Magnonics

Magnetostatic Modes in Samples With Inhomogeneous Internal Fields

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Abstract—Magnetostatic modes in ferromagnetic samples have been studied systematically since the 1950s. They are very well characterized and understood in samples with uniform internal magnetic fields. More recently, interest has shifted to the study of magnetization modes in ferromagnetic samples with inhomogeneous internal fields. This paper shows that under the magnetostatic approximation and for samples of arbitrary shape and/or arbitrary inhomogeneous internal magnetic fields, the modes can be classified as elliptic or hyperbolic, and their associated frequency spectrum and spatial range can be delimited. This results from the analysis of the character of the second-order partial differential equation satisfied by the magnetostatic potential: i.e., if it is hyperbolic or elliptic (parabolic is a limiting case). In elliptic regions, the magnetostatic modes have a smooth monotonic character (with generally decaying or "tunneling" behavior from the surfaces) and in hyperbolic regions an oscillatory wave-like character. A simple local criterion distinguishes hyperbolic from elliptic regions: the sign of a frequency-dependent susceptibility parameter. This study shows that one may control to some extent magnetostatic modes via external fields or geometry. For example, one may imagine propagation along interior regions avoiding surfaces, as is suggested in one case presented here.

Index Terms—Magnonics, magnetodynamics, magnetostatic, modes, particles, internal magnetic fields.

I. INTRODUCTION

Magnetostatic modes, i.e., modes in ferromagnetic samples where retardation effects can be neglected as well as the effect of exchange, have been studied systematically since the 1950s [Herring 1951, Walker 1957, Damon 1961, Joseph 1961, Schlömann 1964, Hurben 1995]. These modes are relevant in the range of scales between the exchange length (nanometers) and the electromagnetic wavelength ($\lambda = 2\pi c/\omega$, of the order of millimeters at gegahertz frequencies), i.e., they belong to the micrometers range: the dipolar and applied magnetic fields are relevant in this scale (anisotropy fields are not considered here).

In this paper, a general result is presented that allows us to classify the types of magnetostatic modes and their associated frequency spectrum and spatial range in samples of arbitrary shape and/or arbitrary nonuniform internal magnetic fields. The criterion to classify the modes follows from the mathematical classification of partial differential equations (PDEs) as hyperbolic, elliptic, parabolic, or with undefined character [Rauch 2012, Sauvigny 2012]. This is a local criterion, i.e., a PDE may change character in different regions, which leads to interesting effects.

A differential equation is derived for the magnetostatic potential of the modes in these samples of arbitrary shapes or with nonuniform internal fields: it is a generalization of Walker's equation [Walker 1957] for samples with homogeneous internal fields. In its derivation, it is considered that an external, static, and in general nonuniform magnetic field, $\vec{H}_o(\vec{x})$ is applied in the region of the sample. This generalized Walker's equation is a linear second-order partial differential equation, with in general spatially dependent coefficients. It is proven to be locally hyperbolic or elliptic type depending on the sign of a susceptibility coefficient $\mu(\vec{x},\omega)$, which depends only on the local magnitude of the internal magnetic field $H_i(\vec{x})$ and the frequency ω; it is parabolic in regions where $\mu(\vec{x},\omega) = 0$, which if they exist in general would be boundary regions. PDEs of hyperbolic type are associated with solutions that have oscillatory or wave-like character, whereas those of elliptic type have a monotonic or smooth character, with decaying or "tunneling" behavior from boundaries [Rauch 2012]. Thus, in the magnetostatic case one may have coexistence between hyperbolic or elliptic regions depending on the local sign of $\mu(\vec{x},\omega)$, with the boundary shifting in general with a changing frequency: a particular region of a certain type may lose its character outside of a frequency range, in this sense the frequency spectrum of these types of modes is delimited.

Previously, magnetostatic modes have in general been classified as volume or surface modes, the former having significant amplitudes throughout the sample, and the latter mainly confined close to the boundaries, i.e., a classification in terms of degree of penetration of the modes. The present classification into elliptic or hyperbolic modes is more general since, for example, one may have hyperbolic regions confined to the edges of the samples (and elliptic regions in the interior, or vice versa), i.e., one could have surface or edge modes of oscillatory character. From their general characteristics, qualitatively hyperbolic and elliptic modes would be "volume" and "surface" modes in their regions of existence, respectively.

