$, with $\boldsymbol{x} \in \Omega$, appears if Ω is axisymmetric with respect to the axis with vector direction $\boldsymbol{b} \in \mathbb{R}^3$; $\langle\cdot,\cdot\rangle_\Omega$ means duality between $(\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega))'$ and $\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)$, and $\langle\cdot,\cdot\rangle_\Gamma$ means duality between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$. Then, the Stokes problem (SNb) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Further, we have the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \le C\left(\|\boldsymbol{f}\|_{(\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right)$$

with $C = C(p,\Omega) > 0$, and where

$$\boldsymbol{\mathcal{K}}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \mathbb{D}(\boldsymbol{v}) = \boldsymbol{0} \text{ in } \Omega,\ \mathrm{div}\ \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}.$$

Remark 3.3.2. The space $\boldsymbol{\mathcal{K}}(\Omega)$ stands for the kernel of the Stokes operator associated with the system (SNb), in the case $\chi = 0$ and $g = 0$. This kernel is equal to span$\{\boldsymbol{\beta}\}$ (a particular case of the space of rigid motions, see (3.3)) if Ω is axisymmetric or, otherwise, this is reduced to the zero vectorial function. Further, this shows that $\boldsymbol{\mathcal{K}}(\Omega)$ does not depend on p.

Proposition 3.3.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{2,1}$. For $1 < p < \infty$, let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\chi \in W^{1,p}(\Omega)$, $g \in W^{2-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ satisfying the compatibility conditions (3.4), (3.5) and (3.6). Then, the Stokes problem (SNb) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ which satisfies the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \le C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\right),$$

with $C = C(p,\Omega) > 0$.

It is worthwhile to note that the assumption on $\boldsymbol{f}$ in Proposition 3.3.1 can be relaxed by considering the space defined for all $1 < r, p < \infty$

$$\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^r(\Omega);\ \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega) \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

which is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|_{L^p(\Omega)}.$$

Note that if $r = p$, then $\boldsymbol{H}_0^{p,p}(\mathrm{div}, \Omega) = \boldsymbol{H}_0^{p}(\mathrm{div}, \Omega)$ and, the norms $\|\cdot\|_{\boldsymbol{H}_0^{p,p}(\mathrm{div},\Omega)}$ and $\|\cdot\|_{\boldsymbol{H}_0^{p}(\mathrm{div},\Omega)}$ are equivalent. Furthermore, it is possible to show that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega)$.

Let us denote by $(\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega))'$ the dual space of $\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega)$. The following lemma gives us a characterization of functionals in $(\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega))'$.

Lemma 3.3.4 (characterization of $(\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega))'$)**.** *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz bounded domain. A distribution $\boldsymbol{f}$ lies in $(\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega))'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{r'}(\Omega)$ and $\varphi \in L^{p'}(\Omega)$ such that*

$$\boldsymbol{f} = \boldsymbol{\psi} + \nabla\varphi.$$

Moreover, we have that

$$\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega))'} = \inf_{\substack{\boldsymbol{\psi} \in \boldsymbol{L}^{r'}(\Omega) \\ \varphi \in L^{p'}(\Omega)}} \max\{\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{r'}(\Omega)}, \|\varphi\|_{L^{p'}(\Omega)}\}.$$

Proof. Suppose that there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{r'}(\Omega)$ and $\varphi \in L^{p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\varphi$. Then, for all $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ we have

$$\langle \boldsymbol{\psi} + \nabla\varphi, \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)} = \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{v} - \varphi\ \mathrm{div}\ \boldsymbol{v}\ \mathrm{d}x.$$

Then, it is clear that $T : \boldsymbol{v} \mapsto \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{v} - \varphi\ \mathrm{div}\ \boldsymbol{v}\ \mathrm{d}x$ is a linear and continuous functional on $\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega)$, and as $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega)$, by density, we conclude that $T = \boldsymbol{\psi} + \nabla\varphi \in (\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega))'$.

Conversely, assume that $\boldsymbol{f} \in (\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega))'$. Let us define the space $\boldsymbol{X} = \boldsymbol{L}^r(\Omega) \times L^p(\Omega)$ endowed with the norm

$$\|(\boldsymbol{\chi_1}, \chi_2)\|_{\boldsymbol{X}} = \|\boldsymbol{\chi_1}\|_{\boldsymbol{L}^r(\Omega)} + \|\chi_2\|_{L^p(\Omega)}.$$

Let us set S as the operator given by

$$\begin{array}{rcl} S : \boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega) & \longrightarrow & \boldsymbol{X} \\ \boldsymbol{v} & \mapsto & S\ \boldsymbol{v} = (\boldsymbol{v}, \mathrm{div}\ \boldsymbol{v}). \end{array}$$

This operator is linear and an isometry, then S is an isometric isomorphism from $\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega)$ onto its range $\mathcal{R}(S)$. Its adjoint operator $S^* : (\mathcal{R}(S))' \longrightarrow (\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega))'$ is also an isometric isomorphism, and then, for all $\boldsymbol{g} \in (\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega))'$, there exists a unique $\boldsymbol{g}^* \in (\mathcal{R}(S))'$ such that

$$\langle \boldsymbol{g}^*, S\ \boldsymbol{v}\rangle_{(\mathcal{R}(S))',\mathcal{R}(S)} = \langle S^*\boldsymbol{g}^*, \boldsymbol{v}\rangle_{(\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega))',\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)} = \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{(\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega))',\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)},$$

for any $\boldsymbol{v} \in \boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega)$. Moreover, since S is an isometry, it is clear that $\|\boldsymbol{g}\|_{(\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega))'} = \|\boldsymbol{g}^*\|_{(\mathcal{R}(S))'}$.

In particular, taking $\boldsymbol{g} = \boldsymbol{f}$ and by using the Hahn-Banach theorem, we have that $\boldsymbol{f}^*$ defined on $\mathcal{R}(S)$ can be extended to the whole space $\boldsymbol{X}$, to an element denoted by $(\boldsymbol{\psi}, -\varphi) \in \boldsymbol{X}'$ and thanks to the fact that $\boldsymbol{X}'$ is isomorphic to $\boldsymbol{L}^{r'}(\Omega) \times L^{p'}(\Omega)$, we have the following identification

$$(\boldsymbol{\psi}, -\varphi) : (\boldsymbol{\rho_1}, \rho_2) \mapsto \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{\rho_1} + (-\varphi)\rho_2 \,\mathrm{d}x,$$

for any $(\boldsymbol{\rho_1}, \rho_2) \in \boldsymbol{X}$, and $\|\boldsymbol{f}^*\|_{(\mathcal{R}(S))'} = \|(\boldsymbol{\psi}, -\varphi)\|_{\boldsymbol{X}'}$. Then, it follows that

$$\begin{aligned}
\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{(\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega))', \boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)} &= \langle \boldsymbol{f}^*, S\,\boldsymbol{v}\rangle_{(\mathcal{R}(S))', \mathcal{R}(S)} \\
&= \langle (\boldsymbol{\psi}, -\varphi), (\boldsymbol{v}, \mathrm{div}\,\boldsymbol{v})\rangle_{\boldsymbol{X}', \boldsymbol{X}} \\
&= \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{v} - \varphi \,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x
\end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega)$. This implies that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\varphi$ and, since the representation of $\boldsymbol{f}$ is not unique, it is clear that

$$\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega))'} = \inf_{\substack{\boldsymbol{\psi} \in \boldsymbol{L}^{r'}(\Omega) \\ \varphi \in L^{p'}(\Omega)}} \max\{\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{r'}(\Omega)}, \|\varphi\|_{L^{p'}(\Omega)}\}.$$

□

Next proposition is about the existence and uniqueness of the solution for the problem (SNb) in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ with the right hand side $\boldsymbol{f}$ weakened.

Proposition 3.3.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{2,1}$. For $1 < r, p < \infty$ such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, let $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))'$, $\chi \in L^p(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ satisfying the compatibility conditions (3.4), (3.5) and (3.6). Then, the Stokes problem (SNb) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Further, we have the following estimate:*

$$\begin{aligned}
\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C \Big(&\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} \\
&+ \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \Big),
\end{aligned}$$

with $C = C(r, p, \Omega) > 0$, and where we replace the duality pairing on Ω by the duality between $(\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))'$ and $\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)$.

Proof. Thanks to Lemma 3.3.4, there exist $\boldsymbol{\psi} \in \boldsymbol{L}^r(\Omega)$ and $\varphi \in L^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\varphi$. Then, by using Proposition 3.3.1, we have that there exists a unique solution $(\boldsymbol{u}_1, \pi_1) \in \boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times L^p(\Omega)/\mathbb{R}$ of the problem

$$\begin{cases}
-\Delta \boldsymbol{u}_1 + \nabla\pi_1 = \nabla\varphi & \text{in } \Omega, \\
\mathrm{div}\,\boldsymbol{u}_1 = \chi & \text{in } \Omega, \\
\boldsymbol{u}_1 \cdot \boldsymbol{n} = g\ ,\ 2\,[\mathbb{D}(\boldsymbol{u}_1)\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma.
\end{cases}$$

This solution satisfies the estimate

$$\|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega)} + \|\pi_1\|_{L^p(\Omega)/\mathbb{R}} \leq C\left(\|\nabla\varphi\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} \right.$$
$$\left. + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right). \tag{3.7}$$

On the other hand, by using Proposition 3.3.3, there exists a unique solution $(\boldsymbol{u}_2, \pi_2) \in \boldsymbol{W}^{2,r}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times W^{1,r}(\Omega)/\mathbb{R}$ of the problem

$$\begin{cases} -\Delta \boldsymbol{u}_2 + \nabla\pi_2 = \boldsymbol{\psi} & \text{in } \Omega, \\ \operatorname{div}\, \boldsymbol{u}_2 = 0 & \text{in } \Omega, \\ \boldsymbol{u}_2\cdot\boldsymbol{n} = 0 \ ,\ 2\left[\mathbb{D}(\boldsymbol{u}_2)\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

which satisfies

$$\|\boldsymbol{u}_2\|_{\boldsymbol{W}^{2,r}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega)} + \|\pi_2\|_{W^{1,r}(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\psi}\|_{\boldsymbol{L}^r(\Omega)}. \tag{3.8}$$

But since $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, we have that $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and $W^{1,r}(\Omega) \hookrightarrow L^p(\Omega)$. Hence, $(\boldsymbol{u}_2, \pi_2) \in \boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Finally, $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1 + \boldsymbol{u}_2, \pi_1 + \pi_2) \in \boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times L^p(\Omega)/\mathbb{R}$ is the unique solution of the problem (SNb), and the estimate is obtained by using (3.7), (3.8) and Lemma 3.3.4. □

Remark 3.3.6. It is not necessary to assume $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ in order to have the existence of a solution for (SNb). In fact, it is enough to assume $\boldsymbol{f}$ in a proper subspace of $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$. This is because if $\boldsymbol{f}$ belongs to $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, thanks to Lemma 3.3.4, we can always write $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\varphi$ for some $\boldsymbol{\psi} \in \boldsymbol{L}^r(\Omega)$ and $\varphi \in L^p(\Omega)$. Then, it is possible to rewrite the first equation of (SNb) to have the following problem: find $(\boldsymbol{u}, \rho) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla\rho = \boldsymbol{\psi} & \text{in } \Omega, \\ \operatorname{div}\, \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \ ,\ 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma, \end{cases} \tag{3.9}$$

with $\rho := \pi - \varphi$. This suggest the idea of finding a solution of (3.9), taking into account that the right hand side $\boldsymbol{\psi}$ just lies in $\boldsymbol{L}^r(\Omega)$. In fact, this solution already exists and is unique, thanks to Proposition 3.3.5. Indeed, if $\boldsymbol{\psi} \in \boldsymbol{L}^r(\Omega)$, then $\boldsymbol{\psi} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, and if we assume that r satisfies $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, then by Proposition 3.3.5, there exists a unique $(\boldsymbol{u}, \rho) \in \boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times L^p(\Omega)/\mathbb{R}$ solution of (3.9). In conclusion, the Stokes problem (SNb) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega) \times L^p(\Omega)/\mathbb{R}$, if we just consider $\boldsymbol{f}$ belongs to $\boldsymbol{L}^r(\Omega)$ with r satisfying $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$. Moreover, since $\boldsymbol{L}^r(\Omega) \hookrightarrow (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, this solution satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{K}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right). \tag{3.10}$$

Remark 3.3.7. (i) If we consider the following Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\, \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \ ,\ 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma, \end{cases} \tag{3.11}$$

with α satisfying (H), it is easy to see that the kernel associated with the Stokes operator under this system is

$$\boldsymbol{\mathcal{I}}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \mathbb{D}(\boldsymbol{v}) = \boldsymbol{0} \text{ in } \Omega,\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v} = 0 \text{ on } \Gamma \right\}.$$

From (3.3), it follows that

$$\boldsymbol{\mathcal{I}}(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \exists\, \boldsymbol{b} \in \mathbb{R}^3 \text{ and } \exists\, \boldsymbol{c} \in \mathbb{R}^3 \text{ such that } \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x} + \boldsymbol{c},\ \text{a.e. } \boldsymbol{x} \in \Omega \text{ and } \\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma \}.$$

But thanks to [16, Problem 6.15-2,p. 411], this kernel is reduced to the zero vector function, i.e., $\boldsymbol{\mathcal{I}}(\Omega) = \{\boldsymbol{0}\}$ independent if Ω is axisymmetric or not.

