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MINIMIZATION OF THE GROUND STATE OF THE MIXTURE OF TWO CONDUCTING MATERIALS IN A SMALL CONTRAST REGIME

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To Sandra and Juan Esteban

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Abstract

In this thesis we consider the resolution of some optimal design problems through the asymptotic procedure of second order. These procedures provide us a simplified formulation of the problem which allows it to be implemented to obtain the numerical results.

The techniques of the first order approximation have been taken into account in order to show the importance of the second order model. Theoretical considerations reflect the differences between the implemented models. In particular, the asymptotic procedure of first order is based on the level-set method, while the procedure of second order is based on relaxation processes.

The second order model is not well-posed in general, so that an optimal solution of the problem might not exist in the class of the admissible sets. So, the relaxed problem is then obtained as the result of the homogenization method which requires, through the processes, the H-measure theory.

The addressed problems in this thesis are focused on finding the optimal configuration of two isotropic materials (characterized by two different physical constants) within a fixed bounded domain Ω in order to minimize an objective function associated to a state equation, under the constraint that the materials are kept to constant volume proportion. Asymptotic expansions are described by the scalar parameter ε which is defined as the "difference" between the two magnitudes that characterize the behavior of the two materials. Additionally, this parameter must be taken small enough to obtain the desired results.

The first problem of this thesis is to minimize the first eigenvalue λ of the diffusion operator $-\text{div}\ (\sigma\nabla\cdot)$ in the $\mathrm{H}_0^1(\Omega)$ space where, as it is well known, σ represents the diffusion density of the materials inside of the domain Ω.

The second problem is devoted to the study of maximizing the *compliance* or the stored elastic energy of an elastic system with a mixed Dirichlet-Newmann condition.

The third and last problem of this thesis combines the obtained results of the two problems above to the partial resolution of the problem of minimizing the first eigenvalue of the elasticity operator $-\text{div}\ Ae(\cdot)$ in the $\mathbf{H}_0^1(\Omega)^d$ space, where A represents the fourth order tensor which describes the elastic behavior of the materials within Ω and e is the differential operator that generates the strain tensor of the system.

Résumé

Cette thèse considère la résolution des problèmes de la conception optimale en utilisant des procédures asymptotiques de deuxième ordre. Ces procédures nous donnent une formulation simplifiée du problème qui nous permet d'obtenir des résultats numériques.

Les procédures asymptotiques de premier ordre ont été prises en compte afin de montrer l'importance du modèle de second ordre. Les considérations théoriques expliquent les différences entre les modèles mis en œuvre. En particulier, la méthode asymptotique du premier ordre est basé sur la technique des ensembles de niveau tandis que le deuxième ordre est basé sur des processus de relaxation.

Le modèle de second ordre est généralement mal conditionnée parce que une solution optimale pour le problème peut ne pas exister dans la classe des ensembles admissibles. Le problème relaxé est alors obtenu à la suite de la méthode d'homogénéisation qui, au moyen de processus, requiert la théorie des H-mesures.

Les problèmes abordées dans cette thèse visent à déterminer la configuration optimale de deux matériaux isotropes (caractérisés par deux constantes physiques différentes) dans un domaine borné fixe Ω afin de minimiser une fonction objectif associé à une équation d'état, sous la contrainte que les matériaux sont maintenus à un rapport de volume fixe. Le paramétre scalaire ε, avec lequel sont effectués les expansions asymptotiques, est défini comme la "différence" entre les quantités qui décrivent le comportement des matériaux, qu'il faut considérer suffisamment petit pour obtenir les résultats désirés

Le premier problème de la thèse consiste à minimiser la première valeur propre λ de l'opérateur de diffusion $-\mathrm{div}\,(\sigma\nabla\cdot)$ dans l'espace $\mathrm{H}_0^1(\Omega)$ où, comme on le sait, σ représente la densité de diffusion des matériaux à l'intérieur du domaine Ω.

Le deuxième problème traite de l'étude de la maximisation de la *compliance* ou l'énergie élastique accumulée d'un système élastique avec des conditions mixtes Dirichlet-Neumann.

Le troisième et dernier problème de cette thèse combine les résultats obtenus du premier et deuxième problème pour la résolution partielle du problème de minimiser la première valeur propre de l'opérateur d'élasticité $-\mathrm{div}\,Ae(\cdot)$ dans l'espace $\mathbf{H}_0^1(\Omega)^d$, où A représente le tenseur de quatrième ordre qui décrit le comportement élastique des matériaux à l'intérieur de Ω et e est l'opérateur différentiel qui génère le tenseur de déformations du système.

Resumen

En esta tesis se considera la resolución de problemas de diseño óptimo mediante procedimientos asintóticos de segundo orden. Estos procedimientos nos proveen de una formulación simplificada del problema que permite ser implementada para la obtención de resultados numéricos.

Los procedimientos asintóticos de primer orden se han tenido en cuenta con el fin de mostrar la importancia del modelo de segundo orden. Las consideraciones teóricas dan cuenta de las diferencias entre los modelos implementados. En particular, el procedimiento asintótico de primer orden está basado en la técnica de los conjuntos de nivel mientras que el de segundo orden se basa en los procesos de relajación.

En términos generales el modelo de segundo orden está mal condicionado debido a que un óptimo para el problema puede no existir en la clase de los conjuntos admisibles. El problema relajado es obtenido entonces como resultado del método de homogeneización que, a través de los procesos, requiere de la teoría de las H-medidas.

Los problemas abordados en esta tesis se enfocan en determinar la configuración óptima de dos materiales isotrópicos (caracterizados por dos magnitudes físicas constantes distintas) dentro de un dominio acotado fijo Ω con el fin de minimizar una función objetivo asociada a una ecuación de estado, bajo la restricción de que los materiales se mantengan a una proporción de volumen fija. El parámetro escalar ε, con respecto al cual se realizan las expansiones asintóticas, se define como la "diferencia" entre las magnitudes que describen el comportamiento de los materiales, el cual se debe tomar lo suficientemente pequeño para la obtención de los resultados.

El primer problema de la tesis consiste en minimizar el primer valor propio λ del operador de difusión $-\mathrm{div}\,(\sigma\nabla\cdot)$ en el espacio $\mathrm{H}^1_0(\Omega)$ donde, como es bien sabido, σ representa la densidad de difusión de los materiales al interior del dominio Ω.

El segundo problema está dedicado al estudio de la maximización de la *"Compliance"* o la energía elástica acumulada de un sistema elástico con condiciones mixtas Dirichlet-Neumann.

El tercer y último problema de esta tesis combina los resultados obtenidos de el primer y segundo problema para la resolución parcial al problema de minimizar el primer valor propio del operador de elasticidad $-\mathrm{div}\, Ae(\cdot)$ en el espacio $\mathbf{H}^1_0(\Omega)^d$, donde A representa el tensor de cuarto orden que describe el comportamiento elástico de los materiales dentro de Ω y e es el operador diferencial que genera el tensor de deformaciones del sistema.

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Chapter 1

Introduction

1.1 Asymptotic with respect to a small parameter

The homogenization techniques, so-called asymptotic analysis, are implemented to solve many interesting and applied problems in Mechanics, Physics, Chemistry and Engineering [5]. Homogenization is the theory that allows us to characterize composite materials. Roughly speaking these techniques consist in finding materials with fixed characteristics obtained as "averages" from other heterogeneous materials. From a mathematical point of view, a composite material is the "limit" of a sequence of geometries with lengthscale going to 0. A composite material is characterized by a volume fraction of material and a microstructure. These methods are based on convergence results and multiple scales asymptotic expansions.

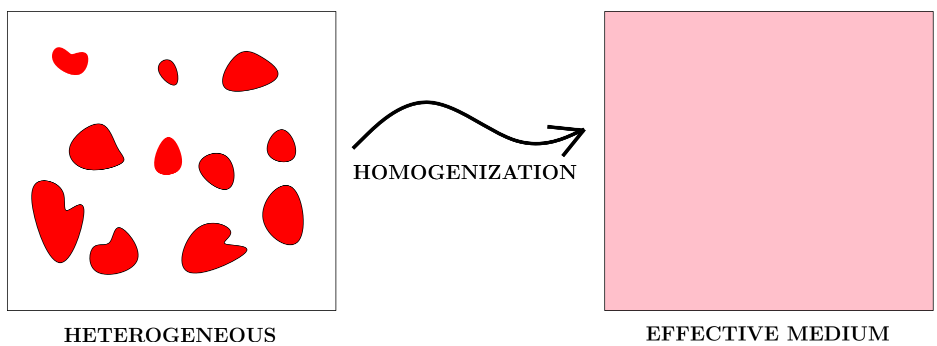


Figure 1.1: Composite material

Some branches of analysis employing homogenization methods are, for example, shape and topology optimization. Physically the idea is to produce the "best" material or structure may be subject to certain constraints, for example structural design and manufacturing resistant wings to many aircraft as well as design of bridges which generally includes specific requirements (see fig.1.2 below).

One of the most important mathematical question in optimal design is the existence of the optimal solution in the "admissible" set of shapes. This question involves some kind of compactness argument (even convexity arguments). Unfortunately a great number of optimal design problems are ill-posed. In particular there is not an optimal shape among the admissible sets that optimize an objective function, i.e., such an optimization problem does not admit a minimizer in general; see F. Murat [18].

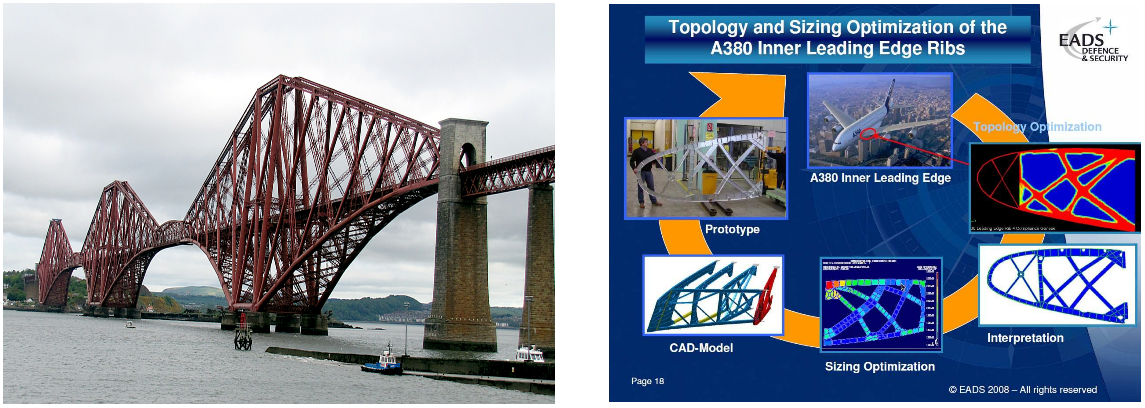


Figure 1.2: Structural designs of bridges and wings[1]

Typically these kinds of problems are described by partial differential equations which model, for instance, physical situations among others. A general presentation on this topic is outlined as follows. A family of partial differential operators L_ε, depending on a "small" parameter ε, is defined in a suitable Sobolev space $\mathrm{H}^s(\Omega)$ ($\Omega \subset \mathbb{R}^N$ satisfying certain regularity conditions; $N = 2, 3$ for applications).

One hence has a boundary value problem:

$$\begin{cases} L_\varepsilon u^\varepsilon = f & \text{in } \Omega, \\ u^\varepsilon \text{ subject to boundary conditions,} \end{cases} \tag{1.1}$$

which is assumed well-possed for $\varepsilon > 0$ fix. In many cases the data f is a regular given function, so that the family of functions u^ε can be uniformly bounded by *a priori* estimatives; that is the case when the family of differential operators are linear and uniformly elliptic in the Sobolev Hilbert space H^1_0. This is often useful to have uniform estimates, depending only on the parameter ε, of the remainders with respect to asymptotic expansions.

The problem is now to obtain, if possible, an expansion of u^ε

$$u^\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j u_j.$$

This series is inserted in the equation (1.1) getting a cascade of auxiliar equations which allows to obtain the coefficients u_j. A typical "result" is: one can construct a differential operator L such that $u_\varepsilon \to u$ (in an appropriate topology) as $\varepsilon \to 0$ where u is the solution of

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u \text{ subject to boundary conditions}^2. \end{cases}$$

The objective of the present work is to consider two-phase optimal design problems in the context of conductivity or linearized elasticity. We use a strong simplifying assumption, namely that the two component phases involved in the optimal design have close coefficients or material properties (the low contrast regime). It shall allow to make an asymptotic expansion of the coefficients in terms of the small parameter that characterizes the variations between

[1]From G. Allaire's talks

[2]which of course will depend on those boundary conditions imposed on u_ε

the two phases. Let us now describe the general situation occurring in the problems that we will consider in the work under these conditions. The main idea is to look for an optimal distribution of the two-phase optimal design problem. More precisely, if Ω represents a fix geometry (bounded domain) and ω is a subset of Ω, then we are interested in minimizing an objective function of the type

$$J(\omega) = \int_\Omega F(x, u_\omega(x), \nabla u_\omega(x))\,\mathrm{d}x, \tag{1.2}$$

where u_ω is solution of the system

$$\begin{cases} L_\omega u = f & \text{in } \Omega, \\ u \text{ subject to boundary conditions.} \end{cases}$$

The dependence of the solution with respect to ω is given by the (linear and elliptic) differential operator who carries with it the two-phase coefficient. Since the problem of minimizing (1.2) is in general not well posed, the problem may be relaxed by introducing composite designs (homogenization method).

We take advantage of the "low contrast regime" assumption in order to simplify the problem, so that we perform an asymptotic procedure with respect to the small parameter. The obtained result is called "small amplitude" optimal design problem.

Next we look for an asymptotic expansion of the form

$$u_\omega^\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j u_{j,\omega}(x).$$

After determining the coefficients of $u_\omega^\varepsilon(x)$ we obtain uniform estimates which allows us to conclude that the problem of minimizing the objective function is not far away from those who we want to approximate when ε is taken to be small. However, the approximation of the objective function is still ill-posed and we must use those already mentioned relaxation process.

At first the relaxation process provides a sequence of inclusions ω_n which are identified with characteristic functions $\chi_{\omega_n} \equiv \chi_n$ in the $\mathrm{L}^\infty(\Omega)$ space. The weak* topology allows to find a weak* limit in the $\mathrm{L}^\infty(\Omega; [0,1])$ space with positive values. On the basis of this, numerical simulations are implemented to show the optimal configuration to the small amplitude optimal design poblem (the relaxation of the approximation of the objective function).

1.2 First and second order approximation

Low contrast regime assumption allows us to greatly simplify optimal design problems as mentioned above. In particular the "order of approximation" obtained in an asymptotic approach can be implemented to solve numerically the associated relaxed problems with considerable precision. The optimal design problems discussed in this thesis have been addressed by the second order approximation.

Other contributions in trying to solve the same optimization problems with the asymptotic analysis are those who have been raised by first order approximation, among them is the paper written by Conca, Laurain and Mahadevan [8]. It shows how is the optimal distribution of two conducting materials with respect to the minimization of the first eigenvalue of a Dirichlet operator for different geometries. In particular that work refutes a conjecture about the optimal distribution when the fixed domain is a ball.

What follows is a general mathematical description of the procedures done in the above article.

One has a problem of the type

$$\begin{array}{l} \text{minimize } J(\omega) = \int_\Omega F(u_\omega, \nabla u_\omega)\,\mathrm{d}x, \\ \text{subject to } |\omega| = m \end{array}$$

where u_ω is solution of

$$\begin{cases} -\mathrm{div}\ \sigma_\omega \nabla u = f & \text{in } \Omega, \\ u \text{ subject to boundary conditions,} \end{cases}$$

and $0 < m < |\Omega|$ is the volume proportion.

One takes the truncated series of u_ω^ε up to first order

$$u_\omega^\varepsilon = u_0 + \varepsilon u_1(\omega) + \underbrace{\mathcal{O}(\varepsilon^2).}_{\text{dropped}}$$

Substituting this into J one also has

$$J(\omega) = \int_\Omega F_1(u_0, \nabla u_0)\,\mathrm{d}x + \varepsilon \int_\Omega F_2(u_1(\omega), \nabla u_1(\omega))\,\mathrm{d}x + \mathcal{O}(\varepsilon^2).$$

From this the initial problem may be reduced to an equivalent (small amplitude) problem of the type

$$\begin{array}{l} \text{minimize } J_{sa}(\omega) = \int_\omega \tilde{F}(\nabla u_0)\,\mathrm{d}x, \\ \text{subject to } |\omega| = m \end{array}$$

which simplifies considerably the problem. Finally, $\tilde{F} \geq 0$ gives a strategy of solution, namely the level-set method.

The methodology of the first order model motivates the study of the second order model in order to increase the precision.

One of the most significant differences between first and second order models lies in the relaxation and homogenization processes that are used in the second order model. More precisely, in the energy estimatives where weak limits have to be calculated through H-measures. This important tool provides us an explicit approximation of the objective function that depends on ε in contrast to the first order approximation. These H-measures are quadratic default measures introduced by Gerard [17] and Tartar [24]. It is a default measure which quantifies the lack of compactness of weakly converging sequence in $\mathrm{L}^2(\mathbb{R}^N)$. More exactly, it indicates where in the physical space, and at which frequency in the Fourier space, are the obstructions to strong convergence. In practice, one has the problem to pass to the limit to the product of two sequences that, one only knows, are weakly convergent, so that H-measures are the right tool in order to be able to pass to the limit. In the end, the limits will depend on the H-measure of the sequence χ_n (which is thus a new variable for optimization). This technique works in the context of pseudo-differential operators. A pseudo-differential operator q is defined through its symbol $(q_{ij}(x,\xi))_{1\leq i,j\leq p} \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ by

$$\big(q(u)\big)_i(x) = \sum_{j=1}^{p} \mathcal{F}^{-1}\big(q_{ij}(x,\cdot)\mathcal{F}u_j(\cdot)\big)(x),$$

where $\mathcal{F}$ symbolizes the Fourier transform. Furthermore, here is assumed that the symbol $(q_{ij}(x,\xi))$ is homogeneous of degree 0 in ξ and with compact support in x.

The mathematical formulation in the application of the H-measures to pass to the limit is as follows. Typically we have weak convergent sequences

$$\chi_n \overset{*}{\rightharpoonup} \theta \quad \text{and} \quad v_n \overset{\mathrm{H}^1}{\rightharpoonup} v.$$

Then we wish to calculate, for instance

$$\lim_{n\to\infty} \int_\Omega \chi_n \nabla v_n \cdot \vec{p}.$$

It is well known that this kind of limits may not exist. However, from the partial differential equation satisfied by v_n, we may prove that ∇v_n is a pseudo-differential operator of order 0 depending on χ_n. According to the H-measure theory, the limit can be computed as

$$\lim_{n\to\infty} \int_\Omega \chi_n \nabla v_n \cdot \vec{p} = \int_\Omega \theta \nabla v \cdot \vec{p} + \text{Corrector } (H\text{-measure}).$$

The corrector term depend on the symbol q by the expression

$$\int_\Omega \int_{\mathbb{S}^{N-1}} q(x,\xi) \cdot \theta(1-\theta)\vec{p}\,\nu(\mathrm{d}x, \mathrm{d}\xi).$$

Here $\mathbb{S}^{N-1}$ is the unit sphere in $\mathbb{R}^N$ and the vectorial measure $\theta(1-\theta)\vec{p}\,\nu(\mathrm{d}x,\mathrm{d}\xi)$ corresponding to the sequence $\chi_n - \theta$ is the so-called H-measure where, for given x, the measure $\nu(\mathrm{d}x,\mathrm{d}\xi)$ is a probability measure with respect to ξ. Furthermore, any such probability measure can be attained as the H-measure of a sequence χ_ϵ of characteristic functions that weakly* converges. Additionally, we will find with the fact that, in our relaxed small ampitude problems, the probability measure ν can be simplified by a Dirac mass (in ξ). In fact, this optimal Dirac mass H-measure will not depend on the density θ. All the details will be complemented through the presentation of the thesis.

We remark that each optimal design problems exhibited in this thesis are subject to stationary partial differential equations.

1.3 Intervals of fluctuation

In order to compare the effective optimal relaxed objective function in both cases first and second order models, we employ the same techniques to solve the counterpart of the optimal design problem. That is to say, if we study the optimization problem

$$\inf_\omega J(\omega), \tag{1.3}$$

its counterpart

$$\sup_\omega J(\omega), \tag{1.4}$$

is then also addressed to obtain the intervals of fluctuation

$$\inf_\omega J(\omega) \leq J(\omega) \leq \sup_\omega J(\omega).$$

The procedure to solve the counterpart problem is, in essence, totally the same to that made for the previous one; the only difference lies in that the optimal Dirac mass H-measure must be taken in a suitable single lamination direction ξ^* which, of course, will depend on the expression in the integral over $\mathbb{S}^{N-1}$ when we consider fixed x.

It is seen, by numerical examples, that for ε suficiently large, first and second order models differ considerably. Moreover, in the optimal design problem of minimizing the first eigenvalue of the two-phase escalar conductivity operator $-\text{div}\,(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla$, perforated domains are built to show those differences between both methods.

For the sake of completeness, we finally use a Monte-Carlo approach reinforcing the obtained comparisons.

To end this introduction let us briefly outline the material of this doctoral thesis.

In section 2 we shall discuss the problem of minimizing, under the proportion volume constraint, the first eigenvalue λ of the conductivity operator $-\text{div}\,\sigma(x)\nabla$ in $\mathrm{H}_0^1(\Omega)$. Here $\sigma(x)$ is the conductivity of the material at x.

In the third section it is addressed to the problem of maximizing the stored elastic energy and the so-called compliance of structures clamped in a part of its boundary which has a free boundary condition and has a load in the remaind boundary. The vectorial elasticity operator is given by div $A_\omega e(\cdot)$ being A_ω a Hooke law fourth order tensor and $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the second order strain tensor for the displacement u.

Section 4 leads with the minimization of the first eigenvalue of the elasticity operator $-\text{div}\,Ae(\cdot)$ in $\mathrm{H}_0^1(\Omega)^N$ which was already mentionated above.

The problems of optimal design described in section 3 and 4 are also subject to the proportion volume constraint.

Chapter 2

Minimization of the Ground State of the mixture of two conducting materials in a small contrast regime

Abstract. We consider the problem of distributing two conducting materials with a prescribed volume ratio in a given domain so as to minimize the first eigenvalue of an elliptic operator with Dirichlet conditions. The gap between the two conductivities is assumed to be small (low contrast regime). For any geometrical configuration of the mixture, we provide a complete asymptotic expansion of the first eigenvalue. We then consider a relaxation approach to minimize the second order approximation with respect to the mixture. We present numerical simulations in dimensions two and three to illustrate optimal distributions and the advantage of using a second order method.

2.1 Introduction

Problems of minimizing the ground state of composite materials appear frequently and are of interest in applications. We refer to Henrot [20], Cox and McLaughlin [12, 13], Cox and Lipton[11] and included references. In this article, we consider the following problem. Given a domain Ω and a subdomain B and two nonnegative numbers α and β, we define the ground state $\lambda(B)$ of the mixture as the infimum of the λ such that there exists $0 \neq u$ such that

$$- \operatorname{div} \big((\alpha + (\beta - \alpha)\chi_B)\nabla u \big) = \lambda u \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\,. \tag{2.1}$$

In other words, $\lambda(B)$ is the smallest eigenvalue of the operator $-\operatorname{div}\big((\alpha + (\beta - \alpha)\chi_B)\nabla .\big)$ on $\mathrm{H}^1_0(\Omega)$. We are then interested in minimizing $\lambda(B)$ with respect to B among the subdomains of Ω of given volume. In general, it is well-known that this problem is not wellposed: the infimum is not usually reached at a given B and we have to consider a relaxed version corresponding to a situation of homogenization (see [11]).

Nevertheless, when Ω is a ball, the infimum is reached on a radially symmetric domain B^* (see [4],[9]). In the recent years, much attention has been put on the determination of the corresponding B^*. First, Conca and al. conjectured in [10] that the global minimizer B^* in Ω should be a concentric ball of the prescribed volume. The conjecture was motivated by the situation in dimension one and by numerical simulations. Then, Dambrine and Kateb reinforced the conjecture by an order two sensitivity analysis in [14] by proving that the concentric ball of prescribed volume is a local strict minimizer of $\lambda(B)$.

However, Conca et. al. proved in [8] that the conjecture is false. Their strategy was the following. They consider the case of small contrast, that is to say, α and β such that the difference of both conductivities is small: $\beta = \alpha(1+\varepsilon)$ and provide the first order asymptotic expansion $\lambda_1(B)$ of $\lambda(B)$ with respect to the small parameter ε for any admissible domain $B \subset \Omega$. Then, they minimize the new objective functional $\lambda_1(B)$ with respect to B and observe that the minimizer B_1 of this approximation is not always the concentric ball of prescribed volume. Finally, thanks to a precise estimate of the remainder in the approximation, they prove that $\lambda(B_1) < \lambda(B^*)$.

Finally, Laurain proved in [21] that the global minimum of the first eigenvalue in low contrast regime is either a centered ball or the union of a centered ball and of a centered ring touching the boundary, depending on the prescribed volume ratio between the two materials. Thus the small contrast case is well understood when the domain is a ball.

We aim in this work to make a precise analysis of the small contrast case in general domains. In Section 2, to begin with, we characterize completely the full asymptotic expansion of $\lambda(B)$ with respect to the small parameter ε. Subsequently, we obtain a second order approximation $\lambda_2(B)$ of $\lambda(B)$ with uniform estimates for the remainder, uniform with respect to B. This means that minimizers for the second order approximation $\lambda_2(B)$ are approximate minimizers for the original objective functional $\lambda(B)$. With this motivation, in Section 3, we study the problem of minimizing λ_2. Unlike the first order approximation $\lambda_1(B)$, the minimization problem for $\lambda_2(B)$ is not, *a priori*, well posed and thus, qualitatively, resembles more closely the minimization problem for $\lambda(B)$. A relaxed formulation for the minimization problem for $\lambda_2(B)$ is obtained using H-measures. It can be seem that the relaxed problem for $\lambda_2(B)$ has a much more simple aspect compared to the relaxed problem for $\lambda(B)$ obtained in Cox and Lipton [11]. Then, in Section 4, the optimality conditions for the relaxed problem for $\lambda_2(B)$ are obtained and the minimization problem is studied numerically using a descent algorithm. Finally, we present a numerical comparison of optimal solution obtained for first and second order.

2.2 Asymptotic expansion of the first eigenvalue with respect to the constrast.

We consider the low contrast regime, that is to say, α and β such that the difference of both conductivities is small: $\beta = \alpha(1+\varepsilon)$. We shall denote the first eigenvalue in the problem (2.1) by $\lambda_\varepsilon(B)$ for a given distribution B of the material with conductivity β and a given value of the contrast parameter $\varepsilon > 0$.

The existence of an asymptotic development for $\lambda_\varepsilon(B)$, for given B, is classical from perturbation theory of simple eigenvalues. By the Krein-Rutman theorem, the first eigenvalue $\lambda_\varepsilon(B)$ in (2.1) is simple. The corresponding normalized eigenfunction, with unit L^2 norm and taken to be non-negative, will be denoted by $u_\varepsilon(B)$. So, by classical results from perturbation theory (see, for instance, Theorem 3, Chapter 2.5 of Rellich [23]), for a given B, the map $\varepsilon \mapsto (\lambda_\varepsilon, u_\varepsilon)$ is analytic in $(\mathbb{R}, \mathrm{H}_0^1(\Omega))$. Therefore there are sequences (λ_i) of real numbers and (u_i) of functions in $\mathrm{H}^1_0(\Omega)$ such that:

$$\lambda_\varepsilon = \sum_{i=0}^{\infty} \lambda_i \varepsilon^i \text{ and } u_\varepsilon = \sum_{i=0}^{\infty} u_i \varepsilon^i. \tag{2.2}$$

As a consequence, there are constants $C_n(B)$ such that

$$\left| \lambda_\varepsilon - \sum_{i=0}^{n} \lambda_i \varepsilon^i \right| \leq C_n(B)\varepsilon^{n+1} \text{ and } \left\| u_\varepsilon - \sum_{i=0}^{n} u_i \varepsilon^i \right\|_{\mathrm{H}^1_0} \leq C_n(B)\varepsilon^{n+1}.$$

In this section, we will first identify the coefficients λ_i, u_i then prove that the constants $C_n(B)$ can be taken uniform in B. This will serve in obtaining an approximate model problem for the eigenvalue minimization problem.

2.2.1 Computation of the coefficients of the asymptotic expansions

The terms in the asymptotic expansions in (2.2) may be identified, formally, by injecting the expansions in the equations defining $(\lambda_\varepsilon, u_\varepsilon)$, that is,

$$\begin{aligned} -\mathrm{div}\left(\alpha(1+\chi_B\varepsilon)\nabla\left(\sum_{i=0}^{\infty} u_i\varepsilon^i\right)\right) &= \left(\sum_{i=0}^{\infty} \lambda_i\varepsilon^i\right)\left(\sum_{i=0}^{\infty} u_i\varepsilon^i\right) \text{ in } \Omega, \\ \sum_{i=0}^{\infty} u_i\varepsilon^i &= 0 \text{ on } \partial\Omega, \\ \int_\Omega \left(\sum_{i=0}^{\infty} u_i\varepsilon^i\right)^2 &= 1. \end{aligned}$$

and we obtain then the following relationships by identifying the coefficients of same order in the previous power series.

$$\left\{ \begin{aligned} -\alpha\Delta u_0 - \lambda_0 u_0 &= 0 \text{ in } \Omega \\ u_0 &= 0 \text{ on } \partial\Omega \quad \forall i \geq 0, \\ \int_\Omega u_0^2 &= 1\,. \end{aligned} \right. \tag{2.3}$$

$$\left\{ \begin{aligned} -\alpha\Delta u_i - \lambda_0 u_i &= \mathrm{div}\,(\alpha\chi_B\nabla u_{i-1}) + \sum_{k=1}^{i} \lambda_k u_{i-k} \text{ in } \Omega \quad \forall i \geq 1, \\ u_i &= 0 \text{ on } \partial\Omega \quad \forall i \geq 0, \\ \sum_{k=0}^{i} \int_\Omega u_k u_{i-k} &= 0 \quad \forall i \geq 1. \end{aligned} \right. \tag{2.4}$$

It is possible to rigorously justify the relations by using the expansions (2.2) in the weak formulation of the partial differential equation in (2.1). We then have an iterative procedure to compute the pair (λ_i, u_i).

The case: $i = 0$**.** By definition, one has:

$$-\alpha\Delta u_0 - \lambda_0 u_0 = 0 \text{ in } \Omega \tag{2.5}$$
$$u_0 = 0 \text{ on } \partial\Omega. \tag{2.6}$$

Hence, the couple (λ_0, u_0) is an eigenpair of $-\alpha\Delta$ with homogeneous Dirichlet boundary condition. Clearly $u_0 \geq 0$ in Ω since $u_\varepsilon \to u_0$ as $\varepsilon \to 0$ and the eigenmodes u_ε are non-negative. Now, by the Krein-Rutman theorem, since all eigenmodes change sign except those associated to the first eigenvalue, we obtain that λ_0 is the ground state of $-\alpha\Delta$ with Dirichlet boundary condition and u_0 is the positive eigenmode with L^2-norm 1.

Now assume that, for a given i, we have knowledge of all the λ_k, u_k for $k < i$. We now then treat

The case: $k = i$. We know that u_i satisfies the equation

$$\begin{aligned} -\alpha\Delta u_i - \lambda_0 u_i &= \operatorname{div}(\alpha\chi_B \nabla u_{i-1}) + \sum_{k=1}^{i} \lambda_k u_{i-k} \text{ in } \Omega, \\ u_i &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.7}$$

Notice that the right hand side has the unknown quantity λ_i. We shall first obtain an expression for λ_i in terms of λ_k's and u_k's for $k < i$ which have been assumed to be calculated previously. The compatibility condition, the Fredholm alternative for the equation (2.7), imposes the orthogonality of the right hand side of the former equation to the kernel of $-\alpha\Delta - \lambda_0 I$ with Dirichlet boundary condition which is spanned by u_0

$$\int_\Omega \left(\operatorname{div}(\alpha\chi_B \nabla u_{i-1}) + \sum_{k=1}^{i} \lambda_k u_{i-k} \right) u_0 = 0.$$

This gives the expression for the eigenvalue λ_i

$$\lambda_i = \int_B \alpha \nabla u_{i-1} \cdot \nabla u_0 - \sum_{k=2}^{i-1} \int_\Omega \lambda_{i-k} u_0 u_k \tag{2.8}$$

taking into account the fact that the L^2 norm of u_0 is 1 and, u_0 and u_1 are orthogonal. In the sequel, whenever there is a sum whose upper limit is less than the lower limit, we shall adopt the convention that the sum is 0.

Now, to end, we note that u_i is not completely determined by the equation (2.7), but only upto the kernel of $-\alpha\Delta - \lambda_0 I$. For $i = 0$, the non-negativity of u_0 and the normalization condition (the third relation in (2.3)) determines uniquely u_0. For general i, having determined uniquely the u_k for $k < i$, the term u_i is determined uniquely using the normalization condition (the third relation in (2.4) which can be written as

$$\int_\Omega u_i u_0 = -\frac{1}{2} \sum_{k=1}^{i-1} \int_\Omega u_k u_{i-k}. \tag{2.9}$$

and should be understood as the orthogonality relation $\int_\Omega u_i u_0 = 0$ when $i = 1$.

2.2.2 Uniform estimate of the remainders

We seek to estimate the remainder in the expansions (2.2), uniformly in B. Our main results in this section are the following estimates.