II. MAGNETOSTATIC MODES IN SAMPLES WITH INHOMOGENEOUS INTERNAL FIELDS: MODE TYPES AND THEIR FREQUENCY SPECTRUM

We will consider the problem of finding in ferromagnetic samples of arbitrary shape the spin wave modes in the magnetostatic approximation. We assume that there is a time-independent nonuniform applied magnetic field $\vec{H}_o(\vec{x})$.

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The latter will induce an equilibrium configuration of the magnetization $\vec{M}_{\mathrm{eq}}(\vec{x})$ and an associated internal magnetic field in equilibrium $\vec{H}_i^{\mathrm{eq}}(\vec{x}) = \vec{H}_o(\vec{x}) + \vec{H}_D(\vec{M}_{\mathrm{eq}})$, with $\vec{H}_D(\vec{M}_{\mathrm{eq}})$ the generally nonuniform demagnetizing field produced by the equilibrium magnetization (only in samples of ellipsoidal shape with uniform applied fields, these fields are indeed uniform [Herring 1951, Walker 1957, Damon 1961, Joseph 1961, Schlömann 1964]). The equilibrium condition is that the magnetization should point locally in the same direction as the internal field where this is nonzero, i.e., $\vec{H}_i^{\mathrm{eq}}(\vec{x}) = \lambda(\vec{x})\vec{M}_{\mathrm{eq}}(\vec{x})$, with $\lambda(\vec{x}) \geq 0$. Indeed $\lambda(\vec{x})$ is a measure of the magnitude of the equilibrium internal field, since $\lambda(\vec{x}) = |\vec{H}_i^{\mathrm{eq}}(\vec{x})|/M_s \equiv H_i^{\mathrm{eq}}(\vec{x})/M_s$. The static homogeneous Maxwell's equations imply $\vec{H}_D(\vec{M}_{\mathrm{eq}}) = -\nabla\Phi_{\mathrm{eq}}$, and $\nabla \cdot \vec{H}_D(\vec{M}_{\mathrm{eq}}) + 4\pi\nabla \cdot \vec{M}_{\mathrm{eq}} = 0$. We consider the equilibrium problem to be solved, i.e., that $\vec{M}_{\mathrm{eq}}(\vec{x})$ and $\vec{H}_i^{\mathrm{eq}}(\vec{x})$ are known.

Now, turning to the magnetization dynamics, this is governed by the Landau–Lifshitz (LL) equation:

$$\frac{d\vec{M}}{dt} = -\gamma\vec{M} \times \vec{H}_i \tag{1}$$

with γ the absolute value of the gyromagnetic factor, and $\vec{H}_i(\vec{x}, t) = \vec{H}_o(\vec{x}) + \vec{H}_D(\vec{M}(\vec{x}, t))$ the local internal magnetic field. In order to find the linear magnetostatic modes in the sample, one approximates $\vec{M}(\vec{x}, t) \simeq \vec{M}_{\mathrm{eq}}(\vec{x}) + Re(\vec{m}_\omega e^{-i\omega t})$ and needs to solve the LL equation to linear order ($\tilde{\omega} \equiv \omega/\gamma$):

$$i\tilde{\omega}\vec{m}_\omega = \vec{M}_{\mathrm{eq}} \times (\vec{H}_D(\vec{m}_\omega) - \lambda(\vec{x})\vec{m}_\omega)\ . \tag{2}$$

Taking the cross product of (2) with $\vec{M}_{\mathrm{eq}}$ and combining it with itself times $i\tilde{\omega}$, one obtains an expression for the oscillating magnetization in terms of its demagnetizing field:

$$\vec{m}_\omega(\vec{x}) = \frac{\vec{M}_{\mathrm{eq}} \times \left(-i\tilde{\omega}\vec{H}_D(\vec{m}_\omega) + \lambda\vec{M}_{\mathrm{eq}} \times \vec{H}_D(\vec{m}_\omega)\right)}{\tilde{\omega}^2 - \lambda^2 M_s^2}\ . \tag{3}$$