(ii) Thanks to $\int_\Omega (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{v}\, \mathrm{d}x = 0$ for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, the kernel of the Navier-Stokes operator with Navier boundary conditions is equal to the kernel of its respective associated Stokes operator (review Remark 3.3.2 and the previous point (i)).

The following result plays an important role in the existence of weak solutions for the problem (BS)-(3.1)-(3.2). The reason is that we need an equivalence of norms between $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and $\|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}$ in order to have coerciveness of the bilinear form associated with the problem (BS)-(3.1)-(3.2).

Proposition 3.3.8. *Let Ω be a bounded domain of class $\mathcal{C}^{1,1}$. Then $\|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}$ is a norm which is equivalent to the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$, for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ solution of the Stokes or Navier-Stokes equations with Navier boundary condition (3.1) as long as α satisfies (H).*

Proof. As $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, thanks to [5, Lemma 3.3], it follows that

$$\inf_{\boldsymbol{v} \in \boldsymbol{\mathcal{I}}(\Omega)} \|\boldsymbol{u} + \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \le C_1 \|\mathbb{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)},$$

for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, where $C_1 = C_1(\Omega) > 0$. On the other hand, we have that

$$\inf_{\boldsymbol{v} \in \boldsymbol{\mathcal{I}}(\Omega)} \|\nabla(\boldsymbol{u} + \boldsymbol{v})\|^2_{\boldsymbol{L}^2(\Omega)} \le C_2 \|\mathbb{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)},$$

with $C_2 = C_2(\Omega) > 0$, see [23]. Then, thanks to Remark 3.3.7, we have that $\boldsymbol{\mathcal{I}}(\Omega) = \{\boldsymbol{0}\}$ and from the last two inequalities, we conclude that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C_3 \|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}$$

where $C_3 = (C_1 + C_2)^{\frac{1}{2}}$. Clearly, we have that

$$\|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} \le C_4 \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$$

for some positive constant C_4. This completes the proof. ☐

Remark 3.3.9. (i) In the case of Dirichlet boundary conditions, thanks to the well-known Poincaré inequality, we know that the seminorm $\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}$ is a norm which is equivalent to the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ for all $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$. This fact is used to get a priori estimates of the velocity $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$. In a similar way, Proposition 3.3.8 gives to us the equivalence of the norms $\|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}$ and $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$. This property is known as Korn's inequality.
(ii) The result given in Proposition 3.3.8 is based on the fact that the kernel for the Stokes operator in this case is reduced to zero.

It is worthwhile to give the following remark concerning the existence of the pressure, which is always present in the study of Stokes, Navier-Stokes and related systems.

Remark 3.3.10. As a classical method in the study of the Stokes and Navier-Stokes equations, the pressure π is obtained thanks to a variant of De Rham's theorem. Indeed, let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. If $\boldsymbol{f} \in \boldsymbol{W}^{-1,p}(\Omega)$ for $1 < p < \infty$, satisfies that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \quad \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = 0,$$

then there exists $\pi \in L^p(\Omega)$ such that $\boldsymbol{f} = \nabla \pi$, see [4, Theorem 2.8] for details.

3.4 Weak solutions

The goal of this section is to establish the existence of a weak solution in the Hilbertian case for the Boussinesq system (BS) with (3.1) and (3.2) as boundary conditions.

Theorem 3.4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{2,1}$ and let*

$$\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega),\ h \in H^{-1}(\Omega),\ \alpha \in L^{2+\varepsilon}(\Gamma) \text{ satisfying (H) for any } \varepsilon > 0 \text{ sufficiently small,}$$

$$\boldsymbol{a} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ such that } \boldsymbol{a} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma,\ \theta_b \in H^{\frac{1}{2}}(\Gamma).$$

Then, problem (BS)-(3.1)-(3.2) has a weak solution $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times H^1(\Omega)$. Further, if $\theta_b = 0$ on Γ, $\boldsymbol{u}$ and θ satisfy the following estimates:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{M_1}{\nu} \left(\nu \|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \right), \tag{3.12}$$

$$\|\nabla \theta\|_{L^2(\Omega)} \leq \frac{M_1}{\kappa} \|h\|_{H^{-1}(\Omega)}, \tag{3.13}$$

with $M_1 = M_1(\Omega) > 0$ independent of α. Moreover, if there exists $\gamma = \gamma(\Omega) > 0$ such that

$$\nu\kappa \geq \gamma \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{3.14}$$

then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{M_2}{\nu} \left(\nu \|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\theta\|_{H^1(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \right), \tag{3.15}$$

$$\|\theta\|_{H^1(\Omega)} \leq M_2 \left[\left(1 + \frac{1}{\kappa} \|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} + \frac{1}{\kappa} \|h\|_{H^{-1}(\Omega)} \right], \tag{3.16}$$

with $M_2 = M_2(\Omega, \gamma) > 0$ independent of α.

Proof. Let us define $\boldsymbol{H} := \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ as the Hilbert space equipped with the norm

$$\|(\boldsymbol{u}, \theta)\|_{\boldsymbol{H}} = \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\theta\|_{H^1(\Omega)}.$$

Let $(\boldsymbol{u}, \theta) \in \boldsymbol{H}$ given. Thanks to the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have that $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ and $\boldsymbol{u} \cdot \nabla \theta = \operatorname{div}(\theta \boldsymbol{u}) \in \boldsymbol{W}^{-1,3}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$. Also, due to the embedding $\boldsymbol{H}^{\frac{1}{2}}(\Gamma) \hookrightarrow \boldsymbol{L}^4(\Gamma)$, it follows that $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \boldsymbol{L}^{\frac{4}{3}}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$. Then, by using Remark 3.3.6, the classical existence and uniqueness result for Poisson equation with Dirichlet boundary condition and the existence and uniqueness result for Stokes equations with Navier boundary condition, there exists a unique $(\boldsymbol{u}^*, \theta^*, \pi^*) \in \boldsymbol{H} \times L^2(\Omega)/\mathbb{R}$ weak solution of the following uncoupled system:

$$\begin{cases} -\nu\Delta \boldsymbol{u}^* + \nabla\pi^* = \theta^* \boldsymbol{g} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & \text{in } \Omega, \\ \text{div } \boldsymbol{u}^* = 0 & \text{in } \Omega, \\ -\kappa\Delta\theta^* = h - \text{div}(\theta\boldsymbol{u}) & \text{in } \Omega, \\ \boldsymbol{u}^*\cdot\boldsymbol{n} = 0 \ , \ 2\left[\mathbb{D}(\boldsymbol{u}^*)\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{a} - \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} & \text{on } \Gamma, \\ \theta^* = \theta_b & \text{on } \Gamma. \end{cases} \tag{3.17}$$

Let $\mathcal{R} : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ be the operator such that $(\boldsymbol{u}^*, \theta^*) = \mathcal{R}(\boldsymbol{u}, \theta)$ is the unique weak solution to (3.17). Let us realize that a fixed point of the operator $\mathcal{R}$ is a weak solution of (BS)-(3.1)-(3.2). In order to find a fixed point of the operator $\mathcal{R}$, we will apply the Leray-Schauder fixed point theorem.

(i) *Let us prove that $\mathcal{R}$ is a compact operator.* Suppose $(\boldsymbol{u}, \theta) \in \boldsymbol{H}$, $(\boldsymbol{u}_n, \theta_n) \in \boldsymbol{H}$, with $n \in \mathbb{N}$ and $(\boldsymbol{u}_n, \theta_n) \rightharpoonup (\boldsymbol{u}, \theta)$, in $\boldsymbol{H}$-weak. Let us define $(\boldsymbol{u}_n^*, \theta_n^*) := \mathcal{R}(\boldsymbol{u}_n, \theta_n)$, for all $n \in \mathbb{N}$. We obtain that $(\boldsymbol{u}_n^* - \boldsymbol{u}^*, \theta_n^* - \theta^*)$ satisfies the following system:

$$\begin{cases} -\nu\Delta(\boldsymbol{u}_n^* - \boldsymbol{u}^*) + \nabla(\pi_n^* - \pi^*) = (\theta_n^* - \theta^*)\boldsymbol{g} - \left[(\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] & \text{in } \Omega, \\ \text{div}(\boldsymbol{u}_n^* - \boldsymbol{u}^*) = 0 & \text{in } \Omega, \\ -\kappa\Delta(\theta_n^* - \theta^*) = -\text{div}(\theta_n\boldsymbol{u}_n - \theta\boldsymbol{u}) & \text{in } \Omega, \\ (\boldsymbol{u}_n^* - \boldsymbol{u}^*)\cdot\boldsymbol{n} = 0 \ , \ 2\left[\mathbb{D}(\boldsymbol{u}_n^* - \boldsymbol{u}^*)\boldsymbol{n}\right]_{\boldsymbol{\tau}} = -\alpha\, (\boldsymbol{u}_n - \boldsymbol{u})_{\boldsymbol{\tau}} & \text{on } \Gamma, \\ \theta_n^* - \theta^* = 0 & \text{on } \Gamma. \end{cases}$$

Note that $\theta_n^* - \theta^* \in H_0^1(\Omega)$, then by multiplying both sides of the Poisson equation by $\theta_n^* - \theta^*$ and integrating by parts, we have

$$\begin{aligned} \kappa\|\nabla(\theta_n^* - \theta^*)\|_{\boldsymbol{L}^2(\Omega)} &\leq C_1\|\text{div}(\theta_n\boldsymbol{u}_n - \theta\boldsymbol{u})\|_{H^{-1}(\Omega)} \\ &\leq C_1\|\theta_n\boldsymbol{u}_n - \theta\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq C_1\left[\|(\theta_n - \theta)\boldsymbol{u}_n\|_{\boldsymbol{L}^2(\Omega)} + \|(\boldsymbol{u}_n - \boldsymbol{u})\theta\|_{\boldsymbol{L}^2(\Omega)}\right] \\ &\leq C_1\left[\|\theta_n - \theta\|_{L^3(\Omega)}\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\|\theta\|_{H^1(\Omega)}\right], \end{aligned}$$

with $C_1 = C_1(\Omega) > 0$. Since $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$, in $\boldsymbol{H}^1(\Omega)$-weak and $\theta_n \rightharpoonup \theta$, in $H^1(\Omega)$-weak, therefore, $\boldsymbol{u}_n \to \boldsymbol{u}$, in $\boldsymbol{L^s(\Omega)}$ and $\theta_n \to \theta$, in $L^s(\Omega)$, for $1 \leq s < 6$. Then

$$\theta_n^* \underset{n\to\infty}{\longrightarrow} \theta^*, \text{ in } H^1(\Omega). \tag{3.18}$$