Proposition 2.1 *There exists a constant C, independent of B, such that*

$$|\lambda_\varepsilon - (\lambda_0 + \varepsilon\lambda_1)| \leq \sqrt{\frac{\lambda_0}{\alpha}} C\varepsilon^2. \tag{2.10}$$

Proposition 2.2 *There is a constant $C > 0$ independent of B such that:*

$$|\lambda_\varepsilon - (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2)| \leq 2\, C\varepsilon^3 \sqrt{\frac{\lambda_0}{\alpha}}. \tag{2.11}$$

The main tool we use for the estimation of the remainders is the notion of h-*quasimode* with $h = \mathcal{O}(\varepsilon^k)$, for $k = 1, 2$ in the sequel. The notion of quasimode is defined as follows.

Definition 1 *Let A be a self-adjoint operator on a Hilbert space H with domain $D(A)$. For a fixed $h > 0$, a pair $(\lambda, u) \in \mathbb{R} \times D(A) \setminus \{0\}$ is called a h-*quasimode *if we have*

$$\|(A - \lambda)u\|_H \leq h\|u\|_H.$$

The interest of such a definition relies on the following fact: if (λ, u) is a h-*quasimode* of A, then the distance from λ to the spectrum of A is less than h and the distance between u and certain eigenspaces of A can be estimated (See Lemma 2-2 in [15]). We will prove that our truncated power series expansions are quasimodes in the Hilbert space $\mathrm{H}^{-1}(\Omega)$.

Remainder of order one.

The first step is to prove a uniform bound in B of $\|u\|_{\mathrm{H}^1(\Omega)}$.

Lemma 2.1 *There exists C, which is independent of B, such that:*

$$\|u_1\|_{\mathrm{H}^1_0(\Omega)} \leq C \text{ and } |\lambda_\varepsilon - \lambda_0| \leq C\varepsilon. \tag{2.12}$$

Proof of Lemma 2.1: By using (2.8), with $i = 1$, we have the following expression and uniform bounds for $\lambda_1(B)$

$$\lambda_1 = \int_B \alpha|\nabla u_0|^2 \leq \alpha \int_\Omega |\nabla u_0|^2 = \lambda_0\,. \tag{2.13}$$

By (2.4), for $i = 1$, u_1 satisfies the following:

$$-\alpha\Delta u_1 - \lambda_0 u_1 = \text{div}\,(\alpha\chi_B\nabla u_0) + \lambda_1 u_0 \text{ in } \Omega, \tag{2.14}$$

$$u_1 = 0 \text{ on } \partial\Omega, \tag{2.15}$$

$$\int_\Omega u_0 u_1 = 0. \tag{2.16}$$

After multiplying the first relation by u_1 and integrating over Ω, by integration by parts, we get

$$\int_\Omega \alpha|\nabla u_1|^2 - \lambda_0 \int_\Omega u_1^2 = -\int_B \alpha\nabla u_0 \cdot \nabla u_1.$$

By the characterization of the spectrum of an elliptic self-adjoint operator using the Rayleigh's quotient, we know that for all v in $\mathrm{H}^1_0(\Omega)$ orthogonal to the first eigenfunction u_0, it holds that

$$\lambda^1 \int_\Omega v^2 \leq \alpha \int_\Omega |\nabla v|^2, \tag{2.17}$$

where $\lambda^1 > \lambda_0$ is the second eigenvalue of $-\alpha\Delta$ in $\mathrm{H}^1_0(\Omega)$. We have used the superscript here to distinguish the second eigenvalue λ^1 from λ_1 which appears in the second term of the expansion (2.2). Since u_1 is orthogonal to u_0, it follows using (2.17) that

$$\alpha\left(1-\frac{\lambda_0}{\lambda^1}\right)\int_\Omega |\nabla u_1|^2 \le \int_\Omega \alpha|\nabla u_1|^2 - \lambda_0\int_\Omega u_1^2 \le \alpha\|u_0\|_{\mathrm{H}^1_0(\Omega)}\|u_1\|_{\mathrm{H}^1_0(\Omega)} \tag{2.18}$$

where at the end we have used (2.14) and followed it by a simple estimation. We have obtained the upper bound for u_1. Finally, using the variational characterization of the first eigenvalue for elliptic self-adjoint operators, we obtain

$$\begin{aligned}\lambda_0 = \int_\Omega \alpha|\nabla u_0|^2 \le \int_\Omega \alpha|\nabla u_\varepsilon|^2 \le \int_\Omega \alpha(1+\chi_B\varepsilon)|\nabla u_\varepsilon|^2 = \lambda_\varepsilon \\ \le \int_\Omega \alpha(1+\chi_B\varepsilon)|\nabla u_0|^2 \le (1+\varepsilon)\int_\Omega \alpha|\nabla u_0|^2 = (1+\varepsilon)\lambda_0\end{aligned}$$

which allows us to conclude that $|\lambda_\varepsilon - \lambda_0| \le C\varepsilon$. ■

To use the quasimode strategy, we compute:

$$\begin{aligned}&-\mathrm{div}\,(\alpha(1+\chi_B\varepsilon)\nabla(u_0+\varepsilon u_1)) - (\lambda_0+\varepsilon\lambda_1)(u_0+\varepsilon u_1)\\ &\quad= -\alpha\Delta u_0 - \lambda_0 u_0 + \varepsilon\left(-\alpha\Delta u_1 - \lambda_0 u_1 - \lambda_1 u_0 - \mathrm{div}\,(\alpha\chi_B\nabla u_0)\right)\\ &\qquad+ \varepsilon^2\left(-\lambda_1 u_1 - \mathrm{div}\,(\alpha\chi_B\nabla u_1)\right)\\ &\quad= \varepsilon^2\left(-\lambda_1 u_1 - \mathrm{div}\,(\alpha\chi_B\nabla u_1)\right)\end{aligned} \tag{2.19}$$

where we have used (2.5) and, (2.7) with $i = 1$.

Proof of Proposition 2.1: We need a uniform bound on the normalized right-hand side: $\lambda_1 u_1 + \mathrm{div}\,(\alpha\chi_B\nabla u_1)$. Obviously, this term is only defined in $\mathrm{H}^{-1}(\Omega)$ hence we have to make the estimation in the $\mathrm{H}^{-1}(\Omega)$ norm. To that end, we use a test function $\varphi \in \mathrm{H}^1_0(\Omega)$ and compute the duality product:

$$\begin{aligned}\langle -\mathrm{div}\,(\alpha\chi_B\nabla u_1), \varphi\rangle_{\mathrm{H}^{-1}(\Omega)\times\mathrm{H}^1_0(\Omega)} = \int_\Omega \alpha\chi_B\nabla u_1\cdot\nabla\varphi = \int_B \alpha\nabla u_1\cdot\nabla\varphi \\ \le \alpha\|u_1\|_{\mathrm{H}^1_0(\Omega)}\|\varphi\|_{\mathrm{H}^1_0(\Omega)}.\end{aligned}$$

This proves that

$$\|\mathrm{div}\,\alpha(\chi_B\nabla u_1)\|_{\mathrm{H}^{-1}(\Omega)} \le \alpha\|u_1\|_{\mathrm{H}^1_0(\Omega)}. \tag{2.20}$$

And

$$\begin{aligned}\langle \lambda_1 u_1, \varphi\rangle_{\mathrm{H}^{-1}(\Omega)\times\mathrm{H}^1_0(\Omega)} = \int_\Omega \lambda_1 u_1\varphi \le \lambda_1\|u_1\|_{\mathrm{L}^2(\Omega)}\|\varphi\|_{\mathrm{L}^2(\Omega)} \\ \le \lambda_1\|u_1\|_{\mathrm{H}^1_0(\Omega)}\|\varphi\|_{\mathrm{H}^1_0(\Omega)} \le C\|\varphi\|_{\mathrm{H}^1_0(\Omega)}\end{aligned}$$

using the estimation (2.13) and the fact that u_1 is bounded independently of B proved in Lemma 2.1. This gives

$$\|\lambda_1 u_1\|_{\mathrm{H}^{-1}(\Omega)} \le C\,. \tag{2.21}$$

Hence, we obtain from (2.19), using (2.20) and (2.21) that there exists a constant C independent of B such that

$$\| -\text{div }(\alpha(1+\chi_B\varepsilon)\nabla(u_0+\varepsilon u_1)) - (\lambda_0+\varepsilon\lambda_1)(u_0+\varepsilon u_1)\|_{\mathrm{H}^{-1}(\Omega)} \leq C\varepsilon^2 \tag{2.22}$$

Moreover, using $u_0 \in \mathrm{H}^1_0$ as test function in the definition of the H^{-1}-norm of $u_0+\varepsilon u_1$, we obtain

$$\begin{aligned}\|u_0+\varepsilon u_1\|_{\mathrm{H}^{-1}(\Omega)} &= \sup_{\varphi\in\mathrm{H}^1_0(\Omega)} \frac{\langle u_0+\varepsilon u_1,\varphi\rangle_{\mathrm{H}^{-1},\mathrm{H}^1_0}}{\|\varphi\|_{\mathrm{H}^1_0(\Omega)}} = \sup_{\varphi\in\mathrm{H}^1_0(\Omega)} \frac{\displaystyle\int_\Omega (u_0+\varepsilon u_1)\varphi}{\|\varphi\|_{\mathrm{H}^1_0(\Omega)}} \\ &\geq \frac{\displaystyle\int_\Omega (u_0+\varepsilon u_1)u_0}{\|u_0\|_{\mathrm{H}^1_0(\Omega)}} = \frac{\displaystyle\int_\Omega u_0^2}{\left(\displaystyle\int_\Omega |\nabla u_0|^2\right)^{\frac12}} = \sqrt{\frac{\alpha}{\lambda_0}}.\end{aligned} \tag{2.23}$$

Hence, by (2.22) and (2.23), we obtain

$$\| -\text{div }(\alpha(1+\chi_B\varepsilon)\nabla(u_0+\varepsilon u_1)) - (\lambda_0+\varepsilon\lambda_1)(u_0+\varepsilon u_1)\|_{\mathrm{H}^{-1}(\Omega)} \leq \sqrt{\frac{\lambda_0}{\alpha}}\, C\varepsilon^2\, \|u_0+\varepsilon u_1\|_{\mathrm{H}^{-1}(\Omega)}$$

As a consequence of the theory of quasi mode, there is an element of the spectrum of the self-adjoint operator $-\text{div }(\alpha(1+\chi_B\varepsilon)\nabla\cdot)$ in $\mathrm{H}^{-1}(\Omega)$ at distance at most $\sqrt{\frac{\lambda_0}{\alpha}}C\varepsilon^2$ from $\lambda_0+\varepsilon\lambda_1$. To finish, we need to argue that this element of the spectrum is λ_ε, the first eigenvalue of $-\text{div }(\alpha(1+\chi_B\varepsilon)\nabla\cdot)$. If these were higher eigenvalues, then as $\varepsilon\to 0$, they would tend to a higher eigenvalue of the operator $-\alpha\Delta$. But this would lead to a contradiction, since this sequence is within a distance $O(\varepsilon^2)$ from the sequence $\lambda_0+\varepsilon\lambda_1$ which tends to λ_0, the first eigenvalue of $-\alpha\Delta$ which is simple. ■

Remainder of order two

We first prove an uniform upper bound for λ_2 and u_2.

Lemma 2.2 *There exists C, which is independent of B, such that:*

$$\|u_2\|_{\mathrm{H}^1_0(\Omega)} \leq C \text{ and } \lambda_2 \leq C. \tag{2.24}$$

Proof of Lemma 2.2: First, notice that by (2.8) applied with $i=2$, we get

$$\lambda_2 = \int_B \alpha\nabla u_0\cdot\nabla u_1 \leq \alpha\|u_0\|_{\mathrm{H}^1_0(\Omega)}\|u_1\|_{\mathrm{H}^1_0(\Omega)} \leq C \tag{2.25}$$

where C is independent of B by the estimate (2.12). In a second step, we search a uniform estimate for u_2. To that end, we follow the strategy already used to estimate u_1. The main change is that u_2 is not orthogonal to u_0 so the adaptation is not straightforward. To overcome the difficulty we introduce the combination u_2+au_0 where

$$a = -\int_\Omega u_2 u_0$$

is chosen such that $u_2 + au_0$ is $\mathrm{L}^2(\Omega)$-orthogonal to u_0.

By (2.9) for $i = 2$ we have

$$\int_\Omega u_2 u_0 = -\frac{1}{2}\int_\Omega u_1^2 \tag{2.26}$$

which gives

$$a = \frac{1}{2}\int_\Omega u_1^2 \le \frac{1}{2}\|u_1\|^2_{\mathrm{H}^1(\Omega)} \le C \tag{2.27}$$

with C independent of B (by (2.12)). We now estimate $u_2 + au_0$. For this, we multiply equation (2.5) by a and add it to equation (2.7) to obtain:

$$\begin{aligned} -\alpha\Delta(u_2 + au_0) - \lambda_0(u_2 + au_0) &= \text{div }(\alpha\chi_B\nabla u_1) + \lambda_1 u_1 + \lambda_2 u_0, \quad \text{in } \Omega \\ u_2 + au_0 &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Using $u_2 + au_0$ as test function, it follows that

$$\begin{aligned} \alpha\int_\Omega |\nabla(u_2 + au_0)|^2 &- \lambda_0\int_\Omega (u_2 + au_0)^2 \\ &= -\int_B \alpha\nabla u_1 \cdot \nabla(u_2 + au_0) + \int_\Omega \lambda_1 u_1(u_2 + au_0) + \int_\Omega \lambda_2 u_0(u_2 + au_0) \\ &\le \left(\alpha\|u_1\|_{\mathrm{H}^1_0(\Omega)} + \lambda_1\|u_1\|_{\mathrm{H}^1_0(\Omega)} + |\lambda_2|\,\|u_0\|_{\mathrm{H}^1_0(\Omega)}\right)\|u_2 + au_0\|_{\mathrm{H}^1_0(\Omega)} \\ &\le C\,\|u_2 + au_0\|_{\mathrm{H}^1_0(\Omega)} \end{aligned} \tag{2.28}$$

where C is independent of B, by estimates (2.13), (2.12) and (2.25). Since $u_2 + au_0$ is orthogonal to u_0, similarly as in the estimation (2.18), we conclude that $u_2 + au_0$ is bounded in $\mathrm{H}^1_0(\Omega)$ uniformly in B. Therefore,

$$\|u_2\|_{\mathrm{H}^1_0(\Omega)} \le C + a\|u_0\|_{\mathrm{H}^1_0(\Omega)} \le C'$$

with C' independent of B by estimate (2.27).

■

Proof of Proposition 2.2: We compute

$$\begin{aligned} &-\text{div }\big(\alpha(1 + \chi_B\varepsilon)\nabla(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\big) - (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2) \\ &= -\alpha\Delta u_0 - \lambda_0 u_0 + \varepsilon\left(-\alpha\Delta u_1 - \lambda_0 u_1 - \lambda_1 u_0 - \text{div }(\alpha\chi_B\nabla u_0)\right) \\ &\quad + \varepsilon^2\left(-\alpha\Delta u_2 - \lambda_0 u_2 - \lambda_1 u_1 - \lambda_2 u_0 - \text{div }(\alpha\chi_B\nabla u_1)\right) \\ &\quad + \varepsilon^3\left(-\lambda_1 u_2 - \lambda_2 u_1 - \text{div }(\alpha\chi_B\nabla u_2)\right) + \varepsilon^4(\lambda_2 u_2) \\ &= \varepsilon^3\left(-\lambda_1 u_2 - \lambda_2 u_1 - \text{div }(\alpha\chi_B\nabla u_2)\right) + \varepsilon^4(\lambda_2 u_2) \end{aligned} \tag{2.29}$$

using equations (2.5), and (2.7) for $i = 1, 2$. Then, since

$$\|-\text{div }(\alpha\chi_B\nabla u_2)\|_{\mathrm{H}^{-1}(\Omega)} \le \alpha\,\|u_2\|_{\mathrm{H}^1_0(\Omega)},$$

it follows from equation (2.29) and estimates (2.13), (2.12), and (2.24), that for $\varepsilon \ll 1$,

$$\begin{aligned} &\|\text{div }\big(\alpha(1 + \chi_B\varepsilon)\nabla(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\big) + (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)\|_{\mathrm{H}^{-1}(\Omega)} \\ &\le \Big((\alpha + \lambda_1)\|u_2\|_{\mathrm{H}^1_0(\Omega)} + |\lambda_2|\,\|u_1\|_{\mathrm{H}^1_0(\Omega)}\Big)\varepsilon^3 + (|\lambda_2|\,\|u_2\|_{\mathrm{H}^1_0(\Omega)})\varepsilon^4 \\ &\le C_1\varepsilon^3 + C_2\varepsilon^4 \le C\varepsilon^3, \end{aligned} \tag{2.30}$$

Moreover, one has

$$\|u_0+\varepsilon u_1+\varepsilon^2 u_2\|_{\mathrm{H}^{-1}(\Omega)} = \sup_{\varphi\in \mathrm{H}^1_0(\Omega)} \frac{\displaystyle\int_\Omega (u_0+\varepsilon u_1+\varepsilon^2 u_2)\varphi}{\|\varphi\|_{\mathrm{H}^1_0(\Omega)}} \geq \frac{\displaystyle\int_\Omega (u_0+\varepsilon u_1+\varepsilon^2 u_2)u_0}{\|u_0\|_{\mathrm{H}^1_0(\Omega)}}$$
$$= \frac{\displaystyle\int_\Omega u_0^2 + \varepsilon^2 \int_\Omega u_0 u_2}{\|u_0\|_{\mathrm{H}^1_0(\Omega)}}.$$

Then, using relation (2.26), we obtain

$$\|u_0+\varepsilon u_1+\varepsilon^2 u_2\|_{\mathrm{H}^{-1}(\Omega)} \geq \frac{1-\dfrac{\varepsilon^2}{2}\displaystyle\int_\Omega u_1^2}{\|u_0\|_{\mathrm{H}^1_0(\Omega)}} \geq \frac{1-\frac{\varepsilon^2}{2}C^2}{\|u_0\|_{\mathrm{H}^1_0(\Omega)}},$$

since u_1 is bounded in $H^1_0(\Omega)$ and consequently, in $L^2(\Omega)$ as shown in (2.12). For $\varepsilon < \frac{1}{C}$, we get

$$\|u_0+\varepsilon u_1+\varepsilon^2 u_2\|_{\mathrm{H}^{-1}(\Omega)} \geq \frac{1}{2\|u_0\|_{\mathrm{H}^1_0(\Omega)}} = \frac{1}{2}\sqrt{\frac{\alpha}{\lambda_0}} \tag{2.31}$$

By (2.30) and (2.31), we then have for $\varepsilon < 1/C$ small enough

$$\begin{aligned}&\|-\mathrm{div}\,\big(\alpha(1+\chi_B\varepsilon)\nabla(u_0+\varepsilon u_1+\varepsilon^2 u_2)\big) - (\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2)(u_0+\varepsilon u_1+\varepsilon^2 u_2)\|_{\mathrm{H}^{-1}(\Omega)}\\ &\leq 2\,C\varepsilon^3\sqrt{\frac{\lambda_0}{\alpha}}\,\|u_0+\varepsilon u_1+\varepsilon^2 u_2\|_{\mathrm{H}^{-1}(\Omega)}.\end{aligned} \tag{2.32}$$

By the quasimode argument, there is an element of the spectrum of $-\mathrm{div}\,(\alpha(1+\chi_B\varepsilon)\nabla\cdot)$ in $H^{-1}(\Omega)$ whose distance from $\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2$ is at most $2\,C\varepsilon^3\sqrt{\frac{\lambda_0}{\alpha}}$. By similar arguments as those at the end of Proposition 2.1, one concludes that such an element is precisely λ_ε, the first eigenvalue of $-\mathrm{div}\,(\alpha(1+\chi_B\varepsilon)\nabla\cdot)$. ■

2.3 Minimization of the first order approximation of the ground state

Although our main interest is to minimize the ground state λ_ε with respect to the set B, given $\varepsilon > 0$, the general feeling is that the optimization problem is not well posed. A relaxed problem which is not so simple to describe was obtained in Cox and Lipton [11]. In order to understand the nature of the problem for small contrasts Conca et. al. used a first order approximation [8]. Indeed, after proving a slightly weaker estimate as compared to Proposition 2.1 using a more ad hoc method of estimation, they conclude that

$$\left|\inf_B \lambda_\varepsilon(B) - \lambda_0 - \varepsilon \inf_B \lambda_1(B)\right| \leq C\varepsilon^{\frac{3}{2}}\,. \tag{2.33}$$

This permits to obtain approximate minimizers for the eigenvalue functional λ_ε by minimizing, instead, the functional $\lambda_0+\varepsilon\lambda_1$. This is a well posed problem and since the original problem may not be well posed it may fail to capture some of the features of the original minimization problem.

2.3.1 Optimality conditions for the first order approximation.

One of the contribution of this thesis is to improve the estimative (2.33). Indeed, Proposition 2.1 shows an uniform bound composed by a power 2 of ε which is sensitive better to that obtained in [8], namely, that shown in (2.33) which has an uniform bound composed by a power 3/2 of ε. Thus, minimizers for λ_1 are nearly optimal for λ_ε in the above sense. The following theorem provides us a numerical approximation of the minimum value of λ_ε (see [8] for more details).

Theorem 2.1 *There exists $c^* \geq 0$ such that whenever B is a measurable subset of Ω satisfying $|B| = m$ and*

$$\{x : |\nabla u_0(x)| < c^*\} \subset B \subset \{x : |\nabla u_0(x)| \leq c^*\},$$

then B is an optimal solution for the problem of minimizing λ_1.

2.3.2 Random inclusions

In order to check the accuracy of the estimated bound obtained by the first order model of Conca and al., we now consider random inclusions and we will evaluate on these random distributions of the two materials the first eigenvalue of the Dirichlet Laplacian. We will first explain how we generate the random inclusions, then present some histograms resulting computations made for the previous algorithm of random domain generation and finaly we will illustrate numerically how sensitive is the result to an important parameter in the algorithm.

Random inclusions generation

Unless the fact that the literature contains many works on the generation of random domains especially union of balls, the main difficulty here is the volume constraint which turns out to be non standard in the cases studied by probabilists. Hence our approach is empirical in particular we do not hope to obtain strong result in terms of probability. In this section, the event is denoted by e.

The idea of our approach is to use a level set point of view. The inclusion will be the set of point where a scalar stochastic field f takes values less or equal than a threshold $\alpha(f)$. More precisely, our random domains B will be calculated through the expression

$$B(e) = \{(x, y), f(x, y, e) \leq \alpha(f)\}.$$

The algorithm has then two steps:

1. *determine a realization of the random field f.* The idea there is to choose a model of random fields that mimic the Karhunen-Loeve decomposition of the random field with a L^2-covariance kernel. In other words, we choose f as:

$$f(x, y, e) = \sum_{k=0}^{N} \alpha_k(e)\ \varphi_k(x, y)$$

where the α_k are independent random variable following the same distribution (in the sequel, we take the uniform distribution on $[-1/2, 1/2]$) and the φ_k are an Hilbert Base of $L^2(\Omega)$ (in the sequel, we choose the spectral basis of the Dirichlet-Laplacian. This is a possibility that presents the advantage to be avaible for all domain Ω), and N is an integer the parameter of the method.

2. *determine the threshold* $\alpha(f)$ *corresponding to that realization.* This is achieved by a binary search by solving the equation

$$|B(e)| = m$$

up to a given small tolerance.

We present in Figure 2.1 some realizations for a fraction of volume fixed to $m/|\Omega| = 0.4$:

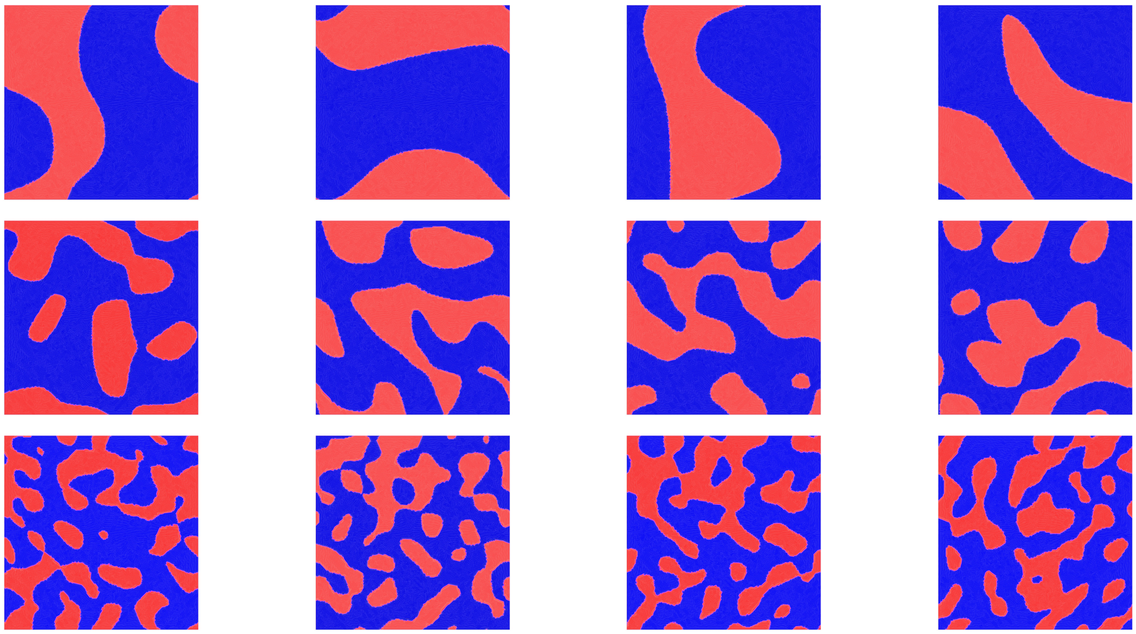


Figure 2.1: Some realizations with $N = 16$ on the first line, $N = 64$ on the second line and $N = 196$ on the third line.

Some experiments.

Let us now present some experimentations performed in order to obtain a rough idea of the distribution of the first eigenvalue of the mixture. The simulations are performed with the Freefem++ library developped at the Laboratoire Jacques Louis Lions of the university Paris 6. We consider the unit square as Ω and choose the parameter N as 36. We plot the histograms of λ_1 obtained thanks to 4000 samples. This size is relatively small but the main behavior already appears as one can see in Figure 2.2, 2.3, 2.4 and 2.5 for various volume fraction.

We also have plotted the distribution of eigenvalues computed for samples of random elements obtained by applying the method of Conca and al. We observe that this lower bound seems to well provide an attained lower bounds for the first eigenvalues but that this lower bound is far from the typical eigenvalue at least for the inclusions under consideration.

On the influence of the parameter of truncation N.

The generated random inclusions clearly varies with the value taken by the parameter N. In Figure 2.1, it appears that the size of each connected componant of the inclusion decrease

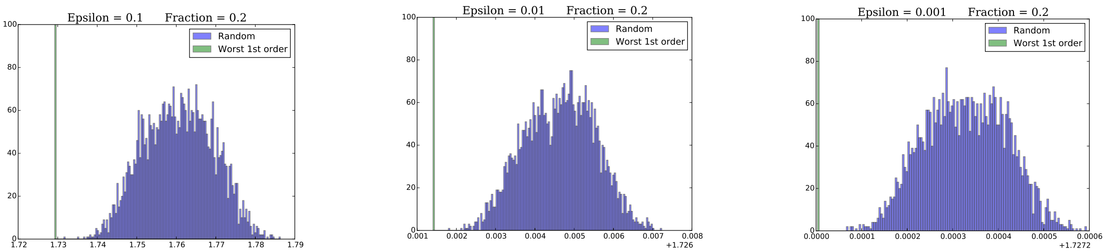


Figure 2.2: Histograms for $m/|\Omega| = 0.2$ and $\varepsilon = 10^{-3}$ (left), 10^{-2} (center) and 10^{-1} (right).

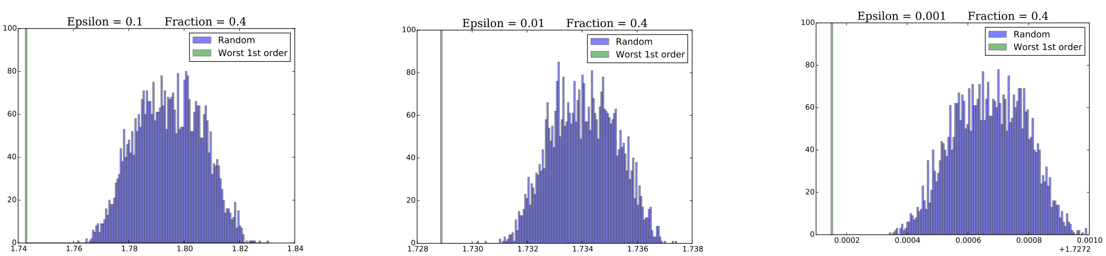


Figure 2.3: Histograms for $m/|\Omega| = 0.4$ and $\varepsilon = 10^{-1}$ (left), 10^{-2} (center) and 10^{-3} (right).

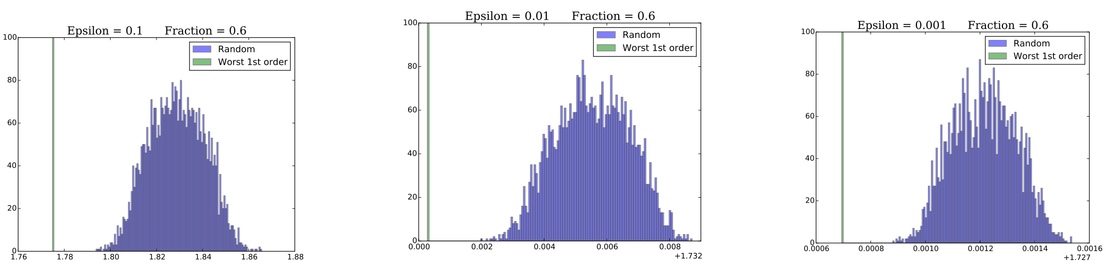


Figure 2.4: Histograms for $m/|\Omega| = 0.6$ and $\varepsilon = 10^{-1}$ (left), 10^{-2} (center) and 10^{-3} (right).

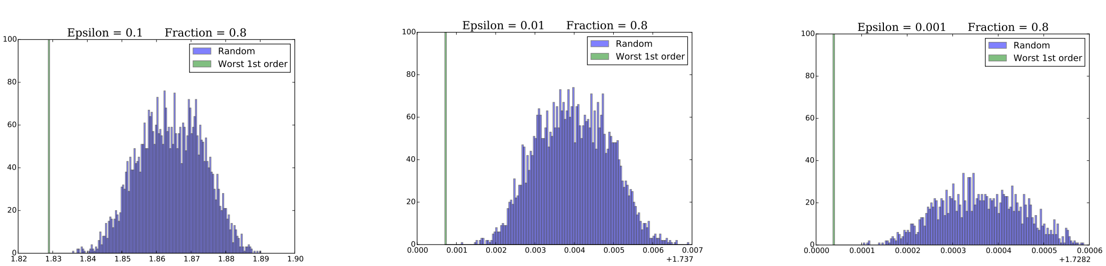


Figure 2.5: Histograms for $m/|\Omega| = 0.8$ and $\varepsilon = 10^{-1}$ (left), 10^{-2} (center) and 10^{-3} (right).

(and of course their number increases) when N increases. This is heuristically due to the higly oscillating functions φ_k that appeared in the definition of the random field f. In order to explore the influence of this observation on our quantity of interest, let us focus on a special case: the contrast $\varepsilon = 0.5$, the volume fraction $m/|\Omega| = 0.4$ and we make vary the number of used modes. The obtained histograms are presented in Figure 2.6.

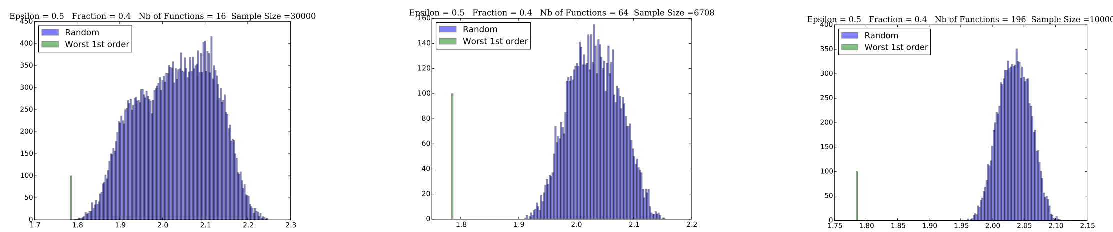


Figure 2.6: Histograms for $N = 2$ (left), 4 (center) and 7 (right).

We observe that the variance of the distribution of the eigenvalues seems to decrease as N increases. Since we are not able to derive any clear probabilistic description of the randoms inclusions (in term of laws for example), we are far from being able to justify the observed behavior. Nevertheless, an heuristic explanation is that with typical inclusions being smaller and smaller, one should observe a passage to the limit in some sort of homogenization limiting the range of possible "material" and hence of the eigenvalues. In the converse, with small N, the two materials do not really mix and drastic choices are made.