Defining coefficients $\nu(\vec{x}) \equiv -4\pi M_s\tilde{\omega}/(H_i^{\mathrm{eq}}(\vec{x})^2 - \tilde{\omega}^2)$, $\chi(\vec{x}) \equiv 4\pi M_s H_i^{\mathrm{eq}}(\vec{x})/(H_i^{\mathrm{eq}}(\vec{x})^2 - \tilde{\omega}^2)$, and the equilibrium magnetization direction as $\hat{\xi}(\vec{x}) \equiv \vec{M}_{\mathrm{eq}}(\vec{x})/M_s$, then the divergence of (3) implies:

$$\begin{aligned} 4\pi\nabla \cdot \vec{m}_\omega = {} & i\nabla\phi_\omega \cdot (\nu\nabla \times \hat{\xi} + \nabla\nu \times \hat{\xi}) \\ & + \nabla \cdot (\chi\hat{\xi} \times (\hat{\xi} \times \nabla\phi_\omega)) \end{aligned} \tag{4}$$

with $\vec{H}_D(\vec{m}_\omega) = -\nabla\phi_\omega \equiv \vec{h}_\omega$, implying $\nabla^2\phi_\omega = 4\pi\nabla \cdot \vec{m}_\omega$. The latter and (4) lead to a single PDE for ϕ_ω:

$$\begin{aligned} 0 = {} & \nabla^2\phi_\omega - i\nabla\phi_\omega \cdot (\nu\nabla \times \hat{\xi} + \nabla\nu \times \hat{\xi}) \\ & - \nabla \cdot (\chi[(\hat{\xi} \cdot \nabla\phi_\omega)\hat{\xi} - \nabla\phi_\omega]) \end{aligned} \tag{5}$$

which is the generalization of Walker's equation for the magnetostatic potential inside the sample [Walker 1957] to the case of an inhomogeneous internal field background. Now, we analyze the type of PDE that (5) is, i.e., if it can be classified as elliptic, parabolic, or hyperbolic. The solutions of elliptic equations are in general smooth, monotonic; while those of hyperbolic equations are in general oscillatory, wave-like [Rauch 2012]. The theory [Sauvigny 2012] states that one should analyze the eigenvalues of the matrix associated with the higher order derivatives, i.e., in this case the second-order terms. Using Einstein notation, the higher order terms of (5) can be written as $A_{ij}\phi_{ij}$, with $\phi_{ij} \equiv \partial^2\phi/\partial x_i\partial x_j$. From (5), one deduces that the matrix A can be written as $A = (1+\chi)U - \chi M$, with U the 3×3 unit matrix and M such that $M_{ij} = \xi_i\xi_j$. Now, the eigenvalue problem for matrix M, i.e., $M \cdot u = \beta u$ can be written as $\hat{\xi}(\hat{\xi} \cdot \vec{u}) = \beta\vec{u}$, i.e., $(\hat{\xi} \cdot \vec{u}) = \beta(\hat{\xi} \cdot \vec{u})$. This means that the eigenvalues of M are $\beta = 1$ if $(\hat{\xi} \cdot \vec{u}) \neq 0$ and $\beta = 0$ if $(\hat{\xi} \cdot \vec{u}) = 0$, with the latter doubly degenerate. Thus, the eigenvalues of matrix A are $\alpha_1 = (1+\chi) - \chi = 1$, $\alpha_{2,3} = (1+\chi) \equiv \mu$. The mathematical theory [Sauvigny 2012] states that if the eigenvalues of A are nonzero and one of them is of a different sign from the rest, then the PDE is hyperbolic, while if all eigenvalues are nonzero and of the same sign it is elliptic. The main result of this paper is then that the sign of the parameter $\mu(\vec{x}, \tilde{\omega})$,

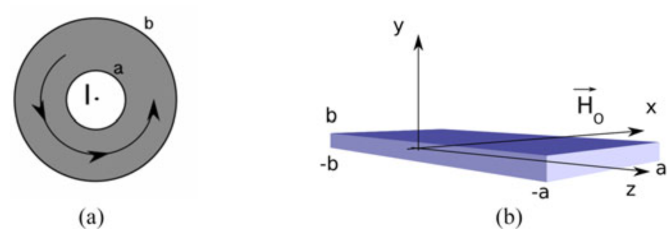

Fig. 1. (a) Ferromagnetic tube: cross section with a current I flowing through a wire along the longitudinal axis of symmetry ($a < \rho < b$); the equilibrium magnetization and the Oersted field circulate inside the tube. (b) Ferromagnetic stripe under a transverse applied magnetic field $H_0\hat{x}$.