On the other hand, note that we can write the term $(\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ of the right hand side of the Stokes equations as

$$(\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \left((\boldsymbol{u}_n - \boldsymbol{u})\cdot\nabla\right)\boldsymbol{u}_n + (\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}_n - \boldsymbol{u}),$$

and since the Stokes operator is linear, it is possible to split the Stokes equations in three parts:

$$\begin{cases} -\nu\Delta\boldsymbol{v}_n + \nabla p_n = (\theta_n^* - \theta^*)\boldsymbol{g} & \text{in } \Omega, \\ \text{div } \boldsymbol{v}_n = 0 & \text{in } \Omega, \\ \boldsymbol{v}_n\cdot\boldsymbol{n} = 0 \ , \ 2\left[\mathbb{D}(\boldsymbol{v}_n)\boldsymbol{n}\right]_{\boldsymbol{\tau}} = -\alpha\, (\boldsymbol{u}_n - \boldsymbol{u})_{\boldsymbol{\tau}} & \text{on } \Gamma, \end{cases} \tag{S_1}$$

$$\left\{\begin{aligned} -\nu\Delta \boldsymbol{w}_n + \nabla q_n &= -((\boldsymbol{u}_n - \boldsymbol{u})\cdot\nabla)\boldsymbol{u}_n && \text{in } \Omega,\\ \operatorname{div}\, \boldsymbol{w}_n &= 0 && \text{in } \Omega,\\ \boldsymbol{w}_n\cdot\boldsymbol{n} = 0\ ,\ 2\left[\mathbb{D}(\boldsymbol{w}_n)\boldsymbol{n}\right]_{\boldsymbol{\tau}} &= \boldsymbol{0} && \text{on } \Gamma, \end{aligned}\right. \tag{S_2}$$

$$\left\{\begin{aligned} -\nu\Delta \boldsymbol{y}_n + \nabla \chi_n &= -(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}_n - \boldsymbol{u}) && \text{in } \Omega,\\ \operatorname{div}\, \boldsymbol{y}_n &= 0 && \text{in } \Omega,\\ \boldsymbol{y}_n\cdot\boldsymbol{n} = 0\ ,\ 2\left[\mathbb{D}(\boldsymbol{y}_n)\boldsymbol{n}\right]_{\boldsymbol{\tau}} &= \boldsymbol{0} && \text{on } \Gamma. \end{aligned}\right. \tag{S_3}$$

Note that $\boldsymbol{u}_n^* - \boldsymbol{u}^* = \boldsymbol{v}_n + \boldsymbol{w}_n + \boldsymbol{y}_n$ and $\pi_n^* - \pi^* = p_n + q_n + \chi_n$. Then, we must study the convergence of $\boldsymbol{u}_n^* - \boldsymbol{u}^*$ and $\pi_n^* - \pi^*$ through the convergence of their respective representations. Since $(\theta_n^* - \theta^*)\boldsymbol{g} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$, from Remark 3.3.6 there exists $(\boldsymbol{v}_n, p_n) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ solution of (S_1) which satisfies the estimate

$$\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} + \|p_n\|_{L^2(\Omega)/\mathbb{R}} \leq C\left(\|(\theta_n^* - \theta^*)\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\alpha\,(\boldsymbol{u}_n - \boldsymbol{u})_{\boldsymbol{\tau}}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right).$$

Because of $\boldsymbol{L}^{\frac{4}{3}}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $\alpha \in L^{2+\varepsilon}(\Gamma)$, we have that

$$\begin{aligned} \|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} + \|p_n\|_{L^2(\Omega)/\mathbb{R}} &\leq C\left(\|(\theta_n^* - \theta^*)\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\alpha\,(\boldsymbol{u}_n - \boldsymbol{u})_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^{\frac{4}{3}}(\Gamma)}\right)\\ &\leq C\left(\|(\theta_n^* - \theta^*)\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\alpha\|_{L^{2+\varepsilon}(\Gamma)}\|(\boldsymbol{u}_n - \boldsymbol{u})_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^{4-\delta}(\Gamma)}\right) \end{aligned}$$

for any $0 < \delta < 1$ and $\varepsilon = \frac{2\delta}{8-3\delta}$. Further, for some $0 < \varepsilon^* < 1$, we have that $\boldsymbol{H}^{\frac{1}{2}-\varepsilon^*}(\Gamma) \hookrightarrow \boldsymbol{L}^{4-\delta}(\Gamma)$, then

$$\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} + \|p_n\|_{L^2(\Omega)/\mathbb{R}} \leq C\left(\|(\theta_n^* - \theta^*)\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\alpha\|_{L^{2+\varepsilon}(\Gamma)}\|(\boldsymbol{u}_n - \boldsymbol{u})_{\boldsymbol{\tau}}\|_{\boldsymbol{H}^{\frac{1}{2}-\varepsilon^*}(\Gamma)}\right).$$

Since $(\boldsymbol{u}_n)_{\boldsymbol{\tau}} \rightharpoonup \boldsymbol{u}_{\boldsymbol{\tau}}$, in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$-weak, and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ is compactly embedded in $\boldsymbol{H}^{\frac{1}{2}-\varepsilon^*}(\Gamma)$, we have that $(\boldsymbol{u}_n)_{\boldsymbol{\tau}} \to \boldsymbol{u}_{\boldsymbol{\tau}}$, in $\boldsymbol{H}^{\frac{1}{2}-\varepsilon^*}(\Gamma)$, and thanks to (3.18), $\boldsymbol{v}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{0}$ in $\boldsymbol{H}^1(\Omega)$ and $p_n \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(\Omega)/\mathbb{R}$.

On the other hand, we have that $((\boldsymbol{u}_n - \boldsymbol{u})\cdot\nabla)\boldsymbol{u}_n \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Then, from Remark 3.3.6 there exists $(\boldsymbol{w}_n, q_n) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ solution of (S_2) which satisfies the estimate

$$\|\boldsymbol{w}_n\|_{\boldsymbol{H}^1(\Omega)} + \|q_n\|_{L^2(\Omega)/\mathbb{R}} \leq C\|((\boldsymbol{u}_n - \boldsymbol{u})\cdot\nabla)\boldsymbol{u}_n\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}.$$

Since $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$, in $\boldsymbol{H}^1(\Omega)$-weak, it follows that $\|\nabla\boldsymbol{u}_n\|_{\boldsymbol{L}^2(\Omega)} \leq M$, for all $n\in\mathbb{N}$, with $M > 0$ independent of n, and $\boldsymbol{u}_n \to \boldsymbol{u}$, in $\boldsymbol{L}^3(\Omega)$. Then, by using Hölder inequality we have that $((\boldsymbol{u}_n - \boldsymbol{u})\cdot\nabla)\boldsymbol{u}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{0}$, in $\boldsymbol{L}^{\frac{6}{5}}(\Omega)$, and hence, $\boldsymbol{w}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{0}$ in $\boldsymbol{H}^1(\Omega)$ and $q_n \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(\Omega)/\mathbb{R}$.

Finally, since $(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}_n - \boldsymbol{u}) \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, thanks to Proposition 3.3.3, there exists $(\boldsymbol{y}_n, \chi_n) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)\times W^{1,\frac{3}{2}}(\Omega)$ solution of (S_3). But $(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}_n - \boldsymbol{u}) \rightharpoonup \boldsymbol{0}$ in $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$-weak, which implies that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{0}$ in $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$-weak. As $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$ is compactly embedded in $\boldsymbol{H}^1(\Omega)$, we have that $\boldsymbol{y}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{0}$ in $\boldsymbol{H}^1(\Omega)$ and $\chi_n \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(\Omega)/\mathbb{R}$.

Therefore, we conclude that

$$\boldsymbol{u}_n^* \underset{n\to\infty}{\longrightarrow} \boldsymbol{u}^*, \text{ in } \boldsymbol{H}^1(\Omega) \text{ and } \pi_n^* \underset{n\to\infty}{\longrightarrow} \pi^*, \text{ in } L^2(\Omega)/\mathbb{R}. \tag{3.19}$$

By (3.18) and (3.19), we deduce that $(\boldsymbol{u}_n^*, \theta_n^*) \to (\boldsymbol{u}^*, \theta^*)$, in $\boldsymbol{H}$. Therefore, $\mathcal{R}$ is a compact operator in $\boldsymbol{H}$.

(ii) *Let us show that the set of fixed points of the operator $\lambda\mathcal{R}$ is bounded for all $\lambda \in [0,1]$.* Let $(\boldsymbol{u},\theta) = \lambda\mathcal{R}(\boldsymbol{u},\theta)$, with $(\boldsymbol{u},\theta) \in \boldsymbol{H}$ and $\lambda \in [0,1]$. As $(\boldsymbol{u},\theta) = \lambda(\boldsymbol{u}^*,\theta^*) = (\lambda\boldsymbol{u}^*,\lambda\theta^*)$, then $(\boldsymbol{u}^*,\theta^*) = \mathcal{R}(\boldsymbol{u},\theta) = \mathcal{R}(\lambda\boldsymbol{u}^*,\lambda\theta^*)$ satisfies the following system:

$$\begin{cases} -\nu\Delta\boldsymbol{u}^* + \nabla\pi^* = \theta^*\boldsymbol{g} - \lambda^2(\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^* & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^* = 0 & \text{in } \Omega, \\ -\kappa\Delta\theta^* = h - \lambda^2\boldsymbol{u}^*\cdot\nabla\theta^* & \text{in } \Omega, \\ \boldsymbol{u}^*\cdot\boldsymbol{n} = 0 \ , \ 2\,[\mathbb{D}(\boldsymbol{u}^*)\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{a} - \lambda\alpha\,\boldsymbol{u}^*_{\boldsymbol{\tau}} & \text{on } \Gamma, \\ \theta^* = \theta_b & \text{on } \Gamma. \end{cases} \tag{3.20}$$

We will consider two cases depending on the values of the boundary data θ_b.

(a) *Case $\theta_b = 0$.* Multiplying by $\boldsymbol{u}^* \in \boldsymbol{V}^2_{\sigma,T}(\Omega)$ and by $\theta^* \in H^1_0(\Omega)$ the first and third equations of (3.20), respectively, and integrating by parts, we have

$$2\nu\int_\Omega |\mathbb{D}(\boldsymbol{u}^*)|^2\,\mathrm{d}x - 2\nu\langle[\mathbb{D}(\boldsymbol{u}^*)\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{u}^*_{\boldsymbol{\tau}}\rangle_\Gamma \;=\; \int_\Omega \theta^*\boldsymbol{g}\cdot\boldsymbol{u}^*\,\mathrm{d}x \tag{3.21}$$

$$\kappa\int_\Omega |\nabla\theta^*|^2\,\mathrm{d}x \;=\; \langle h,\theta^*\rangle_\Omega, \tag{3.22}$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality between $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. From (3.22), we have immediately that

$$\|\nabla\theta^*\|_{\boldsymbol{L}^2(\Omega)} \le \frac{C_2}{\kappa}\|h\|_{H^{-1}(\Omega)} \tag{3.23}$$

with $C_2 = C_2(\Omega) > 0$. By using the Navier boundary condition, from (3.21) it follows that

$$2\nu\int_\Omega |\mathbb{D}(\boldsymbol{u}^*)|^2\,\mathrm{d}x + \lambda\nu\int_\Gamma \alpha\,\boldsymbol{u}^*_{\boldsymbol{\tau}}\cdot\boldsymbol{u}^*_{\boldsymbol{\tau}}\,\mathrm{d}s = \nu\langle\boldsymbol{a},\boldsymbol{u}^*_{\boldsymbol{\tau}}\rangle_\Gamma + \int_\Omega \theta^*\boldsymbol{g}\cdot\boldsymbol{u}^*\,\mathrm{d}x,$$

that is, if α satisfies (H), it follows that

$$\int_\Omega |\mathbb{D}(\boldsymbol{u}^*)|^2\,\mathrm{d}x \le \frac{1}{2}\left|\langle\boldsymbol{a},\boldsymbol{u}^*_{\boldsymbol{\tau}}\rangle_\Gamma\right| + \frac{1}{2\nu}\int_\Omega \left|\theta^*\boldsymbol{g}\cdot\boldsymbol{u}^*\right|\,\mathrm{d}x.$$

By using Proposition 3.3.8, Hölder inequality and Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we obtain that

$$\begin{aligned} \|\boldsymbol{u}^*\|^2_{\boldsymbol{H}^1(\Omega)} &\le C_3\left(\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\|\boldsymbol{u}^*_{\boldsymbol{\tau}}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \frac{1}{\nu}\|\theta^*\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}^*\|_{\boldsymbol{L}^6(\Omega)}\right) \\ &\le C_3\left(\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\|\boldsymbol{u}^*\|_{\boldsymbol{H}^1(\Omega)} + \frac{1}{\nu}\|\nabla\theta^*\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}^*\|_{\boldsymbol{H}^1(\Omega)}\right) \end{aligned}$$

with $C_3 = C_3(\Omega) > 0$. Hence, by using (3.23), we have that

$$\|\boldsymbol{u}^*\|_{\boldsymbol{H}^1(\Omega)} \le \frac{C_4}{\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right) \tag{3.24}$$

with $C_4 = C_3\max\{1, C_2\}$. Finally, it follows from (3.24) and (3.23) that

$$\|(\boldsymbol{u},\theta)\|_{\boldsymbol{H}} = \lambda\|(\boldsymbol{u}^*,\theta^*)\|_{\boldsymbol{H}} \le C_5,$$

where $C_5 = C_5\left(\Omega, \nu, \kappa, \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}, \|h\|_{H^{-1}(\Omega)}, \|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right)$ is a positive constant independent of $(\boldsymbol{u}, \theta)$ and λ.