2.4 Minimization of the second order approximation of the ground state

In order to be more precise, we go further and do a second order approximation. Indeed, Proposition 2.2 allows us to conclude that

$$\left|\inf_B \lambda_\varepsilon(B) - \inf_B(\lambda_0 + \varepsilon\lambda_1(B) + \varepsilon^2\lambda_2(B))\right| \leq C\varepsilon^3 . \tag{2.34}$$

Thus, we can obtain approximate minimizers for the functional λ_ε, for given $\varepsilon > 0$ small enough, by minimizing the functional $\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2$ which is a second order approximation of λ_ε. We then study the problem:

$$\text{minimize } \{\lambda_0 + \varepsilon\lambda_1(B) + \varepsilon^2\lambda_2(B) \ ; \ B \subseteq \Omega, \, |B| = m\}, \quad 0 < m < |\Omega|, \ m \text{ fixed}$$

or equivalently

$$\text{minimize } \{\lambda_1(B) + \varepsilon\lambda_2(B) \ ; \ B \subseteq \Omega, \, |B| = m\},$$

since λ_0 is independent of B and $\varepsilon > 0$ is fixed. From the expressions for $\lambda_1(B), \lambda_2(B)$ computed in the previous section, we finally consider the problem

$$\text{minimize } F(\chi) := \alpha \int_\Omega \chi(\nabla u_0 + \varepsilon \nabla v(\chi)) \cdot \nabla u_0$$

over the class of admissible domains represented by their characteristic functions

$$\mathcal{U}_{ad} := \{\chi \ ; \ \chi = \chi_B, \, B \subseteq \Omega, \, |B| = m\} \subseteq \mathrm{L}^\infty(\Omega),$$

and $v = v(\chi) \in \mathrm{H}_0^1(\Omega)$ satisfisfies

$$-\alpha\Delta v - \lambda_0 v = \lambda_1(\chi)u_0 + \mathrm{div}\,(\alpha\chi\nabla u_0), \tag{2.35}$$
$$\lambda_1(\chi) := \int_\Omega \alpha\chi|\nabla u_0|^2, \tag{2.36}$$
$$v \perp u_0 \text{ in } \mathrm{L}^2(\Omega).$$

2.4.1 Relaxation of the minimization problem

The functional F is lower-semicontinuous for the weak-$*$ topology on $L^\infty(\Omega)$, being quadratic with respect to χ, but the admissible set $\mathcal{U}_{ad}$ is not closed for this topology. In order to have a well-posed minimization problem we need to work on the closure $\overline{\mathcal{U}_{ad}}$ and calculate the *lower semicontinuous envelope* of F with respect to the weak-$*$ topology on $L^\infty(\Omega)$.

$$\bar{F}(\theta) := \inf\{\liminf F(\chi_n) \ : \ \chi_n \rightharpoonup \theta \text{ in } \mathrm{L}^\infty(\Omega)^*\},\ \theta \in \overline{\mathcal{U}_{ad}},$$

where

$$\overline{\mathcal{U}_{ad}} = \overline{\mathcal{U}_{ad}}^{L^\infty(\Omega)^*} = \{\theta \in L^\infty(\Omega)\ ;\ 0 \le \theta \le 1,\ \int_\Omega \theta = m\}.$$

We shall follow the general procedure to compute $\bar{F}$ and obtain the following theorem.

Theorem 2.2 *For any $\theta \in \overline{\mathcal{U}_{ad}}$, we have*

$$\bar{F}(\theta) = \alpha\int_\Omega \theta\left[\nabla u_0 + \varepsilon\nabla v_\infty(\theta)\right].\nabla u_0 - \varepsilon\theta(1-\theta)|\nabla u_0|^2,$$

where $v_\infty(\theta) \in \mathrm{H}_0^1(\Omega)$ is solution of

$$-\alpha\Delta v - \lambda_0 v = \lambda_1(\theta)u_0 + \mathrm{div}\,(\alpha\theta\nabla u_0), \tag{2.37}$$
$$\lambda_1(\theta) := \int_\Omega \alpha\theta|\nabla u_0|^2, \tag{2.38}$$
$$v \perp u_0 \ \textit{in } \mathrm{L}^2(\Omega).$$

The proof of the Theorem 2.2 will use some results on H-measures. This tool was introduced by P. Gérard in [17] and L. Tartar in [24] to understand the obstruction to compactness via a matrix of complex-valued Radon measures $(\mu_{ij}(x,\xi))_{1\le i,j\le p}$ on $\mathbb{R}^N \times \mathbb{S}^{N-1}$ on the space-frequency domain associated to weakly convergent sequences. We refer to the two previous references for a complete presentation of H-measures and to [3] for their applications in small contrast homogenization. We will need the two following results (Theorem 2-2 and Lemma 2-3 in [3]).

Theorem 2.3 [3] *Let u_ε be a sequence which weakly converges to 0 in $\mathrm{L}^2(\mathbb{R}^N)^p$. There exists a subsequence and a H-measure μ such that*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} q(u_\varepsilon).\bar{u}_\varepsilon = \int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}\sum_{i,j=1}^p q_{ij}(x,\xi)\mu_{ij}(\mathrm{d}x,\mathrm{d}\xi)$$

for any polyhomogeneous pseudo-differential operator q of degree 0 with symbol $(q_{ij}(x,\xi))$.

We shall also use the following lemma due to Kohn and Tartar that deals with the special case of sequences of characteristic functions.

Lemma 2.3 [3] *Let χ_ε be a sequence of characteristic functions that weakly-$*$ converges to some θ in $\mathrm{L}^\infty(\Omega,[0,1])$. Then the corresponding H-measure μ for the sequence $(\chi_\varepsilon-\theta)$ is necessarily of the type*

$$\mu(dx,d\xi)=\theta(x)(1-\theta(x))\nu(\mathrm{d}x,\mathrm{d}\xi),$$

where, for a given x, the measure $\nu(dx,d\xi)$ is a probability measure with respect to ξ.

Conversely, for any such probability measure $\nu\in\mathcal{P}(\Omega,\mathbb{S}^{N-1})$, there exists a sequence χ_ε of characteristic functions which weakly-$*$ converges to $\theta\in\mathrm{L}^\infty(\Omega,[0,1])$ such that $\theta(1-\theta)\nu$ is the H-measure of $(\chi_\varepsilon-\theta)$.

Proof of Theorem 2.2: Let $\theta\in\overline{\mathcal{U}_{ad}}$. Let $\{\chi_n\}$ be a sequence in $\mathcal{U}_{ad}$ such that

$$\chi_n\overset{\star}{\rightharpoonup}\theta\in\overline{\mathcal{U}_{ad}}. \tag{2.39}$$

We then analyze the limit of

$$F(\chi_n)=\underbrace{\alpha\int_\Omega\chi_n|\nabla u_0|^2}_{A_n}+\underbrace{\alpha\,\varepsilon\int_\Omega\chi_n\nabla v_n\cdot\nabla u_0}_{B_n},$$

with $v_n:=v(\chi_n)\in\mathrm{H}^1_0(\Omega)$ such that

$$-\alpha\Delta v_n-\lambda_0 v_n=\lambda_1(\chi_n)u_0+\operatorname{div}\,(\alpha\chi_n\nabla u_0), \tag{2.40}$$

$$\lambda_1(\chi_n)=\int_\Omega\alpha\chi_n|\nabla u_0|^2,$$

$$v_n\perp u_0 \text{ in } \mathrm{L}^2(\Omega). \tag{2.41}$$

STEP 1: Passing to the limit in A_n is easy. By the convergence (2.39), we have

$$A_n=\lambda_1(\chi_n)\longrightarrow\alpha\int_\Omega\theta\,|\nabla u_0|^2=\lambda_1(\theta). \tag{2.42}$$

STEP 2: Now we study the limit of the sequence v_n. By (2.41), we know that

$$\left(1-\frac{\lambda_0}{\lambda^1}\right)\int_\Omega|\nabla v_n|^2\le C,$$

using a similar estimation as (2.18). Then $\|v_n\|^2_{\mathrm{H}^1_0}\le C$ and hence,

$$v_n\rightharpoonup v_\infty=v_\infty(\theta)\quad\text{weak-}\mathrm{H}^1_0(\Omega)$$

up to a subsequence. Since $\mathrm{H}^1_0(\Omega)$ is compactly embedded in $\mathrm{L}^2(\Omega)$,

$$v_n\longrightarrow v_\infty\quad\text{in } \mathrm{L}^2(\Omega)$$

up to a subsequence. Therefore, we can pass to variational limit from (2.40) to obtain,

$$-\alpha\Delta v_\infty-\lambda_0 v_\infty=\lambda_1(\theta)u_0+\operatorname{div}\,(\alpha\,\theta\nabla u_0). \tag{2.43}$$

Moreover, passing to the limit from (2.41), we have

$$v_\infty \perp u_0 \text{ in } \mathrm{L}^2(\Omega),$$

accordingly, since $\|u_0\|_{\mathrm{L}^2} = 1$, $v_\infty = v_\infty(\theta)$ is uniquely defined in (2.43) and v_∞ depends (linearly) only on θ and not on the convergent subsequence of $\{v_n\}$.

Step 3: The main difficulty is to pass to the limit in B_n which is quadratic with respect to χ_n. First, we can rewrite B_n as

$$B_n = \int_\Omega \chi_n \nabla w_n \cdot \nabla u_0 + \int_\Omega \chi_n \nabla z_n \cdot \nabla u_0, \tag{2.44}$$

$w_n, z_n \in \mathrm{H}^1_0(\Omega)$ such that

$$-\alpha \Delta w_n = \lambda_0 v_n + \lambda_1(\chi_n) u_0, \tag{2.45}$$

$$-\Delta z_n = \mathrm{div}\ (\chi_n \nabla u_0). \tag{2.46}$$

On the one hand, since

$$\lambda_0 v_n + \lambda_1(\chi_n) u_0 \longrightarrow \lambda_0 v_\infty + \lambda_1(\theta) u_0 \quad \text{in } \mathrm{L}^2(\Omega),$$

(2.45) implies

$$w_n \longrightarrow w \quad \text{in } \mathrm{H}^1_0(\Omega),$$

where $w \in \mathrm{H}^1_0(\Omega)$ satisfies the equation

$$-\alpha \Delta w = \lambda_0 v_\infty + \lambda_1(\theta) u_0$$

and, in consequence,

$$\int_\Omega \chi_n \nabla w_n \cdot \nabla u_0 \longrightarrow \int_\Omega \theta\, \nabla w \cdot \nabla u_0. \tag{2.47}$$

The difficulty is now to calculate the limit in the second term of B_n in (2.44). We observe that div $\chi_n \nabla u_0 \rightharpoonup$ div $\theta \nabla u_0$ weakly in $\mathrm{H}^{-1}(\Omega)$ and since $(-\Delta)^{-1}$ is a isomorphism from $\mathrm{H}^{-1}(\Omega)$ into $\mathrm{H}^1_0(\Omega)$, we get L^2-weak convergence of ∇z_n. However, this is not enough for passing to the limit in the second term of B_n because, in the product $\chi_n \nabla z_n$, both sequences χ_n and ∇z_n only converge weakly. For handling this convergence problem we use the results on H-convergence stated before. To this end, we calculate heuristically the associated symbol q of the pseudo-differential operator Q as the gradient of the solution of (2.46) with θ replaced by χ_n in whole space $\mathbb{R}^N$. As a result of this, let us state the following lemma

Lemma 2.4 *Let $\bar{u}_0 \in H^1(\mathbb{R}^N)$ be the usual extension of u_0 by zero in the whole space $\mathbb{R}^N$. For $\theta \in L^\infty(\mathbb{R}^N)$, if $z \in H^1(\mathbb{R}^N)$ is such that $\lim_{|x|\to\infty} z(x) = 0$ and verifies*

$$-\Delta z = \mathrm{div}\ (\theta \nabla \bar{u}_0) \quad \text{in } \mathbb{R}^N, \tag{2.48}$$

then $\theta \mapsto Q(\theta) := \nabla z$ defines a linear pseudo-differential operator with symbol

$$q(x,\xi) = -\frac{\xi \cdot \nabla \bar{u}_0(x)}{|\xi|^2}\, \xi. \tag{2.49}$$

Note that q is homogenous of degree 0 in ξ.

Heuristic calculation of q :

Denote by $\widehat{\ }$ the Fourier transform and assume that the variable x is fixed in $\nabla\bar{u}_0(x)$. Then, using Fourier calculus and starting from Equation (2.46), formally we can calculate as follows

$$\begin{aligned}(-\Delta z)\widehat{\ }(\xi) &= (\operatorname{div}\,(\theta\nabla\bar{u}_0(x)))\widehat{\ }(\xi)\\ -(-|\xi|^2\widehat{z}) &= -i\xi\cdot\nabla\bar{u}_0(x)\widehat{\theta}\\ \widehat{z} &= -\frac{i\xi\cdot\nabla\bar{u}_0(x)}{|\xi|^2}\,\widehat{\theta},\end{aligned}$$

which gives

$$\widehat{\nabla z}(\xi) = -i\xi\widehat{z}(\xi) = -i\xi\left(-\frac{i\xi\cdot\nabla\bar{u}_0(x)}{|\xi|^2}\,\widehat{\theta}\right) = \underbrace{\left(-\frac{\xi\cdot\nabla\bar{u}_0(x)}{|\xi|^2}\,\xi\right)}_{q(x,\xi)}\widehat{\theta}.$$

For a more rigorous obtention of q we refer the reader to [24], Sect. 4.2, or [17], Thm. 1.
Step 4: For simplicity if Ω is $\mathbb{R}^N$, in view of Theorem 2.3 and Lemma 2.3, the limit of the second term in (2.44) can be written as

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N}\chi_n\nabla z_n\cdot\nabla\bar{u}_0 &= \lim_{n\to\infty}\int_{\mathbb{R}^N}(\chi_n-\theta+\theta)\,Q(\chi_n-\theta+\theta)\cdot\nabla\bar{u}_0\\ &= \int_{\mathbb{R}^N}\theta\,Q(\theta)\cdot\nabla\bar{u}_0 + \lim_{n\to\infty}\int_{\mathbb{R}^N}(\chi_n-\theta)\,Q(\chi_n-\theta)\cdot\nabla\bar{u}_0\\ &= \int_{\mathbb{R}^N}\theta\,Q(\theta)\cdot\nabla\bar{u}_0 + \int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}q(x,\xi)\cdot\nabla\bar{u}_0\,\theta(1-\theta)\,\nu(\mathrm{d}x,\mathrm{d}\xi)\\ &= \int_{\mathbb{R}^N}\theta\,Q(\theta)\cdot\nabla\bar{u}_0 - \int_{\mathbb{R}^N}\theta(1-\theta)M\nabla\bar{u}_0\cdot\nabla\bar{u}_0,\end{aligned}$$

where the pseudo-differential operator Q is defined in Lemma 2.4 and its symbol q has been calculated therein and,

$$M = \int_{\mathbb{S}^{N-1}}\xi\otimes\xi\,\nu(x,\mathrm{d}\xi),$$

$\nu = \nu(x,\xi)$ is a probability measure with respect to ξ that depends on the sequence $\{\chi_n\}$ and $Q(\theta) = \nabla z$ with $z\in\mathrm{H}^1(\Omega)$ verifies the equation

$$-\Delta z = \operatorname{div}\,(\theta\,\nabla\bar{u}_0),\qquad \text{in } \mathbb{R}^N.$$

Step 5: But we need to work on Ω bounded. To that end, we use a localization procedure. This argument proceeds as follows. Let (ζ_k) be a sequence of smooth compactly supported functions in $C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\zeta_k\subset\Omega$ for all k and ζ_k converges to 1 strongly in $\mathrm{L}^2(\Omega)$. Then the second term on the right hand side of (2.44) can be written as

$$\int_\Omega\chi_n\nabla z_n\cdot\nabla u_0 = \int_{\mathbb{R}^N}\zeta_k\chi_n\nabla z_n\cdot\nabla u_0 + \int_{\mathbb{R}^N}(1-\zeta_k)\chi_n\nabla z_n\cdot\nabla u_0. \tag{2.50}$$

Note that the last term in (2.50) converges to 0 uniformly with respect to n when k tends to infinity because z_n is bounded in $\mathrm{H}^1(\Omega)$. We now fix k and consider another smooth compactly supported function $\psi_k \in C_0^\infty$ such that $\psi_k \equiv 1$ inside the support of ζ_k. The first term on the right hand side of (2.50) is thus equal to

$$\int_\Omega \zeta_k(\psi_k\chi_n)\nabla(\psi_k z_n)\cdot\nabla u_0. \tag{2.51}$$

Rewriting the equation (2.46) in $\mathbb{R}^N$ as

$$-\Delta(\psi_k z_n) - \Delta((1-\psi_k)z_n) = \text{div}\ (\psi_k\chi_n\nabla u_0) + \text{div}\ ((1-\psi_k)\chi_n\nabla u_0),$$

we can show that the function $\psi_k z_n$ is the sum of $\tilde{z}_n, \check{z}_n$ on the support of ζ_k being $\tilde{z}_n, \check{z}_n$ solutions of the following equations in the whole space $\mathbb{R}^N$

$$\begin{aligned} -\Delta\tilde{z}_n &= \text{div}\ \psi_k\chi_n\nabla u_0 \quad \text{in } \mathbb{R}^N, \\ \Delta\check{z}_n &= \text{div}\ z_n\nabla\psi_k + \nabla\psi_k\cdot(\chi_n\nabla u_0 + \nabla z_n) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

We then notice that

$$\text{div}\ \psi_k\chi_n\nabla u_0 \rightharpoonup \text{div}\ \psi_k\theta\nabla u_0 \quad \text{weakly in } \mathrm{H}^{-1}(\mathbb{R}^N) \text{ and}$$

$$\text{div}\ z_n\nabla\psi_k + \nabla\psi_k\cdot(\chi_n\nabla u_0+\nabla z_n) \to \text{div}\ z\nabla\psi_k + \nabla\psi_k\cdot(\theta\nabla u_0 + \nabla z) \quad \text{strongly in } \mathrm{H}^{-1}(\mathbb{R}^N)$$

since this last term clearly converges weak-$\mathrm{L}^2(\Omega)$. Using the fact that $(-\Delta)^{-1}$ is an isomorphism from $\mathrm{H}^{-1}(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N)$, we thus have

$$\tilde{z}_n \rightharpoonup \tilde{z} \quad \text{weakly in } H^1(\mathbb{R}^N)$$

and

$$\check{z}_n \to \check{z} \quad \text{strongly in } H^1(\mathbb{R}^N)$$

where $\tilde{z}, \check{z}$ verify

$$\begin{aligned} -\Delta\tilde{z} &= \text{div}\ \psi_k\theta\nabla u_0 \quad \text{in } \mathbb{R}^N, \\ \Delta\check{z} &= \text{div}\ z\nabla\psi_k + \nabla\psi_k\cdot(\theta\nabla u_0+\nabla z) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

Obviously $z = \tilde{z} + \check{z}$ on the support of ζ_k.

Now noting that the integral (2.51) has close relationship with the formulation of the H-measures, we see that, as in the whole space case, $\nabla\tilde{z}_n$ depends linearly on $(\psi_k\chi_n)$ through the pseudo-differential operator Q of symbol (2.49). Therefore applying Theorem 2 of [17], we conclude that the limit of the first term on the right hand side of (2.50) is equal to

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N}\zeta_k(\psi_k\chi_n)\nabla(\check{z}_n+\tilde{z}_n)\cdot\nabla u_0 &= \int_{\mathbb{R}^N}\zeta_k(\psi_k\theta)\nabla\check{z}\cdot\nabla u_0 \\ &\quad + \lim_{n\to\infty}\int_{\mathbb{R}^N}\zeta_k(\psi_k\chi_n)\nabla\tilde{z}_n\cdot\nabla u_0 \\ = \int_{\mathbb{R}^N}\zeta_k(\psi_k\theta)\nabla\check{z}\cdot\nabla u_0 + \int_{\mathbb{R}^N}\zeta_k(\psi_k\theta)\nabla\tilde{z}\cdot\nabla u_0 &- \int_{\mathbb{R}^N}\zeta_k\psi_k\theta(1-\theta)M\nabla u_0\cdot\nabla u_0 \\ &= \int_\Omega\zeta_k\theta\nabla z\cdot\nabla u_0 - \int_\Omega\zeta_k\theta(1-\theta)M\nabla u_0\cdot\nabla u_0. \end{aligned}$$

Finally, making k tends to ∞, we obtain the desired bounded domain case.

We go back to the calculation of the limit in (2.44). Indeed, gathering the limit (2.47) and limit calculated above, it follows that

$$\lim_{n\to\infty} B_n = \int_\Omega \theta\,\nabla v_\infty(\theta)\cdot\nabla u_0 - \int_\Omega \theta(1-\theta)\int_{\mathbb{S}^{N-1}} (\xi\cdot\nabla u_0)^2\,\nu(\mathrm{d}x,\mathrm{d}\xi). \tag{2.52}$$

From (2.42) and (2.52), finally one has

$$\begin{aligned}\lim_{n\to\infty} F(\chi_n) &= \lim_{n\to\infty} A_n + \varepsilon \lim_{n\to\infty} B_n \\ &= \frac{1}{\alpha}\,\lambda_1(\theta) + \varepsilon\int_\Omega \theta\,\nabla v_\infty(\theta)\cdot\nabla u_0 - \varepsilon\int_\Omega \theta(1-\theta)\int_{\mathbb{S}^{N-1}} (\xi\cdot\nabla u_0)^2\,\nu(\mathrm{d}x,\mathrm{d}\xi).\end{aligned}$$

Step 6: Now we calculate

$$\bar F(\theta) = \inf_\nu \lim F(\chi_n).$$

To that end, we notice that

$$\int_{\mathbb{S}^{N-1}} (\xi\cdot\nabla u_0)^2\,\nu(\mathrm{d}x,\mathrm{d}\xi) \le |\nabla u_0|^2(x) \quad a.e.\ x\in\Omega,$$

since ν is a probability measure with respect to ξ *a.e.* $x\in\Omega$. Moreover, this value is reached when we take the Dirac measure $\delta_{\nabla u_0(x)}$, i.e., when

$$\nu(x,\xi) = \delta_{\xi_x}\mathrm{d}x, \quad \xi_x = \nabla u_0(x)/|\nabla u_0(x)|.$$

From the converse part of Lemma 2.3 in [3], the minimum for

$$\inf_\nu \lim F(\chi_n)$$

is also achieved. So, finally we can conclude

$$\bar F(\theta) = \int_\Omega \theta\,(\nabla u_0 + \varepsilon\nabla v_\infty(\theta))\cdot\nabla u_0 - \varepsilon\int_\Omega \theta\,(1-\theta)\,|\nabla u_0|^2. \tag{2.53}$$

Recall that $v_\infty = v_\infty(\theta)$ depends linearly on θ. ∎

2.4.2 Optimality conditions for the relaxed problem.

The relaxed functional $\bar F$ achieves its minimum of $\overline{\mathcal{U}_{ad}}$ since it is lower-semicontinuous and the constraint set is compact for the weak-$*$ topology. We first investigate the differentiability properties of $\bar F$ in order to obtain optimality conditions for a minimizer of $\bar F$ on the compact convex set $\overline{\mathcal{U}_{ad}}$.

Proposition 2.3 *The functional $\bar F$ is Fréchet differentiable of every order and we have the following expressions for the Gateaux derivatives of first and second order*

$$\bar F'(\theta)\varphi = \int_\Omega \Big[2\varepsilon(\nabla v_\infty(\theta) + \theta\nabla u_0) + (1-\varepsilon)\nabla u_0\Big]\cdot\nabla u_0\,\varphi. \tag{2.54}$$

and

$$\bar F''(\theta)(\varphi,\varphi) = 2\varepsilon\int_\Omega (\nabla v_\infty(\varphi) + \varphi\nabla u_0)\cdot\nabla u_0\,\varphi. \tag{2.55}$$

Proof: The linearity of the application $\theta \mapsto v_\infty(\theta)$ and the expression for $\bar{F}$ show clearly that it is quadratic with respect to θ. So, the Fréchet derivatives exist. In order to calculate the first order derivative, we rewrite (2.53) as

$$\bar{F}(\theta) = \varepsilon \int_\Omega \theta \nabla v_\infty(\theta) \cdot \nabla u_0 + \varepsilon \int_\Omega \theta^2 |\nabla u_0|^2 + (1-\varepsilon) \int_\Omega \theta |\nabla u_0|^2.$$

But, using $v_\infty(\theta)$ as test function in (2.43), we get

$$\bar{F}(\theta) = -\varepsilon \int_\Omega |\nabla v_\infty(\theta)|^2 - \frac{\lambda_0}{\alpha} v_\infty^2(\theta) + \varepsilon \int_\Omega \theta^2 |\nabla u_0|^2 + (1-\varepsilon) \int_\Omega \theta |\nabla u_0|^2.$$

A simple calculation gives us

$$\bar{F}'(\theta)\varphi = -2\varepsilon \int_\Omega \nabla v_\infty(\theta) \cdot \nabla v_\infty(\varphi) - \frac{\lambda_0}{\alpha} v_\infty(\theta) v_\infty(\varphi) + 2\varepsilon \int_\Omega \theta |\nabla u_0|^2 \varphi + (1-\varepsilon) \int_\Omega |\nabla u_0|^2 \varphi.$$

We now notice that $v_\infty(\varphi)$ satisfies (2.43). Then, again taking $v_\infty(\theta)$ as test function, we can explicitly write the above expression in terms of φ to obtain (2.54) then (2.55). ∎

We wish to investigate the critical points for the constrained minimization problem of minimizing $\bar{F}$ over $\overline{\mathcal{U}_{ad}}$. To that end, we use the Lagrange's multipliers method with the constraint

$$C(\theta) := \int_\Omega \theta = m, \quad \theta \in \overline{\mathcal{U}_{ad}} \text{ hence } C'(\theta)\varphi = \int_\Omega \varphi.$$

Therefore, the critical points satisfy the Euler-Lagrange equation: for all admissible φ

$$[\bar{F}'(\theta) + \Lambda C'(\theta)]\varphi = 0$$

for some $\Lambda \in \mathbb{R}$; i.e.

$$\int_\Omega \Big[2\varepsilon(\nabla v_\infty(\theta) + \theta \nabla u_0) + (1-\varepsilon)\nabla u_0 \Big] \cdot \nabla u_0 \, \varphi + \Lambda \int_\Omega \varphi = 0 \quad \forall \varphi.$$

Consequently the density of $\overline{\mathcal{U}_{ad}}$ in $\mathrm{L}^2(\Omega)$ implies

$$2\varepsilon \nabla v_\infty(\theta) \cdot \nabla u_0 + (2\varepsilon\theta + 1 - \varepsilon)|\nabla u_0|^2 = \Lambda \quad \text{on } \Omega.$$

Proposition 2.4 *If θ^* is optimal in the relaxed formulation, then there is real Λ such that:*

$$2\varepsilon \nabla v_\infty(\theta) \cdot \nabla u_0 + (2\varepsilon\theta + 1 - \varepsilon)|\nabla u_0|^2 = \Lambda \quad \textit{in } \Omega.$$

Integrating over Ω and considering u_0 as test function in (2.43), we get the following consequence

$$\int_\Omega \nabla v_\infty(\theta) \cdot \nabla u_0 = 0 \text{ and } 2\varepsilon \int_\Omega \theta |\nabla u_0|^2 + \frac{1-\varepsilon}{\alpha} \lambda_0 = \Lambda |\Omega|.$$

2.5 Maximization of the ground state by the first and second order approximation in the low contrast regime

This section was intended as an attempt to complement the minimization of the first eigenvalue $\lambda^\varepsilon(B)$ of (2.1) in order to know whether the minimization by asymptiotic approximation has some validity in some sense to be specified. Actually, we are first interested in investigating the range of variation of the first eigenvalue in the respecive first and second order approximation. That is to say, we want to find the interval of fluctuation of $\lambda^\varepsilon(B)$ in those orders of approximation. But this holds if we only maximize $\lambda^\varepsilon(B)$.

The procedure is then to perform the maximization process for the ground state in the first and second asymptotic order cases.

2.5.1 Maximization by the first order approximation

We know that $\lambda_\varepsilon(B) = \lambda_0 + \varepsilon\lambda_1(B) + \mathcal{O}(\varepsilon^2)$ where

$$\lambda_0 = \int_\Omega \alpha|\nabla u_0|^2,$$

(λ_0, u_0) is eigenpair of $-\alpha\Delta$ in $\mathrm{H}^1_0(\Omega)$ with λ_0 the first eigenvalue and $\|u_0\|_{\mathrm{L}^2(\Omega)} = 1$. Additionally

$$\lambda_1(B) = \int_B \alpha|\nabla u_0|^2 \le \lambda_0.$$

Furthermore, for $\varepsilon < 1$,

$$\lambda_\varepsilon(B) = \inf_{\substack{u\in\mathrm{H}^1_0(\Omega)\\ \|u\|_{\mathrm{L}^2(\Omega)}=1}} \int_\Omega \alpha(1+\varepsilon\chi_B)|\nabla u|^2 \le 2 \inf_{\substack{u\in\mathrm{H}^1_0(\Omega)\\ \|u\|_{\mathrm{L}^2(\Omega)}=1}} \int_\Omega \alpha|\nabla u|^2 = 2\lambda_0.$$

Since $|\lambda_\varepsilon(B) - (\lambda_0 + \varepsilon\lambda_1(B))| \le C\varepsilon^2$, we get

$$\left|\sup_{|B|=m} \lambda_\varepsilon(B) - \lambda_0 - \varepsilon \sup_{|B|=m} \lambda_1(B)\right| \le C\varepsilon^2.$$

Theorem 2.4 *There exists $c^* \ge 0$ such that whenever B is a measurable subset of Ω satisfying*

$$\{x : |\nabla u_0(x)|^2 > c^*\} \subseteq B \subseteq \{x : |\nabla u_0(x)|^2 \ge c^*\}$$

and $|B| = m$, then B is an optimal solution for the problem of maximizing $\lambda_1(B)$.

Proof: Set $f(c) := |\{x \in \Omega : |\nabla u_0(x)|^2 \ge c\}|$. Then f is decreasing function and $0 \le f(c) \le |\Omega|$. Set $c^* := \sup\{c : f(c) \ge m\}$. Notice that, on one hand, $f(c^*) \ge m$ and, on the other hand, $|\{x : |\nabla u_0(x)|^2 > c^*\}| \le m$. Indeed, let $c_k < c^*$ be a increasing sequence such that $c_k \nearrow c^*$. As $f(c_k) \ge m$, $\lim_k f(c_k) \ge m$. But

$$\begin{aligned}\lim_k f(c_k) &= \lim_k |\{x \in \Omega : |\nabla u_0(x)|^2 \ge c_k\}| \\ &= |\cap_k \{x \in \Omega : |\nabla u_0(x)|^2 \ge c_k\}| \\ &= |\{x \in \Omega : |\nabla u_0(x)|^2 \ge c^*\}| = f(c^*).\end{aligned}$$

Taking now $c_k \searrow c^*$ $(c_k > c^*)$, we have $f(c_k) < m$, so that $\lim_k f(c_k) \leq m$. But

$$\begin{aligned}\lim_k f(c_k) &= \lim_k |\{x \in \Omega : |\nabla u_0(x)|^2 \geq c_k\}| \\ &= |\cup_k \{x \in \Omega : |\nabla u_0(x)|^2 \geq c_k\}| \\ &= |\{x \in \Omega : |\nabla u_0(x)|^2 > c^*\}|.\end{aligned}$$

So, this allows to consider the subsets $B \subseteq \Omega$ with the property $|B| = m$ and $\{x : |\nabla u_0(x)|^2 > c^*\} \subseteq B \subseteq \{x : |\nabla u_0(x)|^2 \geq c^*\}$ in order to prove that these sets are optimal solutions of maximization of $\lambda_1(B)$. In fact, if $D \subset\subset \Omega$ with $|D| = m$, then

$$\begin{aligned}\int_D |\nabla u_0|^2 &= \int_{D\cap B} |\nabla u_0|^2 + \int_{D\setminus B} |\nabla u_0|^2 \\ &\leq \int_{D\cap B} |\nabla u_0|^2 + \int_{D\setminus B} c^* \\ &= \int_{D\cap B} |\nabla u_0|^2 + \int_{B\setminus D} c^* \\ &\leq \int_{D\cap B} |\nabla u_0|^2 + \int_{B\setminus D} |\nabla u_0|^2 \\ &= \int_B |\nabla u_0|^2,\end{aligned}$$

so that $\lambda_1(D) \leq \lambda_1(B)$. ■

Theorem 2.4 provide us of a similar algorithm to that made in [8]. This will be useful numerically to compare with the second order approach.

2.5.2 Maximization by the second order approximation

In this part, we know that $\lambda^\varepsilon(B) = \lambda_0 + \varepsilon\lambda_1(B) + \varepsilon^2\lambda_2(B) + \mathcal{O}(\varepsilon^3)$, where

$$\lambda_2(B) = \int_\Omega \alpha\chi_B \nabla u_0 \cdot \nabla u_1(B)$$

and $u_1(B)$ is the solution in $\mathrm{H}_0^1(\Omega)$ of the system

$$\begin{gathered}-\alpha\Delta u_1 - \lambda_0 u_1 = \lambda_1 u_0 + \operatorname{div} \alpha\chi_B \nabla u_0, \\ \int_\Omega u_1 u_0 = 0.\end{gathered}$$

By (2.18), we obtain

$$|\lambda_2(B)| \leq \int_\Omega \alpha|\nabla u_0|\, |\nabla u_1(B)| \leq \sqrt{\alpha\lambda_0}\, \|u_1(B)\|_{\mathrm{H}_0^1(\Omega)} \leq C\sqrt{\alpha\lambda_0},$$

where C is a constant independent of B.

Since $|\lambda_\varepsilon(B) - (\lambda_0 + \varepsilon\lambda_1(B) + \varepsilon^2\lambda_2(B))| \leq C\varepsilon^3$, it holds

$$\left| \sup_{|B|=m} \lambda_\varepsilon(B) - \lambda_0 - \varepsilon\alpha \sup_{|B|=m} \int_\Omega \chi_B \nabla u_0 \cdot (\nabla u_0 + \varepsilon\nabla u_1(B)) \right| \leq C\varepsilon^3.$$

In order to maximize $\int_\Omega \chi_B \nabla u_0 \cdot (\nabla u_0 + \varepsilon \nabla u_1(B))$ we consider the relaxation procedure as follows: we define the general problem

$$\sup_{\chi \in \mathcal{U}_{\rm ad}} \left[F(\chi) := \int_\Omega \chi \nabla u_0 \cdot (\nabla u_0 + \varepsilon \nabla v(\chi)) \right],$$

where $v = v(\chi)$ is the solution in $\mathrm{H}^1_0(\Omega)$ of the system

$$\begin{aligned} -\alpha \Delta v - \lambda_0 v &= \lambda_1 u_0 + \operatorname{div} \alpha \chi \nabla u_0, \\ \lambda_1 = \lambda_1(\chi) &= \int_\Omega \alpha \chi |\nabla u_0|^2, \\ \int_\Omega v u_0 &= 0. \end{aligned}$$

Here $\mathcal{U}_{\rm ad} = \{\chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) : \int_\Omega \chi = m\}$.