$$\mu(\vec{x}, \tilde{\omega}) = \frac{H_i(\vec{x})B_i(\vec{x}) - \tilde{\omega}^2}{H_i(\vec{x})^2 - \tilde{\omega}^2} \tag{6}$$

determines the types of solutions, with $B_i(\vec{x}) \equiv H_i(\vec{x}) + 4\pi M_s$. If $\mu(\vec{x}, \tilde{\omega}) < 0$ that region is an hyperbolic region with expected "oscillatory, wave-like" solutions, while if $\mu(\vec{x}, \tilde{\omega}) > 0$ that region is an elliptic region with expected "monotonic, smooth" solutions ($\mu(\vec{x}, \tilde{\omega}) = 0$ corresponds to a parabolic region, which generally would be a boundary region between elliptic and hyperboic regions, and a limiting case).

From (6), one deduces that at a particular position $\vec{x}$, the range of frequencies over which the PDE for the magnetostatic potential is hyperbolic, corresponds to

$$\gamma H_i(\vec{x}) < \omega < \gamma\sqrt{H_i(\vec{x})B_i(\vec{x})} \tag{7}$$

while the PDE is elliptic outside this frequency range.

III. EXAMPLES OF MAGNETOSTATIC MODES UNDER INHOMOGENEOUS INTERNAL MAGNETIC FIELDS

We present two examples of application of the theory just introduced. Fig. 1 shows the geometries of these examples.

A. Cylindrical Ferromagnetic Tube Under an Oersted Field

This is an example with an inhomogeneous internal field and an equilibrium magnetization with rotational symmetry, both simple enough that the behavior of the modes and their frequencies can be determined: the results agree with the theory just presented. There is a long cylindrical ferromagnetic tube ($a < \rho < b$), and a wire centered along the longitudinal axis of

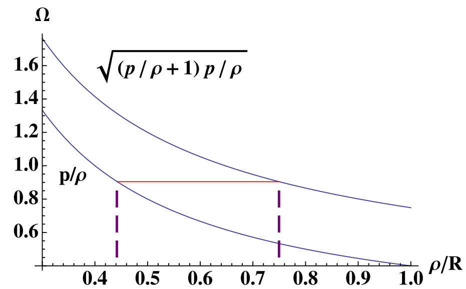

Fig. 2. Frequency limits of the hyperbolic region in a ferromagnetic tube, $p = 0.4$. Also, the spatial range of an eventual eigenmode is shown, with frequency at the red level.

symmetry $\hat{z}$, with a current I flowing through it. Fig. 1(a) shows a cross-sectional view of the geometry.

The current produces an "Oersted" magnetic field, that takes the form $\vec{H}_o(\vec{x}) = \hat{\varphi}(2I)/(c\rho) \equiv \hat{\varphi}4\pi M_s p/\rho$. The equilibrium magnetization is $\vec{M}_{\rm eq} = M_s\hat{\varphi}$ and does not have a demagnetizing field associated with it (no surface nor volume charges associated); thus, the internal field is equal to the Oersted field, i.e., $\vec{H}_i(\vec{x}) = \vec{H}_o(\vec{x})$. Then, according to (5), if one looks for magnetostatic modes without z dependence, the magnetostatic potential satisfies the following equation:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\mu(\rho)\frac{\partial\phi_\omega}{\partial\rho}\right] + \frac{1}{\rho^2}\frac{\partial^2\phi_\omega}{\partial\varphi^2} = 0. \quad (8)$$

From (6), $\mu(\rho) = ((p+\rho)p - \Omega^2\rho^2)/(p^2 - \Omega^2\rho^2)$, with $\Omega \equiv \tilde{\omega}/4\pi M_s$. Using the separation of variables, the modes are $\phi_\omega = \phi_\omega^{(m)}(\rho)\exp(im\varphi)$, m an integer. Thus, (8) leads to:

$$\rho\mu(\rho)\frac{\partial}{\partial\rho}\left[\rho\mu(\rho)\frac{\partial\phi_\omega^{(m)}}{\partial\rho}\right] - m^2\mu(\rho)\phi_\omega^{(m)} = 0. \quad (9)$$