(b) *Case* $\theta_b \neq 0$. Let us define $\theta^*_\eta = \theta^* - \theta^\eta_b$, where θ^η_b is the lift function of the boundary condition θ_b such that for all $\eta > 0$

$$\|\theta^\eta_b\|_{L^3(\Omega)} \leq \eta \|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{3.25}$$

see [1, Lemma 3.5]. Then, we get the following system:

$$\begin{cases} -\nu\Delta \boldsymbol{u}^* + \nabla \pi^* = \theta^* \boldsymbol{g} - \lambda^2 (\boldsymbol{u}^* \cdot \nabla)\boldsymbol{u}^* & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^* = 0 & \text{in } \Omega, \\ -\kappa\Delta\theta^*_\eta - \kappa\Delta\theta^\eta_b = h - \lambda^2 \boldsymbol{u}^* \cdot \nabla\theta^*_\eta - \lambda^2 \boldsymbol{u}^* \cdot \nabla \theta^\eta_b & \text{in } \Omega, \\ \boldsymbol{u}^* \cdot \boldsymbol{n} = 0 \ , \ 2\left[\mathbb{D}(\boldsymbol{u}^*)\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{a} - \lambda\alpha\, \boldsymbol{u}^*_{\boldsymbol{\tau}} & \text{on } \Gamma, \\ \theta^*_\eta = 0 & \text{on } \Gamma. \end{cases} \tag{3.26}$$

If we multiply by $\boldsymbol{u}^* \in \boldsymbol{V}^2_{\sigma,T}(\Omega)$ and by $\theta^*_\eta \in H^1_0(\Omega)$ the first and third equations of (3.26), respectively, and by integrating by parts, it follows that

$$2\nu \int_\Omega |\mathbb{D}(\boldsymbol{u}^*)|^2 \,\mathrm{d}x - 2\nu \langle [\mathbb{D}(\boldsymbol{u}^*)\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{u}^*_{\boldsymbol{\tau}}\rangle_\Gamma = \int_\Omega \theta^* \boldsymbol{g} \cdot \boldsymbol{u}^* \,\mathrm{d}x, \tag{3.27}$$

$$\kappa \int_\Omega |\nabla\theta^*_\eta|^2 \,\mathrm{d}x + \kappa \int_\Omega \nabla\theta^\eta_b \cdot \nabla\theta^*_\eta \,\mathrm{d}x = \langle h, \theta^*_\eta\rangle_\Omega + \lambda^2 \int_\Omega (\boldsymbol{u}^* \cdot \nabla\theta^*_\eta)\theta^\eta_b \,\mathrm{d}x. \tag{3.28}$$

Then, by using Hölder inequality, Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$ and (3.25), we have from (3.28) that

$$\|\nabla\theta^*_\eta\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_6}{\kappa}\left(\kappa\|\theta^\eta_b\|_{H^1(\Omega)} + \|h\|_{H^{-1}(\Omega)} + \eta\|\boldsymbol{u}^*\|_{\boldsymbol{H}^1(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right) \tag{3.29}$$

with $C_6 = C_6(\Omega) > 0$. In a similar way as we did for the case $\theta_b = 0$, by using the Navier boundary condition, from (3.27) it follows that

$$2\nu \int_\Omega |\mathbb{D}(\boldsymbol{u}^*)|^2 \,\mathrm{d}x + \lambda\nu \int_\Gamma \alpha\, \boldsymbol{u}^*_{\boldsymbol{\tau}} \cdot \boldsymbol{u}^*_{\boldsymbol{\tau}} \,\mathrm{d}s = \nu\langle \boldsymbol{a}, \boldsymbol{u}^*_{\boldsymbol{\tau}}\rangle_\Gamma + \int_\Omega \theta^* \boldsymbol{g} \cdot \boldsymbol{u}^* \,\mathrm{d}x,$$

and since α is a non negative function satisfying (H), it follows that

$$\int_\Omega |\mathbb{D}(\boldsymbol{u}^*)|^2 \,\mathrm{d}x \leq \frac{1}{2}\,|\langle \boldsymbol{a}, \boldsymbol{u}^*_{\boldsymbol{\tau}}\rangle_\Gamma| + \frac{1}{2\nu}\int_\Omega \theta^* \boldsymbol{g} \cdot \boldsymbol{u}^* \,\mathrm{d}x.$$

Therefore, by using Proposition 3.3.8, Hölder inequality and Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, it follows that

$$\begin{aligned} \|\boldsymbol{u}^*\|^2_{\boldsymbol{H}^1(\Omega)} &\leq \frac{C_7}{\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\|\boldsymbol{u}^*_{\boldsymbol{\tau}}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \|\theta^*\|_{L^6(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}^*\|_{\boldsymbol{L}^6(\Omega)}\right) \\ &\leq \frac{C_7}{\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\theta^*\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right)\|\boldsymbol{u}^*\|_{\boldsymbol{H}^1(\Omega)} \end{aligned}$$

with $C_7 = C_7(\Omega) > 0$, which implies the following estimate

$$\|\boldsymbol{u}^*\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_7}{\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\theta^*\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right). \tag{3.30}$$

Now, from (3.29), it follows that

$$\|\theta^*\|_{\boldsymbol{H}^1(\Omega)} \leq (C_6+1)\|\theta_b^\eta\|_{H^1(\Omega)} + \frac{C_6}{\kappa}\|h\|_{H^{-1}(\Omega)} + \frac{C_6C_7}{\kappa}\eta\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \frac{C_6C_7}{\kappa\nu}\eta\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta^*\|_{\boldsymbol{H}^1(\Omega)}.$$

Taking $\eta = \frac{\kappa\nu}{2C_6C_7\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}}$, we have

$$\|\theta^*\|_{\boldsymbol{H}^1(\Omega)} \leq 2(C_6+1)\left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \frac{\nu}{\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}}\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\theta_b^\eta\|_{H^1(\Omega)}\right). \tag{3.31}$$

By using (3.31), we deduce from (3.30) that

$$\|\boldsymbol{u}^*\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{(2C_6+3)C_7}{\nu}\left[\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \left(\frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)} + \|\theta_b^\eta\|_{H^1(\Omega)}\right)\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right]. \tag{3.32}$$

Finally, it follows from (3.32) and (3.31) that

$$\|(\boldsymbol{u},\theta)\|_{\boldsymbol{H}} = \lambda\|(\boldsymbol{u}^*,\theta^*)\|_{\boldsymbol{H}} \leq C_8,$$

where $C_8 = C_8\left(\Omega,\nu,\kappa,\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)},\|h\|_{H^{-1}(\Omega)},\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)},\|\theta_b^\eta\|_{H^1(\Omega)}\right)$ is a positive constant independent of $(\boldsymbol{u},\theta)$ and λ.

By using Leray-Schauder fixed point theorem and Remark 3.3.10, there exists at least one $(\boldsymbol{u},\theta,\pi) \in \boldsymbol{H}\times L^2(\Omega)$ such that problem (BS)-(3.1)-(3.2) is satisfied.

(iii) *Proof of estimates* (3.12) *and* (3.13). From the Leray-Schauder fixed point theorem, we have that $\boldsymbol{u}^* = \boldsymbol{u}$ and $\theta^* = \theta$, and hence if $\theta_b = 0$, we have directly the desired estimates from (3.24) and (3.23).

(iv) *Proof of estimates* (3.15) *and* (3.16). Let us define $\hat{\theta} = \theta - \sigma$, where $\sigma \in H^1(\Omega)$ such that $\sigma = \theta_b$ on Γ and

$$\|\sigma\|_{H^1(\Omega)} \leq C_1\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{3.33}$$

with $C_1 = C_1(\Omega) > 0$. Then, we have

$$\begin{cases} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega,\\ -\kappa\Delta\hat{\theta} + \boldsymbol{u}\cdot\nabla\hat{\theta} = h + \kappa\Delta\sigma - \boldsymbol{u}\cdot\nabla\sigma & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0\ ,\ 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\,\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{a} & \text{on } \Gamma,\\ \hat{\theta} = 0 & \text{on } \Gamma. \end{cases} \tag{3.34}$$

Multiplying by $\boldsymbol{u} \in \boldsymbol{V}^2_{\sigma,T}(\Omega)$ the first equation of (3.34), integrating by parts and using Hölder inequality, it is easy to see that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_2}{2\nu}\left(\nu\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\theta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\right), \tag{3.35}$$

with $C_2 = C_2(\Omega) > 0$. In the same way, multiplying by $\hat{\theta} \in H^1_0(\Omega)$ the third equation of (3.34), integrating by parts, by using (3.33) and Hölder inequality, it follows that

$$\|\nabla\hat{\theta}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_3}{\kappa}\left(\kappa\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} + \|h\|_{H^{-1}(\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right),$$

with $C_3 = C_3(\Omega) > 0$. Then, as $\theta = \hat{\theta} + \sigma$, it follows that

$$\|\theta\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_4}{\kappa}\left(\|h\|_{H^{-1}(\Omega)} + \kappa\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} + \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)}\right), \tag{3.36}$$

with $C_4 = C_4(\Omega) > 0$. By using (3.35) in (3.36), and taking $\gamma = C_2C_4$ in (3.14), it follows that

$$\|\theta\|_{H^1(\Omega)} \leq C_5\left[\left(1 + \frac{1}{\kappa}\|\boldsymbol{a}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right)\|\theta_b\|_{H^{\frac{1}{2}}(\Gamma)} + \frac{1}{\kappa}\|h\|_{H^{-1}(\Omega)}\right],$$

with $C_5 = C_5(\Omega, \gamma) > 0$. Then, the theorem is totally proved. □

Remark 3.4.2. (i) Thanks to (3.12) and (3.13), or (3.15) and (3.16), it follows that the estimate for π does not depend on α because

$$\|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\left(\|\theta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \nu\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\right).$$

(ii) As Proposition 3.3.1 and Remark 3.3.2 tell us, in the case when $\alpha = 0$ and Ω is axisymmetric, it is the unique case when the kernel $\boldsymbol{\mathcal{K}}(\Omega) \neq \{\boldsymbol{0}\}$, and then the solution velocity $\boldsymbol{u}$ given by Theorem 3.4.1 belongs to $\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{K}}(\Omega)$.