Since

$$\sup_\chi F(\chi) = - \inf_\chi (-F(\chi)),$$

it is sufficient to study

$$\inf_{\chi \in \mathcal{U}_{\rm ad}} \left[G(\chi) := - \int_\Omega \chi \nabla u_0 \cdot (\nabla u_0 + \varepsilon \nabla v(\chi)) \right].$$

To that end, we introduce a usual relaxation. In this direction is useful to consider the weak-* topology of $\mathrm{L}^\infty(\Omega)$ and so we pass to calculate

$$\overline{G}(\theta) = \inf\{\liminf G(\chi_k) : \exists \chi_k \overset{*}{\rightharpoonup} \theta\}$$

with $\theta \in \{\varphi \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \varphi = m\}$.

Proposition 2.5 *For any $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$ such that $\int_\Omega \theta = m$, it holds*

$$\overline{G}(\theta) = - \int_\Omega \theta \, \nabla u_0 \cdot (\nabla u_0 + \varepsilon \nabla v(\theta)),$$

$v = v(\theta) \in \mathrm{H}^1_0(\Omega)$ solution of

$$\begin{aligned} -\alpha \Delta v - \lambda_0 v &= \lambda_1 u_0 + \operatorname{div} \alpha \theta \nabla u_0, \\ \lambda_1 = \lambda_1(\theta) &= \int_\Omega \alpha \theta |\nabla u_0|^2, \\ \int_\Omega v u_0 &= 0. \end{aligned} \tag{2.56}$$

Proof: Following the same ideas made in section 2.4, we can assert that

$$\overline{G}(\theta) = \inf_\nu \varepsilon \int_\Omega \theta(1-\theta) \int_{\mathbb{S}^{N-1}} |\xi \cdot \nabla u_0|^2 \nu(x,\xi) - \int_\Omega \theta \, \nabla u_0 \cdot (\nabla u_0 + \varepsilon \nabla v(\theta)),$$

where $\nu = \nu(x, \xi)$ is a probability measure in the sphere $\mathbb{S}^{N-1}$ for a.e. $x \in \Omega$ and $v = v(\theta)$ is the solution in $\mathrm{H}^1_0(\Omega)$ of the system

$$-\alpha\Delta v - \lambda_0 v = \lambda_1 u_0 + \operatorname{div} \alpha\theta\, \nabla u_0,$$
$$\lambda_1 = \lambda_1(\theta) = \int_\Omega \alpha\theta|\nabla u_0|^2,$$
$$\int_\Omega v u_0 = 0.$$

According to Lemma 2.3 in [3] and taking into account $|\xi \cdot \nabla u_0|^2 \geq 0$, we may consider the probability measure $\xi \mapsto \nu(\cdot, \xi)$ as

$$\nu(x, \xi) = \delta_{\xi_x} \mathrm{d}\xi$$

with $\xi_x \in \mathbb{S}^{N-1}$, $\xi_x \perp \nabla u_0$. Therefore we finally obtain

$$\overline{G}(\theta) = -\int_\Omega \theta\, \nabla u_0 \cdot (\nabla u_0 + \varepsilon \nabla v(\theta)).$$

■

Proposition 2.6 *The Fréchet derivative of $\overline{G}$ is given by the expression*

$$\overline{G}'(\theta) = -\nabla u_0 \cdot (\nabla u_0 + 2\varepsilon \nabla v(\theta)).$$

Proof: It is clear that

$$\overline{G}'(\theta)\varphi = -\int_\Omega \varphi \nabla u_0 \cdot (\nabla u_0 + \varepsilon \nabla v(\theta)) - \int_\Omega \theta\, \nabla u_0 \cdot \varepsilon \nabla v(\varphi) \quad \forall \varphi \in \mathrm{L}^\infty(\Omega; [0,1]),$$

since ∇v is linear in θ.

From the system (2.56), we can see

$$\int_\Omega \theta \nabla u_0 \cdot \varepsilon \nabla v(\varphi) = \int_\Omega \varphi \nabla u_0 \cdot \varepsilon \nabla v(\theta).$$

Indeed, multiplying the first equation in (2.56) for $v(\varphi)$ and integrating by parts, yields

$$\int_\Omega \alpha \nabla v(\theta) \cdot \nabla v(\varphi) - \lambda_0 \int_\Omega v(\theta) v(\varphi) = -\int_\Omega \alpha\theta\, \nabla u_0 \cdot \nabla v(\varphi).$$

The symmetry in the left hand side shows the result and the proof of proposition 2.6 is completed. ■

2.6 Numerical illustration

Let us emphasize that the original problem of minimizing the leading eigenvalue with respect to the inclusion B is not well posed; usually, it admits no solution.

In this section we shall illustrate the behavior of the solution of the approximated problem through numerical simulations. To that end, we place ourselves under assumption of low contrast regime, i.e. $\beta = \alpha(1 + \varepsilon)$ for small ε. In the following examples, we will consider $\varepsilon = 0.1$ and $\varepsilon = 10^{-6}$.

We use an optimization algorithm to minimize $\bar{F}$: we have implemented a gradient-based steepest descend numerical algorithm for the local proportion θ. At each step of the optimization algorithm, we update the local proportion with a step $\rho_i > 0$ by

$$\theta_i = \min(1, \max(0, \tilde{\theta}_i)) \text{ with } \tilde{\theta}_i = \theta_{i-1} - \rho_i(\bar{F}'(\theta_{i-1}) + \Lambda_i)$$

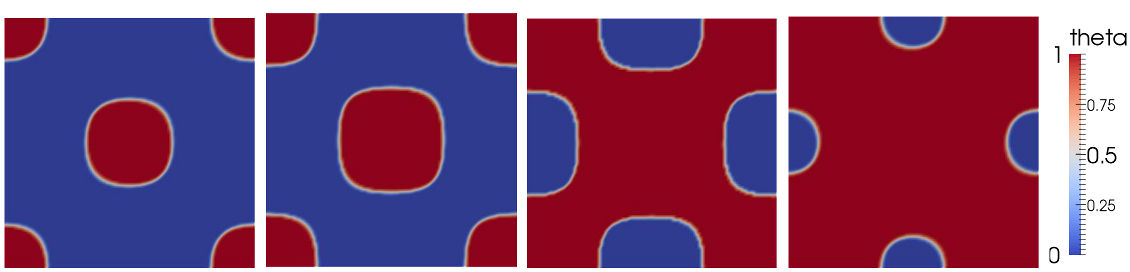
where Λ_i is the Lagrange multiplier for the volume constraint. The Lagrange multipliers Λ_i are approximated at each iteration by simple dichotomy in order to get the constraint $\int_\Omega \theta_i = m$ corresponding to a fixed proportion.

The optimization procedure is coupled with finite elements approximations of the boundary values problems needed to compute both $\bar{F}$ and its derivative $\bar{F}'$. To calculate the eigenpair (λ_0, u_0) and all the states $v_{i,\infty}$, we use $\mathbb{P}_2$ finite elements while the local proportions θ_i have been discretized with $\mathbb{P}_1$.

We will present examples in dimension two and three. The computations have been made with the FEM library FreeFem++ [19]. The subsequent figures show the local proportion of the material with higher conductivity. We do a comparative analysis in dimension two and three for square and cube cases respectively confirming the mentioned properties in [8] with respect to the distribution of the material with higher conductivity that depends on the shape of the domain Ω. The volume always refers to the percentage of volume occupied by the higher conductivity material.

2.6.1 The square and the cube.

The computations are made on the unit square $[0,1]^2$ with a regular mesh of 80000 triangles. For a very small value of ε here 10^{-6}, we have obtained the optimal designs displayed into Figure 2.7 for different volume proportions. The dark red region corresponds to B and material β, the local proportion is then 1. The blue region correspond to material α, the local proportion is then 0.

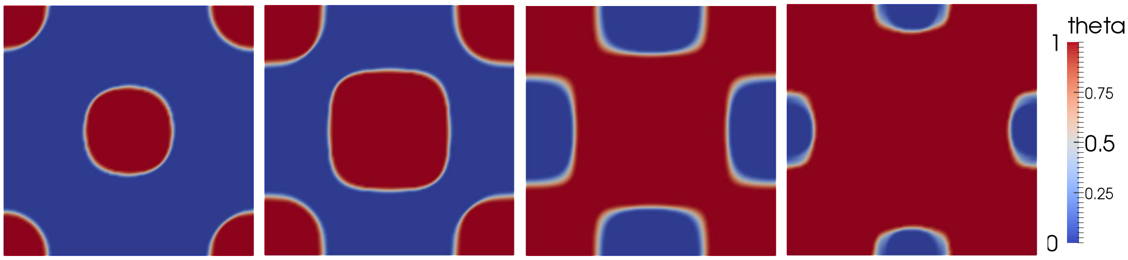


(a) $m/|\Omega| = 0.2$ (b) $m/|\Omega| = 0.4$ (c) $m/|\Omega| = 0.7$ (d) $m/|\Omega| = 0.9$

Figure 2.7: Nearly optimal distribution B in the square case for $\varepsilon = 10^{-6}$.

The numerically computed optimal region B contains neighborhoods around corners and the center always is also included. Similar results were obtained by Conca, Laurain and Mahadevan in [8] with a first order approximation only. Nevertheless, the local proportion is very often either 0 or 1. Let us now consider the same cases with a much larger parameter ε. In Figure 2.8, we present the results obtained with $\varepsilon = 0.1$.

We observe that the mixture is much more important: the interest of higher order approximation appear, as one can expect, for reasonably large value of the small parameter ε.

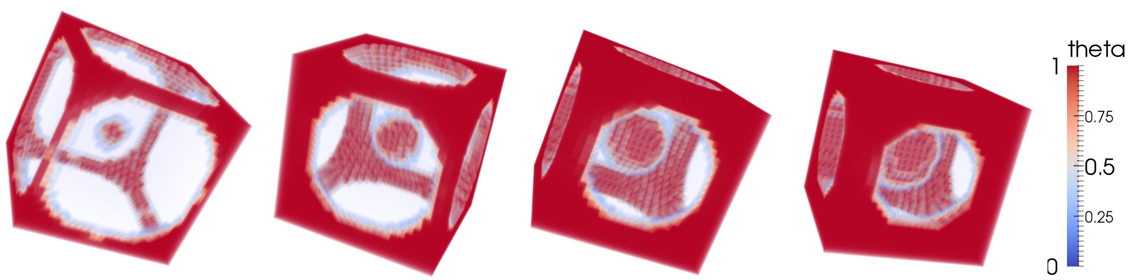


(a) $m/|\Omega| = 0.2$ (b) $m/|\Omega| = 0.4$ (c) $m/|\Omega| = 0.7$ (d) $m/|\Omega| = 0.9$

Figure 2.8: Nearly optimal distribution B in the square case for $\varepsilon = 0.1$.

For very small values of ε like 10^{-6}, the first order model should already provide a very good approximation.

Let us now present in Figure 2.9 simulations on the unit cube $[0, 1]^3$. For the visualisation, we have remove the phase where $\theta = 0$. Since the computation have been made on a Laptop, the resolution is coarser in these simulations in dimension three, we kept the same numbers of degree of freedom.



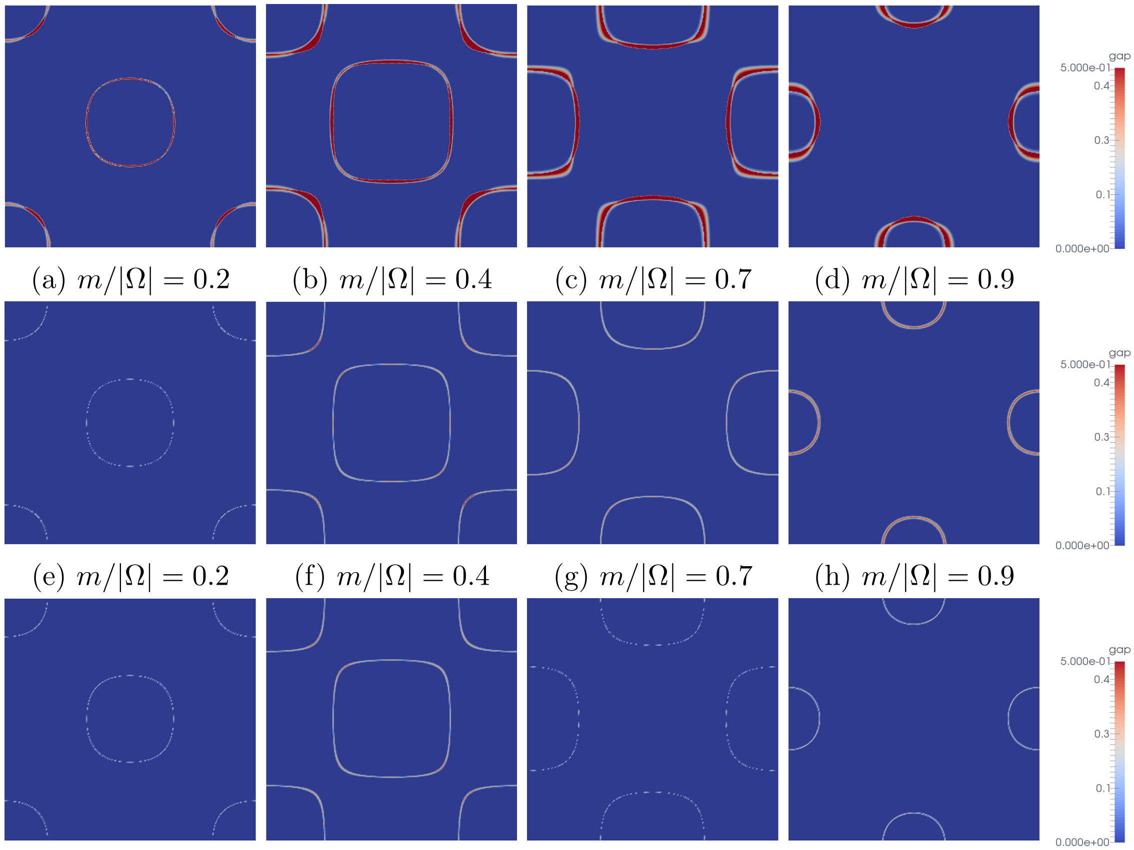
(a) $m/|\Omega| = 0.125$ (b) $m/|\Omega| = 0.25$ (c) $m/|\Omega| = 0.375$ (d) $m/|\Omega| = 0.5$

Figure 2.9: Nearly optimal distribution B in the cube case for $\varepsilon = 10^{-6}$.

2.6.2 Numerical comparison of first and second order model.

Since the original problem does not have in general a solution, it makes little sense to test how good are the optimal domains obtained by the first order approximation method (see [8]) and by the second order method. Nevertheless, one may wonder if the optimal solution obtained by the first and second order methods really differ. We claim that the optimal design really differ in general. Of course, it depends on the range of the small parameter ε. In order to illustrate and defend this claim, we have plotted in Figure 2.10 the absolute value of the difference between the characteristic function of the optimal domain computed by the method of [8] (based on the first order approximation) and the optimal density computed with the method presented in this work. In order to catch more precisely this gap, we use a refined mesh made of 180 000 triangles and $\mathbb{P}_2$ finite elements.

It appears that the second order method really differ for rather large values of ε and brings a real gain in decreasing the leading eigenvalue by generating mixture in a transition zone between the two phases as it is expected. The size of the mesh appears in the last line when $\varepsilon = 10^{-6}$, it corresponds to the width of the light curved lines. In the contrary, the mixing zone has the size of several element when ε take larger values.



(a) $m/|\Omega| = 0.2$ (b) $m/|\Omega| = 0.4$ (c) $m/|\Omega| = 0.7$ (d) $m/|\Omega| = 0.9$

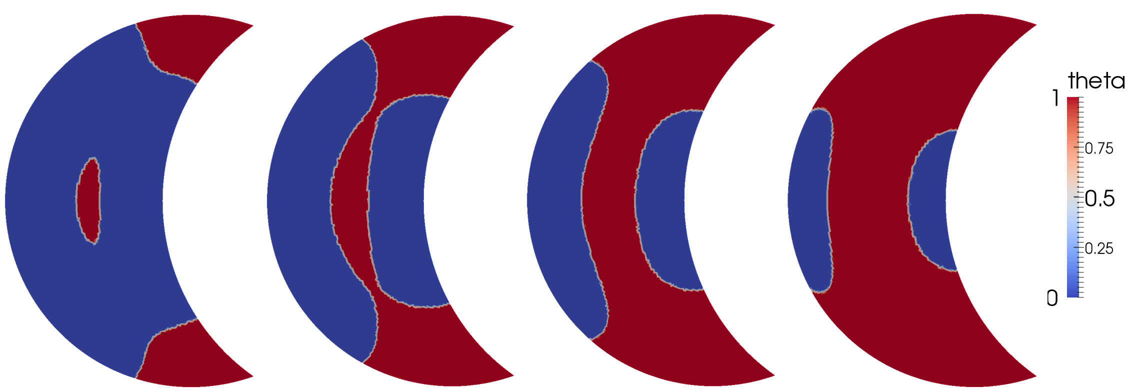
(e) $m/|\Omega| = 0.2$ (f) $m/|\Omega| = 0.4$ (g) $m/|\Omega| = 0.7$ (h) $m/|\Omega| = 0.9$

(i) $m/|\Omega| = 0.2$ (j) $m/|\Omega| = 0.4$ (k) $m/|\Omega| = 0.7$ (l) $m/|\Omega| = 0.9$

Figure 2.10: Absolute value of the gap between optimal design for first and second order models. The parameter ε takes the value 10^{-1} on the first line, 5.10^{-3} on the second line and 10^{-6} on the third line.

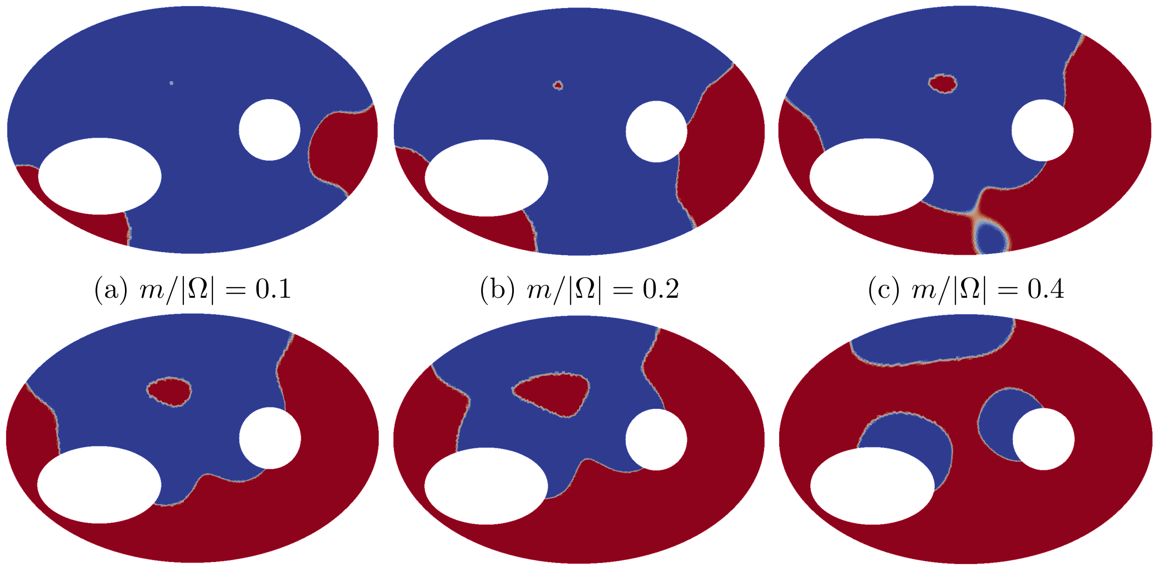
2.6.3 Others domains

For the sake of completeness, we present computations in other plane domains for the comparison with [8]: a crescent in Figure 2.11 and a perforated ellipse in Figure 2.12.



(a) $m/|\Omega| = 0.2$ (b) $m/|\Omega| = 0.4$ (c) $m/|\Omega| = 0.6$ (d) $m/|\Omega| = 0.8$

Figure 2.11: Nearly optimal distribution B in a crescent for $\varepsilon = 10^{-6}$.



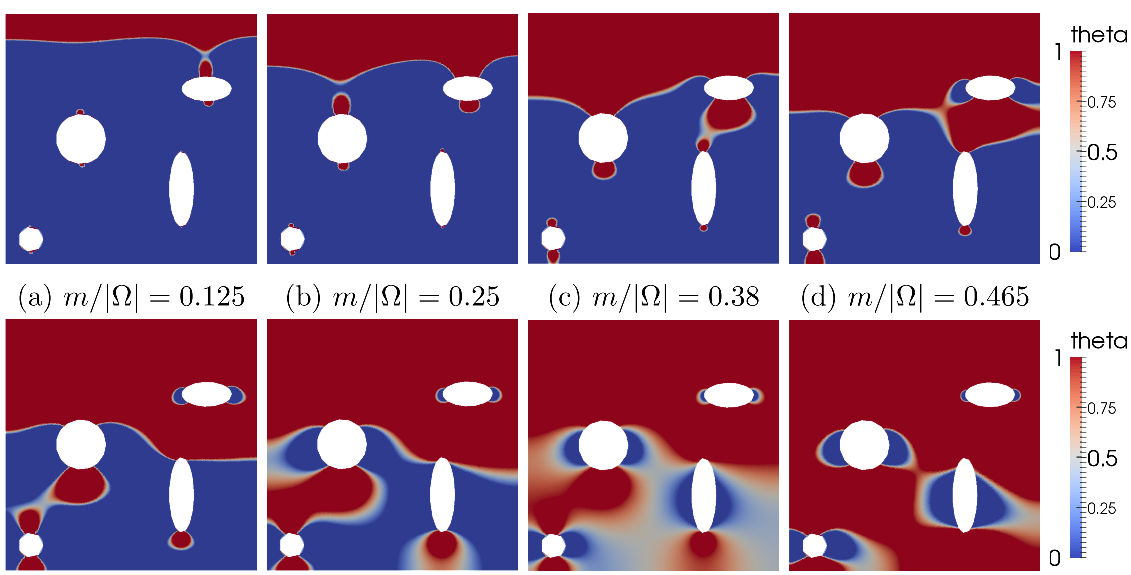
(a) $m/|\Omega| = 0.1$ (b) $m/|\Omega| = 0.2$ (c) $m/|\Omega| = 0.4$

(d) $m/|\Omega| = 0.5$ (e) $m/|\Omega| = 0.6$ (f) $m/|\Omega| = 0.8$

Figure 2.12: Nearly optimal distribution B in a perforated ellipse for $\varepsilon = 10^{-6}$.

Let us emphasize that in the last case, even for $\varepsilon = 10^{-6}$, we observe clearly in Figure 2.12(c) a small area where θ takes values strictly between 0 and 1 where we see the effect of the modelling with a second order approximation. This can be explained by the fact that the first eigenmode is more oscillating in a more complex geometrical configuration.

In order to enlight this observation let us consider a perforated square. The need of relaxation appears clearly in Figure 2.13 and Figure 2.14 for various configurations of perforated squares.

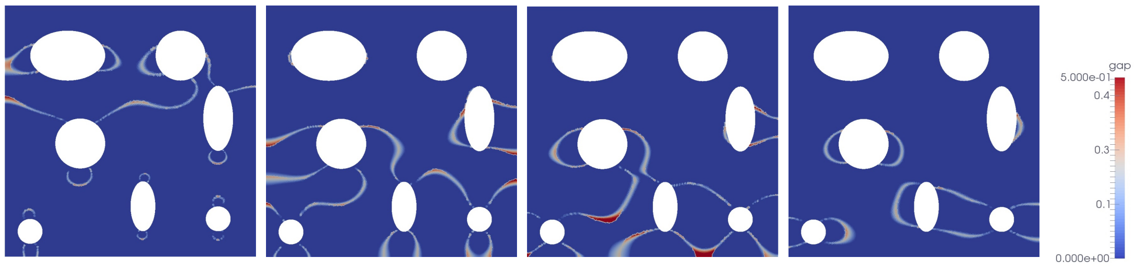


(a) $m/|\Omega| = 0.125$ (b) $m/|\Omega| = 0.25$ (c) $m/|\Omega| = 0.38$ (d) $m/|\Omega| = 0.465$

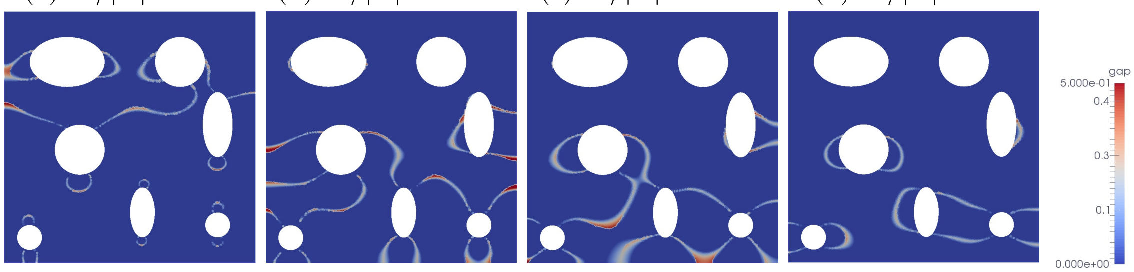
(e) $m/|\Omega| = 0.55$ (f) $m/|\Omega| = 0.665$ (g) $m/|\Omega| = 0.789$ (h) $m/|\Omega| = 0.9$

Figure 2.13: Optimal design for second order model for various fractions. The parameter ε takes the value 10^{-1}.

The results illustrated through Figures 2.13 and 2.14 for domains with many inclusions provide a convincing case for the use of the second order approximation. The used meshes involve around 110 000 triangles.



(a) $m/|\Omega| = 0.38$ (b) $m/|\Omega| = 0.665$ (c) $m/|\Omega| = 0.789$ (d) $m/|\Omega| = 0.9$

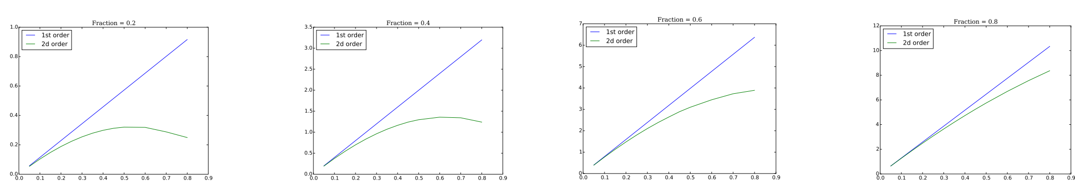


(e) $m/|\Omega| = 0.38$ (f) $m/|\Omega| = 0.665$ (g) $m/|\Omega| = 0.789$ (h) $m/|\Omega| = 0.9$

Figure 2.14: Absolute value of the gap between optimal design for first and second order models. The parameter ε takes the value 10^{-1} on the first line, 10^{-3} on the second line.

2.6.4 Numerical comparison of the methods

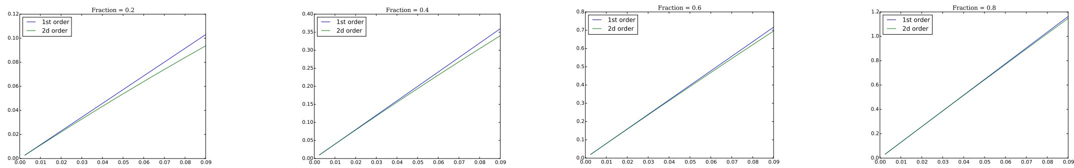
Next we illustrate the differences between both models first and second order approximation. We plot the gap between the original value λ_0 and the approximated values of the infimum by first and second model. From the Figure 2.15 below we can check how close are the models for ε larger than 0.1 for different volume constraints in the unit square case. It seems that for such a large value of ε the second order model really differs from the first order's one.



(a) volume fraction $m/|\Omega| = 0.2$ (b) volume fraction $m/|\Omega| = 0.4$ (c) volume fraction $m/|\Omega| = 0.6$ (d) volume fraction $m/|\Omega| = 0.8$

Figure 2.15: Gap between the approximated model and λ_0 of the homogeneous operator for different volume constraints with interpolation points for $\varepsilon = 0.05, 0.1, 0.2, \ldots, 0.9$: unit square case

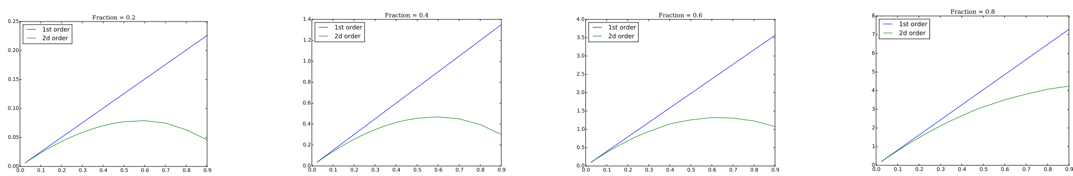
Let us now illustrate the same comparison curves but in the small range of values of ε, $[0.01, 0.1]$. This allows us to observe more accurately the small differences between the both approximation methods for ε small (see Figure 2.16 below).



(a) volume fraction $m/|\Omega| = 0.2$ (b) volume fraction $m/|\Omega| = 0.4$ (c) volume fraction $m/|\Omega| = 0.6$ (d) volume fraction $m/|\Omega| = 0.8$

Figure 2.16: Gap between the approximated model and λ_0 of the homogeneous operator for different volume constraints in the unit square case: $\varepsilon \in [0.01, 0.1]$

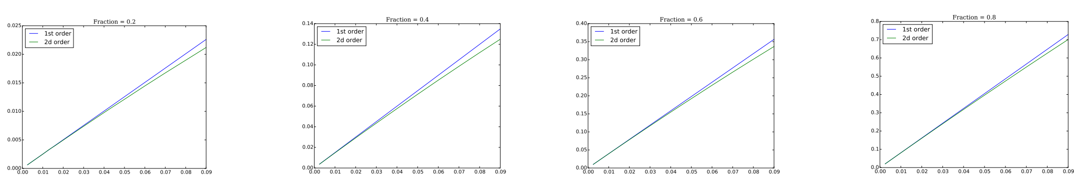
In the Figure 2.17 the drawn curves represent the made comparisons in the case of the square with holes for the same volume fractions and same values for ε as before (same domain as in Figure 2.14).



(a) volume fraction $m/|\Omega| = 0.2$ (b) volume fraction $m/|\Omega| = 0.4$ (c) volume fraction $m/|\Omega| = 0.6$ (d) volume fraction $m/|\Omega| = 0.8$

Figure 2.17: Gap between the approximated model and λ_0 of the homogeneous operator for different volume constraints: case of the unit square with holes, $\varepsilon \in [0.05, 0.9]$

The Figure 2.18 below shows the small differences between first and second order approximations for $\varepsilon \in [0.01, 0.09]$ in the case of the unit square with holes



(a) volume fraction $m/|\Omega| = 0.2$ (b) volume fraction $m/|\Omega| = 0.4$ (c) volume fraction $m/|\Omega| = 0.6$ (d) volume fraction $m/|\Omega| = 0.8$

Figure 2.18: Gap between the approximated model and λ_0 of the homogeneous operator for different volume constraints in the case of the unit square with holes: $\varepsilon \in [0.01, 0.1]$

Chapter 3

A second order approach for worst-case analysis of the compliance for a mixture of two materials in the low contrast regime

3.1 Introduction

In this chapter, we are interested in fundamental quantity of interest in linear elasticity: the compliance or work done by the loads or the stored elastic energy. The compliance is an appropriate indicator for the global rigidity of structure. In general, one considers structures made of a single material: homogeneous concrete or steel. Concrete structure can been made in different steps with diverse mixture in concrete mixers and the assumption of an homogeneous material is questionable in practice. Knowing how a structure is sensitive to material perturbation and performing a “worst-case" analysis is then of high engineering interest. Our motivation in this work is to study the optimal design problem under this uncertainty of material characteristics.