A new radial variable $r = r(\rho)$ is defined through $\partial/\partial r = \rho\mu(\rho)\partial/\partial\rho$, i.e., $dr/d\rho = 1/\rho\mu(\rho)$. Then, (9) becomes

$$\frac{\partial^2\phi_\omega^{(m)}}{\partial r^2} - m^2\mu(\rho(r))\phi_\omega^{(m)} = 0. \quad (10)$$

This previous equation (10) may be compared with the well-known Schrodinger equation in 1-D for the eigenstate wave function of energy E of a particle of mass M in a potential $V(x)$,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2M}{\hbar^2}(E - V(x))\psi(x) = 0. \quad (11)$$

The solutions of the Schrodinger equation (11) have oscillatory character in a region where $E - V(x) > 0$ and a decaying exponential behavior where $E - V(x) < 0$ (the particle "tunnels" into these regions). Equivalently, (10) will have oscillatory solutions in regions where $\mu < 0$ and decaying exponential behavior in regions where $\mu > 0$: all this agrees with the predicted behavior of the general theory in regions that are hyperbolic ($\mu < 0$) and elliptic ($\mu > 0$), respectively. According to (7), at a radial position ρ, the frequencies of the hyperbolic modes lie in the range $p/\rho < \Omega < \sqrt{(p/\rho+1)p/\rho}$. Fig. 2 shows the previous limiting frequencies as the smooth curves (p is taken as 0.4). Also, it shows an eventual eigenfrequency at the red level with its associated spatial range in the hyperbolic region, interestingly showing that the mode is localized in the inner region of the tube avoiding its surfaces. By using the appropriate boundary conditions at $\rho = a, b$ (magnetostatic potential and normal induction continuous), one could obtain the exact eigenfrequencies and eigenmodes: this requires a numerical calculation that was not done here, since we derived the types of solutions to be expected and their associated ranges of frequencies, which was the goal here.

B. Transversely Magnetized Ferromagnetic Stripe

In the last decade or so, a problem that has attracted a deal of attention from the experimental and theoretical point of views corresponds to the spin wave modes of a transversely magnetized ferromagnetic stripe [Park 2002, Bailleul 2003, Bayer 2004, Roussigné 2004, Bayer 2006, McMichael 2006]. The picture that has emerged from these works is that, due to the inhomogeneity of the internal magnetic field close to the edges and corners, edge modes appear and also the volume modes are affected close to the edge regions: they avoid this region, their quantization conditions are modified, as well as their shapes. Most of these studies include the exchange interaction in their analysis.

We approach the problem of finding the spin wave modes in these stripes in the magnetostatic regime, in an approximate way, assuming the thickness of the stripes is small enough so that the magnetization may be considered uniform across the thickness (effectively a 1-D problem, i.e., the effect of the corners is not treated exactly). We consider stripes of rectangular cross section, with sides $2a, 2b$ ($a > b$, see Fig. 1(b)), and with a magnetic field $H_o\hat{x}$ applied parallel to the long axis of the cross section, assumed strong enough to magnetize uniformly the sample in its direction. The approximate magnetization to linear order is $\vec{M}(\vec{x},t) \simeq M_s\hat{x} + Re[(m_y^\omega\hat{y} + m_z^\omega\hat{z})e^{-i\omega t}]$, and the LL equation for the linear magnetization dynamics is:

$$i\tilde{\omega}m_y^\omega = (H_o - 4\pi M_s n_x(X))m_z^\omega \quad (12)$$

$$i\tilde{\omega}m_z^\omega = -(H_o - 4\pi M_s n_x(X))m_y^\omega + M_s < \vec{H}_D(\vec{m}) >_y \quad (13)$$