3.5 Regularity of the weak solution

In order to study the regularity of the weak solution for the problem (BS)-(3.1)-(3.2), we are going to take advantage of the regularity results for the Poisson equation and Stokes problem (3.11). Then, we can rewrite (BS)-(3.1)-(3.2), in the following way:

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla\pi = \theta\boldsymbol{g} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ -\kappa\Delta\theta = h - \boldsymbol{u}\cdot\nabla\theta & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0\ ,\ 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\,\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{a} & \text{on } \Gamma, \\ \theta = \theta_b & \text{on } \Gamma. \end{cases}$$

Theorem 3.5.1 (regularity $W^{1,p}(\Omega)$ with $p > 2$)**.** *Let us suppose that*

$$\boldsymbol{g} \in \boldsymbol{L}^r(\Omega), \quad h \in W^{-1,p}(\Omega), \quad \alpha \in L^{t^*(p)}(\Gamma) \quad \textit{satisfying} \quad \text{(H)}$$

$$\textit{with } t^*(p) \textit{ is defined by (4.14) and} \quad (\boldsymbol{a}, \theta_b) \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times W^{1-\frac{1}{p},p}(\Gamma)$$

with

$$p > 2, \quad r = \max\left\{\frac{3}{2}, \frac{3p}{3+p}\right\} \quad \textit{if} \quad p \neq 3 \quad \textit{and} \quad r = \frac{3}{2} + \varepsilon \quad \textit{if} \quad p = 3$$

for any $\varepsilon > 0$ sufficiently small. Then the weak solution for (BS)-(3.1)-(3.2) given by Theorem 3.4.1 satisfies

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R} \times W^{1,p}(\Omega).$$

Proof. Since $p > 2$, we have that $\boldsymbol{L}^r(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, $W^{-1,p}(\Omega) \hookrightarrow H^{-1}(\Omega)$, $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $W^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$. Thanks to Theorem 3.4.1, there exists $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times H^1(\Omega)$ weak solution for (BS)-(3.1)-(3.2). By using the embedding

$H^1(\Omega) \hookrightarrow L^6(\Omega)$, it follows that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{u}\cdot\nabla\theta = \operatorname{div}(\theta\boldsymbol{u}) \in W^{-1,3}(\Omega)$. Realize that $W^{-1,3}(\Omega) \hookrightarrow W^{-1,p}(\Omega)$ if $p \le 3$, then we have three cases:

(i) *Case* $2 < p < 3$: Note that $h - \boldsymbol{u}\cdot\nabla\theta \in W^{-1,p}(\Omega)$, then by regularity of the Poisson equation we have $\theta \in W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. Since $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, it follows that $\theta\boldsymbol{g} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega)$, by which $\theta\boldsymbol{g} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega)$. Consequently, thanks to the regularity of the Stokes equations with Navier boundary conditions, see Theorem 4.5.1, we have that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)/\mathbb{R}$.

(ii) *Case* $p = 3$: In view of $\operatorname{div}(\theta\boldsymbol{u}) \in W^{-1,3}(\Omega)$ and $h \in W^{-1,3}(\Omega)$, by regularity of the Poisson equation, we have that $\theta \in W^{1,3}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 < q < \infty$. Since $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$, we have $\theta\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and then $\theta\boldsymbol{g} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$. Thus, by regularity of the Stokes equations with Navier boundary conditions (Theorem 4.5.1) we have that $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$ and $\pi \in L^3(\Omega)/\mathbb{R}$.

(iii) *Case* $p > 3$: From the previous case, we have that $(\boldsymbol{u}, \theta) \in \boldsymbol{W}^{1,3}(\Omega) \times W^{1,3}(\Omega)$. Therefore, $(\boldsymbol{u}, \theta) \in \boldsymbol{L}^q(\Omega) \times L^q(\Omega)$, for any $1 < q < \infty$, and then $\theta\boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$ for any $1 < q < \infty$. Consequently, $\operatorname{div}(\theta\boldsymbol{u}) \in W^{-1,p}(\Omega)$ and by regularity of the Poisson equation, we have $\theta \in W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. Further, $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^t(\Omega)$ for all $1 \le t < 3$. In particular, taking $\frac{1}{t} = \frac{1}{p} + \frac{1}{3}$ with $\frac{3}{2} < t < 3$, we have that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega)$. As $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega)$, it follows that $\theta\boldsymbol{g} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega)$. Then, by regularity of the Stokes equations with Navier boundary conditions (Theorem 4.5.1) we have that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)/\mathbb{R}$.

□

Theorem 3.5.2 (regularity $W^{2,p}(\Omega)$ with $p \ge \frac{6}{5}$). *Let us suppose that* $\boldsymbol{g} \in \boldsymbol{L}^r(\Omega)$, $h \in L^p(\Omega)$,

$$\alpha \in H^{\frac{1}{2}}(\Gamma) \quad \textit{if} \quad \frac{6}{5} \le p \le 2; \quad \alpha \in H^{\frac{1}{2}+\varepsilon}(\Gamma) \quad \textit{if} \quad 2 < p < 3; \quad \alpha \in W^{1-\frac{1}{p},p}(\Gamma) \quad \textit{if} \quad p \ge 3$$

satisfying (H) *and*

$$(\boldsymbol{a}, \theta_b) \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \times W^{2-\frac{1}{p},p}(\Gamma)$$

with

$$p \ge \frac{6}{5}, \quad r = \max\left\{\frac{3}{2}, p\right\} \quad \textit{if} \quad p \ne \frac{3}{2} \quad \textit{and} \quad r = \frac{3}{2} + \varepsilon \quad \textit{if} \quad p = \frac{3}{2}$$

for any $\varepsilon > 0$ *sufficiently small. Then the solution for* (BS)-(3.1)-(3.2) *given by Theorem 3.4.1 satisfies*

$$(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R} \times W^{2,p}(\Omega).$$

Proof. By hypothesis $p \ge \frac{6}{5}$, then we have that $\boldsymbol{L}^r(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$, $\alpha \in L^2(\Gamma)$, $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $W^{2-\frac{1}{p},p}(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$. Thanks to Theorem 3.4.1, there exists $(\boldsymbol{u}, \pi, \theta) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times H^1(\Omega)$ weak solution for (BS)-(3.1)-(3.2). Since $H^1(\Omega) \hookrightarrow L^6(\Omega)$, it follows that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{u}\cdot\nabla\theta \in L^{\frac{3}{2}}(\Omega)$. Note that $L^{\frac{3}{2}}(\Omega) \hookrightarrow L^p(\Omega)$ if $p \le \frac{3}{2}$, then we have three cases:

(i) *Case* $\frac{6}{5} \le p < \frac{3}{2}$: Since $h - \boldsymbol{u}\cdot\nabla\theta \in L^p(\Omega)$, by regularity of the Poisson equation we have $\theta \in W^{2,p}(\Omega) \hookrightarrow L^{p^{**}}(\Omega)$ with $\frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{3}$. We have that $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, therefore, $\theta\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ by which $\theta\boldsymbol{g} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Consequently, thanks to the existence of strong solutions for the Stokes equations with Navier boundary conditions (see Theorem 4.5.3), we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)/\mathbb{R}$.

(ii) *Case* $p = \frac{3}{2}$: We have that $\boldsymbol{u} \cdot \nabla\theta \in L^{\frac{3}{2}}(\Omega)$ and $h \in L^{\frac{3}{2}}(\Omega)$, then by regularity of the Poisson equation, it follows that $\theta \in W^{2,\frac{3}{2}}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 < q < \infty$. Since $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$, we deduce that $\theta\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and then $\theta\boldsymbol{g} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$. Thanks to Theorem 4.5.3, we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$ and $\pi \in W^{1,\frac{3}{2}}(\Omega)/\mathbb{R}$.

(iii) *Case* $p > \frac{3}{2}$: From the previous case, we have that $(\boldsymbol{u}, \theta) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{2,\frac{3}{2}}(\Omega)$. Therefore, $(\boldsymbol{u}, \theta) \in \boldsymbol{L}^q(\Omega) \times L^q(\Omega)$, for any $1 < q < \infty$, then $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^t(\Omega)$ and $\boldsymbol{u} \cdot \nabla\theta \in L^t(\Omega)$ for all $1 \leq t < 3$. Thus, we must regard the following cases:

(a) If $\frac{3}{2} < p < 3$, we have that $h - \boldsymbol{u} \cdot \nabla\theta \in L^p(\Omega)$, and by regularity of the Poisson equation, it follows that $\theta \in W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. As $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, we have that $\theta\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, hence, $\theta\boldsymbol{g} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then, by Theorem 4.5.3, we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)/\mathbb{R}$.

(b) Suppose now that $p \geq 3$. From the above result, we have that $(\boldsymbol{u}, \theta) \in \boldsymbol{W}^{2,3-\delta}(\Omega) \times W^{2,3-\delta}(\Omega)$ for all $0 < \delta < \frac{3}{2}$. This implies that $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$ and since $\nabla\theta \in \boldsymbol{W}^{1,3-\delta}(\Omega)$, it follows that $\boldsymbol{u} \cdot \nabla\theta \in L^p(\Omega)$. By using the regularity of the Poisson equation, we conclude that $\theta \in W^{2,p}(\Omega)$. As $\nabla\boldsymbol{u} \in \boldsymbol{W}^{1,3-\delta}(\Omega)$, we deduce $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and since $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, we have that $\theta\boldsymbol{g} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Finally, by applying Theorem 4.5.3, we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)/\mathbb{R}$. $\square$

Chapter 4

Stokes equations with Navier boundary condition

Abstract

This chapter deals with the stationary Stokes equations with non-homogeneous Navier boundary condition in a bounded domain $\Omega \subset \mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. We prove the existence and uniqueness of a weak solution in the Hilbert case. Moreover, we analyze the L^p-regularity for this solution.

Keywords: Stokes equations, non-homogeneous Navier boundary condition, weak solution, L^p-regularity

4.1 Introduction

We are interested in studying the existence, uniqueness and regularity of the solution for the following stationary Stokes equations with Navier boundary condition:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g & \text{on } \Gamma, \\ 2\left[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma, \end{cases} \tag{S}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain of class $\mathcal{C}^{1,1}$, Γ is the boundary of Ω, $\boldsymbol{u}$ and π are the velocity and pressure of the fluid, respectively, $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$ is the strain tensor associated with the velocity field $\boldsymbol{u}$, $\boldsymbol{n}$ is the unit outward normal vector, $\boldsymbol{\tau}$ is the corresponding unit tangent vector, $\boldsymbol{f}$ is an external force acting on the fluid, χ and g stand for the compressibility and permeability conditions, respectively, α is a friction scalar function and $\boldsymbol{h}$ is a tangential vector field on the boundary. In the case $\alpha > 0$, the Navier boundary condition is said to be a boundary condition with linear friction.

4.2 Main results

The main results of this chapter are presented in this section. The first theorem is concerned with the existence and uniqueness of a weak solution for the Stokes equations with Navier boundary condition in the Hilbert case.

Theorem 4.2.1 (weak solution in $H^1(\Omega)$). *Let us suppose $\chi = 0$ and $g = 0$. Let*

$$\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega), \quad \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ such that } \boldsymbol{h}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \quad \alpha \in L^2(\Gamma)$$

with α verifying the hypothesis (4.3), (H_1) and (H_2). Then the Stokes problem (S) has a unique solution $(\boldsymbol{u}, \pi)$ belonging to $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ which satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega, \alpha_*) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

where

$$C(\Omega, \alpha_*) = \begin{cases} C(\Omega) & \text{under the hypothesis } (H_1) \\ \frac{C(\Omega)}{\min\{1, \alpha_*\}} & \text{under the hypothesis } (H_2) \text{ with } \alpha \geq \alpha_* > 0 \\ C(\Omega) & \text{under the hypothesis } (H_2) \text{ with } \alpha(x) > 0 \text{ a.e. } x \in \Gamma. \end{cases}$$

The following theorems deal with the L^p regularity of the weak solution for the Stokes equations with Navier boundary condition. We consider $p > 2$ for regularity in $W^{1,p}(\Omega)$ and $p \geq \frac{6}{5}$ for regularity in $W^{2,p}(\Omega)$.

Theorem 4.2.2 (generalized solutions in $W^{1,p}(\Omega)$ with $p > 2$). *Let us suppose $\chi = 0$, $g = 0$,*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega) \text{ with } r(p) \text{ is defined by (4.1)}, \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ such that } \boldsymbol{h}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,$$

and $\alpha \in L^{t^*(p)}(\Gamma)$ satisfying (4.3), (H_1) and (H_2) with

$$t^*(p) = \begin{cases} 2 + \varepsilon & \text{if } 2 < p \leq 3, \\ \frac{2}{3}p + \varepsilon & \text{if } p > 3, \end{cases}$$

where $\varepsilon > 0$ is an arbitrary number sufficiently small. Then the Stokes problem (S) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

Theorem 4.2.3 (strong solutions in $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$). *Let us suppose $\chi = 0$, $g = 0$,*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ such that } \boldsymbol{h}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,$$

and

$$\alpha \in H^{\frac{1}{2}}(\Gamma) \quad \text{if} \quad \frac{6}{5} \leq p \leq 2; \quad \alpha \in H^{\frac{1}{2}+\varepsilon}(\Gamma) \quad \text{if} \quad 2 < p < 3; \quad \alpha \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{if} \quad p \geq 3$$

where $\varepsilon > 0$ is an arbitrary number sufficiently small and α satisfies (4.3), (H_1) and (H_2). Then the solution given by Theorem 4.2.1 satisfies that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$.