To make things clear, let $d = 2, 3$ be the dimension and take a domain Ω in $\mathbb{R}^d$ as regular as we need. Its boundary $\partial\Omega$ is decomposed in three complementary parts Γ_D, Γ_N and Γ where homogeneous Dirichlet (respectively Neumann) boundary conditions are imposed. Let $\mathbf{H}^1_D(\Omega)^d$ the space of vector fields in $\mathbf{H}^1(\Omega)^d$ which satisfy the homogeneous Dirichlet boundary conditions[1] on Γ_D. The domain Ω is filled with two isotropic materials of different fourth-order Hooke elasticity tensors A_0 and A_1: the second material lays in the open subset ω of Ω and the background material fills the complement so that the Hooke law is written in Ω as

$$A_\omega(x) = A_0 + (A_1 - A_0)\chi_\omega(x) = A_1\chi_\omega(x) + A_0\chi_{\Omega\setminus\omega}(x),$$

where χ_ω denotes the indicator function of ω. The displacement field $\boldsymbol{u}_\omega$ in Ω is the solution

[1] it's well known that continuity of the trace operator implies that this set is closed in $\mathbf{H}^1$ and thus it's a Hilbert space.

of the linear elasticity system[2]

$$\begin{cases} -\text{div } A_\omega e(\boldsymbol{u}_\omega) = \boldsymbol{0} \text{ in } \Omega, \\ \boldsymbol{u}_\omega = \boldsymbol{0} \text{ on } \Gamma_D, \\ (A_\omega e(\boldsymbol{u}_\omega))\boldsymbol{n} = \boldsymbol{g} \text{ on } \Gamma_N, \\ (A_\omega e(\boldsymbol{u}_\omega))\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \end{cases} \tag{3.1}$$

where $\boldsymbol{g} \in \mathbf{H}^{-1/2}(\Gamma_N)$ is a given surface load, $e(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$ is the strain tensor and

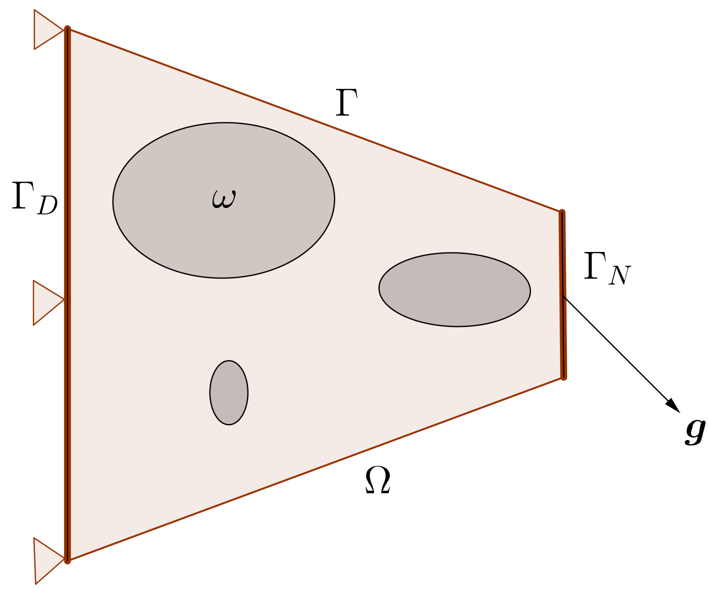


Figure 3.1: Model of Compliance

$\boldsymbol{n}$ the outer unitary normal field. The compliance is then defined as

$$C(\Omega, \omega, \boldsymbol{g}) := \langle \boldsymbol{g}, \boldsymbol{u}_\omega \rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_N} = E(\Omega, \boldsymbol{g}, \boldsymbol{u}_\omega) = \underset{\boldsymbol{v} \in H^1_D(\Omega)^d}{\text{Max}} E(\Omega, \boldsymbol{g}, \boldsymbol{v}),$$

where E is the elastic energy

$$E(\Omega, \boldsymbol{g}, \boldsymbol{v}) = -\int_\Omega A_\omega e(\boldsymbol{v}) : e(\boldsymbol{v}) + 2\langle \boldsymbol{g}, \boldsymbol{v} \rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_N}.$$

For a given fraction $\alpha \in (0,1)$, we consider the worst case or robust compliance as

$$J(\Omega) = \underset{\substack{\omega \subset \Omega, \\ |\omega| = \alpha|\Omega|}}{\text{Sup}} \; C(\Omega, \omega, \boldsymbol{g}). \tag{3.2}$$

The concept of robust compliance is not new: the case of uncertain loadings has been studied in topology structural optimization [6, 7] and in the context of the level set methods [16]. In this work, the uncertain parameter is not the applied forces but the material parameter through the mixture distribution.

The aim in this work is try to approximate the maximization problem (3.2) with respect to the material parameter when it is small.

[2] with A_ω satisfying the respective coercivity conditions.

3.2 Asymptotic analysis of the robust compliance problem in the low contrast regime

In this section, the domain Ω is fixed and we are interested in finding ω^* solving the optimization problem:

$$\begin{aligned} &\text{maximize } C(\Omega,\omega,\boldsymbol{g}) = \int_\Omega A_\omega e(\boldsymbol{u}_\omega) : e(\boldsymbol{u}_\omega) \\ &\text{subject to } \omega \subset \Omega,\ |\omega| = \alpha|\Omega|. \end{aligned} \tag{3.3}$$

In general, this question is a difficult shape optimization problem: the objective involves $\boldsymbol{u}_\omega$ the solution of the boundary value problem (3.1) depending on ω. Here, we take advantage of the low contrast assumption which allows to highly simplify the problem by performing the asymptotic expansion of the solution $\boldsymbol{u}_\omega$ and then of the cost function itself in term of the contrast parameter ε. Here, this assumption is written as

$$A_1 = A_0 + \varepsilon B \text{ which leads to } A_\omega = A_0 + \varepsilon B\chi_\omega,$$

where B is a given fourth order tensor. Since we make no positive assumptions on B (the only restriction is that A_1 is nonnegative). We assume in the sequel that $\varepsilon > 0$.

3.2.1 Asymptotic analysis and approximated robust compliance

Next, we study the boundary value problem

$$\begin{cases} -\text{div } (A_0 + \varepsilon B\chi_\omega)e(\boldsymbol{u}_\varepsilon) = \mathbf{0} \text{ in } \Omega, \\ \boldsymbol{u}_\varepsilon = \mathbf{0} \text{ on } \Gamma_D, \\ ((A_0 + \varepsilon B\chi_\omega)e(\boldsymbol{u}_\varepsilon))\boldsymbol{n} = \boldsymbol{g} \text{ on } \Gamma_N, \\ ((A_0 + \varepsilon B\chi_\omega)e(\boldsymbol{u}_\varepsilon))\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma, \end{cases}$$

where the displacement $\boldsymbol{u}_\varepsilon$ is written as a power series in ε

$$\boldsymbol{u}_\varepsilon = \sum_{k\geq 0} \varepsilon^k \boldsymbol{u}_k.$$

In order to get the coefficients $\boldsymbol{u}_k$, we plug the ansatz in the equation to obtain

$$\begin{aligned} -\text{div}\left(\sum_{k\geq 0}\varepsilon^k A_0 e(\boldsymbol{u}_k) + \sum_{k\geq 0}\varepsilon^{k+1}\chi_\omega B e(\boldsymbol{u}_k)\right) &= \mathbf{0} \text{ in } \Omega, \\ \sum_{k\geq 0}\varepsilon^k \boldsymbol{u}_k &= \mathbf{0} \text{ on } \Gamma_D, \\ \sum_{k\geq 0}\varepsilon^k (A_0 e(\boldsymbol{u}_k))\boldsymbol{n} + \sum_{k\geq 0}\varepsilon^{k+1}\chi_\omega (B e(\boldsymbol{u}_k))\boldsymbol{n} &= \boldsymbol{g} \text{ on } \Gamma_N, \\ \sum_{k\geq 0}\varepsilon^k (A_0 e(\boldsymbol{u}_k))\boldsymbol{n} + \sum_{k\geq 0}\varepsilon^{k+1}\chi_\omega (B e(\boldsymbol{u}_k))\boldsymbol{n} &= \mathbf{0} \text{ on } \Gamma. \end{aligned}$$

We now proceed iteratively starting with the first order $k = 0$:

$$\begin{cases} -\text{div } A_0 e(\boldsymbol{u}_0) = \mathbf{0} \text{ in } \Omega, \\ \boldsymbol{u}_0 = \mathbf{0} \text{ on } \Gamma_D, \\ (A_0 e(\boldsymbol{u}_0))\boldsymbol{n} = \boldsymbol{g} \text{ on } \Gamma_N, \\ (A_0 e(\boldsymbol{u}_0))\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma, \end{cases} \tag{3.4}$$

and for $k \geq 1$ it holds the iterative scheme

$$\begin{cases} -\text{div } A_0 e(\boldsymbol{u}_k) = \text{div } \chi_\omega B e(\boldsymbol{u}_{k-1}) \text{ in } \Omega, \\ \boldsymbol{u}_k = \mathbf{0} \text{ on } \Gamma_D, \\ (A_0 e(\boldsymbol{u}_k))\boldsymbol{n} = -\chi_\omega (B e(\boldsymbol{u}_{k-1}))\boldsymbol{n} \text{ on } \Gamma_N \cup \Gamma. \end{cases} \tag{3.5}$$

Here, the meaning of the above system is seen through its variational formulation

$$\int_\Omega (A_0 e(\boldsymbol{u}_k) + \chi_\omega B e(\boldsymbol{u}_{k-1})) : e(\boldsymbol{\varphi}) = 0 \quad \forall \boldsymbol{\varphi} \in \mathrm{H}^1_D(\Omega)^d \tag{3.6}$$

Note that all the coefficients $\boldsymbol{u}_k$ are obtained as solutions of boundary value problem (3.5) where only the ground Hooke tensor A_0 appears in the whole domain Ω. We take advantage of this to state a first result on the quality of the second order approximation[3] $\boldsymbol{u}^2 := \boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1 + \varepsilon^2 \boldsymbol{u}_2$ of the displacement field $\boldsymbol{u}_\varepsilon$.

Lemma 3.1 *There is a constant C independent of ω such that*

$$\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}^2\|_{\mathrm{H}^1(\Omega)^d} \leq C\varepsilon^3.$$

Proof of Lema 3.1: The proof is straightforward and can be directly adapted to any order k. For that, first of all consider the test function $\boldsymbol{\varphi} = \boldsymbol{u}_k$ in the variational formulation (3.6). The classical *a priori* estimates gives

$$C\|\boldsymbol{u}_k\|^2_{1,\Omega} \leq \int_\Omega A_0 e(\boldsymbol{u}_k) : e(\boldsymbol{u}_k) \leq C(B)\|\boldsymbol{u}_{k-1}\|_{1,\Omega}\|\boldsymbol{u}_k\|_{1,\Omega},$$

where the constant C in the left hand side depends only on[4] Ω and A_0. Thus

$$\|\boldsymbol{u}_k\|_{1,\Omega} \leq \mathcal{C}\|\boldsymbol{u}_{k-1}\|_{1,\Omega} \quad k \geq 1.$$

Since $\boldsymbol{u}_0$ does not depend on ω, the above scheme produces a uniform bound for $\boldsymbol{u}_k$. From (3.6) we check that the committed error $\boldsymbol{r}^2 := \boldsymbol{u}_\varepsilon - \boldsymbol{u}^2$ is the solution of

$$\begin{cases} -\text{div } A_\omega e(\boldsymbol{r}^2) = \varepsilon^3 \text{div } (\chi_\omega B e(\boldsymbol{u}_2)) \text{ in } \Omega, \\ \boldsymbol{r}^2 = \mathbf{0} \text{ on } \Gamma_D, \\ (A_\omega e(\boldsymbol{r}^2))\boldsymbol{n} = -\varepsilon^3 \chi_\omega (B e(\boldsymbol{u}_2))\boldsymbol{n} \text{ on } \Gamma_N \cup \Gamma. \end{cases}$$

The usual *a priori* estimates and the uniform bound of $\boldsymbol{u}_2$ in $\mathbf{H}^1_D(\Omega)^d$ end the proof. ∎

We now plug the expansion of $\boldsymbol{u}_\varepsilon$ into the expression of the compliance:

$$\begin{aligned} C(\Omega,\omega,\boldsymbol{g}) &= \int_\Omega A_\omega e(\boldsymbol{u}_\varepsilon) : e(\boldsymbol{u}_\varepsilon) \\ &= \int_\Omega \Big(\sum_{k\geq 0} \varepsilon^k A_0 e(\boldsymbol{u}_k)\Big) : \Big(\sum_{k\geq 0} \varepsilon^k e(\boldsymbol{u}_k)\Big) + \int_\Omega \Big(\sum_{k\geq 0} \varepsilon^{k+1} \chi_\omega B e(\boldsymbol{u}_k)\Big) : \Big(\sum_{k\geq 0} \varepsilon^k e(\boldsymbol{u}_k)\Big) \\ &= \sum_{k\geq 0} \varepsilon^k \sum_{p=0}^{k} \int_\Omega A_0 e(\boldsymbol{u}_p) : e(\boldsymbol{u}_{k-p}) + \sum_{k\geq 0} \varepsilon^{k+1} \sum_{p=0}^{k} \int_\Omega \chi_\omega B e(\boldsymbol{u}_p) : e(\boldsymbol{u}_{k-p}). \end{aligned}$$

[3] the study of the first order approach is done using level sets methods.
[4] by Korn's inequality.

Hence the above expression is written to second order terms as:

$$C(\Omega,\omega,\boldsymbol{g}) = \int_\Omega A_0 e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) + \varepsilon\Big(2\int_\Omega A_0 e(\boldsymbol{u}_1) : e(\boldsymbol{u}_0) + \int_\Omega \chi_\omega B e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0)\Big) \\ + \varepsilon^2\Big(2\int_\Omega A_0 e(\boldsymbol{u}_2) : e(\boldsymbol{u}_0) + 2\int_\Omega \chi_\omega B e(\boldsymbol{u}_1) : e(\boldsymbol{u}_0) + \int_\Omega A_0 e(\boldsymbol{u}_1) : e(\boldsymbol{u}_1)\Big) \\ + \mathcal{O}(\varepsilon^3) \quad (3.7)$$

where the meaning of the remainder $\mathcal{O}(\varepsilon^3)$ will be made precise at the final calculus.

Using $\boldsymbol{u}_0$ as test function in (3.6), $k=1$, we observe that:

$$\int_\Omega A_0 e(\boldsymbol{u}_1) : e(\boldsymbol{u}_0) = -\int_\Omega \chi_\omega B e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0). \quad (3.8)$$

Similar argument applies to the case $k=2$ to obtain

$$\int_\Omega A_0 e(\boldsymbol{u}_2) : e(\boldsymbol{u}_0) + \int_\Omega \chi_\omega B e(\boldsymbol{u}_1) : e(\boldsymbol{u}_0) = 0. \quad (3.9)$$

Replacing (3.8) and (3.9) in (3.7) yields

$$C(\Omega,\omega,\boldsymbol{g}) = \int_\Omega A_0 e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon\int_\Omega \chi_\omega B e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) \\ + \varepsilon^2\int_\Omega A_0 e(\boldsymbol{u}_1) : e(\boldsymbol{u}_1) + \mathcal{O}(\varepsilon^3). \quad (3.10)$$

Nevertheless, using $\boldsymbol{u}_1$ as test function in the variational formulation (3.6), $k=1$, gives

$$\int_\Omega A_0 e(\boldsymbol{u}_1) : e(\boldsymbol{u}_1) = -\int_\Omega \chi_\omega B e(\boldsymbol{u}_0) : e(\boldsymbol{u}_1).$$

Finally (3.10) becomes

$$C(\Omega,\omega,\boldsymbol{g}) = C^2(\Omega,\omega,\boldsymbol{g}) + \mathcal{O}(\varepsilon^3),$$

where the “second order approximate" robust compliance $C^2(\Omega,\omega,\boldsymbol{g})$ is defined by

$$C^2(\Omega,\omega,\boldsymbol{g}) = \int_\Omega A_0 e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon\int_\Omega \chi_\omega B e(\boldsymbol{u}_0) : \big(e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u}_1)\big). \quad (3.11)$$

To measure the perpetrated error, we can use an alternative definition of the approximate robust compliance $C^2(\Omega,\omega,\boldsymbol{g})$ obtained by approximating the compliance

$$C(\Omega,\omega,\boldsymbol{g}) = \langle \boldsymbol{g}, \boldsymbol{u}_\varepsilon\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_N}$$

by

$$C^2(\Omega,\omega,\boldsymbol{g}) = \langle \boldsymbol{g}, \boldsymbol{u}^2\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_N}.$$

The systematic error we commit in such an approximation is

$$|C(\Omega,\omega,\boldsymbol{g}) - C^2(\Omega,\omega,\boldsymbol{g})| = \Big|\langle \boldsymbol{g}, \boldsymbol{u}_\varepsilon - \boldsymbol{u}^2\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_N}\Big| \le C\varepsilon^3,$$

where the constant C is independent of ω.

3.2.2 Solving the approximated worst case problem

The question is now to find ω^* solution of the maximization problem:

$$\begin{aligned} &\text{maximize } C^2(\Omega, \omega, \boldsymbol{g}) \\ &\text{subject to } \omega \subset \Omega,\ |\omega| = \alpha|\Omega|. \end{aligned}$$

In view of (3.11) this maximization problem is equivalent to

$$\begin{aligned} &\text{minimize } \int_\Omega \chi_\omega Be(\boldsymbol{u}_0) : \big(e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u}_1)\big) \\ &\text{subject to } \omega \subset \Omega,\ |\omega| = \alpha|\Omega|, \end{aligned} \tag{3.12}$$

where $\boldsymbol{u}_1 = \boldsymbol{u}_1(\omega)$ depend on ω as the solution of the system[5]

$$\begin{cases} -\text{div } A_0 e(\boldsymbol{u}_1) = \text{div } \chi_\omega Be(\boldsymbol{u}_0) \text{ in } \Omega, \\ \boldsymbol{u}_1 = \boldsymbol{0} \text{ on } \Gamma_D, \\ (A_0 e(\boldsymbol{u}_1))\boldsymbol{n} = -\chi_\omega (Be(\boldsymbol{u}_0))\boldsymbol{n} \text{ on } \Gamma_N \cup \Gamma. \end{cases}$$

We now study the above problem through the general relaxed problem. For this purpose, we set

$$\mathcal{U}_{ad} := \{\chi;\ \chi = \chi_\omega,\ \omega \subset \Omega,\ |\omega| = \alpha|\Omega|\}.$$

Then, our optimization problem over the last admissible set is formulated as

$$\text{minimize}_{\chi \in \mathcal{U}_{ad}} F(\chi)$$

where

$$F(\chi) := \int_\Omega \chi Be(\boldsymbol{u}_0) : \big(e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u})\big)$$

and $\boldsymbol{u} = \boldsymbol{u}(\chi) \in \mathbf{H}^1_D(\Omega)^d$ is the solution of the system

$$\begin{cases} -\text{div } A_0 e(\boldsymbol{u}) = \text{div } \chi Be(\boldsymbol{u}_0) \text{ in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D, \\ (A_0 e(\boldsymbol{u}))\boldsymbol{n} = -\chi (Be(\boldsymbol{u}_0))\boldsymbol{n} \text{ on } \Gamma_N \cup \Gamma. \end{cases} \tag{3.13}$$

3.2.3 Relaxation of the minimization problem.

In order to derive optimization results, we employ the relaxation approach and we thus equip to the admissible set, $\mathcal{U}_{\text{ad}}$, of the weak-$\star$ topology where it is possible to get a minimum for the *lower semicountinuos envelop* of F.

It is well know that

$$\overline{\mathcal{U}}^\star_{\text{ad}} = \{\theta \in \mathrm{L}^\infty(\Omega; [0,1]), \int_\Omega \theta = \alpha|\Omega|\}.$$

From now on, we assume that the fourth order elasticity tensor A_0 is isotropic, i.e., there exist two positive constants μ and λ such that

$$A_0 \eta = 2\mu\, \eta + \lambda (\text{Tr}\, \eta) I_d, \tag{3.14}$$

where λ is a Lamé coefficient, I_d is the identity matrix of order $d \times d$ and $\text{Tr}\, \eta$ is the trace of the matrix η.

[5]recall that this system makes sense from its variational formulation (3.6).

Proposition 3.1 *Let us denote by $\overline{F}$ the lower semicountinuous envelop of F, i.e. for any $\theta \in \overline{\mathcal{U}}^*_{\mathrm{ad}}$*

$$\overline{F}(\theta) := \inf\Big\{\liminf F(\chi_k) : \exists \text{ a sequence } \chi_k \text{ in } \mathrm{L}^\infty(\Omega;\{0,1\}) \text{ s.t. } \chi_k \overset{*}{\rightharpoonup} \theta\Big\}.$$

Then the explicit expression for $\overline{F}(\theta)$, depending only on θ, is given by

$$\overline{F}(\theta) = \int_\Omega \theta B e(\boldsymbol{u}_0) : \big(e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{v}(\theta))\big) - \varepsilon \int_\Omega \theta(1-\theta) h(\xi^*), \tag{3.15}$$

where

$$h(\xi^*) = \max_{\xi \in \mathbb{S}^{d-1}} \left[h(\xi) \equiv -\frac{(\sigma^0\xi)^2}{\mu} + \frac{(\mu+\lambda)(\sigma^0\xi\cdot\xi)^2}{\mu(2\mu+\lambda)} \right], \quad \sigma^0 := Be(\boldsymbol{u}_0).$$

ξ^* and $h(\xi^*)$ will be made precise in Lemma 3.3.

Remark 3.1 *The proof of Proposition (3.1) is based on the context of the H-measures which is formulated through the symbols of the pseudo-differential operators (see Th. 2.2 and Lem. 2.3 in [1]). Therefore, as we did in the heuristic obtention of the symbol in the scalar case (see Lemma 2.4), before proving Proposition 3.1, we state the following lemma.*

Lemma 3.2 *If $\boldsymbol{v}$ solves:*

$$-\operatorname{div} A_0 e(\boldsymbol{v}) = \operatorname{div} \chi\sigma^0 \text{ in } \mathbb{R}^d, \tag{3.16}$$

then

$$e(\boldsymbol{v}) = P(\chi),$$

where P is a pseudo-differential operator, the symbol of which is (homogenous of degree 0 in ξ)

$$q(x,\xi) = -\frac{\sigma^0\xi\otimes\xi + \xi\otimes\sigma^0\xi}{2\mu|\xi|^2} + \frac{(\mu+\lambda)(\sigma^0\xi\cdot\xi)\xi\otimes\xi}{\mu(2\mu+\lambda)|\xi|^4}. \tag{3.17}$$

Heuristic computation of the symbol q of the pseudo-differential operator P :

Fixing $x \in \mathbb{R}^d$ in $\sigma^0(x)$ and taking Fourier transform in (3.16) we get

$$-A_0\widehat{e(\boldsymbol{v})}(\xi)\xi = \sigma^0\xi\,\hat{\chi}(\xi).$$

The definition of the strain tensor $e(\boldsymbol{v})$ yields

$$\widehat{e(\boldsymbol{v})}(\xi) = \tfrac{1}{2}(\widehat{\partial_j v^i}(\xi) + \widehat{\partial_i v^j}(\xi)) = \tfrac{1}{2}c\,(\xi_j\widehat{v^i} + \xi_i\widehat{v^j}) = \frac{c}{2}(\hat{\boldsymbol{v}}\otimes\xi + \xi\otimes\hat{\boldsymbol{v}}) = c\,(\hat{\boldsymbol{v}}(\xi)\odot\xi),$$

where $c = 2\pi i$ and $\hat{\boldsymbol{v}}(\xi)\odot\xi := {}^1\!/_2\,(\hat{\boldsymbol{v}}\otimes\xi + \xi\otimes\hat{\boldsymbol{v}})$.

Taking into account the expression (3.14) and the above calculus, it holds

$$-c\Big(2\mu(\hat{\boldsymbol{v}}(\xi)\odot\xi) + \lambda\mathrm{Tr}\,(\hat{\boldsymbol{v}}(\xi)\odot\xi)\Big)\xi = \sigma^0\xi\,\hat{\chi}(\xi).$$

Since

$$(\hat{\boldsymbol{v}}(\xi)\odot\xi)\xi = \frac{\widehat{v^i}\xi^j + \widehat{v^j}\xi^i}{2}\xi^i = \tfrac{1}{2}(\xi\otimes\xi + |\xi|^2 I_d)\hat{\boldsymbol{v}}(\xi)$$

and

$$\mathrm{Tr}\,(\hat{\boldsymbol{v}}(\xi)\odot\xi)\xi = \tfrac{1}{2}\mathrm{Tr}\,(\widehat{v^i}\xi^j + \widehat{v^j}\xi^i)\xi = \widehat{v^i}\xi^i\xi^j = (\xi\otimes\xi)\hat{\boldsymbol{v}}(\xi)$$

we have

$$-c\Big((\mu+\lambda)\xi\otimes\xi+\mu|\xi|^2 I_d\Big)\,\hat{\boldsymbol{v}}(\xi)=\sigma^0\xi\,\hat{\chi}(\xi). \tag{3.18}$$

We are interested in finding expressions like

$$\hat{\boldsymbol{v}}(\xi)=q\,\hat{\chi}(\xi),$$

where the coefficient q is to be determined. For this purpose, let us first analyse the symmetric matrix $\xi\otimes\xi$. Note that $\mathcal{I}m(\xi\otimes\xi)\subset\text{Span}\,\{\xi\}$ and, if $\xi\neq 0$, $|\xi|^2$ is the only non-zero eigenvalue, the associated eigenvector of which is ξ. Therefore, we can decompose $\xi\otimes\xi$ as

$$\xi\otimes\xi=Q\begin{pmatrix}|\xi|^2&&&\\&0&&\\&&\ddots&\\&&&0\end{pmatrix}Q^T$$

for certain orthogonal matrix Q which will be described below. According to the above decomposition, (3.18) becomes

$$-c|\xi|^2 Q\begin{pmatrix}2\mu+\lambda&&&\\&\mu&&\\&&\ddots&\\&&&\mu\end{pmatrix}Q^T\hat{\boldsymbol{v}}(\xi)=\sigma^0\xi\,\hat{\chi}(\xi).$$

It follows that

$$\begin{aligned}
\hat{\boldsymbol{v}}(\xi)&=-\frac{1}{c|\xi|^2}Q\begin{pmatrix}1/(2\mu+\lambda)&&&\\&1/\mu&&\\&&\ddots&\\&&&1/\mu\end{pmatrix}Q^T\sigma^0\xi\,\hat{\chi}(\xi)\\
&=-Q\left\{\frac{1}{(2\mu+\lambda)|\xi|^2}\begin{pmatrix}|\xi|^2&&&\\&0&&\\&&\ddots&\\&&&0\end{pmatrix}+\frac{1}{\mu}I_2-\frac{1}{\mu}\begin{pmatrix}1&&&\\&0&&\\&&\ddots&\\&&&0\end{pmatrix}\right\}Q^T\frac{\sigma^0\xi}{c|\xi|^2}\,\hat{\chi}(\xi)\\
&=-\frac{1}{c|\xi|^2}\left\{\frac{\xi\otimes\xi}{(2\mu+\lambda)|\xi|^2}+\frac{1}{\mu}I_d-\frac{1}{\mu}Q\begin{pmatrix}1&&&\\&0&&\\&&\ddots&\\&&&0\end{pmatrix}Q^T\right\}\sigma^0\xi\,\hat{\chi}(\xi).
\end{aligned}$$

Since Q is a orthogonal matrix associated with the decomposition of $\xi\otimes\xi$, we can write

$$Q=\begin{pmatrix}|&|&&|\\ \xi/|\xi|&\theta^2&\cdots&\theta^d\\ |&|&&|\end{pmatrix}$$

where $\{\xi/|\xi|,\theta^2,\dots,\theta^d\}$ is an orthonormal basis of $\mathbb{R}^d$. It is easy to check that

$$Q\begin{pmatrix}1&&&\\&0&&\\&&\ddots&\\&&&0\end{pmatrix}Q^T=\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}=\frac{\xi\otimes\xi}{|\xi|^2}.$$

Therefore

$$\hat{\boldsymbol{v}}(\xi) = -\frac{1}{c|\xi|^2}\left[\frac{1}{\mu}I_d - \frac{\mu+\lambda}{\mu(2\mu+\lambda)|\xi|^2}\xi\otimes\xi\right]\sigma^0\xi\,\hat{\chi}(\xi).$$

Finally,

$$\begin{aligned}\widehat{e(\boldsymbol{v})}(\xi) &= c\,\hat{\boldsymbol{v}}(\xi)\odot\xi\\ &= \left(-\frac{1}{|\xi|^2}\left[\frac{1}{\mu}I_d - \frac{\mu+\lambda}{\mu(2\mu+\lambda)|\xi|^2}\xi\otimes\xi\right]\sigma^0\xi\,\hat{\chi}(\xi)\right)\odot\xi\\ &= \left(-\frac{\sigma^0\xi\odot\xi}{\mu|\xi|^2} + \frac{(\mu+\lambda)(\xi\otimes\xi\,\sigma^0\xi)\odot\xi}{\mu(2\mu+\lambda)|\xi|^4}\right)\hat{\chi}(\xi).\end{aligned}$$

The definition of $\odot$ implies

$$(\xi\otimes\xi(\sigma^0\xi))\odot\xi = [(\xi\otimes I_2\xi)(\sigma^0\xi)]\odot\xi = [\xi(I_2\xi)^T(\sigma^0\xi)]\odot\xi = \xi\xi^T(I_2\sigma^0\xi\odot\xi) = (\sigma^0\xi\cdot\xi)\xi\otimes\xi,$$

we then obtain

$$\widehat{e(\boldsymbol{v})}(\xi) = \underbrace{\left(-\frac{\sigma^0\xi\otimes\xi+\xi\otimes\sigma^0\xi}{2\mu|\xi|^2} + \frac{(\mu+\lambda)(\sigma^0\xi\cdot\xi)\xi\otimes\xi}{\mu(2\mu+\lambda)|\xi|^4}\right)}_{q(x,\xi)}\hat{\chi}(\xi)$$

which gives the desired symbol $q(x,\xi)$.

Proof of Proposition 3.1: Let $\{\chi_k\}$ be a sequence of characteristic functions in $\mathcal{U}_{ad}$ (not necessarily minimizing) which converges weakly-$\star$ to a limit density θ. We denote by $\boldsymbol{v}_k$ the solution in $\mathbf{H}^1_D(\Omega)^d$ of (3.13) associated to χ_k. Then, the *a priori* estimates in (3.13) shows that the sequence $\boldsymbol{v}_k$ is bounded in $\mathbf{H}^1_D(\Omega)^d$ and, up to a subsequence, it converges weakly to a limit $\boldsymbol{v}$ in $\mathbf{H}^1_D(\Omega)^d$ which is solution of

$$\begin{cases} -\text{div}\ A_0e(\boldsymbol{v}) = \text{div}\ \theta Be(\boldsymbol{u}_0) \text{ in } \Omega,\\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D,\\ (A_0e(\boldsymbol{v}))\boldsymbol{n} = -\theta(Be(\boldsymbol{u}_0))\boldsymbol{n} \text{ on } \Gamma_N\cup\Gamma.\end{cases} \tag{3.19}$$

We now turn to calculate

$$\lim_{k\to\infty} F(\chi_k) = \lim_{k\to\infty}\int_\Omega \chi_k Be(\boldsymbol{u}_0):e(\boldsymbol{u}_0) + \varepsilon\int_\Omega \chi_k Be(\boldsymbol{u}_0):e(\boldsymbol{v}_k). \tag{3.20}$$

It is straightforward to calculate the limit of the first term on the right hand side in (3.20). In fact, since $\chi_k \overset{*}{\rightharpoonup} \theta$, then

$$\lim_{k\to\infty}\int_\Omega \chi_k Be(\boldsymbol{u}_0):e(\boldsymbol{u}_0) = \int_\Omega \theta Be(\boldsymbol{u}_0):e(\boldsymbol{u}_0). \tag{3.21}$$

The limit of the second term is more tricky and we use H-measure results, since this term contains the quadratic form $\chi_k e(\boldsymbol{v}_k)$ in the context of pseudo-differential operators.

Since the above Lemma 3.2 is referred to the solution of (3.16) in whole space $\mathbb{R}^d$ and our problem is referred to the bounded domain Ω, we divide the proof into the following steps:

Step 1: The expression (3.16) holds for $\Omega = \mathbb{R}^d$. Indeed, let σ^0 be defined by $\sigma^0 = Be(\boldsymbol{u}_0)$. Then, from Theorem 2.3 and Lemma 2.3, it follows that

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \chi_k Be(\boldsymbol{u}_0) : e(\boldsymbol{v}_k) &= \lim_{k\to\infty}\int_\Omega e(\boldsymbol{v}_k) : \chi_k\sigma^0 \\ &= \int_\Omega e(\boldsymbol{v}) : \theta\sigma^0 + \int_\Omega\int_{\mathbb{S}^{d-1}} q(x,\xi) : \theta(1-\theta)\sigma^0\,\nu(\mathrm{d}x,\mathrm{d}\xi),\end{aligned} \tag{3.22}$$

where $\nu = \nu(x,\xi)$ is a probability measure on the sphere $\mathbb{S}^{d-1}$ for a.e. $x \in \Omega$ and q is the symbol of the pseudo-differential operator P; i.e., q is that given in Lemma 3.2.

Let us now define the function h on $\mathbb{S}^{d-1}$ by $h(\xi) = q(x,\xi) : \sigma^0$. Hence

$$h(\xi) = -\frac{(\sigma^0\xi\otimes\xi) : \sigma^0 + (\xi\otimes\sigma^0\xi) : \sigma^0}{2\mu|\xi|^2} + \frac{(\mu+\lambda)(\sigma^0\xi\cdot\xi)(\xi\otimes\xi) : \sigma^0}{\mu(2\mu+\lambda)|\xi|^4}.$$

Using the facts: $|\xi| = 1$ on $\mathbb{S}^{d-1}$, $v\otimes\xi : A = v\cdot A\xi = \xi\otimes v : A$ and $A : \xi\otimes\xi = A\xi\cdot\xi$ for any symmetric matrix A and any vector v, yields

$$h(\xi) = -\frac{(\sigma^0\xi)\cdot(\sigma^0\xi)}{\mu} + \frac{(\mu+\lambda)(\sigma^0\xi\cdot\xi)(\sigma^0\xi\cdot\xi)}{\mu(2\mu+\lambda)}. \tag{3.23}$$

Note that h does not depend on θ and

$$\bar{F}(\theta) = \inf_{\nu}\lim_{k\to\infty} F(\chi_k). \tag{3.24}$$

So, if we take an optimal direction ξ^* which maximizes $h(\xi)$ on $\mathbb{S}^{d-1}$ and we choose the measure $\nu(x,\xi)$ as the Dirac mass in the optimal direction ξ^*, then, gathering (3.21) and (3.22), we finally obtain (3.15) in the case $\Omega = \mathbb{R}^d$.

Step 2: In the case of a bounded domain Ω we use a further localization argument.

