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A distribution formula for Kashio's p-adic log-gamma function

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ABSTRACT

We study a special case of Kashio's p-adic $\log \Gamma$-function, that we call $\mathrm{Log}\Gamma_p$, which combines these of Morita and Diamond. It agrees with each of these on large parts of its domain and has the advantage of being a locally analytic function. We prove a distribution formula for $\mathrm{Log}\Gamma_p$ which generalizes and links the known distribution formulas for Diamond's and Morita's functions.

1. Introduction

Let p be a fixed prime number, and let $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ and $\mathcal{Z}_p$ denote, respectively, the ring of p-adic integers, the field of fractions of $\mathbb{Z}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, and the ring of integers of $\mathbb{C}_p$. For any $x \in \mathcal{Z}_p$, let $\overline{x}$ be its natural image in the residue field of $\mathbb{C}_p$, which is isomorphic to the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$. Also, if $x \in \mathcal{Z}_p$ and $\overline{x} \in \mathbb{F}_p$ we write $\ell(x)$ for the unique natural number satisfying $1 \leq \ell(x) \leq p$ and $\overline{x} = \overline{\ell(x)}$. Finally, we will use the convention $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

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In the mid 1970's, two p-adic analogues of the classical $\log\Gamma$-function were defined by Yasuo Morita [5] and Jack Diamond [3]. Here $\log\Gamma$ is the logarithm of the classical Γ-function satisfying the difference equation

$$\log\Gamma(x+1) - \log\Gamma(x) = \log(x) \qquad (x > 0). \tag{1}$$

The function $\log\Gamma$ satisfies the distribution formula

$$\sum_{k=0}^{n-1} \log\Gamma\left(\frac{x+k}{n}\right) = \log\Gamma(x) + \frac{n-1}{2}\log(2\pi) + \left(\frac{1}{2} - x\right)\log(n) \quad (x > 0, n \in \mathbb{N}). \tag{2}$$

Also, $\log\Gamma$ is the unique convex function defined on $(0, \infty)$ satisfying $\log\Gamma(1) = 0$ and the difference equation (1).

Morita [5] defined a p-adic analogue of Γ, which we will call Γ_{M}, having $\mathbb{Z}_p$ as its domain and taking values in the units $\mathbb{Z}_p^*$. For positive integers n, Γ_{M} is defined as

$$\Gamma_{\mathrm{M}}(n) := (-1)^n \prod_{\substack{1 \le j < n \\ p \nmid j}} j.$$

Morita proved that this function is continuous on $\mathbb{N}$ (with the p-adic topology) and extended to a continuous function on $\mathbb{Z}_p$. He also showed that $\Gamma_{\mathrm{M}} : \mathbb{Z}_p \to \mathbb{Z}_p^*$ satisfies the functional equation

$$\frac{\Gamma_{\mathrm{M}}(x+1)}{\Gamma_{\mathrm{M}}(x)} = \begin{cases} -x & \text{if } x \in \mathbb{Z}_p^*, \\ -1 & \text{if } x \in p\mathbb{Z}_p. \end{cases} \tag{3}$$

Since Γ_{M} is continuous on $\mathbb{Z}_p$, it is completely characterized by (3) and by its value $\Gamma_{\mathrm{M}}(1) = -1$.

To prove analytic properties of his function, Morita actually worked with the Iwasawa p-adic logarithm $\log_p$ of Γ_{M} [6, §V.4.5], [7, §45]. We will write this function $\mathrm{Log}\Gamma_{\mathrm{M}}$, i.e.,

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) := \log_p \Gamma_{\mathrm{M}}(x) \qquad\qquad (x \in \mathbb{Z}_p).$$

An important property of $\mathrm{Log}\Gamma_{\mathrm{M}}$ is that it has a power series expansion around 0, valid for all $x \in p\mathbb{Z}_p$, and this power series actually defines an analytic function on the open unit ball $B(0; 1^-) := \left\{x \in \mathbb{C}_p \,\middle|\, |x|_p < 1\right\} \subset \mathbb{C}_p$ [7, Lemma 58.2]. Hence, we can extend the domain of $\mathrm{Log}\Gamma_{\mathrm{M}}$ to $B(0; 1^-) \cup \mathbb{Z}_p$. It can be shown that with this extended definition $\mathrm{Log}\Gamma_{\mathrm{M}}$ is no longer the Iwasawa logarithm of a p-adic function.