4.3 Notations and some useful results

Before studying the problem (S), we start with some preliminaries. Throughout this work, up to mention the contrary, we suppose $\Omega \subset \mathbb{R}^3$ a bounded domain with boundary Γ of class $\mathcal{C}^{1,1}$. We use *domain* to stand for a nonempty open and connected set. Later, we will use the term *axisymmetric* to stand for a nonempty set which is generated by rotation around an axis. Bold font for spaces means vector (or matrix) valued spaces, and their elements will be denoted with bold font also. We will denote by $\boldsymbol{n}$ and $\boldsymbol{\tau}$ the unit outward normal vector and the unit tangent vector on Γ, respectively. Unless otherwise stated or unless the context otherwise requires, we will write with the same positive constant all the constants which depend on the same arguments in the estimations that will appear along this work.

We will denote by $\mathcal{D}(\Omega)$ the set of smooth functions (infinitely differentiable functions) with compact support in Ω and by $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ the subspace of $\boldsymbol{\mathcal{D}}(\Omega)$ formed by divergence-free vector functions in Ω. If $1 < p < \infty$, then p' will denote the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. For this p, we introduce the spaces

$$\boldsymbol{V}^p_{\sigma,T}(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \Gamma \right\},$$
$$\boldsymbol{E}^p(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta \boldsymbol{v} \in \boldsymbol{L}^{r(p)}(\Omega) \right\},$$

with

$$r(p) = \frac{3p}{p+3} \quad \text{if} \quad p > \frac{3}{2}, \quad \text{or} \quad r(p) > 1 \quad \text{if} \quad 1 < p \le \frac{3}{2}. \tag{4.1}$$

Lemma 4.3.1. **(i)** *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz bounded open set. We have that*

$$\boldsymbol{\mathcal{D}}(\overline{\Omega}) \quad \textit{is dense in} \quad \boldsymbol{E}^p(\Omega).$$

(ii) *The linear mapping $\boldsymbol{v} \mapsto [\mathbb{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear and continuous mapping from $\boldsymbol{E}^p(\Omega)$ to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Moreover, the following Green formula is satisfied: for all $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,T}(\Omega)$*

$$-\int_\Omega \Delta \boldsymbol{v} \cdot \boldsymbol{\varphi} \,\mathrm{d}x = 2\int_\Omega \mathbb{D}(\boldsymbol{v}) : \mathbb{D}(\boldsymbol{\varphi}) \,\mathrm{d}x - 2\langle [\mathbb{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_\Gamma$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$.

Proof. We will simply prove **(i)** because the Green formula follows immediately from the one established for smooth functions. Let $P : \boldsymbol{W}^{1,p}(\Omega) \longrightarrow \boldsymbol{W}^{1,p}(\mathbb{R}^3)$ be the extension mapping such that $P\boldsymbol{u}\big|_\Omega = \boldsymbol{u}$. Then, for all $\boldsymbol{\ell} \in [\boldsymbol{E}^p(\Omega)]'$, there exists $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \boldsymbol{W}^{-1,p'}(\mathbb{R}^3) \times \boldsymbol{L}^{r'}(\Omega)$ such that

$$\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle_\Omega = \langle \boldsymbol{\varphi}, P\boldsymbol{v} \rangle_{\mathbb{R}^3} + \int_\Omega \boldsymbol{\psi} \cdot \Delta \boldsymbol{v} \,\mathrm{d}x,$$

for all $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$, with $\langle \cdot, \cdot \rangle_\Omega$ the duality pairing between $[\boldsymbol{E}^p(\Omega)]'$ and $\boldsymbol{E}^p(\Omega)$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ the duality pairing between $\boldsymbol{W}^{-1,p'}(\mathbb{R}^3)$ and $\boldsymbol{W}^{1,p}(\mathbb{R}^3)$.

Let us suppose that $\boldsymbol{\ell} = \boldsymbol{0}$ in $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ and let $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{L}^{r'}(\mathbb{R}^3)$ the extension by zero of $\boldsymbol{\psi}$. Then, for all $\boldsymbol{\xi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$, we have

$$\langle \boldsymbol{\varphi}, \boldsymbol{\xi} \rangle_{\mathbb{R}^3} + \int_{\mathbb{R}^3} \widetilde{\boldsymbol{\psi}} \cdot \Delta \boldsymbol{\xi} \,\mathrm{d}x = 0,$$

since $\langle \boldsymbol{\varphi}, \boldsymbol{\xi} \rangle_{\mathbb{R}^3} = \langle \boldsymbol{\varphi}, P\boldsymbol{v} \rangle_{\mathbb{R}^3}$ with $\boldsymbol{v} = \boldsymbol{\xi}\big|_\Omega$. It follows that

$$\boldsymbol{\varphi} + \Delta \widetilde{\boldsymbol{\psi}} = \boldsymbol{0}, \quad \text{in } \mathbb{R}^3.$$

In this way, $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{L}^{r'}(\mathbb{R}^3)$ and $\Delta \widetilde{\boldsymbol{\psi}} \in \boldsymbol{W}^{-1,p'}(\mathbb{R}^3)$. Because $\Delta \widetilde{\boldsymbol{\psi}}$ is a compact support, it follows that $\Delta \widetilde{\boldsymbol{\psi}} \in \boldsymbol{W}^{-1,q}(\mathbb{R}^3)$ for all $1 < q \le p'$. Here $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, then $\frac{1}{r'} = \frac{1}{p'} - \frac{1}{3}$.

Consequently, $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{L}^{p'}(\mathbb{R}^3)$ and therefore $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{W}^{1,p'}(\mathbb{R}^3)$. As $\widetilde{\boldsymbol{\psi}}|_\Omega = \boldsymbol{\psi} \in \boldsymbol{W}^{1,p'}(\Omega)$, we deduce that $\boldsymbol{\psi} \in \boldsymbol{W}_0^{1,p'}(\Omega)$ and then there exists $(\boldsymbol{\psi}_k)_k \subset \boldsymbol{\mathcal{D}}(\Omega)$ such that $\boldsymbol{\psi}_k \underset{k\to\infty}{\longrightarrow} \boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p'}(\Omega)$.

Finally, for all $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$

$$\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle_\Omega = \lim_{k\to\infty} [\langle -\Delta\widetilde{\boldsymbol{\psi}}_k, P\boldsymbol{v}\rangle_{\mathbb{R}^3} + \int_\Omega \boldsymbol{\psi}_k \cdot \Delta \boldsymbol{v} \, \mathrm{d}x] = 0.$$

□

Since we are interested in generalized weak solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for (S), we consider the following assumptions for the data.

Let us suppose that

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega),\ \chi \in L^p(\Omega),\ g \in W^{1-\frac{1}{p},p}(\Gamma),\ \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

with $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ, $r(p)$ defined by (4.1) and $\alpha \in L^{t(p)}(\Gamma)$ with

$$t(p) = \begin{cases} \frac{2}{3}\, p' & \text{if} \quad p < \frac{3}{2}, \\ 2 & \text{if} \quad \frac{3}{2} < p < 3, \\ \frac{2}{3}\, p & \text{if} \quad p > 3, \\ 2+\varepsilon & \text{if} \quad p = \frac{3}{2} \quad \text{or} \quad p = 3, \end{cases} \tag{4.2}$$

where $\varepsilon > 0$ is an arbitrary number sufficiently small. Also, we can suppose that there exists a real number α_* such that

$$\alpha \geq \alpha_* \geq 0 \tag{4.3}$$

with

$$\alpha_* \geq 0 \quad \text{if } \Omega \text{ is not axisymmetric} \tag{H_1}$$

or

$$\alpha_* > 0 \quad (\text{or even, } \alpha(x) > 0 \text{ a.e. } x \in \Gamma) \text{ otherwise.} \tag{H_2}$$

Remark 4.3.2. With the above hypotheses (4.3), (H_1) and (H_2) over α, we have that the kernel of the Stokes operator with Navier boundary condition is always zero, see Remark 3.3.2 and Remark 3.3.7.

Remark 4.3.3. The relation $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ has sense in $W^{-\frac{1}{p},p}(\Gamma)$. Indeed, since Ω is of class $\mathcal{C}^{1,1}$, then $\boldsymbol{n} \in \boldsymbol{W}^{1,\infty}(\Gamma)$ and for all $\varphi \in W^{1-\frac{1}{p'},p'}(\Gamma)$, we have that

$$\langle \boldsymbol{h} \cdot \boldsymbol{n}, \varphi\rangle = \langle \boldsymbol{h}, \varphi \boldsymbol{n}\rangle_\Gamma,$$

where $\varphi \boldsymbol{n} \in \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$.

Lemma 4.3.4. *Let $\alpha \in L^{t(p)}(\Gamma)$ with $t(p)$ defined by (4.2) and $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$.*
(i) *If $p \neq \frac{3}{2}$ and $p \neq 3$, then $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q(p)}(\Gamma)$ with $q(p) = \max\{1, \frac{2}{3}\, p\}$. Moreover, $\boldsymbol{L}^{q(p)}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and*

$$\|\alpha \boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^{q(p)}(\Gamma)} \leq C \|\alpha\|_{L^{t(p)}(\Gamma)} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}$$

with $C = C(\Omega, p) > 0$.

(ii) *If $p = 3$, then $\alpha \boldsymbol{u_\tau} \in \boldsymbol{L}^2(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{3},3}(\Gamma)$ and we have*

$$\|\alpha \boldsymbol{u_\tau}\|_{\boldsymbol{L}^2(\Gamma)} \leq C\|\alpha\|_{L^{2+\varepsilon}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,3}(\Omega)},$$

with $C = C(\Omega) > 0$.
(iii) *If $p = \frac{3}{2}$, then $\alpha \boldsymbol{u_\tau} \in \boldsymbol{L}^{1+\varepsilon'}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{2}{3},\frac{3}{2}}(\Gamma)$ for some $\varepsilon' > 0$ depending on ε, and we have*

$$\|\alpha \boldsymbol{u_\tau}\|_{\boldsymbol{L}^{1+\varepsilon'}(\Gamma)} \leq C\|\alpha\|_{L^{2+\varepsilon}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)},$$

with $C = C(\Omega) > 0$.

Proof. We know that if $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, then $\boldsymbol{u_\tau} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{L}^s(\Gamma)$ with

$$\frac{1}{s} = \begin{cases} \frac{1}{p} - \frac{1-\frac{1}{p}}{2} & \text{if } p < 3, \\ \text{any positive real number} & \text{if } p = 3, \\ 0 & \text{if } p > 3. \end{cases}$$

(i) By applying Hölder inequality, we deduce that

$$\alpha \boldsymbol{u_\tau} \in \boldsymbol{L}^q(\Gamma), \quad \text{with} \quad \frac{1}{q} = \frac{1}{t(p)} + \frac{1}{s},$$

that is,

$$\frac{1}{q} = \begin{cases} \frac{3}{2p'} + \frac{3}{2p} - \frac{1}{2} & \text{if } p < \frac{3}{2}, \\ \frac{1}{2} + \frac{3}{2p} - \frac{1}{2} & \text{if } \frac{3}{2} < p < 3, \\ \frac{3}{2p} & \text{if } p > 3, \end{cases}$$

hence,

$$q = \begin{cases} 1 & \text{if } p < \frac{3}{2}, \\ \frac{2}{3}p & \text{if } p > \frac{3}{2} \text{ and } p \neq 3. \end{cases}$$

Therefore, the conclusion follows immediately.
(ii) We know that if $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$, then $\boldsymbol{u_\tau} \in \boldsymbol{W}^{\frac{2}{3},3}(\Gamma)$ and $\boldsymbol{W}^{\frac{2}{3},3}(\Gamma) \hookrightarrow \boldsymbol{L}^r(\Gamma)$ for all $1 < r < \infty$. Then, the complete result is clearly obtained.
(iii) We know that if $\boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega)$, then $\boldsymbol{u_\tau} \in \boldsymbol{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma) \hookrightarrow \boldsymbol{L}^2(\Gamma)$. Then, the result follows.