We focus in the second term in (3.20). In order to simplify the calculus, let us consider $f_{A_0}(\xi)$ as a fourth-order tensor which is defined, for any symmetric matrices σ, σ', by

$$f_{A_0}(\xi)\sigma : \sigma' = \frac{\sigma\xi\cdot\sigma'\xi}{\mu} - \frac{(\mu+\lambda)(\sigma\xi\cdot\xi)(\sigma'\xi\cdot\xi)}{\mu(2\mu+\lambda)}.$$

Hence (3.22) can be rewritten as

$$\lim_{k\to\infty}\int_\Omega \chi_k Be(\boldsymbol{v}_k) : e(\boldsymbol{u}_0) = \int_\Omega e(\boldsymbol{v}) : \theta\sigma^0 - \int_\Omega\int_{\mathbb{S}^{d-1}} \theta(1-\theta) f_{A_0}(\xi)\sigma^0 : \sigma^0\,\nu(\mathrm{d}x,\mathrm{d}\xi). \tag{3.25}$$

Moreover, the introduction of a fourth-order tensor $M(x)$ defined, for any symmetric matrix σ, by

$$M\sigma : \sigma = \int_{\mathbb{S}^{d-1}} f_B(\xi)\sigma : \sigma\,\nu(x,\mathrm{d}\xi),$$

allows to rewrite (3.25) as

$$\lim_{k\to\infty}\int_\Omega \chi_k Be(\boldsymbol{v}_k) : e(\boldsymbol{u}_0) = \int_\Omega \theta Be(\boldsymbol{v}) : e(\boldsymbol{u}_0) - \int_\Omega \theta(1-\theta)BMBe(\boldsymbol{u}_0) : e(\boldsymbol{u}_0).$$

Consider a sequence $(\zeta_m)_{m\geq 1}$ of smooth compactly support functions in $C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\zeta_m \subset \Omega$ and ζ_m converges strongly to 1 in $\mathrm{L}^2(\Omega)$. Then

$$\int_\Omega \chi_k Be(\boldsymbol{v}_k) : e(\boldsymbol{u}_0) = \int_{\mathbb{R}^d} \zeta_m \chi_k Be(\boldsymbol{v}_k) : e(\boldsymbol{u}_0) - \int_{\mathbb{R}^d} (\zeta_m - 1)\chi_k Be(\boldsymbol{v}_k) : e(\boldsymbol{u}_0). \tag{3.26}$$

Clearly the second term on the right hand side goes to 0 (when m goes to $+\infty$) uniformly with respect to k since $\boldsymbol{v}_k$ is bounded in $\mathbf{H}^1(\Omega)^d$. Set ψ_m another smooth compactly supported function in $C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\psi_m \subset \Omega$ and $\psi_m \equiv 1$ inside the support of ζ_m. Note that

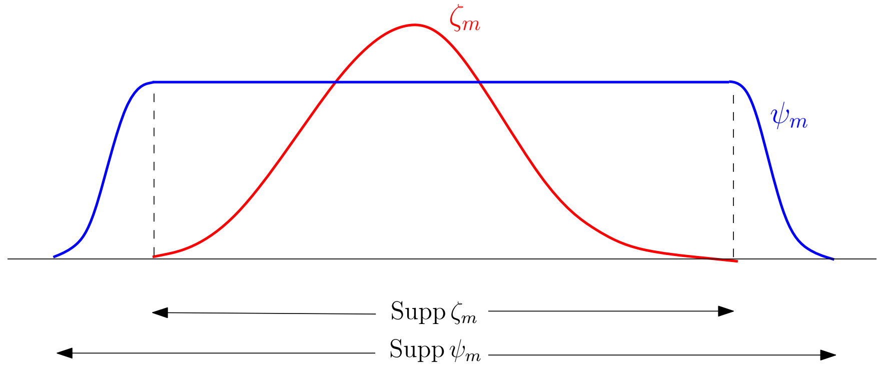


Figure 3.2: Localization argument

$$\int_{\mathbb{R}^d} \zeta_m \chi_k Be(\boldsymbol{v}_k) : e(\boldsymbol{u}_0) = \int_{\mathbb{R}^d} \zeta_m (\psi_m \chi_k) Be(\psi_m \boldsymbol{v}_k) : e(\boldsymbol{u}_0).$$

Now, from (3.13), we can write

$$-\text{div } A_0 e(\psi_m \boldsymbol{v}_k) - \text{div } A_0 e((1-\psi_m)\boldsymbol{v}_k) = \text{div } \psi_m \chi_k Be(\boldsymbol{u}_0) + \text{div } (1-\psi_m)\chi_k Be(\boldsymbol{u}_0)$$

in $\mathbb{R}^d$. So, computing the second terms on both sides of the last equation, we get

$$\begin{aligned} -\text{div } A_0 e(\psi_m \boldsymbol{v}_k) + A_0 e(\boldsymbol{v}_k)\nabla\psi_m - (1-\psi_m)\text{div } A_0 e(\boldsymbol{v}_k) + \text{div } A_0(\boldsymbol{v}_k \odot \nabla\psi_m) \\ = \text{div } \psi_m \chi_k Be(\boldsymbol{u}_0) + (1-\psi_m)\text{div } \chi_k Be(\boldsymbol{u}_0) - \chi_k Be(\boldsymbol{u}_0)\nabla\psi_m. \end{aligned}$$

Thus

$$-\text{div } A_0 e(\psi_m \boldsymbol{v}_k) = \text{div } \psi_m \chi_k Be(\boldsymbol{u}_0) - \text{div } A_0(\boldsymbol{v}_k \odot \nabla\psi_m) - \big(A_0 e(\boldsymbol{v}_k) + \chi_k Be(\boldsymbol{u}_0)\big)\nabla\psi_m$$

in $\mathbb{R}^d$ which allows to show that $\psi_m \boldsymbol{v}_k = \tilde{\boldsymbol{v}}_k + \check{\boldsymbol{v}}_k$ on the support of ζ_m, where the last two functions are the solutions of the following equation in the whole space $\mathbb{R}^d$

$$-\text{div } A_0 e(\tilde{\boldsymbol{v}}_k) = \text{div } (\psi_m \chi_k) Be(\boldsymbol{u}_0) \quad \text{in } \mathbb{R}^d, \tag{3.27}$$

$$-\text{div } A_0 e(\check{\boldsymbol{v}}_k) = -\text{div } A_0(\boldsymbol{v}_k \odot \nabla\psi_m) - \big(A_0 e(\boldsymbol{v}_k) + \chi_k Be(\boldsymbol{u}_0)\big)\nabla\psi_m \quad \text{in } \mathbb{R}^d. \tag{3.28}$$

According to the compact imbedding of the space $\mathbf{H}_0^1(\Omega)^d$ into the space $\mathrm{L}^2(\Omega)^d$, the weak convergence in $\mathbf{H}_0^1(\Omega)^d$ of the sequence $\boldsymbol{v}_k$ to the limit $\boldsymbol{v}$ can be assumed L^2-strong convergence up to a subsequence. Then, since the function ψ_m is fixed, we notice that the term on the right hand side in (3.28) is L^2-strongly convergent and so, the elliptical regularity of the equation implies $\check{\boldsymbol{v}}_k$ converges strongly in $\mathbf{H}^1(\mathbb{R}^d)^d$ to a limit $\check{\boldsymbol{v}}$. Furthermore, clearly the term on the right hand side in (3.27) is $\mathbf{H}^{-1}$-convergent and, again, in view of the elliptical regularity of the equation, $\tilde{\boldsymbol{v}}_k$ converges merely weakly in $\mathbf{H}^1(\mathbb{R}^d)^d$ to a limit $\tilde{\boldsymbol{v}}$ which satisfies $\boldsymbol{v} = \check{\boldsymbol{v}} + \tilde{\boldsymbol{v}}$ on the support of ζ_m. However, as in the case $\Omega = \mathbb{R}^d$, $e(\tilde{\boldsymbol{v}}_k)$ depends linearly on $\psi_m\chi_k$ through the pseudo-differential operator P of symbol (3.17). Therefore, we can apply Theorem 2 of [17] to the product of χ_k and $e(\tilde{\boldsymbol{v}}_k)$.

Finally, the limit of the first term on the right hand side of (3.26) becomes

$$\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}^d}\zeta_m(\psi_m\chi_k)Be(\tilde{\boldsymbol{v}}_k+\check{\boldsymbol{v}}_k):e(\boldsymbol{u}_0) &= \int_{\mathbb{R}^d}\zeta_m(\psi_m\theta)Be(\check{\boldsymbol{v}}):e(\boldsymbol{u}_0)\\ &\quad+\lim_{k\to\infty}\int_{\mathbb{R}^d}\zeta_m(\psi_m\chi_k)Be(\tilde{\boldsymbol{v}}_k):e(\boldsymbol{u}_0)\\ =\int_{\mathbb{R}^d}\zeta_m(\psi_m\theta)Be(\boldsymbol{v}):e(\boldsymbol{u}_0) &- \int_{\mathbb{R}^d}\zeta_m\psi_m\theta(1-\theta)BMBe(\boldsymbol{u}_0):e(\boldsymbol{u}_0)\\ =\int_{\Omega}\zeta_m\theta Be(\boldsymbol{v}):e(\boldsymbol{u}_0) &- \int_{\Omega}\zeta_m\theta(1-\theta)BMBe(\boldsymbol{u}_0):e(\boldsymbol{u}_0),\end{aligned}$$

and making $m \to +\infty$, (3.22) holds for the case Ω bounded, which proves Proposition 3.1. ■

In the next section we shall present the numerical implementation for the relaxed problem obtained in this section, for this reason, it is of particular interest to have a completely explicit formula rather than (3.15). For that purpose we enunciate and prove the following lemma (see [1], Lem. 2.3.21).

Lemma 3.3 *Let $\eta_1 \leq \cdots \leq \eta_d$ be the eigenvalues of the symmetrical matrix $\sigma^0 = Be(\boldsymbol{u}_0)$. Then if $d = 2$ or $\lambda \geq 0$, the maximum value of h on $\mathbb{S}^{d-1}$ is given by*

$$h(\xi^*) = \begin{cases} \dfrac{(\eta_1-\eta_d)^2}{4\mu} + \dfrac{(\eta_1+\eta_d)^2}{4(\lambda+\mu)} & \text{if } \eta_d > \dfrac{2\mu+\lambda}{2(\mu+\lambda)}(\eta_1+\eta_d) > \eta_1\\ \dfrac{\eta_1^2}{2\mu+\lambda} & \text{if } \eta_1 \geq \dfrac{2\mu+\lambda}{2(\mu+\lambda)}(\eta_1+\eta_d)\\ \dfrac{\eta_d^2}{2\mu+\lambda} & \text{if } \eta_d \leq \dfrac{2\mu+\lambda}{2(\mu+\lambda)}(\eta_1+\eta_d). \end{cases} \tag{3.29}$$

ξ^* is a combination of the eigenvectors associated with the extremal eigenvalues in the first case and ξ^* is an eigenvector associated to the corresponding eigenvalue $\eta_{1,d}$ in the remaining cases.

Proof of Lemma 3.3: Let us first rewrite (3.23) as

$$h(\xi) = \frac{1}{\mu}\Big(|\sigma^0\xi|^2 - (\sigma^0\xi\cdot\xi)^2\Big) + \frac{1}{2\mu+\lambda}(\sigma^0\xi\cdot\xi)^2.$$

Then, applying the Lagrange multipliers method in ξ^* to the function h with constraint $g(\xi) = |\xi|^2 - 1$ and combining the derivatives of h and g

$$\begin{cases} Dh(\xi) = \dfrac{2}{\mu}\Big((\sigma^0)^2\xi - 2(\sigma^0\xi\cdot\xi)\sigma^0\xi\Big) + \dfrac{4}{2\mu+\lambda}(\sigma^0\xi\cdot\xi)\sigma^0\xi,\\ Dg(\xi) = 2\xi, \end{cases}$$

we have

$$\frac{1}{\mu}\Big((\sigma^0)^2\xi^* - 2(\sigma^0\xi^*\cdot\xi^*)\sigma^0\xi^*\Big) + \frac{2}{2\mu+\lambda}(\sigma^0\xi^*\cdot\xi^*)\sigma^0\xi^* = \ell\xi^*, \tag{3.30}$$

for some real number ℓ.

From (3.30) we see that $(\sigma^0)^2\xi^*$ is a linear combination of ξ^* and $\sigma^0\xi^*$. Let V be the subspace of $\mathbb{R}^d$ spanned by ξ^* and $\sigma^0\xi^*$. Then, V is clearly stable under the action of the symmetrical matrix σ^0 and we can diagonalize σ^0 on V. Indeed, if we set $e \in V$ and $\eta \in \mathbb{R}$ such that $\sigma^0 e = \eta e$, then there exist constants c_1, c_2 such that $e = c_1\xi^* + c_2\sigma^0\xi^*$ from which it follows that

$$\begin{cases}\sigma^0 e = c_1\sigma^0\xi^* + c_2(\sigma^0)^2\xi^* = c_1\sigma^0\xi^* + c_2(\alpha_1\xi^* + \alpha_2\sigma^0\xi^*)\\ \eta e = c_1\eta\xi^* + c_2\eta\sigma^0\xi^*,\end{cases}$$

where α_1, α_2 come from (3.30), so that the condition $\sigma^0 e = \eta e$ leads to an eigenvalue problem

$$Bc = \eta c, \quad c := \begin{pmatrix}c_1\\ c_2\end{pmatrix}.$$

Without loss of generality we can assume B symmetric (it is enough to assume $\{\xi^*, \sigma^0\xi^*\}$ is orthonormal base of V), since σ^0 is symmetric. Hence σ^0 has a diagonalization on V.

Let $e_i, e_j \in V$ be two orthogonal unit eigenvectors of σ^0 with corresponding eigenvalues η_i, η_j. Then, the critical point ξ^* is a linear combination of e_i, e_j, e.i.

$$\xi^* = c_i e_i + c_j e_j$$

for some real constants c_i, c_j such that $c_i^2 + c_j^2 = 1$. Replacing this combination in (3.30) yields

$$\frac{\eta_p^2 c_p - 2(\eta_i c_i^2 + \eta_j c_j^2)\eta_p c_p}{\mu} + \frac{2(\eta_i c_i^2 + \eta_j c_j^2)\eta_p c_p}{2\mu+\lambda} = \ell c_p, \quad p = i, j. \tag{3.31}$$

It is straightforward to see that if $c_{i,j} = 0$ or $\eta_i = \eta_j$, then ξ^* is an eigenvector of σ^0 associated to the eigenvalue $\eta_{j,i}$ respectively, and the corresponding value of $h(\xi^*)$ is

$$h(\xi^*) = \frac{\eta_{j,i}^2}{2\mu+\lambda}. \tag{3.32}$$

Now, if $c_i \neq 0 \neq c_j$ and $\eta_i \neq \eta_j$, then simplifying by c_p and subtracting of the two components of (3.31) we obtain

$$\eta_i c_i^2 + \eta_j c_j^2 = \frac{2\mu+\lambda}{2(\mu+\lambda)}(\eta_i + \eta_j).$$

This equation together with $c_i^2 + c_j^2 = 1$ gives a linear system, the solution of which is

$$\begin{cases} c_i^2 = \dfrac{(2\mu+\lambda)\eta_i - \lambda\eta_j}{2(\mu+\lambda)(\eta_i - \eta_j)}\\[2ex] c_j^2 = \dfrac{(2\mu+\lambda)\eta_j - \lambda\eta_i}{2(\mu+\lambda)(\eta_j - \eta_i)}.\end{cases} \tag{3.33}$$

The assumption of no nullity of c_i, c_j yields its squares are positive, which is equivalent to the condition

$$\eta_j > \frac{2\mu+\lambda}{2(\mu+\lambda)}(\eta_i + \eta_j) > \eta_i. \tag{3.34}$$

Here we suppose that $\eta_j > \eta_i$. Under these conditions the associated value of $h(\xi^*)$ is

$$h(\xi^*) = \frac{(\eta_i - \eta_j)^2}{4\mu} + \frac{(\eta_i + \eta_j)^2}{4(\mu + \lambda)}. \tag{3.35}$$

Since

$$\frac{(\eta_i - \eta_j)^2}{4\mu} + \frac{(\eta_i + \eta_j)^2}{4(\mu + \lambda)} - \frac{\eta_{j,i}^2}{2\mu + \lambda} = \frac{\big((2\mu + \lambda)\eta_{i,j} - \lambda\eta_{j,i}\big)^2}{4\mu(\mu + \lambda)(2\mu + \lambda)} \geq 0$$

we see that (3.35) is always larger than both values of (3.32). Therefore, a maximum of $h(\xi)$ is either equal to (3.32), or to (3.35) if the condition (3.34) is fulfilled. Let us now analyze the two-dimensional case $d = 2$. Then, considering $\eta_1 \leq \eta_2$ the eigenvalues of σ^0 and assuming condition (3.34) is not satisfied, we then have either

$$\eta_1 \geq \frac{2\mu + \lambda}{2(\mu + \lambda)}(\eta_1 + \eta_2) \tag{3.36}$$

or

$$\eta_2 \leq \frac{2\mu + \lambda}{2(\mu + \lambda)}(\eta_1 + \eta_2) \tag{3.37}$$

Since $\mu > 0$ and $\mu + \lambda > 0$ (μ is a shear moduli and λ is a Lamé coefficient), we can see that (3.36) yields $\eta_1^2 \geq \eta_2^2$. Indeed, (3.36) implies $\eta_1 \leq 0$ because if $\eta_1 > 0$, from (3.36)

$$1 \geq \frac{2\mu + \lambda}{2(\mu + \lambda)}\left(1 + \frac{\eta_2}{\eta_1}\right) \geq \frac{2\mu + \lambda}{\mu + \lambda} > 1.$$

A similar contradiction is obtained when we use (3.37) and suppose that $\eta_2 < 0$. Since $\frac{2\mu+\lambda}{2(\mu+\lambda)} > 0$, the first case (3.36) shows that $\eta_1 \leq \eta_2 \leq -\eta_1$, i.e., $|\eta_2| \leq -\eta_1$. Thus $\eta_2^2 \leq \eta_1^2$. Similarly, the second case (3.37) shows that $|\eta_1| \leq \eta_2$ and consequently $\eta_1^2 \leq \eta_2^2$. This gives the desired formula for $h(\xi^*)$ in the case $d = 2$. To analyse the higher dimensional case $d \geq 3$, we assume that $\lambda \geq 0$ which allows us to perform the computations in a greatly simplified way. This assumption shows easily that if a couple of eigenvalues (η_i, η_j) verifies (3.34), then the same is true for the extremal eigenvalues (η_1, η_d). It follows that ξ^* is a linear combination of the eigenvectors e_1, e_d and additionally, the value

$$h(\xi^*) = \frac{(\eta_1 - \eta_d)^2}{4\mu} + \frac{(\eta_1 + \eta_d)^2}{4(\mu + \lambda)}$$

is larger than value

$$f(\eta_i, \eta_j) := \frac{(\eta_i - \eta_j)^2}{4\mu} + \frac{(\eta_i + \eta_j)^2}{4(\mu + \lambda)}$$

since f is an increasing function of η_j and a decreasing function of η_i when condition (3.34) is satisfied. Therefore, a maximum of $h(\xi)$ is attained for a combination of the extreme eigenvalues and we get the desired formula (3.29) for $h(\xi^*)$ in the higher dimensional case $d \geq 3$. ∎

3.2.4 Optimality conditions for the relaxed problem.

Let us now conclude this section investigating the differentiablity properties of $\overline{F}$ which will allow to obtain optimality conditions for a minimizer of $\overline{F}$ on the compact convex set $\overline{\mathcal{U}}^*_{\mathrm{ad}}$. This will be crucial for the numerical implementation and will be reflected in the gradient-based steepest descent algorithm that we will discuss in a posterior section.

Proposition 3.2 *The Fréchet derivative of $\overline{F}$ is equal to*

$$\overline{F}'(\theta) = Be(\boldsymbol{u}_0) : \big(e(\boldsymbol{u}_0) + 2\varepsilon\, e(\boldsymbol{v}(\theta))\big) + \varepsilon(2\theta - 1)h(\xi^*). \tag{3.38}$$

Proof: In view of the linearity of $\theta \longmapsto \boldsymbol{v}(\theta)$ we see from (3.15) that $\overline{F}(\theta)$ is quadratic in θ. Hence $\overline{F}$ is *Fréchet differentiable* and its derivative can be computed by the *Gâteaux derivative*. In order to get the derivative in θ we rewrite $\overline{F}(\theta)$ as

$$\bar{F}(\theta) = \int_\Omega \theta\Big(Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon h(\xi^*)\Big) + \varepsilon\int_\Omega \theta e(\boldsymbol{v}(\theta)) : Be(\boldsymbol{u}_0) + \varepsilon\int_\Omega \theta^2 h(\xi^*).$$

Using $\boldsymbol{v}(\theta)$ as test function in (3.19) and integrating by parts, the second term in the right hand side can be rewritten as

$$\int_\Omega \theta e(\boldsymbol{v}(\theta)) : Be(\boldsymbol{u}_0) = \int_\Omega A_0 e(\boldsymbol{v}(\theta)) : e(\boldsymbol{v}(\theta))$$

and we thus have

$$\overline{F}'(\theta)\varphi = \int_\Omega \varphi\Big(Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon h(\xi^*)\Big) - 2\varepsilon\int_\Omega A_0 e(\boldsymbol{v}(\theta)) : e(\boldsymbol{v}(\varphi)) + 2\varepsilon\int_\Omega \theta h(\xi^*)\varphi.$$

But $\boldsymbol{v}(\varphi)$ also verifies (3.19), so taking again $\boldsymbol{v}(\theta)$ as test function and integrating by parts, it follows that

$$\int_\Omega A_0 e(\boldsymbol{v}(\theta)) : e(\boldsymbol{v}(\varphi)) = \int_\Omega \varphi Be(\boldsymbol{u}_0) : e(\boldsymbol{v}(\theta)).$$

Thus, we can explicitly write the derivative in terms of φ as

$$\bar{F}'(\theta)\varphi = \int_\Omega \Big(Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon h(\xi^*) + 2\varepsilon Be(\boldsymbol{u}_0) : e(\boldsymbol{v}(\theta)) + 2\varepsilon\theta h(\xi^*)\Big)\varphi. \tag{3.39}$$

Consequently the density of $\overline{\mathcal{U}}^*_{\mathrm{ad}}$ in $\mathrm{L}^2(\Omega)$ establishes the formula (3.38). ∎

3.3 Minimization of the Compliance by the first and second order approximation in low contrast regime

This section is devoted to find the minimum of the compliance in the first and second order approximation. We employ the same ideas to that done in the above chapter, specifically in the maximization of the first eigenvalue of the diffusion escalar operator. As consequence, we will obtain (again as before) the fluctuation interval which will be used in the numerical implementations to compare the different models.

Let us recall that in physic applications, the compliance is the store energy elasticity which is an appropriate measure of the rigidity of structures (see fig 3.1). Mathematically, it is defined through the solution of the elasticity system:

$$\begin{cases} -\text{div } A_\omega e(\boldsymbol{u}_\omega) = \mathbf{0} \text{ in } \Omega, \\ \boldsymbol{u}_\omega = \mathbf{0} \text{ on } \Gamma_D, \\ (A_\omega e(\boldsymbol{u}_\omega))\boldsymbol{n} = \boldsymbol{g} \text{ on } \Gamma_N, \\ (A_\omega e(\boldsymbol{u}_\omega))\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma, \end{cases} \tag{3.40}$$

by the expression

$$C(\Omega, \omega, \boldsymbol{g}) := \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}_\omega,$$

where $\boldsymbol{g} \in \mathbf{H}^{-1/2}(\Gamma_N)$ is a given surface load, A_ω is a fourth-order Hooke elasticity tensor which represents the elasticity density inside the material $\Omega \subset \mathbb{R}^d$, $e(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$ is the second-order strain tensor and $\boldsymbol{n}$ is the outer normal vector. $\omega \subset \Omega$ is a variable subset, in the minimization process, to be set to fixed proportion with respect to Ω.

We assume that the condition of low contrast regime is satisfied, that is to say, we will consider $A_\omega = A_0 + \varepsilon \chi_\omega B$ where A_0 is the fourth-order Hooke tensor that was considered in the beginning of this problem and which represents the material in the complement of ω, and B is a given fourth-order tensor. $\varepsilon > 0$ is the minimization contrast parameter.

Since the variational formulation of (3.40) with test function $\boldsymbol{u}_\omega$ is given by

$$\int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}_\omega = \int_\Omega A_\omega e(\boldsymbol{u}_\omega) : e(\boldsymbol{u}_\omega),$$

we only need to minimize its right hand side. We now implement the asymptotic process with respect to ε rewriting the displacement $\boldsymbol{u}_\omega$ as

$$\boldsymbol{u}_\omega = \boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1 + \varepsilon^2 \boldsymbol{u}_2 + \cdots$$

Replacing this ansatz in (3.40) we get

$$\begin{cases} -\text{div } A_0 e(\boldsymbol{u}_0) = \mathbf{0} \text{ in } \Omega, \\ \boldsymbol{u}_0 = \mathbf{0} \text{ on } \Gamma_D, \\ (A_0 e(\boldsymbol{u}_0))\boldsymbol{n} = \boldsymbol{g} \text{ on } \Gamma_N, \\ (A_0 e(\boldsymbol{u}_0))\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma, \end{cases} \tag{3.41}$$

and for $k \geq 1$,

$$\begin{cases} -\text{div } A_0 e(\boldsymbol{u}_k) = \text{div } \chi_\omega B e(\boldsymbol{u}_{k-1}) \text{ in } \Omega, \\ \boldsymbol{u}_k = \mathbf{0} \text{ on } \Gamma_D, \\ (A_0 e(\boldsymbol{u}_k))\boldsymbol{n} = -\chi_\omega (B e(\boldsymbol{u}_{k-1}))\boldsymbol{n} \text{ on } \Gamma_N \cup \Gamma. \end{cases} \tag{3.42}$$

The meaning of (3.42) is giving by its variational formulation

$$\int_\Omega A_0 e(\boldsymbol{u}_k) : e(\boldsymbol{\varphi}) = -\int_\Omega \chi_\omega B e(\boldsymbol{u}_{k-1}) : e(\boldsymbol{\varphi})$$

for all $\boldsymbol{\varphi} \in \mathbf{H}^1_D(\Omega)^d$, i.e. the test functions being in the Hilbert space $\mathbf{H}^1(\Omega)^d$ with null Dirichlet condition on Γ_N.

We are interested in studying the minimization process for the compliance in the first and second asymptotically order cases.

The *a priori* estimatives applied to the variational formulation of (3.41)-(3.42) yield

$$\|\boldsymbol{u}_\varepsilon - (\boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1)\|_{\mathbf{H}^1(\Omega)^d} \leq C\varepsilon^2 \tag{3.43}$$

as well as

$$\|\boldsymbol{u}_\varepsilon - (\boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1 + \varepsilon^2 \boldsymbol{u}_2)\|_{\mathbf{H}^1(\Omega)^d} \leq C\varepsilon^3 \tag{3.44}$$

3.3.1 Minimization of the first order approximation

The first *a priori* estimative (3.43) allows us to focus in minimizing

$$\inf_{|\omega|=\alpha|\Omega|} \int_\Omega (A_0 + \varepsilon \chi_\omega B) e(\boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1) : e(\boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1)$$

in order to minimize the compliance with an ε^2-error. Here $\alpha \in (0,1)$ is the proportion between ω and Ω.

Combining (3.41) and (3.42) and dropping the terms of higher order than 1, the above minimization problem can be simplified as

$$\inf_{|\omega|=\alpha|\Omega|} \int_\Omega A_0 e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon \int_\omega B e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0),$$

but since $\boldsymbol{u}_0$ is independent of ω and ε is positive, this problem is reduced to

$$\sup_{|\omega|=\alpha|\Omega|} \int_\omega B e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0). \tag{3.45}$$

The following theorem provide admissible sets ω that maximize (3.45).

Theorem 3.1 *Let φ be the density function $Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0)$. Then, there exists $c \geq 0$ such that any measurable subset ω of Ω with $|\omega| = \alpha|\Omega|$ satisfying*

$$\{x \in \Omega : \varphi(x) > c\} \subset \omega \subset \{x \in \Omega : \varphi(x) \geq c\}$$

is an optimal solution for the maximization of (3.45).

Proof: Set $f(t) := |\{x \in \Omega : \varphi(x) \geq t\}|$. Then f is nonincreasing function with values in $[0, |\Omega|]$, so that the supremum $c := \sup\{t, f(t) \geq \alpha|\Omega|\}$ is well defined. Notice that the definition of c, for all a, b such that $a < c < b$ one has:

$$f(b) < \alpha|\Omega| \leq f(a).$$

In fact, $f(c) \geq \alpha|\Omega|$ and $|\{x \in \Omega : \varphi(x) > c\}| \leq \alpha|\Omega|$. Indeed, let $c_k < c$ be an increasing sequence such that $c_k \nearrow c$. As $f(c_k) \geq \alpha|\Omega|$, $\lim_k f(c_k) \geq \alpha|\Omega|$. But, since the level sets of φ are nested,

$$\begin{aligned}
\lim_k f(c_k) &= \lim_k |\{x \in \Omega : \varphi(x) \geq c_k\}| \\
&= |\cap_k \{x \in \Omega : \varphi(x) \geq c_k\}| \\
&= |\{x \in \Omega : \varphi(x) \geq c\}| = f(c),
\end{aligned}$$

so that $f(c) \geq \alpha|\Omega|$.

Taking now $c_k \searrow c$ $(c_k > c)$, we have $f(c_k) < \alpha|\Omega|$, so that $\lim_k f(c_k) \leq \alpha|\Omega|$. But again, by the fact that the level sets of φ are nested, we get

$$\begin{aligned}\lim_k f(c_k) &= \lim_k |\{x \in \Omega : \varphi(x) \geq c_k\}| \\ &= |\cup_k \{x \in \Omega : \varphi(x) \geq c_k\}| \\ &= |\{x \in \Omega : \varphi(x) > c\}|.\end{aligned}$$

So, in view of the previous estimatives, we can consider subsets $\omega \subset \Omega$ with the properties $|\omega| = \alpha|\Omega|$ and $\{x \in \Omega : \varphi(x) > c\} \subset \omega \subset \{x \in \Omega : \varphi(x) \geq c\}$. Let us now to show that those sets are optimal sets for the maximization problem (3.45).

Let D be a subset of Ω with $|D| = \alpha|\Omega|$. Since $|\omega| = |D|$, it is clear that $|\omega \setminus D| = |D \setminus \omega|$. Therefore

$$\begin{aligned}\int_D \varphi &= \int_{D\cap\omega} \varphi + \int_{D\setminus\omega} \varphi \leq \int_{D\cap\omega} \varphi + \int_{D\setminus\omega} c \\ &= \int_{D\cap\omega} \varphi + c\,|D \setminus \omega| = \int_{D\cap\omega} \varphi + c\,|\omega \setminus D| \\ &= \int_{D\cap\omega} \varphi + \int_{\omega\setminus D} c \leq \int_{D\cap\omega} \varphi + \int_{\omega\setminus D} \varphi \\ &= \int_\omega \varphi,\end{aligned}$$

this means that ω is an optimal solution of (3.45). ∎

Notice that the numerical implementations for this problem does not pose any new changes with respect to that done in the maximization of the ground state, so that we will take advantage of this in order to apply the same algorithms already obtained.

3.3.2 Minimization by the second order approximation

From *a priori* estimative (3.44) we can see that the minimization for the compliance may be studied with a small error ('cubic error' for ε small) by replacing the second order expresion into the elastic energy. To that end, we study the expresion

$$\inf_{|\omega|=\alpha|\Omega|} \int_\Omega (A_0 + \varepsilon\chi_\omega B)e(\boldsymbol{u}_0 + \varepsilon\boldsymbol{u}_1 + \varepsilon^2\boldsymbol{u}_2) : e(\boldsymbol{u}_0 + \varepsilon\boldsymbol{u}_1 + \varepsilon^2\boldsymbol{u}_2)$$

which reduces to

$$\inf_{|\omega|=\alpha|\Omega|} \int_\Omega A_0 e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon \int_\Omega \chi_\omega B e(\boldsymbol{u}_0) : (e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u}_1)).$$

Here $\boldsymbol{u}_1 = \boldsymbol{u}_1(\chi_\omega) \in \mathbf{H}^1(\Omega)^d$ verifies

$$\begin{cases} -\text{div } A_0 e(\boldsymbol{u}_1) = \text{div } \chi_\omega B e(\boldsymbol{u}_0) \text{ in } \Omega, \\ \boldsymbol{u}_1 = \mathbf{0} \text{ on } \Gamma_D, \\ (A_0 e(\boldsymbol{u}_1))\boldsymbol{n} = -\chi_\omega (B e(\boldsymbol{u}_0))\boldsymbol{n} \text{ on } \Gamma_N \cup \Gamma. \end{cases} \tag{3.46}$$

The above minimization problem is then reduced to the maximization problem

$$\max_{|\omega|=\alpha|\Omega|} \int_\Omega \chi_\omega Be(\boldsymbol{u}_0) : (e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u}_1)).$$

Relaxation procedure: Let us consider the general problem

$$\sup_{\chi\in\mathcal{U}_{\rm ad}} \left[F(\chi) := \int_\Omega \chi Be(\boldsymbol{u}_0) : (e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{v}))\right],$$

where $\boldsymbol{v} = \boldsymbol{v}(\chi)$ is the solution in $\mathbf{H}^1(\Omega)^d$ of the system (3.46) with

$$\mathcal{U}_{\rm ad} := \{\chi \in \mathrm{L}^\infty(\Omega; \{0,1\}), \int_\Omega \chi = \alpha|\Omega|\}.$$

Since

$$\sup_\chi F(\chi) = -\inf_\chi(-F(\chi))$$

we focus in the problem

$$\inf_{\chi\in\mathcal{U}_{\rm ad}} \left[G(\chi) := -\int_\Omega \chi Be(\boldsymbol{u}_0) : (e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{v}))\right].$$

At this moment we are able to use the relaxation procedure. As usual, here we consider the weak* topology of $\mathrm{L}^\infty(\Omega)$. Then, the procedure is to find

$$\overline{G}(\theta) = \inf\{\liminf G(\chi_k) : \chi_k \overset{*}{\rightharpoonup} \theta\}$$

with $\theta \in \{\varphi \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \varphi = \alpha|\Omega|\}$.