Taking the Iwasawa logarithm on both sides of (3), we find that $\mathrm{Log}\Gamma_{\mathrm{M}}$ satisfies the difference equation

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x+1) - \mathrm{Log}\Gamma_{\mathrm{M}}(x) = \begin{cases} \log_p(x) & \text{if } x \in \mathbb{Z}_p^*, \\ 0 & \text{if } x \in p\mathbb{Z}_p, \end{cases} \tag{4}$$

in analogy to (1). It is again immediate that $\mathrm{Log}\Gamma_{\mathrm{M}}$ is uniquely determined by the value $\mathrm{Log}\Gamma_{\mathrm{M}}(1) = 0$ and by the difference equation (4). Morita's function satisfies the reflection formula [1, Prop. 11.5.13.(2)]

$$\mathrm{Log}\Gamma_{\mathrm{M}}(1-x) + \mathrm{Log}\Gamma_{\mathrm{M}}(x) = 0 \qquad (x \in \mathbb{Z}_p), \tag{5}$$

and can be given by the integral formula [7, §58]

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \int\limits_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\chi_{\mathbb{Z}_p^*}(x+t)dt \qquad (x \in \mathbb{Z}_p). \tag{6}$$

Here $\chi_{\mathbb{Z}_p^*}$ denotes the characteristic function of $\mathbb{Z}_p^*$, and the integral on the right is the Volkenborn integral: if $g : \mathbb{Z}_p \to \mathbb{C}_p$ then g is Volkenborn integrable if the limit

$$\int\limits_{\mathbb{Z}_p} g(t)dt := \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} g(j)$$

exists, and this value is called the Volkenborn integral of g [6, §V.5], [7, §55].

Morita's function satisfies, in analogy with (2), the distribution formula [1, Prop. 11.5.13.(4)]

$$\sum_{0\le j<n} \mathrm{Log}\Gamma_{\mathrm{M}}\left(\frac{x+j}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{M}}(x) - \left(x - \left[\frac{x}{p}\right]\right)\log_p(n) \quad (x \in \mathbb{Z}_p, n \in \mathbb{N}, p \nmid n), \tag{7}$$

where $\left[\frac{x}{p}\right]$ is defined as the p-adic limit of $\left[\frac{x_n}{p}\right]$ as $x_n \to x$ in $\mathbb{N}_0$. (This function is known as Dwork's shift map and it is also written by $x \mapsto x'$.)

Diamond [3] defined his p-adic analogue of the classical $\log\Gamma$-function, which we will write $\mathrm{Log}\Gamma_{\mathrm{D}}$, by the Volkenborn integral

$$\mathrm{Log}\Gamma_{\mathrm{D}}(x) := \int\limits_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)dt \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p). \tag{8}$$

Diamond showed that his function is locally analytic on $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ and that it satisfies the difference equation [3, Theorem 5]

$$\mathrm{Log}\Gamma_{\mathrm{D}}(x+1) - \mathrm{Log}\Gamma_{\mathrm{D}}(x) = \log_p(x) \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p), \tag{9}$$

in analogy to (1), and the reflection formula [3, Theorem 8]

$$\mathrm{Log}\Gamma_{\mathrm{D}}(1-x) + \mathrm{Log}\Gamma_{\mathrm{D}}(x) = 0 \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p). \tag{10}$$

Diamond's function also satisfies the distribution formula [3, Theorem 7], [7, Theorem 60.2.(iii)]

$$\sum_{0\le j<n} \mathrm{Log}\Gamma_{\mathrm{D}}\left(\frac{x+j}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{D}}(x) - \left(x-\frac{1}{2}\right)\log_p(n) \quad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p, n \in \mathbb{N}). \tag{11}$$

The functions of Morita and Diamond are connected by the distribution formula [1, Prop. 11.5.17.(4)]

$$\sum_{\substack{0\le j<n \\ p\nmid(x+j)}} \mathrm{Log}\Gamma_{\mathrm{D}}\left(\frac{x+j}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{M}}(x) - \left(x-\left\lceil\frac{x}{p}\right\rceil\right)\log_p(n) \qquad (x \in \mathbb{Z}_p, p\,|\,n). \tag{12}$$