□

Next proposition shows that the generalized solution for the Stokes problem (S) is in fact a solution of a corresponding variational problem, and vice-versa.

Proposition 4.3.5. *Let us suppose $\chi = 0$ and $g = 0$. Let*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ such that } \boldsymbol{h} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \text{ and } \quad \alpha \in L^{t(p)}(\Gamma)$$

with $r(p)$ defined by (4.1) and $t(p)$ defined by (4.2). Then, the following problems:
(i) *find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying (S) in the sense of distributions, and*
(ii) *find $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,T}(\Omega)$ such that for all $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,T}(\Omega)$*

$$2\int_\Omega \mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{\varphi})\,\mathrm{d}x + \int_\Gamma \alpha \boldsymbol{u_\tau} \cdot \boldsymbol{\varphi_\tau}\,\mathrm{d}s = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}x + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma, \tag{4.4}$$

are equivalents.

Proof. It is enough to show that the integral over the boundary, in the variational formulation (4.4), is well-defined because the result follows from Lemma 4.3.1. Then, realize that $\boldsymbol{\varphi_\tau} \in \boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma) \hookrightarrow \boldsymbol{L}^m(\Gamma)$ with

$$\frac{1}{m} = \begin{cases} \frac{3}{2p'} - \frac{1}{2} & \text{if } p > \frac{3}{2}, \\ \text{any positive real number} & \text{if } p = \frac{3}{2}, \\ 0 & \text{if } p < \frac{3}{2}. \end{cases}$$

Then, by using Lemma 4.3.4 we have the following four cases:
(i) $1 < p < \frac{3}{2}$: $\frac{1}{q(p)} + \frac{1}{m} = \frac{1}{1} + \frac{1}{\infty} = 1$.
(ii) $p > \frac{3}{2}$ and $p \neq 3$: $\frac{1}{q(p)} + \frac{1}{m} = \frac{3}{2p} + \frac{3}{2p'} - \frac{1}{2} = 1$.
(iii) $p = \frac{3}{2}$: $\frac{1}{1+\varepsilon'} + \frac{1}{m} = 1$ if m is big enough.
(iv) $p = 3$: $\frac{1}{2} + \frac{1}{2} = 1$. □

The following remark deals with the recovery of the pressure under a suitable condition. This result is known as De Rham's theorem.

Remark 4.3.6. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. If $\boldsymbol{f} \in \boldsymbol{W}^{-1,p}(\Omega)$ for $1 < p < \infty$, satisfies that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \quad \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = 0,$$

then there exists $\pi \in L^p(\Omega)$ such that $\boldsymbol{f} = \nabla \pi$, see [4, Theorem 2.8].

4.4 Weak solution in the Hilbert case

Before proving the existence and uniqueness of a weak solution for (S), we have to show the following proposition which shows a useful result about equivalence of norms for the velocity field $\boldsymbol{u}$.

Proposition 4.4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ and $\alpha \in L^2(\Gamma)$ satisfying (H_2). Then, we have for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ the following equivalence of norms:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} + \|\alpha^{\frac{1}{2}} \boldsymbol{u_\tau}\|_{\boldsymbol{L}^2(\Gamma)}, \tag{4.5}$$

if $\alpha(x) > 0$ a.e. $x \in \Gamma$, or

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u_\tau}\|_{\boldsymbol{L}^2(\Gamma)}, \tag{4.6}$$

if $\alpha \geq \alpha_* > 0$.

Proof. First of all, it is enough to prove (4.5) because (4.6) is proved analogously. And second, we only prove that there exists $C = C(\Omega) > 0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C \left(\|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} + \|\alpha^{\frac{1}{2}} \boldsymbol{u_\tau}\|_{\boldsymbol{L}^2(\Gamma)} \right),$$

because the other inequality is clearly obtained.

To prove our aim, we will do it by contradiction. Indeed, let us suppose that for all $n \in \mathbb{N}$, there exists $\boldsymbol{u}_n \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{u}_n \cdot \boldsymbol{n} = 0$ on Γ and

$$\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} > n \left(\|\mathbb{D}(\boldsymbol{u}_n)\|_{\boldsymbol{L}^2(\Omega)} + \|\alpha^{\frac{1}{2}}(\boldsymbol{u}_n)_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma)} \right). \tag{4.7}$$

Since $\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} > 0$, we can define $\boldsymbol{v}_n := \frac{\boldsymbol{u}_n}{\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)}}$. Then, $\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} = 1$ for all $n \in \mathbb{N}$ and from (4.7), we have that

$$\left(\|\mathbb{D}(\boldsymbol{v}_n)\|_{\boldsymbol{L}^2(\Omega)} + \|\alpha^{\frac{1}{2}}(\boldsymbol{v}_n)_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma)} \right) < \frac{1}{n}.$$

This inequality implies that

$$\mathbb{D}(\boldsymbol{v}_n) \underset{n\to\infty}{\longrightarrow} \boldsymbol{0} \quad \text{in} \quad \boldsymbol{L}^2(\Omega) \tag{4.8}$$

$$\alpha^{\frac{1}{2}}(\boldsymbol{v}_n)_{\boldsymbol{\tau}} \underset{n\to\infty}{\longrightarrow} \boldsymbol{0} \quad \text{in} \quad \boldsymbol{L}^2(\Gamma). \tag{4.9}$$

As $\boldsymbol{v}_n \cdot \boldsymbol{n} = 0$ on Γ, then (4.9) implies that

$$\alpha^{\frac{1}{2}}\boldsymbol{v}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{0} \ \text{ in } \ \boldsymbol{L}^2(\Gamma). \tag{4.10}$$

On the other hand, since $(\boldsymbol{v}_n)_n$ is bounded in $\boldsymbol{H}^1(\Omega)$, there exist a subsequence $(\boldsymbol{v}_{n_j})_j \subset (\boldsymbol{v}_n)_n$ and $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{v}_{n_j} \rightharpoonup \boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$-weak. Then, thanks to (4.8), (4.10) and $\alpha > 0$, we conclude that $\mathbb{D}(\boldsymbol{v}) = \boldsymbol{0}$ in Ω and $\boldsymbol{v} = \boldsymbol{0}$ on Γ. By using Korn's inequality (see [16, Theorem 6.15-4, p. 409]), it follows that $\boldsymbol{v} = \boldsymbol{0}$ in Ω which is a contradiction with the fact that $\|\boldsymbol{v}_{n_j}\|_{\boldsymbol{H}^1(\Omega)} = 1$ for all $j \in \mathbb{N}$. □

Now, we are ready to establish the existence and uniqueness for the weak solution in the Hilbert case for the Stokes problem (S).

Theorem 4.4.2. *Let us suppose $\chi = 0$ and $g = 0$. Let*

$$\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega), \quad \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ such that } \boldsymbol{h} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma, \quad \alpha \in L^2(\Gamma)$$

with α verifying the hypothesis (4.3), (H_1) and (H_2). Then the Stokes problem (S) has a unique solution $(\boldsymbol{u}, \pi)$ belonging to $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ which satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega, \alpha_*) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \tag{4.11}$$

where

$$C(\Omega, \alpha_*) = \begin{cases} C(\Omega) & \text{under the hypothesis } (H_1) \\ \frac{C(\Omega)}{\min\{1, \alpha_*\}} & \text{under the hypothesis } (H_2) \text{ with } \alpha \geq \alpha_* > 0 \\ C(\Omega) & \text{under the hypothesis } (H_2) \text{ with } \alpha(x) > 0 \text{ a.e. } x \in \Gamma. \end{cases}$$

Proof. Note that under the hypothesis (H_1) we can choose $\alpha_* = 0$ and then it clearly follows that

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)}, \tag{4.12}$$

for all $\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,T}(\Omega)$. On the other hand, under the hypothesis (H_2), we have for all $\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,T}(\Omega)$ the equivalence of norms given in Proposition 4.4.1.

It follows that the existence of the unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ is a consequence of the Lax-Milgram theorem (because from above we can prove the coercivity of the bilinear form associated to the variational formulation (4.4)) besides the De Rham's theorem (to recover the pressure), see Remark 4.3.6.

The estimate follows from the fact that the solution $\boldsymbol{u}$ verifies

$$2\int_\Omega |\mathbb{D}(\boldsymbol{u})|^2\,\mathrm{d}x + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2\,\mathrm{d}s \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)},$$

that is, if α satisfies (H_1) or (H_2) with $\alpha \geq \alpha_* > 0$, then

$$2\|\mathbb{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \alpha_*\|\boldsymbol{u}_{\boldsymbol{\tau}}\|^2_{\boldsymbol{L}^2(\Gamma)} \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)},$$

and if α satisfies (H_2) with $\alpha(x) > 0$ a.e. $x \in \Gamma$, then

$$\|\mathbb{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \|\alpha^{\frac{1}{2}}\boldsymbol{u}_{\boldsymbol{\tau}}\|^2_{\boldsymbol{L}^2(\Gamma)} \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

In this way, if α satisfies (H_1), we can choose $\alpha_* = 0$, and from (4.12) we obtain that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C_1(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right);$$

if α satisfies (H_2) with $\alpha \geq \alpha_* > 0$, from (4.6), we have

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_2(\Omega)}{\min\{1,\alpha_*\}}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right),$$

and if α satisfies (H_2) with $\alpha(x) > 0$ a.e. $x \in \Gamma$, from (4.5), it follows that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C_3(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right).$$

On the other hand,

$$\begin{aligned}\|\pi\|_{L^2(\Omega)/\mathbb{R}} &\leq \|\nabla\pi\|_{\boldsymbol{H}^{-1}(\Omega)}\\ &\leq C_4(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}\right)\\ &\leq C_4(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\right).\end{aligned}$$

Hence,

$$\|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C_4(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right). \tag{4.13}$$

The estimate (4.11) follows immediately from the estimates for $\boldsymbol{u}$ in each case and (4.13). □

Remark 4.4.3. (i) In the case $\alpha(x) > 0$ a.e. $x \in \Gamma$, we have another equivalence of norms for the velocity field thanks to Proposition 3.3.8: for all $\boldsymbol{u} \in \boldsymbol{V}^2_{\sigma,T}(\Omega)$

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}.$$

(ii) The case $\alpha = 0$ and Ω axisymmetric is analyzed in [5].

4.5 Regularity of the weak solution

The regularity $\boldsymbol{W}^{1,p}(\Omega)$ of the weak solution for (S) is given in the following result. We begin for the case $p > 2$.

Theorem 4.5.1 (regularity $W^{1,p}(\Omega)$ with $p > 2$)**.** *Let us suppose $\chi = 0$, $g = 0$,*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega) \textit{ with } r(p) \textit{ is defined by } (4.1), \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \textit{ such that } \boldsymbol{h}\cdot\boldsymbol{n} = 0 \textit{ on } \Gamma,$$

and $\alpha \in L^{t^*(p)}(\Gamma)$ satisfying (4.3), (H_1) and (H_2) with

$$t^*(p) = \begin{cases} 2+\varepsilon & \textit{if} \quad 2 < p \leq 3, \\ \frac{2}{3}\, p + \varepsilon & \textit{if} \quad p > 3, \end{cases} \tag{4.14}$$

where $\varepsilon > 0$ is an arbitrary number sufficiently small. Then the Stokes problem (S) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

Proof. Since $p > 2$, we have that $\boldsymbol{L}^{r(p)}(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{6}{5}}(\Omega)$, $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $L^{t^*(p)}(\Gamma) \hookrightarrow L^2(\Gamma)$, and thanks to Theorem 4.4.2, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ weak solution for (S).