Proposition 3.3 *For any $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$ such that $\int_\Omega \theta = \alpha|\Omega|$, it holds*

$$\overline{G}(\theta) = -\int_\Omega \theta Be(\boldsymbol{u}_0) : \Big(e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{v}(\theta))\Big) + \varepsilon\int_\Omega \theta(1-\theta)h(\xi^*), \tag{3.47}$$

where $\boldsymbol{v} = \boldsymbol{v}(\theta) \in \mathbf{H}^1(\Omega)^d$ is the solution of (3.46) when we replace χ_ω by θ, and $h(\xi^*)$ is given by

$$h(\xi^*) = \frac{1}{2\mu+\lambda}\min\left(\eta_1^2, \ldots, \eta_d^2\right)$$

being $\eta_1 \leq \cdots \leq \eta_d$ the eigenvalues of the symmetric matrix $\sigma^0 = Be(\boldsymbol{u}_0)$ and ξ^* an eigenvector of σ^0.

Proof: Following the same ideas as in the maximization of the compliance by the second order approximation, we get

$$\overline{G}(\theta) = \inf_\nu -\int_\Omega \theta Be(\boldsymbol{u}_0) : \Big(e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{v}(\theta))\Big) + \varepsilon\int_\Omega\int_{\mathbb{S}^{d-1}} \theta(1-\theta)h(\xi)\,\nu(\mathrm{d}x, \mathrm{d}\xi)$$

where $\nu = \nu(x,\xi)$ is a probability measure on the sphere $\mathbb{S}^{N-1}$ for a.e. $x \in \Omega$, $\boldsymbol{v} = \boldsymbol{v}(\theta)$ is the solution in $\mathbf{H}^1(\Omega)^d$ of the system

$$\begin{cases} -\mathrm{div}\, A_0 e(\boldsymbol{v}) = \mathrm{div}\, \theta Be(\boldsymbol{u}_0) \text{ in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D, \\ (A_0 e(\boldsymbol{v}))\boldsymbol{n} = -\theta(Be(\boldsymbol{u}_0))\boldsymbol{n} \text{ on } \Gamma_N \cup \Gamma, \end{cases}$$

and

$$h(\xi) = \frac{1}{\mu}|\sigma^0\xi|^2 - \frac{\mu+\lambda}{\mu(2\mu+\lambda)}(\sigma^0\xi\cdot\xi)^2,$$

where μ is a shear moduli and λ is a Lammé coefficient which come from the Hooke law:

$$A_0 = 2\mu I_4 + \lambda I_2 \otimes I_2.$$

According to Lemma 2.3.21 in [1] (see p. 151), the minimum of h is attained in an eigenvector ξ^* of σ^0 and its value is given by

$$h(\xi^*) \equiv \min_{\xi\in\mathbb{S}^{d-1}} h(\xi) = \frac{1}{2\mu+\lambda}\min\Big(\eta_1^2,\dots,\eta_d^2\Big)$$

$\eta_1 \leq \dots \leq \eta_d$ are the eigenvalues of σ^0. In consequence, after taking ν as a Dirac mass in the optimal direction ξ^*, $\overline{G}(\theta)$ becomes in (3.47), which ends the proof. ■

Proposition 3.4 *The Fréchet derivative of $\overline{G}$ is given by*

$$\overline{G}'(\theta) = \varepsilon(1-2\theta)h(\xi^*) - Be(\boldsymbol{u}_0) : \big(e(\boldsymbol{u}_0) + 2\varepsilon e(\boldsymbol{v}(\theta))\big),$$

Proof: Let us first rewrite (3.47) as

$$\int_\Omega \theta\Big(\varepsilon h(\xi^*) - Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0)\Big) - \varepsilon \underbrace{\int_\Omega \theta Be(\boldsymbol{u}_0) : e(\boldsymbol{v}(\theta))}_{=\int_\Omega A_0 e(\boldsymbol{v}(\theta)):e(\boldsymbol{v}(\theta))} - \int_\Omega \varepsilon\theta^2 h(\xi^*).$$

From this is easy to check that

$$\overline{G}'(\theta)\varphi = \int_\Omega \varphi\Big(\varepsilon h(\xi^*) - Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0)\Big) - 2\varepsilon \underbrace{\int_\Omega A_0 e(\boldsymbol{v}(\varphi)) : e(\boldsymbol{v}(\theta))}_{=\int_\Omega \varphi Be(\boldsymbol{u}_0):e(\boldsymbol{v}(\theta))} - 2\varepsilon\int_\Omega \theta\varphi h(\xi^*)$$

for all $\varphi \in \mathrm{L}^\infty(\Omega;[0,1])$. Thus

$$\overline{G}'(\theta) = \varepsilon h(\xi^*) - Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - 2\varepsilon Be(\boldsymbol{u}_0) : e(\boldsymbol{v}(\theta)) - 2\varepsilon\theta h(\xi^*).$$

Rearranging terms, we obtain the desired expression and the proof of Proposition 3.4 is completed. ■

The algorithm of "gradient-based steepest descend method" is adapted to $\overline{G}$ and $\overline{G}'$ for the local proportion θ in the numerical implementations so as to get the optimal distribution which will be shown by graphics output.

3.4 Implementation and numerical results

In this section we are going to show the numerical tests that reinforce the obtained theoretic results for the maximization of the compliance. All of them will be performed only in dimension 2. In fact, we will employ similar simulations to that made in the second chapter, so that we take advantage of this to rewrite the algorithms that were employed in the scalar case for the ground state optimization problem. In addition to this, we have taken into account the same comparisons beteween the first and second order models in order to emphasize the differences between both methods. Those comparisons are done just as we did in the numerical presentation of the previous chapter, namely the states $\boldsymbol{u}_i = (u_i^1, u_i^2)$ are discretized with $\mathbb{P}_2 \times \mathbb{P}_2$ finite elements and the local proportions θ_i with $\mathbb{P}_1$ finite elements. At first, by simplicity, the numerical results are carried out discretizing the states with $\mathbb{P}_1 \times \mathbb{P}_1$ finite elements and local proportions with $\mathbb{P}_0$. In all the following examples we take the material $B = A_0$. Furthermore, the Poisson ratio is taken as $\nu = 0.33$ and Young's modulus equal to 30.000. The blue color represents to the stiffer material in all graphics while the red color represents the weakest material with proportion volume α.

3.4.1 The bar

We consider the bar problem whose geometry is the rectangle of dimensions 5×1, it is clamped on its left boundary and the vertical traction of 1 N is applied on its right boundary. We observe that the stiffer material is disconnected and concentrated around the clamped part of the boundary.

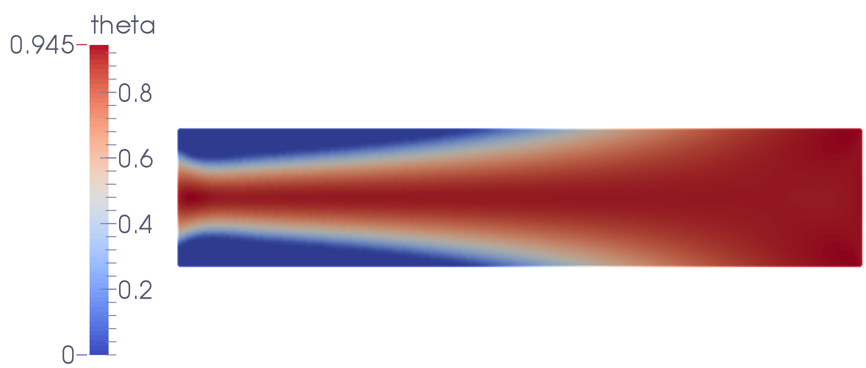


Figure 3.3: Compliance maximization for the bar: $\varepsilon = 10^{-6}$, $\alpha = 0.4$

In this case, the difference between first and second order models is more important than in the scalar case. This emphasizes the interest of the second order method that catch more details than the first order one even in the case of a very small value of the parameter ε.

Different configurations for the bar are now shown when the gap ε takes the values $10^{-1}, 10^{-3}$ and 10^{-6}. In turn, the comparison with the first order is taking into account in order to illustrate how sensitive are the differences between both models. We can observe that the small contrast regime (for $\varepsilon = 10^{-6}$, see Figure 3.6) in these examples does not have significant differences for the two approximation methods. Every numerical implementation here is made to a volume ratio of 25%.

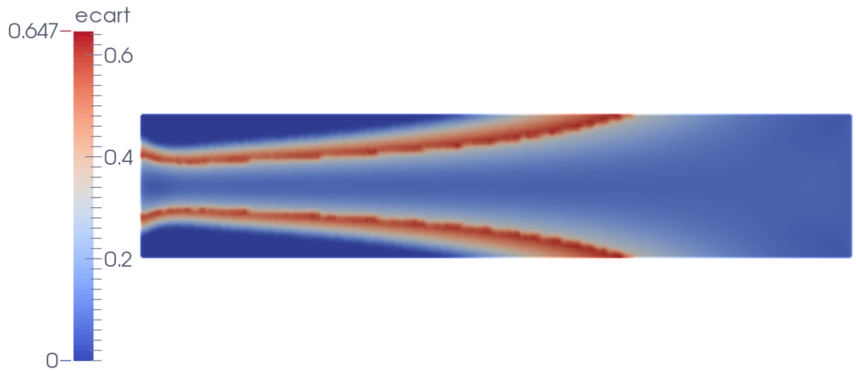
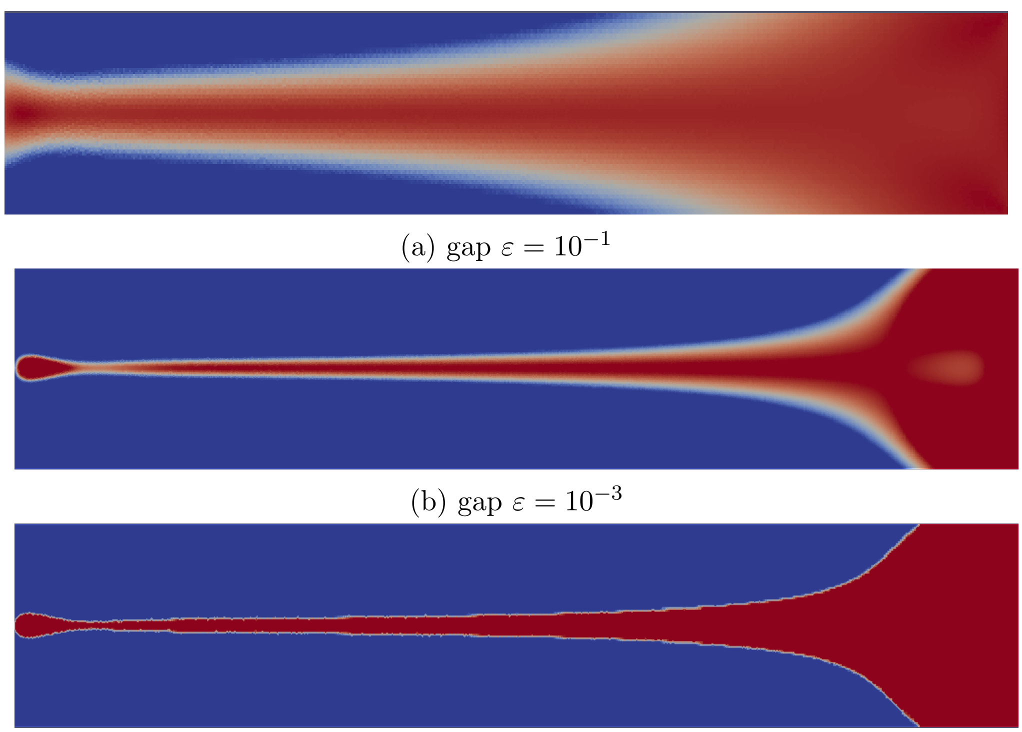


Figure 3.4: Difference between first and second order models for the Compliance maximization for the bar: $\varepsilon = 10^{-6}$, $\alpha = 0.4$



(a) gap $\varepsilon = 10^{-1}$

(b) gap $\varepsilon = 10^{-3}$

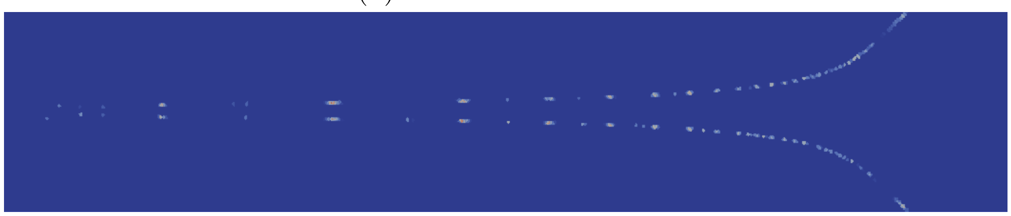
(c) gap $\varepsilon = 10^{-6}$

Figure 3.5: Second order model for the bar for different gaps to a volume ratio $\alpha = 0.25$

Let us now illustrate the different configurations of the first order model and second order model to small contrast regime $\varepsilon = 10^{-6}$.

(a) First order model

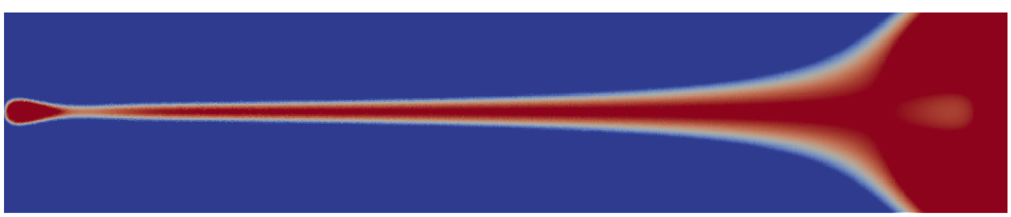
(b) Second order model



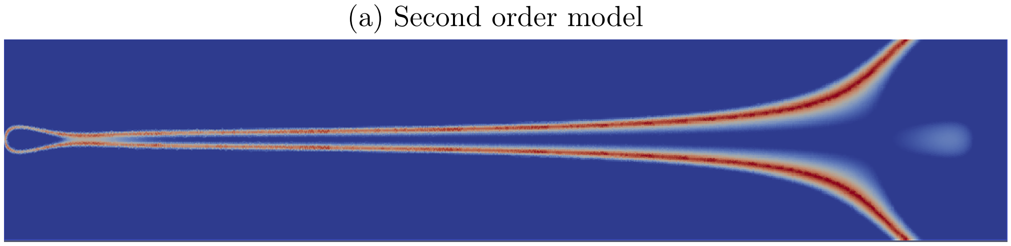
(c) Difference first and second order

Figure 3.6: First and second order models to a small contrast regime $\varepsilon = 10^{-6}$ to a volume ratio $\alpha = 0.25$

To see the contrast between the two above simulations we end with $\varepsilon = 10^{-3}$ and the respective difference between the methods.



(a) Second order model



(b) Difference first and second order

Figure 3.7: Second order model and difference with first order model with $\varepsilon = 10^{-3}$ to a volume ratio $\alpha = 0.25$

3.4.2 The long cantilever

With respect to the long cantilever, we observe that the weak material is distributed in the exterior of the domain, see Figure 3.8. That suggests that only the interior part of the material plays a significant role. This explain why usually a cantilever present no interior or many holes in order to get the lightest structure.

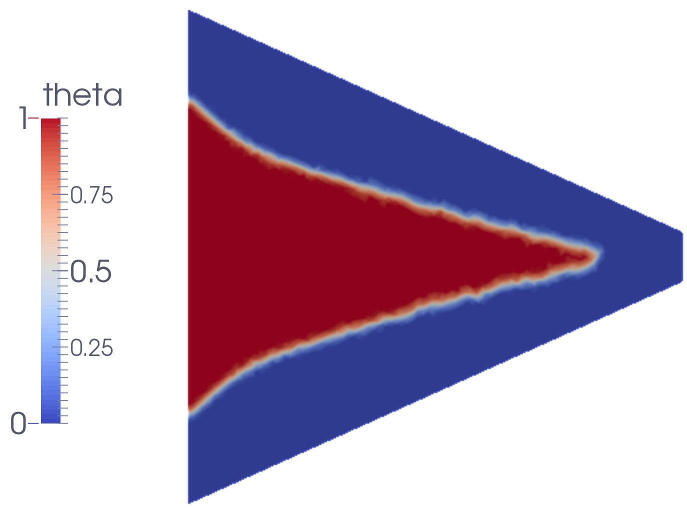


Figure 3.8: Compliance maximization of the long cantilever: $\varepsilon = 10^{-6}$, $\alpha = 0.4$

In this example we observe (see Figure 3.9) differences between first and second order models that are supported in a single triangle of the mesh. This means that they are not significant and may be only caused by the interpolation's error.

Finally, we show in Figure 3.10 and 3.11 the configuration for the long cantilever for $\varepsilon = 0.1$ and how this affects the transition between the stiffer and weakest material.

3.4.3 The short cantilever

We first consider the numerical illustrations for the short cantilever considering the optimal distribution only in the first order approximation for different volume fraction (see Figure 3.12), namely $\alpha = 0.25, 0.5$ and 0.75.

The following distribution of the weakest material in the domain for the short cantilever, in the second order model, is placed in a v-form, which disconnects the strong material in three parts, namely on the corners and on the middle of the clamped side, see Figure 3.13.

For the comparison of the first and second order models, from the Figure 3.14 we can see that, as before, there are not significant and may be only by the interpolation's error.

Figure 3.15 shows the configuration for the short cantilever when ε is taken to be larger and the volume ratio is $\alpha = 0.25$. As one expects, the transition zone between the materials is broader. However, it is of interest to note that this is not so different to the above configuration even for ε smaller. Compare with Figure 3.13.

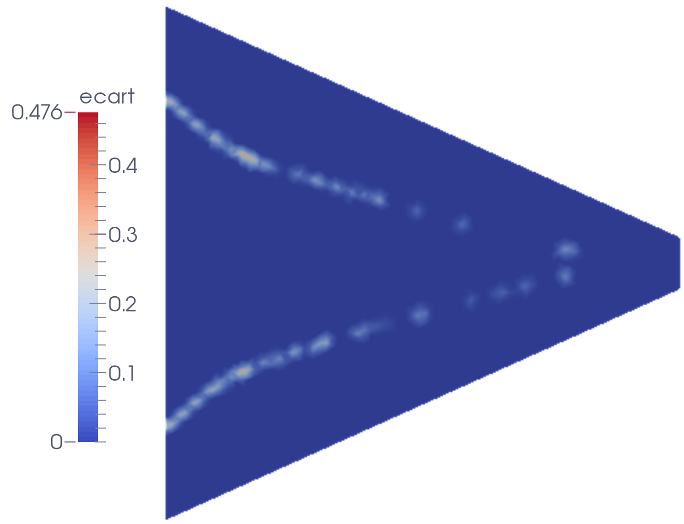
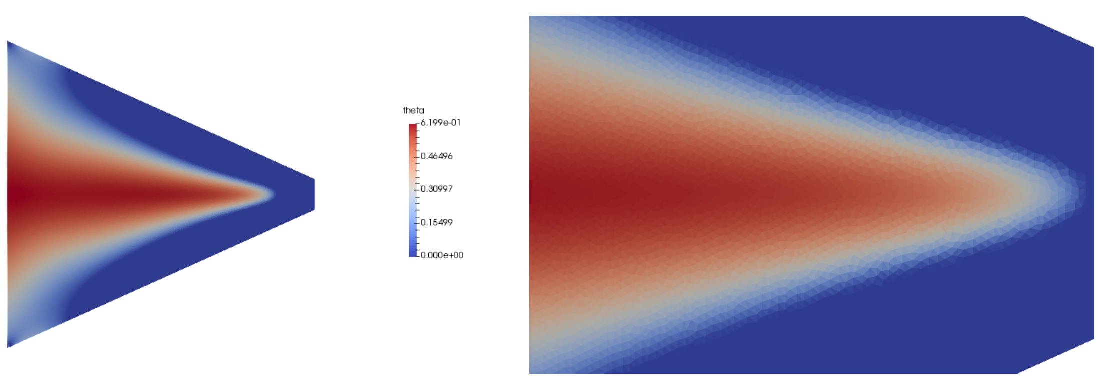


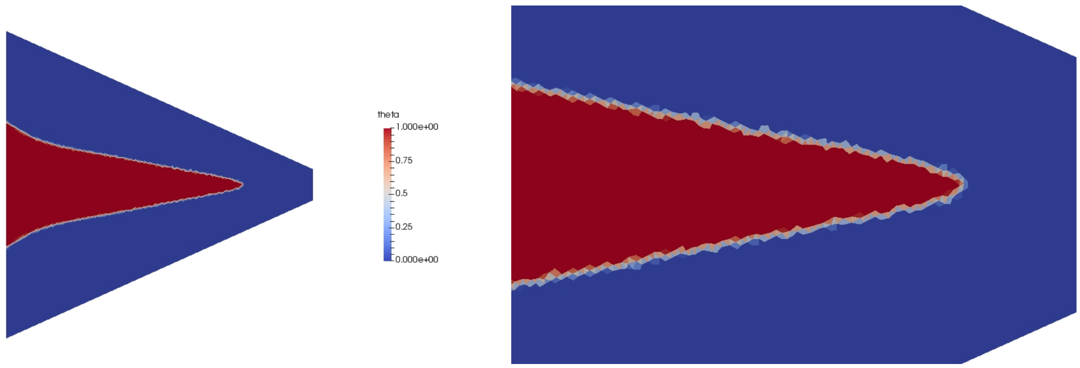
Figure 3.9: Difference between first and second order models for the Compliance maximization of the long cantilever: $\varepsilon = 10^{-6}$, $\alpha = 0.4$



(a) Second order model

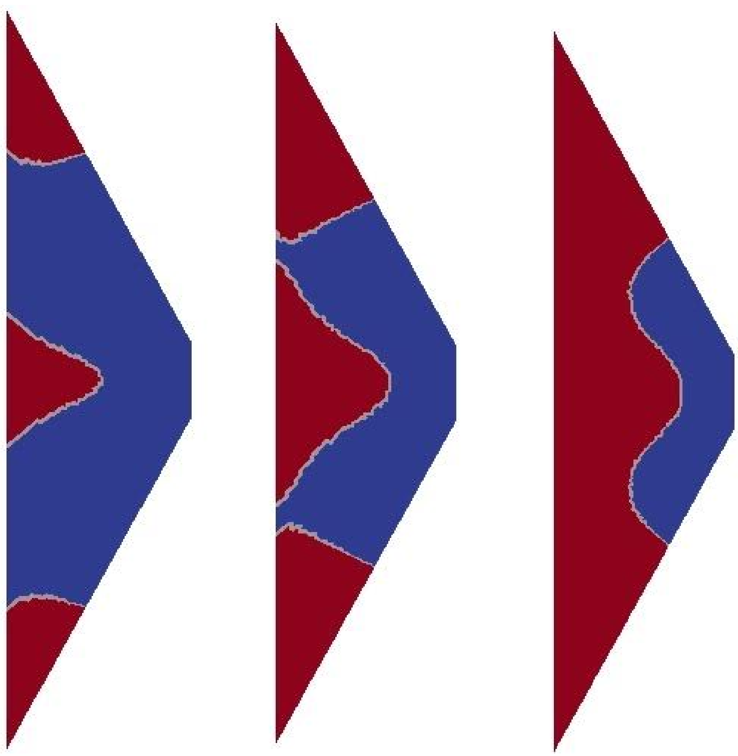
(b) Transition

Figure 3.10: Second order model for the long cantilever with $\varepsilon = 10^{-1}$ to a volume ratio $\alpha = 0.25$



(a) Second order model (b) Transition

Figure 3.11: Second order model for the long cantilever with $\varepsilon = 10^{-3}$ to a volume ratio $\alpha = 0.25$



(a) volume ratio: 25% (b) volume ratio: 50% (c) volume ratio: 75%

Figure 3.12: First order model for the short cantilever for different volume fractions

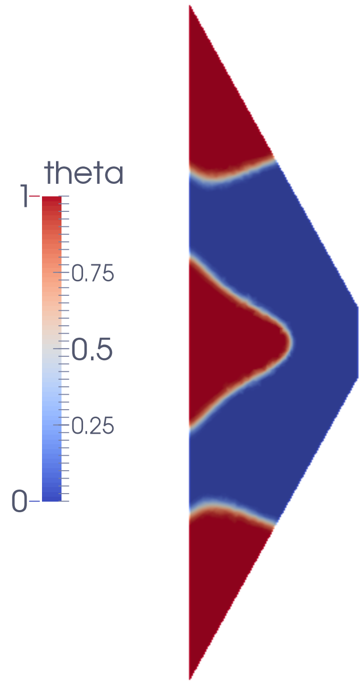


Figure 3.13: Compliance maximization of the short cantilever for the second order model: $\varepsilon = 10^{-6}$, $\alpha = 0.4$

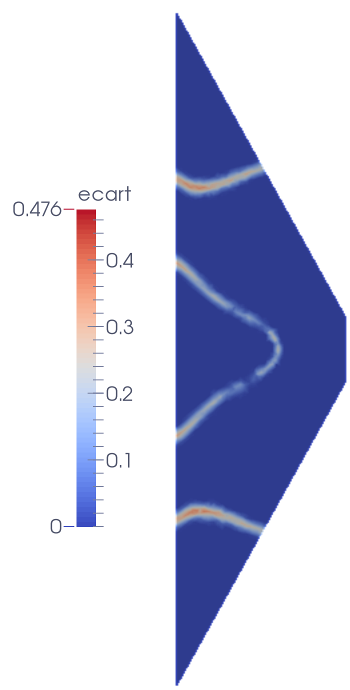


Figure 3.14: Difference between first and second order models for the Compliance maximization of the short cantilever: $\varepsilon = 10^{-6}$, $\alpha = 0.4$

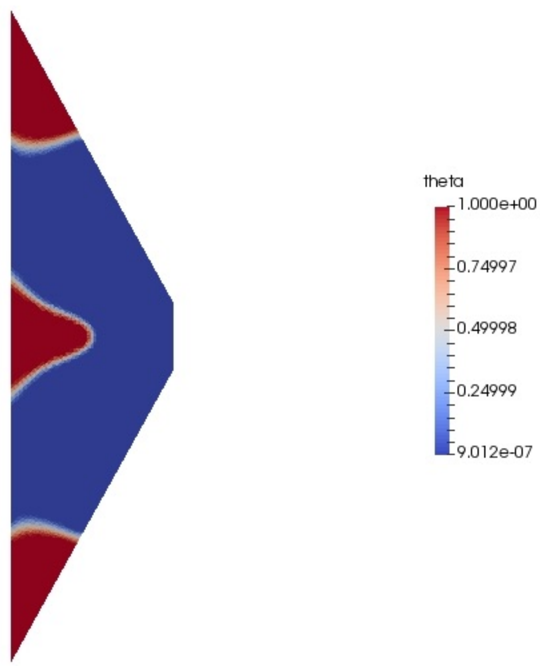


Figure 3.15: Compliance maximization of the short cantilever for the second order model: $\varepsilon = 0.1$, $\alpha = 0.25$

Chapter 4

Minimization of the First Eigenvalue of the Elasticity System in Low Contrast Regime

4.1 Introduction

In this chapter we consider the problem of minimizing the first eigenvalue associated to the linear elasticity system with homogeneous Dirichlet boundary condition in a bounded domain with a fixed volume, which is conformed by two materials distributed to fixed proportions. It is well known that this kind of problem does not have solution in a general context. However, we apply a strategy of approximation by a second order asymptotic procedure in order to get possible outcomes in the relaxed problem for the second order approximation.

As usual, $d = 2, 3$ is the dimension of our euclidean space. We are interested in studying the optimization problem

$$\inf_{\substack{\omega\subset\Omega \\ |\omega|=\alpha|\Omega|}} \lambda(\omega) \tag{4.1}$$

where ω represents a Lebesgue measurable set to be determined in the optimization process which is always contained in a fixed bounded domain Ω with a regular boundary $\partial\Omega$, $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^d$, α is a positive constant that gives the proportions between ω and Ω, so that $\alpha \in (0,1)$, and finally λ denotes the first eigenvalue of the problem

$$\begin{cases} -\text{div } A_\omega e(\boldsymbol{u}_\omega) = \lambda(\omega)\boldsymbol{u}_\omega \text{ in } \Omega, \\ \boldsymbol{u}_\omega = \mathbf{0} \text{ on } \partial\Omega \end{cases} \tag{4.2}$$

being A_ω a fourth order tensor conformed by two different fourth-order Hooke elasticity tensors A^0 and A^1 which represent the distribution of two isotropic materials into Ω, namely $\Omega \setminus \omega$ and ω respectively. $\boldsymbol{u}_\omega$ is a vectorial eigenfunction, $\boldsymbol{u}_\omega \in \mathbf{H}_0^1(\Omega)$, and $e(\boldsymbol{u})$ is the strain tensor which is defined by

$$e(\boldsymbol{u}) = \frac{1}{2}(D\boldsymbol{u} + D\boldsymbol{u}^T).$$

Here $D\boldsymbol{u}$ is the Jacobian matrix of $\boldsymbol{u}$ and $D\boldsymbol{u}^T$ its transpose.

In order to get asymptotic results, we shall make the "*low contrast regime*" assumption:

$$A^1 = A^0 + \varepsilon B$$

with the only restriction that A^1 is nonnegative and B is a fixed fourth order tensor. Thus we may assume hereinafter that $\varepsilon > 0$. Note that,

$$A_\omega = A^0 + \varepsilon B \chi_\omega \tag{4.3}$$

since $A_\omega = A^0 + (A^1 - A^0)\chi_\omega$.

From the variational formulation, it is well known that $\lambda(\omega)$ can be expressed by the Rayleigh quotient:

$$\lambda(\omega) = \min_{\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)\setminus\{\mathbf{0}\}} \frac{\displaystyle\int_\Omega A_\omega e(\boldsymbol{u}) : e(\boldsymbol{u})}{\displaystyle\int_\Omega \|\boldsymbol{u}\|^2} = \frac{\displaystyle\int_\Omega A_\omega e(\boldsymbol{u}_\omega) : e(\boldsymbol{u}_\omega)}{\displaystyle\int_\Omega \|\boldsymbol{u}_\omega\|^2}$$

From now on we shall consider the normalization condition for the fixed vectorial eigenfunction: $\|\boldsymbol{u}_\omega\|_{\mathbf{L}^2(\Omega)} = 1$. Hence, $\lambda(\omega)$ becomes

$$\lambda(\omega) = \int_\Omega A_\omega e(\boldsymbol{u}_\omega) : e(\boldsymbol{u}_\omega) \tag{4.4}$$

4.2 Asymptotic development of order 2 with respect to the gap

In this section our aim is to describe $\lambda(\omega)$ and $\boldsymbol{u}_\omega$ "*nearly*" to a second-order representation with respect to ε. To that end, let us consider the ansätze

$$\lambda^\varepsilon(\omega) := \sum_{j\geq 0} \varepsilon^j \lambda_j(\omega) = \lambda_0(\omega) + \varepsilon\lambda_1(\omega) + \varepsilon^2\lambda_2(\omega) + \cdots \tag{4.5}$$

$$\boldsymbol{u}_\omega^\varepsilon := \sum_{j\geq 0} \varepsilon^j \boldsymbol{u}_j(\omega) = \boldsymbol{u}_0(\omega) + \varepsilon\boldsymbol{u}_1(\omega) + \varepsilon^2\boldsymbol{u}_2(\omega) + \cdots \tag{4.6}$$

In the sequel $\lambda_\varepsilon^2(\omega)$ and $\boldsymbol{u}_\varepsilon^2(\omega)$ denote the truncated series of (4.5) and (4.6) in their order 2 respectively. The goal is then to show that $|\lambda(\omega) - \lambda_\varepsilon^2(\omega)|$ and $\|\boldsymbol{u}_\omega - \boldsymbol{u}_\varepsilon^2(\omega)\|_{\mathbf{H}^1(\Omega)}$ are small with respect only to ε for suitable coefficients; i.e., we wish to show uniform bounds with respect to ω for the last remainders when ε is small enough.

Replacing $\lambda(\omega)$, $\boldsymbol{u}_\omega$ by (4.5), (4.6) in the equation (4.2) and using (4.3), we get the following relationships by identifying the coefficients of same order, namely

$$\begin{cases} -\text{div } A^0 e(\boldsymbol{u}_0) = \lambda_0 \boldsymbol{u}_0 \text{ in } \Omega, \\ \boldsymbol{u}_0 = \mathbf{0} \text{ on } \partial\Omega \end{cases} \tag{4.7}$$

$$\begin{cases} -\text{div } A^0 e(\boldsymbol{u}_1(\omega)) - \lambda_0 \boldsymbol{u}_1(\omega) = \lambda_1(\omega)\boldsymbol{u}_0 + \text{div } \chi_\omega B e(\boldsymbol{u}_0) \text{ in } \Omega, \\ \boldsymbol{u}_1(\omega) = \mathbf{0} \text{ on } \partial\Omega \end{cases} \tag{4.8}$$

$$\begin{cases} -\text{div } A^0 e(\boldsymbol{u}_2(\omega)) - \lambda_0 \boldsymbol{u}_2(\omega) = \lambda_2(\omega)\boldsymbol{u}_0 + \lambda_1(\omega)\boldsymbol{u}_1(\omega) + \text{div } \chi_\omega B e(\boldsymbol{u}_1(\omega)) \text{ in } \Omega, \\ \boldsymbol{u}_2(\omega) = \mathbf{0} \text{ on } \partial\Omega \end{cases} \tag{4.9}$$

From (4.7) we see that the eigen-pair $(\lambda_0, \boldsymbol{u}_0)$ does not depend on ω, since A^0 is a given constant tensor independent of ω. Moreover, it is well known that λ_0 is nonsimple in general (see for instance [2]). We analyze the case when eigenvalue λ_0 is simple.