with $n_x(X)$ a space-dependent average demagnetizing factor ($X \equiv x/a$), and $< \vec{H}_D(\vec{m}) >_y$ the y component of the demagnetizing field due to the dynamic magnetization averaged over the thickness (X dependent). The averaged internal field is $\vec{H}_i(X) = (H_o - 4\pi M_s n_x(X))\hat{x}$. In order to solve the 1-D integro-differential equations (12), (13) for the modes, we considered either free or pinned boundary conditions: in both cases the types of modes and their frequencies agree with the general theory, i.e., for our purposes the boundary conditions are secondary. To solve (12) and (13), we used a Fourier representation of the modes that satisfies the boundary conditions, and then these equations become a linear eigenvalue problem for the Fourier expansion coefficients and the modes' frequencies. The mode solutions may be classified into three categories: "edge," "comb," and "volume" modes. These modes are exemplified in Fig. 3 (we used free boundary conditions in this case), for a stripe of aspect ratio $b/a = 0.1$: it shows plots of m_y for specific modes that belong to these groups (the average "height" at which these magnetostatic modes are plotted coincides with the respective frequency of the mode in ques-

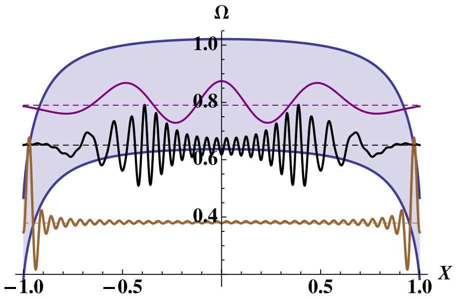

Fig. 3. Frequency limits of the hyperbolic region for a thin stripe (region between smooth curves) and examples of modes of the "edge", "comb" and "volume" types (free boundary conditions). Transverse applied field parallel to the plane $H_0/4\pi M_s = 0.7$, $b/a = 0.1$.

tion: dashed lines in Fig. 3). This figure also shows the frequency range of the hyperbolic region of (7), i.e., $h_i(X) < \Omega < \sqrt{h_i(X)(h_i(X)+1)}$, with $h_i(X) \equiv H_i(X)/4\pi M_s$ (this range is the filled region between the smooth curves in Fig. 3, that depends on X). The lower limit curve $\Omega(X) = h_i(X)$ of the hyperbolic range of frequencies has a "flat" portion far from the edges and a fast decaying part close to the edges (produced by the space-dependent demagnetizing factor). The "edge" modes appear at frequencies below this "flat" region, and are confined mainly to the region close to the edges that has hyperbolic character at the particular frequency of the given mode, as is seen qualitatively on the lower mode of Fig. 3. Now, the "comb" modes are modes that appear at frequencies close but slightly over the flat region frequency (taken as $h_i(0)$, which is equal to 0.637 for the aspect ratio of the figure, i.e., $b/a = 0.1$): the name follows the "comb" modes introduced by Puszkarski [2007]. These "comb" modes show short wavelength oscillations in the flat region and decay smoothly to zero in the edge regions: this may be explained since close to the lower curve characterized by $\Omega(X) = h_i(X)$ one may expect highly localized modes precessing at the local value of the internal field $h_i(X)$ (they would be loosely coupled via the dynamic demagnetizing field and the boundary conditions). Furthermore, in the flat region there is a high degeneracy of these localized modes over an extended region, so one may expect these "comb" modes to be linear combinations of those and to appear at $\Omega \simeq h_i(0)$ as highly oscillating collective modes over the flat region that lose their amplitude close to the edges since there are no localized modes at that frequency in that region. Finally, the "volume" modes appear at frequencies well within the hyperbolic range, and they lose amplitude close to the edges since at those frequencies there they enter into elliptic regions (this can be seen qualitatively for the upper mode of Fig. 3).

Thus, the previous results agree with the general theory, in the sense that the oscillating behavior of the modes appears in the hyperbolic regions (whether extended or not), and there is decay of the modes in elliptic regions.

IV. EXPERIMENTAL AGREEMENT

Also, we mention an experimental work [Rezende 1991] that agrees with the theory. In a film of rectangular shape, these authors studied magnetostatic modes under an applied magnetic field that varies linearly along one direction. They found that there are regions with real wavevectors and imaginary wavevectors that correspond respectively with regions with μ negative or positive: the modes look wavy in the former region and monotonic in the latter, i.e., it coincides with the theory (they obtained experimentally the shapes of the modes by a Brillouin light scattering technique). Other studies of magnetostatic modes under inhomogeneous fields are those by Morgenthaler [1985] and Stancil [2009].

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