(i) *Case* $2 < p \leq 3$: Since $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^4(\Gamma)$ and $\alpha \in L^{2+\varepsilon}(\Gamma)$, we have $\alpha\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q_1}(\Gamma)$ where $\frac{1}{q_1} = \frac{1}{4} + \frac{1}{2+\varepsilon}$. But, $\boldsymbol{L}^{q_1}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p_1},p_1}(\Gamma)$ with $q_1 = \frac{2}{3}p_1$. It follows that

$$\frac{1}{p_1} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon}\right).$$

If $p_1 \geq p$, then thanks to the regularity of the Stokes equations (see Remark 3.3.6), we have that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)/\mathbb{R}$. Otherwise, $\boldsymbol{u} \in \boldsymbol{W}^{1,p_1}(\Omega)$ which implies that $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{s_1}(\Gamma)$ where $\frac{1}{s_1} = \frac{1}{p_1} - \frac{1-\frac{1}{p_1}}{2} = \frac{3}{2p_1} - \frac{1}{2}$ (since $p_1 < p \leq 3$). Then $\alpha\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q_2}(\Gamma)$ where $\frac{1}{q_2} = \frac{1}{s_1} + \frac{1}{2+\varepsilon}$. But, $\boldsymbol{L}^{q_2}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p_2},p_2}(\Gamma)$ with $q_2 = \frac{2}{3}p_2$. It follows that

$$\frac{1}{p_2} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon} - \frac{1}{2} + \frac{1}{2+\varepsilon}\right) = \frac{2}{3}\left(\frac{2}{2+\varepsilon} - \frac{1}{2} + \frac{1}{4}\right).$$

If $p_2 \geq p$, then as before $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)/\mathbb{R}$. Otherwise, $\boldsymbol{u} \in \boldsymbol{W}^{1,p_2}(\Omega)$ which implies that $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{s_2}(\Gamma)$ where $\frac{1}{s_2} = \frac{1}{p_2} - \frac{1-\frac{1}{p_2}}{2} = \frac{3}{2p_2} - \frac{1}{2}$ (since $p_2 < p \leq 3$). Then $\alpha\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q_3}(\Gamma)$ where $\frac{1}{q_3} = \frac{1}{s_2} + \frac{1}{2+\varepsilon}$. But, $\boldsymbol{L}^{q_3}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p_3},p_3}(\Gamma)$ with $q_3 = \frac{2}{3}p_3$. It follows that

$$\frac{1}{p_3} = \frac{2}{3}\left(\frac{3}{2+\varepsilon} - \frac{2}{2} + \frac{1}{4}\right).$$

If $p_3 \geq p$, then as before $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)/\mathbb{R}$. Otherwise, proceeding in a similar way, we can show that $\alpha\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q_{k+1}}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p_{k+1}},p_{k+1}}(\Gamma)$ with

$$\frac{1}{p_{k+1}} = \frac{2}{3}\left(\frac{k+1}{2+\varepsilon} - \frac{k}{2} + \frac{1}{4}\right)$$

(in order to get this, we use the fact that $p_k < 3$). Then, $\boldsymbol{u} \in \boldsymbol{W}^{1,p_{k+1}}(\Omega)$ and choosing $k = \lfloor \frac{1}{\varepsilon} - \frac{1}{2} \rfloor + 1$ (where $\lfloor a \rfloor$ stands for the greatest integer less than or equal to a) we have that $p_{k+1} \geq 3 \geq p$. Finally $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)/\mathbb{R}$.

(ii) *Case* $p > 3$: From the previous case, we have that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,3}(\Omega) \times L^3(\Omega)$. This implies that $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^q(\Gamma)$ for all $1 < q < \infty$, and since $\alpha \in L^{\frac{2p}{3}+\varepsilon}(\Gamma)$, we obtain that $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{\frac{2p}{3}}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Finally, by using the regularity of the Stokes equations (Remark 3.3.6), we have that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$. $\square$

Remark 4.5.2. **(i)** The regularity $\boldsymbol{W}^{1,p}(\Omega)$ when $\alpha = 0$ is given in [5].
(ii) Note that for $p > 3$, we can not consider $\alpha \in L^s(\Gamma)$ with $s \leq 2 + \varepsilon$. Indeed, let us suppose that $\alpha \in L^{2+\varepsilon}(\Gamma)$ and since $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^q(\Gamma)$ for $1 < q < \infty$, we have that $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^2(\Gamma)$. Then, we have $L^2(\Gamma) \hookrightarrow W^{-\frac{1}{p},p}(\Gamma)$ if and only if $p \leq 3$, which is not the case.
(iii) We will study later the case of generalized solutions in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for $p < 2$.

Now, we study the case of strong solutions for the Stokes problem (S).

Theorem 4.5.3 (regularity $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$)**.** *Let us suppose* $\chi = 0$, $g = 0$,

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \textit{ such that } \boldsymbol{h} \cdot \boldsymbol{n} = 0 \textit{ on } \Gamma,$$

and

$$\alpha \in H^{\frac{1}{2}}(\Gamma) \quad \textit{if} \quad \frac{6}{5} \leq p \leq 2; \quad \alpha \in H^{\frac{1}{2}+\varepsilon}(\Gamma) \quad \textit{if} \quad 2 < p < 3; \quad \alpha \in W^{1-\frac{1}{p},p}(\Gamma) \quad \textit{if} \quad p \geq 3$$

where $\varepsilon > 0$ *is an arbitrary number sufficiently small and* α *satisfies* (4.3), (H_1) *and* (H_2). *Then the solution given by Theorem 4.4.2 satisfies that* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$.

Proof. Due to $p \geq \frac{6}{5}$, it follows that $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{6}{5}}(\Omega)$, $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $\alpha \in L^2(\Gamma)$, then thanks to Theorem 4.4.2, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ weak solution for (S).

(i) *Case* $\frac{6}{5} \leq p \leq 2$: Since $\alpha \in H^{\frac{1}{2}}(\Gamma)$, we can extend α to the inside of the domain Ω, hence we can consider $\alpha \in H^1(\Omega)$. Then, for all $i, j = 1, 2, 3$ we have that $\frac{\partial \alpha}{\partial x_j} u_i \in L^{q_1}(\Omega)$ and $\alpha \frac{\partial u_i}{\partial x_j} \in L^{q_1}(\Omega)$ where $\frac{1}{q_1} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$. Then, $\frac{\partial}{\partial x_j}(\alpha u_i) = \frac{\partial \alpha}{\partial x_j} u_i + \alpha \frac{\partial u_i}{\partial x_j} \in L^{\frac{3}{2}}(\Omega)$ which implies that $\alpha \boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega)$ and therefore, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{2}{3},\frac{3}{2}}(\Gamma)$. By using Proposition 3.3.3, we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$ and $\pi \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega)/\mathbb{R}$.

From above we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^s(\Omega)$ for all $1 < s < \infty$ and $\nabla \boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Then, for all $i, j = 1, 2, 3$ we have that $\frac{\partial \alpha}{\partial x_j} u_i \in L^{q_2}(\Omega)$ where $\frac{1}{q_2} = \frac{1}{2} + \frac{1}{s}$ and $\alpha \frac{\partial u_i}{\partial x_j} \in L^{q_3}(\Omega)$ where $\frac{1}{q_3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$. Clearly, $q_2 < q_3$ and then $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^{q_2}(\Omega)$ which implies that $\alpha \boldsymbol{u} \in \boldsymbol{W}^{1,q_2}(\Omega)$ and therefore, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{q_2},q_2}(\Gamma)$. By using Proposition 3.3.3, we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,q_2}(\Omega)$ and $\pi \in W^{1,q_2}(\Omega)/\mathbb{R}$ with $\frac{3}{2} < q_2 < 2$.

Finally, since $\boldsymbol{u} \in \boldsymbol{W}^{2,q_2}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $\nabla \boldsymbol{u} \in \boldsymbol{W}^{1,q_2}(\Omega) \hookrightarrow \boldsymbol{L}^{q_2^*}(\Omega)$ with $\frac{1}{q_2^*} = \frac{1}{q_2} - \frac{1}{3}$, we have for all $i, j = 1, 2, 3$ that $\frac{\partial \alpha}{\partial x_j} u_i \in L^2(\Omega)$ and $\alpha \frac{\partial u_i}{\partial x_j} \in L^{q_4}(\Omega)$ where $\frac{1}{q_4} = \frac{1}{6} + \frac{1}{q_2^*} = \frac{1}{q_2} - \frac{1}{6}$. Clearly, $2 < q_4$ and then $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^2(\Omega)$ which implies that $\alpha \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and therefore, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. By using Proposition 3.3.3, we have that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and $\pi \in H^1(\Omega)/\mathbb{R}$. Then, we have showed that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ for $\frac{6}{5} \leq p \leq 2$.

(ii) *Case* $2 < p < 3$: From the previous case, we have that $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$, hence, $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$ and $\nabla \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$. Since $\alpha \in H^{\frac{1}{2}+\varepsilon}(\Gamma)$, we can regard $\alpha \in H^{1+\varepsilon}(\Omega) \hookrightarrow L^s(\Omega)$ where $\frac{1}{s} = \frac{1}{2} - \frac{1+\varepsilon}{3}$, assuming that $\varepsilon < \frac{1}{2}$. Also, $\nabla\alpha \in \boldsymbol{H}^\varepsilon(\Omega) \hookrightarrow \boldsymbol{L}^t(\Omega)$ where $\frac{1}{t} = \frac{1}{2} - \frac{\varepsilon}{3}$. For all $i, j = 1, 2, 3$ we have that $\frac{\partial\alpha}{\partial x_j} u_i \in L^t(\Omega)$ and $\alpha \frac{\partial u_i}{\partial x_j} \in L^r(\Omega)$ where $\frac{1}{r} = \frac{1}{s} + \frac{1}{6} = \frac{1}{3} - \frac{\varepsilon}{3}$. Clearly, $t < r$ and then $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^t(\Omega)$ which implies that $\alpha\boldsymbol{u} \in \boldsymbol{W}^{1,t}(\Omega)$ and therefore, $\alpha\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{t},t}(\Gamma)$ where $2 < t < 3$. By using Proposition 3.3.3, we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,t}(\Omega)$ and $\pi \in W^{1,t}(\Omega)/\mathbb{R}$. Then, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ for $2 < p < 3$.
(ii) *Case* $p \geq 3$: From above we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,t}(\Omega)$ for $2 < t < 3$, hence, $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$ and $\nabla\boldsymbol{u} \in \boldsymbol{W}^{1,t}(\Omega) \hookrightarrow \boldsymbol{L}^r(\Omega)$ where $\frac{1}{r} = \frac{1}{t} - \frac{1}{3}$. Since $\alpha \in W^{1-\frac{1}{p},p}(\Gamma)$, we can regard $\alpha \in W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ where $1 < s < \infty$ for $p = 3$ and $s = \infty$ for $p > 3$. Further, $\nabla\alpha \in \boldsymbol{L}^p(\Omega)$.

If $p = 3$, then for all $i, j = 1, 2, 3$ we have that $\frac{\partial\alpha}{\partial x_j} u_i \in L^3(\Omega)$ and $\alpha\frac{\partial u_i}{\partial x_j} \in L^{r^*}(\Omega)$ where $\frac{1}{r^*} = \frac{1}{s} + \frac{1}{r}$ for any $1 < s < \infty$, then $\frac{1}{r^*} = \frac{1}{s} + \frac{1}{2} - \frac{\varepsilon}{3} - \frac{1}{3} = \frac{1}{s} + \frac{1}{6} - \frac{\varepsilon}{3}$. Choosing s sufficiently big such that $s - \frac{\varepsilon}{3} \leq 0$, we have that $r^* \geq 6$. Clearly, $3 < r^*$ and then $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^3(\Omega)$ which implies that $\alpha\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$ and therefore, $\alpha\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{3},3}(\Gamma)$. By using Proposition 3.3.3, we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,3}(\Omega)$ and $\pi \in W^{1,3}(\Omega)/\mathbb{R}$.

On the other hand, if $p > 3$, we have $\boldsymbol{u} \in \boldsymbol{W}^{2,3}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $\nabla\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^{s^*}(\Omega)$ for any $1 < s^* < \infty$. Further, $\alpha \in L^\infty(\Omega)$ and $\nabla\alpha \in \boldsymbol{L}^p(\Omega)$. Then, for all $i, j = 1, 2, 3$ we have that $\frac{\partial\alpha}{\partial x_j} u_i \in L^p(\Omega)$ and $\alpha\frac{\partial u_i}{\partial x_j} \in L^{s^*}(\Omega)$. For s^* sufficiently large, we have that $p \leq s^*$. It follows that $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^p(\Omega)$ which implies that $\alpha\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and therefore, $\alpha\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. By using Proposition 3.3.3, we have that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)/\mathbb{R}$. $\square$

Remark 4.5.4. **(i)** The regularity $\boldsymbol{W}^{2,p}(\Omega)$ when $\alpha = 0$ is given in [5].
(ii) We will study later the case of strong solutions in $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ for $1 < p < \frac{6}{5}$.

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