4.2.1 Second order asymptotic analysis with respect to the simple eigenvalue λ_0

We notice, from the normalization condition of $\boldsymbol{u}_\omega$, that $\|\boldsymbol{u}_0\|_{\mathbf{L}^2(\Omega)} = 1$. So, $\boldsymbol{u}_0$ may be uniquely defined.

We now compute the coefficients $\lambda_0, \lambda_1(\omega), \lambda_2(\omega)$. To that end, on the one hand, we see at once that

$$\lambda_0 = \int_\Omega A^0 e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0), \tag{4.10}$$

which is clear from (4.7).

On the other hand, the *Fredholm Alternative* applied to (4.8) yields $\boldsymbol{u}_1(\omega)$ is solution if and only if the right hand side is orthogonal to $\boldsymbol{u}_0$. Thus $\boldsymbol{u}_1(\omega)$ is solution of (4.8) whenever

$$\lambda_1(\omega) = \int_\Omega \chi_\omega B e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0). \tag{4.11}$$

An easy computation, applying again the normalization of $\boldsymbol{u}_\omega$, shows that $\int_\Omega \boldsymbol{u}_1(\omega) \cdot \boldsymbol{u}_0 = 0$, so this allows to write the solution $\boldsymbol{u}_1(\omega)$ univocally.

We can now proceed analogously to the above computation to obtain, from the equation (4.9),

$$\lambda_2(\omega) = \int_\Omega \chi_\omega B e(\boldsymbol{u}_1(\omega)) : e(\boldsymbol{u}_0). \tag{4.12}$$

Moreover, we can write the solution $\boldsymbol{u}_2(\omega)$ of (4.9) univocally since the normalization condition gives $2\int_\Omega \boldsymbol{u}_2(\omega) \cdot \boldsymbol{u}_0 + \int_\Omega \boldsymbol{u}_1(\omega)^2 = 0$.

Estimate of the remainders

In this part we are interesting in finding an estimate for the remainder $|\lambda(\omega) - \lambda^2_\varepsilon(\omega)|$ uniformly in ω. For this purpose, we use the notion of *h-quasimode* developed in [15] which is formulated as follows

Definition 2 *Let A be a self-adjoint operator on a Hilbert space H with domain $D(A)$. For a fixed $h > 0$, a pair $(\lambda, u) \in \mathbb{R} \times D(A) \setminus \{0\}$ is called a h-*quasimode *if we have*

$$\|(A - \lambda)u\|_H \leq h\|u\|_H.$$

The main interest of such a definition relies on the following fact: if (λ, u) is a *h-quasimode* of A, then the distance from λ to the spectrum of A is less than h and the distance between u and certain eigenspaces of A can be estimated.

Below we prove that the approximations of order 2 actually form an *h-quasimode* in the Hilbert space $\mathbf{H}^{-1}(\Omega)$ with $h = \mathcal{O}(\varepsilon^3)$. Indeed, let us show first of all a uniform bound in ω for $\boldsymbol{u}_1(\omega)$. Let us denote by λ^1 the second eigenvalue associated to $-\mathrm{div}\, A^0 e(\cdot)$. Then, λ^1 is characterized by its variational formulation:

$$\lambda^1 = \min_{\substack{\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)\setminus\{\mathbf{0}\} \\ \boldsymbol{u} \perp_{\mathbf{L}^2} \boldsymbol{u}_0}} \frac{\displaystyle\int_\Omega A^0 e(\boldsymbol{u}) : e(\boldsymbol{u})}{\displaystyle\int_\Omega \|\boldsymbol{u}\|^2} > \lambda_0.$$

Using $\boldsymbol{u}_1(\omega)$ as test function in (4.8) and integrating by parts, yields

$$\begin{aligned}\int_\Omega A^0 e(\boldsymbol{u}_1(\omega)) : e(\boldsymbol{u}_1(\omega)) - \lambda_0 \int_\Omega \|\boldsymbol{u}_1(\omega)\|^2 &= -\int_\omega Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_1(\omega)) \\ &\leq C_B \|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} \|\boldsymbol{u}_1(\omega)\|_{\mathbf{H}^1(\Omega)}.\end{aligned} \tag{4.13}$$

On the other hand, considering that $\boldsymbol{u}_1(\omega)$ is $\mathbf{L}^2(\Omega)$-orthogonal to $\boldsymbol{u}_0$ and applying the Korn's inequality, we see that

$$\begin{aligned}\int_\Omega A^0 e(\boldsymbol{u}_1(\omega)) : e(\boldsymbol{u}_1(\omega)) - \lambda_0 \int_\Omega \|\boldsymbol{u}_1(\omega)\|^2 &\geq \left(1 - \frac{\lambda_0}{\lambda^1}\right) \int_\Omega A^0 e(\boldsymbol{u}_1(\omega)) : e(\boldsymbol{u}_1(\omega)) \\ &\geq C_{\Omega, A^0} \left(1 - \frac{\lambda_0}{\lambda^1}\right) \|\boldsymbol{u}_1(\omega)\|^2_{\mathbf{H}^1(\omega)}.\end{aligned} \tag{4.14}$$

Combining (4.13) with (4.14), we can conclude that $\|\boldsymbol{u}_1(\omega)\|_{\mathbf{H}^1(\Omega)} \leq C$, where C is a constant independent of ω. Hence, it follows inmediately that $|\lambda_2(\omega)| \leq C$, which is clear from (4.12). Note that $|\lambda_1(\omega)| \leq C$ it is also clear from (4.11). To estimate $\|\boldsymbol{u}_2(\omega)\|_{\mathbf{H}^1(\Omega)}$, we take advantage of the λ^1-argument applied to estimate $\boldsymbol{u}_1(\omega)$. However we observe that $\boldsymbol{u}_2(\omega)$ is not $\mathbf{L}^2(\Omega)$-orthogonal to $\boldsymbol{u}_0$ since $\int_\Omega \boldsymbol{u}_2(\omega) \cdot \boldsymbol{u}_0 = -\frac{1}{2}\int_\Omega \|\boldsymbol{u}_1(\omega)\|^2$ and $\boldsymbol{u}_1(\omega)$ is not null, which is clear from (4.8). Therefore, the strategy to consider is to look for $\widetilde{\boldsymbol{u}}_2(\omega)$ such that

$$\widetilde{\boldsymbol{u}}_2(\omega) = \boldsymbol{u}_2(\omega) + a\boldsymbol{u}_0$$

and $a \in \mathbb{R}$ is taken so that $\widetilde{\boldsymbol{u}}_2(\omega)$ is $\mathbf{L}^2(\Omega)$-orthogonal to $\boldsymbol{u}_0$. Such $\mathbf{L}^2$-orthogonal condition gives

$$a = a(\omega) = \frac{1}{2}\|\boldsymbol{u}_1(\omega)\|^2_{\mathbf{L}^2(\Omega)} \leq C.$$

It is easy to check that $\widetilde{\boldsymbol{u}}_2(\omega)$ verifies the equation (4.9). Analysis similar to that in the estimate of $\|\boldsymbol{u}_1(\omega)\|_{\mathbf{H}_1(\Omega)}$ shows that $\|\widetilde{\boldsymbol{u}}_2(\omega)\|_{\mathbf{H}^1(\Omega)} \leq C$ with C independent of ω. Therefore

$$\|\boldsymbol{u}_2(\omega)\|_{\mathbf{H}^1(\Omega)} \leq a(\omega)\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} + C \leq C'.$$

We are now in a position to show

Proposition 4.1 *There exists a constant C, independent of ω, such that*

$$|\lambda(\omega) - \lambda^2_\varepsilon(\omega)| \leq C\varepsilon^3.$$

Proof of Proposition 4.1: By virtue of the assertion done below of the definition (2) it is sufficient to prove that

$$\| - \operatorname{div} A_\omega e(\boldsymbol{u}^2_\varepsilon(\omega)) - \lambda^2_\varepsilon(\omega)\boldsymbol{u}^2_\varepsilon(\omega)\|_{\mathbf{H}^{-1}(\Omega)} \leq C\varepsilon^3 \|\boldsymbol{u}^2_\varepsilon(\omega)\|_{\mathbf{H}^{-1}(\Omega)}. \tag{4.15}$$

Indeed, an easy computation shows that

$$\begin{aligned}&- \operatorname{div} A_\omega e(\boldsymbol{u}^2_\varepsilon(\omega)) - \lambda^2_\varepsilon(\omega)\boldsymbol{u}^2_\varepsilon(\omega) = \\ &\qquad \Big(-\operatorname{div} \chi_\omega Be(\boldsymbol{u}_2(\omega)) - \lambda_1(\omega)\boldsymbol{u}_2(\omega) - \lambda_2(\omega)\boldsymbol{u}_1(\omega)\Big)\varepsilon^3 + \Big(-\lambda_2(\omega)\boldsymbol{u}_2(\omega)\Big)\varepsilon^4.\end{aligned}$$

Consequently, according to the above estimates, the canonical injections $\mathbf{H}_0^1 \hookrightarrow \mathbf{L}^2 \hookrightarrow \mathbf{H}^{-1}$, and, considering $\varepsilon \ll 1$, we may obtain

$$\| - \operatorname{div} A_\omega e(\boldsymbol{u}_\varepsilon^2(\omega)) - \lambda_\varepsilon^2(\omega)\boldsymbol{u}_\varepsilon^2(\omega)\|_{\mathbf{H}^{-1}(\Omega)} \leq C\varepsilon^3. \tag{4.16}$$

In order to finish the proof, we make the following estimate:

$$\begin{aligned}\|\boldsymbol{u}_\varepsilon^2(\omega)\|_{\mathbf{H}^{-1}} = \sup_{\boldsymbol{\varphi}\in\mathbf{H}_0^1\setminus\{\mathbf{0}\}} \frac{\langle \boldsymbol{u}_\varepsilon^2(\omega), \boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}_0^1}}{\|\boldsymbol{\varphi}\|_{\mathbf{H}^1}} \geq \frac{\langle \boldsymbol{u}_\varepsilon^2(\omega), \boldsymbol{u}_0\rangle_{\mathbf{H}^{-1}\times\mathbf{H}_0^1}}{\|\boldsymbol{u}_0\|_{\mathbf{H}^1}} \\ = \frac{1+\varepsilon^2\displaystyle\int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{u}_2(\omega)}{\|\boldsymbol{u}_0\|_{\mathbf{H}^1}} = \frac{1-\varepsilon^2\displaystyle\int_\Omega \|\boldsymbol{u}_1(\omega)\|^2}{\|\boldsymbol{u}_0\|_{\mathbf{H}^1}} \geq \frac{1-\varepsilon^2 C}{\|\boldsymbol{u}_0\|_{\mathbf{H}^1}}.\end{aligned}$$

Finally, taking $\varepsilon^2 \leq 1/2C$ we thus get

$$\|\boldsymbol{u}_\varepsilon^2(\omega)\|_{\mathbf{H}^{-1}} \geq \frac{1}{2\|\boldsymbol{u}_0\|_{\mathbf{H}^1}}. \tag{4.17}$$

Gathering (4.16) and (4.17) we deduce (4.15) and hence the proof of proposition (4.1) is completed. ∎

Minimization of the second order approximation by relaxation process

Proposition 4.1 motivates the study of the following optimization problem

$$\inf_{\substack{\omega\subset\Omega\\|\omega|=\alpha|\Omega|}} \lambda_\varepsilon^2(\omega). \tag{4.18}$$

Since ε is taken positively and λ_0 is independent of ω, the optimization problem (4.18) may be reduced equivalently to the optimization problem

$$\inf_{\substack{\omega\subset\Omega\\|\omega|=\alpha|\Omega|}} \left[\lambda_1(\omega) + \varepsilon\lambda_2(\omega) = \int_\Omega \chi_\omega B e(\boldsymbol{u}_0) : (e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u}_1(\omega)))\right] \tag{4.19}$$

where $\boldsymbol{u}_1(\omega)$ is the solution of the system (4.8) with $\lambda_1(\omega)$ given by (4.11) and $\boldsymbol{u}_1(\omega)$ $\mathbf{L}^2(\Omega)$-orthogonal to $\boldsymbol{u}_0$.

Since the infimum in (4.19) is not usually reached at a given ω, we have to consider a relaxed version corresponding a situation of homogenisation (see, for instance, [11]). Let us then consider the following optimization problem

$$\inf_{\chi\in\mathcal{U}_{ad}} \left[F(\chi) = \int_\Omega \chi B e(\boldsymbol{u}_0) : (e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u}(\chi)))\right] \tag{4.20}$$

where, as usual in this type of problems, the admissible set $\mathcal{U}_{ad}$ is expressed by

$$\mathcal{U}_{ad} = \{\chi \in L^\infty(\Omega\,; \{0,1\}) \mid \textstyle\int_\Omega \chi = \alpha|\Omega|\}.$$

and $\boldsymbol{u}(\chi)$, which we denote briefly by $\boldsymbol{u}$, satisfies the conditions

$$\begin{cases} -\text{div } A^0 e(\boldsymbol{u}) - \lambda_0 \boldsymbol{u} = \lambda \boldsymbol{u}_0 + \text{div } \chi B e(\boldsymbol{u}_0) \text{ in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega \\ \int_\Omega \boldsymbol{u} \cdot \boldsymbol{u}_0 = 0. \end{cases} \tag{4.21}$$

Here $\lambda = \lambda(\chi) = \int_\Omega \chi B e(\boldsymbol{u}_0) : e(\boldsymbol{u}_0)$.

Below we perform the general relaxation process. In this direction, we first introduce the weak* topology of $L^\infty(\Omega)$ and then we compute the *lower semi-continuous envelope* (*l.s.e.* for short) of the functional F, denoting by $\bar{F}$, in the weak* topology.

The sequential characterization of the *l.s.e.* (see, for instance, [22]) allows us to argue by sequence limits.

Let χ_n be an arbitrary but fixed sequence in $\mathcal{U}_{ad}$ such that $\chi_n \overset{*}{\rightharpoonup} \theta \in \overline{\mathcal{U}_{ad}}^* = \{\phi \in L^\infty(\Omega\,;[0,1]) \mid \int_\Omega \phi = \alpha|\Omega|\}$, and let $\boldsymbol{u}_n$ be the solution of (4.21) with χ_n replaced by χ and $\lambda_n := \lambda(\chi_n)$.

In order to obtain an explicit limit of $F(\chi_n)$, we use the isotropic assumption for the symmetric tensor A^0, that is,

$$A^0 = 2\mu_{A^0} I_4 + \lambda_{A^0} I_2 \otimes I_2,$$

where μ_{A^0} is a shear moduli, λ_{A^0} is a Lammé coefficient, I_2 is the identity matrix of order $d \times d$ and $I_2 \otimes I_2$ is the tensor product of the matrix I_2.

Under the above assumptions, we have the following

Proposition 4.2 *The explicit computation for the l.s.e. in the limit density θ is given by*

$$\bar{F}(\theta) = \int_\Omega \theta B e(\boldsymbol{u}_0) : \big(e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u}(\theta))\big) - \varepsilon \int_\Omega \theta(1-\theta) g(\xi^*) \tag{4.22}$$

where:

- *$\boldsymbol{u}(\theta)$ is the unique solution of the system (4.21) with θ replaced by χ and $\lambda = \lambda(\theta)$.*
- *$g(\xi^*)$ is a function in $\mathscr{C}^0(\bar{\Omega})$ which depends on an optimal direction $\xi^* = \xi^*(x) \in \mathbb{S}^{d-1}$ (the unit sphere in $\mathbb{R}^d$), $x \in \Omega$, by the expression*

$$g(\xi^*) = \max_{\xi \in \mathbb{S}^{d-1}} \left[g(\xi) := \frac{1}{\mu_{A^0}} \Big(|\sigma^0 \xi|^2 - (\sigma^0 \xi \cdot \xi)^2 \Big) + \frac{1}{2\mu_{A^0} + \lambda_{A^0}} (\sigma^0 \xi \cdot \xi)^2 \right]$$

with $\sigma^0 := B e(\boldsymbol{u}_0)$.

Remark 4.1 *[1] shows the explicit formulae to $g(\xi^*)$ (see Prop. 2.3.20 and Lemma 2.3.21, pg. 151) which depends on the optimal direction ξ^* being a combination of the eigenvectors associated with the respective extremal eigenvalues of σ^0. Indeed, denoting by $\eta_1 \le \cdots \le \eta_d$ the eigenvalues of the symmetrical matrix σ^0 yields if $d = 2$ or $\lambda_{A^0} \ge 0$, the maximum value of g on $\mathbb{S}^{d-1}$ is given by*

$$g(\xi^*) = \begin{cases} \dfrac{(\eta_1 - \eta_d)^2}{4\mu_{A^0}} + \dfrac{(\eta_1 + \eta_d)^2}{4(\lambda_{A^0} + \mu_{A^0})} & \text{if } \eta_d > \dfrac{2\mu_{A^0} + \lambda_{A^0}}{2(\mu_{A^0} + \lambda_{A^0})}(\eta_1 + \eta_d) > \eta_1 \\ \dfrac{\eta_1^2}{2\mu_{A^0} + \lambda_{A^0}} & \text{if } \eta_1 \ge \dfrac{2\mu_{A^0} + \lambda_{A^0}}{2(\mu_{A^0} + \lambda_{A^0})}(\eta_1 + \eta_d) \\ \dfrac{\eta_d^2}{2\mu_{A^0} + \lambda_{A^0}} & \text{if } \eta_d \le \dfrac{2\mu_{A^0} + \lambda_{A^0}}{2(\mu_{A^0} + \lambda_{A^0})}(\eta_1 + \eta_d). \end{cases} \tag{4.23}$$

ξ^* is a combination of the eigenvectors associated with the extremal eigenvalues in the first case and ξ^* is an eigenvector associated to respective eigenvalue $\eta_{1,d}$ in the other cases.

Proof of Proposition 4.2: The $\mathbf{L}^2$-orthogonal condition in (4.21) applied to $\boldsymbol{u}_n$ implies $\boldsymbol{u}_n$ is bounded in $\mathbf{H}_0^1(\Omega)$. Consequently (up to subsequence) $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$, $\mathbf{H}_0^1$-weak, and hence $\boldsymbol{u}_n \to \boldsymbol{u}$, $\mathbf{L}^2$-strong. Therefore, letting $n \to \infty$ in the variational formulation of (4.21), we see that $\boldsymbol{u} = \boldsymbol{u}(\theta)$ is the solution of (4.21) with θ replaced by χ.

We now compute the limit of $F(\chi_n)$. To that end, we divide $F(\chi_n)$ as

$$F(\chi_n) = \lambda_n + \varepsilon \int_\Omega e(\boldsymbol{w}_n) : \chi_n \sigma^0 + \varepsilon \int_\Omega e(\boldsymbol{z}_n) : \chi_n \sigma^0, \tag{4.24}$$

where $\boldsymbol{w}_n \in \mathbf{H}_0^1(\Omega)$ is the solution of

$$- \operatorname{div} A^0 e(\boldsymbol{w}_n) = \lambda_0 \boldsymbol{u}_n + \lambda_n \boldsymbol{u}_0 \quad \text{ in } \Omega, \tag{4.25}$$

and $\boldsymbol{z}_n \in \mathbf{H}_0^1(\Omega)$ is the solution of

$$- \operatorname{div} A^0 e(\boldsymbol{z}_n) = \operatorname{div} \chi_n \sigma^0 \quad \text{ in } \Omega. \tag{4.26}$$

From the weak* convergence of χ_n it is clear that

$$\lim_{n\to\infty} \lambda_n = \lambda(\theta) = \int_\Omega e(\boldsymbol{u}_0) : \theta\, \sigma^0 \tag{4.27}$$

as well as, from (4.25) and the $\mathbf{L}^2$-strong convergence of $\boldsymbol{u}_n$ $\boldsymbol{w}_n \to \boldsymbol{w}$, $\mathbf{H}_0^1$-strong, and thus

$$\lim_{n\to\infty} \int_\Omega e(\boldsymbol{w}_n) : \chi_n \sigma^0 = \int_\Omega e(\boldsymbol{w}) : \theta\, \sigma^0, \tag{4.28}$$

where $\boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$ verifies (4.25) with $\boldsymbol{u}(\theta), \lambda(\theta)$ replaced by $\boldsymbol{u}_n, \lambda_n$ respectively.

The limit of the third term in (4.24) must be calculated carefully since the product $\chi_n e(\boldsymbol{z}_n)$ does not give a direct convergence. The key is to consider the H-measure approach (see [3] and the references given there).

In the context of the pseudo-differential operators we easily see from (4.26) that $e(\boldsymbol{z}_n)$ depends linearly on χ_n and, moreover, $e(\boldsymbol{z}_n) = Q(\chi_n)$ with Q a pseudo-differential operator, homogeneous of order 0, which has the following associated symbol

$$q(x,\xi) = -\frac{\sigma^0\xi \otimes \xi + \xi \otimes \sigma^0 \xi}{2\mu_{A^0}|\xi|^2} + \frac{(\mu_{A^0} + \lambda_{A^0})(\sigma^0\xi \cdot \xi)\xi \otimes \xi}{\mu_{A^0}(2\mu_{A^0} + \lambda_{A^0})|\xi|^4}.$$

Therefore we can assert that the limit of the third term in (4.24) is given by (see [3], Th. 2.2 and Lemma 2.3)

$$\lim_{n\to\infty} \int_\Omega Q(\chi_n) : \chi_n \sigma^0 = \int_\Omega Q(\theta) : \theta\, \sigma^0 - \int_\Omega \int_{\mathbb{S}^{d-1}} q(x,\xi) : \theta(1-\theta)\, \sigma^0\, \nu(\mathrm{d}x, \mathrm{d}\xi) \tag{4.29}$$

where $Q(\theta)$ is the respective pseudo-differential operator of $e(\boldsymbol{z})$, being $\boldsymbol{z} \in \mathbf{H}_0^1(\Omega)$ the solution of (4.26) with θ replaced by χ_n, and, for given x, $\nu(\mathrm{d}x, \mathrm{d}\xi)$ is a probability measure with respect to ξ.

An easy computation shows that

$$g(\xi) := q(x,\xi) : \sigma^0 = \frac{1}{\mu_{A^0}}\Big(|\sigma^0\xi|^2 - (\sigma^0\xi\cdot\xi)^2\Big) + \frac{1}{2\mu_{A^0}+\lambda_{A^0}}(\sigma^0\xi\cdot\xi)^2 \tag{4.30}$$

for any $\xi \in \mathbb{S}^{d-1}$.

Gathering (4.27), (4.28), (4.29) and (4.30), we finally get the limit of (4.24), namely

$$\lim_{n\to\infty} F(\chi_n) = \int_\Omega \big(e(\boldsymbol{u}_0) + \varepsilon e(\boldsymbol{u}(\theta))\big) : \theta\,\sigma^0 - \varepsilon\int_\Omega \theta(1-\theta)\int_{\mathbb{S}^{d-1}} g(\xi)\,\nu(x,\mathrm{d}\xi)\mathrm{d}x. \tag{4.31}$$

We notice that the objective function $\bar{F}$ may be expressed as $\inf_\nu \lim F(\chi_n)$ (ν depends on the sequence χ_n). Therefore we can conclude that the objective function is that given by (4.22) when we "*eliminate*" the measure ν in (4.31) by considering the optimal Dirac mass concentrated on $\xi^*(x)$ which maximizes the function $g(\xi)$ in $\mathbb{S}^{d-1}$ and thus the proof of Proposition 4.2 is completed. ■

Remark 4.2 *From the definition of the function $g(\xi)$ in (4.30) it is evident that ξ^* does not depend on θ and hence the objective function depends only on θ.*

The following proposition is aimed at giving an explicit directional derivative of the objective function $\bar{F}(\theta)$ which is clearly differentiable with respect to θ.

Proposition 4.3 *Let $\varphi \in L^\infty(\Omega;[0,1])$ be any directional limit density of the objective function $\bar{F}$. Then the directional derivative of $\bar{F}$ at φ is given by*

$$\bar{F}'(\theta)\varphi = \int_\Omega \varphi\,\theta\Big(Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - g(\xi^*)\Big) + 2\varepsilon\int_\Omega \varphi\,\Big(\theta g(\xi^*) + Be(\boldsymbol{u}_0) : e(\boldsymbol{u}(\theta))\Big). \tag{4.32}$$

Remark 4.3 *By (4.32) it is obvious that the Fréchet derivative of $\bar{F}$ is nothing more than*

$$\bar{F}'(\theta) = \theta\,\Big(Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - g(\xi^*)\Big) + 2\varepsilon\Big(\theta\, g(\xi^*) + Be(\boldsymbol{u}_0) : e(\boldsymbol{u}(\theta))\Big). \tag{4.33}$$

Proof of Proposition 4.3: Writing

$$\bar{F}(\theta) = \int_\Omega \theta\Big(Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon g(\xi^*)\Big) + \varepsilon\int_\Omega \theta^2 g(\xi^*) + \varepsilon\int_\Omega \theta Be(\boldsymbol{u}_0) : e(\boldsymbol{u}(\theta))$$

yields

$$\bar{F}'(\theta) = \int_\Omega \varphi\,\theta\Big(Be(\boldsymbol{u}_0) : e(\boldsymbol{u}_0) - \varepsilon g(\xi^*)\Big) + 2\varepsilon\int_\Omega \varphi\,\theta g(\xi^*) \\ + \varepsilon\int_\Omega \theta Be(\boldsymbol{u}_0) : e(\boldsymbol{u}(\varphi)) + \varepsilon\int_\Omega \varphi Be(\boldsymbol{u}_0) : e(\boldsymbol{u}(\theta))$$

since $\boldsymbol{u}$ depends linearly on the limit density.

Finally we check that

$$\int_\Omega \theta Be(\boldsymbol{u}_0) : e(\boldsymbol{u}(\varphi)) = \int_\Omega \varphi Be(\boldsymbol{u}_0) : e(\boldsymbol{u}(\theta)),$$

which concludes the proof of Proposition 4.3. Indeed, taking into account (4.21) with both θ, φ instead of χ and integrating by parts, we deduce that

$$\begin{aligned}
\int_\Omega \theta B e(\boldsymbol{u}_0) : e(\boldsymbol{u}(\varphi)) &= -\int_\Omega \operatorname{div}\, \theta B e(\boldsymbol{u}_0) \cdot \boldsymbol{u}(\varphi) \\
&= -\int_\Omega \Big[-\operatorname{div}\, A^0 e(\boldsymbol{u}(\theta)) - \lambda_0 \boldsymbol{u}(\theta) \Big] \cdot \boldsymbol{u}(\varphi) \\
&= \int_\Omega \operatorname{div}\, A^0 e(\boldsymbol{u}(\theta)) \cdot \boldsymbol{u}(\varphi) + \lambda_0 \int_\Omega \boldsymbol{u}(\varphi) \cdot \boldsymbol{u}(\theta) \\
&= -\int_\Omega A^0 e(\boldsymbol{u}(\varphi)) : e(\boldsymbol{u}(\theta)) + \lambda_0 \int_\Omega \boldsymbol{u}(\varphi) \cdot \boldsymbol{u}(\theta) \\
&= \int_\Omega \Big[\operatorname{div}\, A^0 e(\boldsymbol{u}(\varphi)) + \lambda_0 \boldsymbol{u}(\varphi) \Big] \cdot \boldsymbol{u}(\theta) \\
&= \int_\Omega -\operatorname{div}\, \varphi B e(\boldsymbol{u}_0) \cdot \boldsymbol{u}(\theta) \\
&= \int_\Omega \varphi B e(\boldsymbol{u}_0) : e(\boldsymbol{u}(\theta)).
\end{aligned}$$

■

4.3 Numerical illustrations

4.3.1 The bar

The numerical implementations for the bar have been performed under the following considerations: the dimensions of the bar are 3 units in the base and 1 unit in hight; the proportion volume is 0.4 in all cases and ε is always 10^{-6}. The first ten calculated eigenvalues for the first eigenvalue, λ_0, of the operator $-\alpha\Delta$ in the bar, are: 0.760116, 1.71331, 2.9541, 3.24305, 3.643, 3.91941, 5.55363, 6.11333, 7.23639, 9.5585 which confirms that this is a simple eigenvalue from a numerical point of view.

4.3.2 The long cantilever

4.3.3 The short cantilever

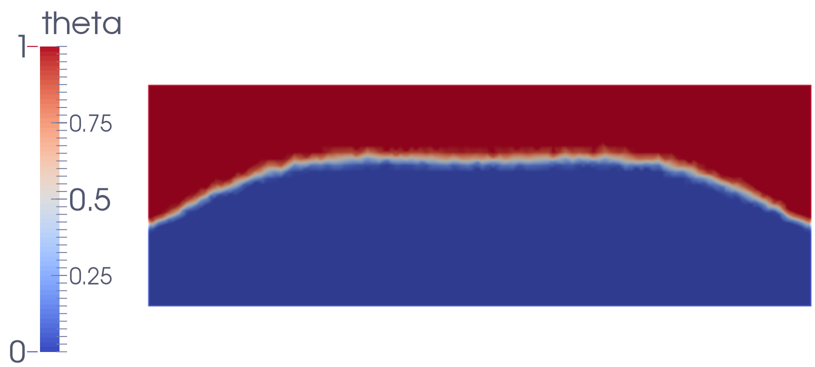


Figure 4.1: Minimization of the second order approximation of the first elasticity eigenvalue: bar, $\varepsilon = 10^{-6}$, $\alpha = 0.4$

Figure 4.2: Difference between first and second order models for the minimization of the first elasticity eigenvalue: bar, $\varepsilon = 10^{-6}$, $\alpha = 0.4$

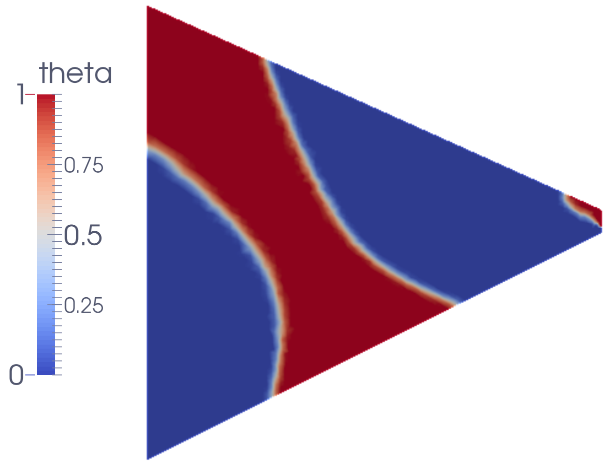


Figure 4.3: Minimization of the second order approximation of the first elasticity eigenvalue: long cantilever, $\varepsilon = 10^{-6}$, $\alpha = 0.4$

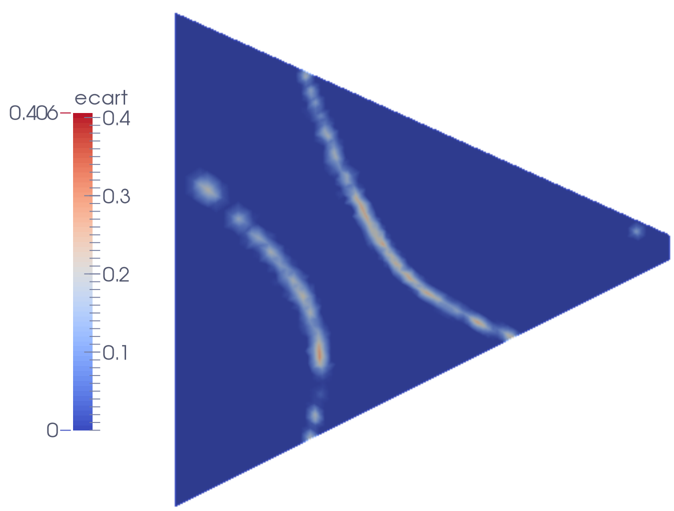


Figure 4.4: Difference between first and second order models for the minimization of the first elasticity eigenvalue: long cantilever, $\varepsilon = 10^{-6}$, $\alpha = 0.4$

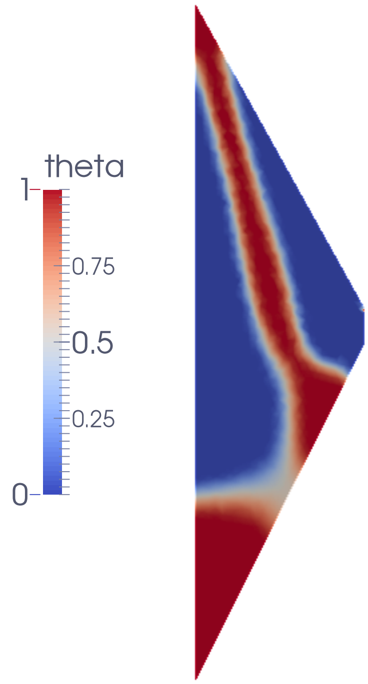


Figure 4.5: Minimization of the second order approximation of the first elasticity eigenvalue: short cantilever, $\varepsilon = 10^{-6}$, $\alpha = 0.4$

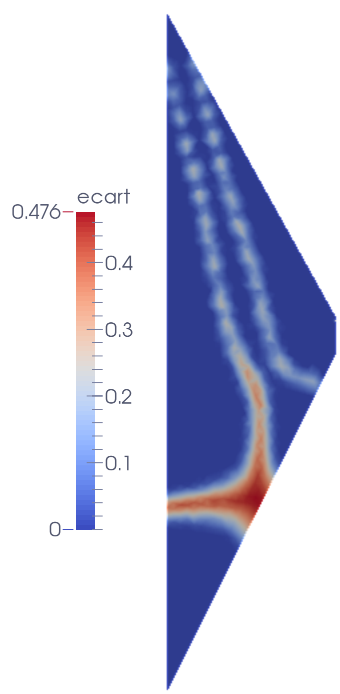


Figure 4.6: Difference between first and second order models for the minimization of the first elasticity eigenvalue: short cantilever, $\varepsilon = 10^{-6}$, $\alpha = 0.4$

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