Comparing formulas (6) and (8), we notice that $\mathrm{Log}\Gamma_{\mathrm{M}}$ and $\mathrm{Log}\Gamma_{\mathrm{D}}$ have very similar expressions involving a Volkenborn integral. Equations (5) and (10) show that $\mathrm{Log}\Gamma_{\mathrm{M}}$ and $\mathrm{Log}\Gamma_{\mathrm{D}}$ have identical reflection formulas, and (4) and (9) show that these functions satisfy similar difference equations. Equation (12) hints at a connection between these two functions. Also, the domains of $\mathrm{Log}\Gamma_{\mathrm{M}}$ and $\mathrm{Log}\Gamma_{\mathrm{D}}$ are disjoint and complementary in $\mathbb{C}_p$. Finally, we mention that if we could extend $\mathrm{Log}\Gamma_{\mathrm{D}}$ to $\mathbb{C}_p$, then the difference equation would force $\mathrm{Log}\Gamma_{\mathrm{D}}$ to be discontinuous at either the positive integers or the negative integers. Since both these sets are dense in $\mathbb{Z}_p$, $\mathrm{Log}\Gamma_{\mathrm{D}}$ cannot be extended continuously to any point of $\mathbb{Z}_p$. In particular, we do not obtain a continuous function on $\mathbb{C}_p$ by extending the domain of $\mathrm{Log}\Gamma_{\mathrm{D}}$ defining $\mathrm{Log}\Gamma_{\mathrm{D}}(x) := \mathrm{Log}\Gamma_{\mathrm{M}}(x)$ for $x \in \mathbb{Z}_p$.

In 2005, Tomokazu Kashio [4] defined a p-adic $\log\Gamma$-function which combines these of Morita and Diamond. His definition is actually very general and here we work with its simplest case. We take the approach due to Diamond, that is, we work with locally analytic functions and with the Volkenborn integral. It is worth mentioning that Kashio's definition involves the Volkenborn integral but without mentioning it.

Kashio's definition, in terms of the Volkenborn integral, is as follows. Define $\mathrm{Log}\Gamma_p : \mathbb{C}_p \to \mathbb{C}_p$ by the Volkenborn integral

$$\mathrm{Log}\Gamma_p(x) := \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t)-1\big)\chi(x+t)dt,$$

where χ is the characteristic function of the complement of the open unit ball $B(0;1^-)^{\mathsf{c}} := \left\{x \in \mathbb{C}_p \,\middle|\, |x|_p \ge 1\right\}$. This function will be proved to be locally analytic on $\mathbb{C}_p$, and to satisfy the difference equation (Proposition 3.2)

$$\mathrm{Log}\Gamma_p(x+1) - \mathrm{Log}\Gamma_p(x) = \chi(x)\log_p(x) \qquad (x \in \mathbb{C}_p),$$

as well as the reflection formula (Proposition 3.4)

$$\mathrm{Log}\Gamma_p(1-x) + \mathrm{Log}\Gamma_p(x) = 0 \qquad (x \in \mathbb{C}_p).$$

On certain sub-domains of $\mathbb{C}_p$, $\mathrm{Log}\Gamma_p$ coincides with Morita's function, on other with Diamond's. Namely, in §5 we will show that

$$\mathrm{Log}\Gamma_p(x) = \mathrm{Log}\Gamma_{\mathrm{M}}(x) \qquad (x \in \mathbb{Z}_p \text{ or } |x|_p < 1),$$
$$\mathrm{Log}\Gamma_p(x) = \mathrm{Log}\Gamma_{\mathrm{D}}(x) \qquad (|x|_p > 1).$$

For $|x|_p = 1$ the relation between $\mathrm{Log}\Gamma_p$ and Diamond's function is more subtle (see Propositions 5.1 and 5.2).

In §4 we will prove a distribution formula for $\mathrm{Log}\Gamma_p$ (Proposition 4.8). Let $x \in \mathbb{C}_p$, and fix $n = mp^r \in \mathbb{N}$. We define next a finite sequence $x_i \in \mathbb{C}_p$ of length at most $r + 1$. Write $x_0 := x$. If $x_0 \notin \mathcal{Z}_p$, or $x_0 \in \mathcal{Z}_p$ but $\overline{x_0} \notin \mathbb{F}_p$, then the sequence stops. If $x_0 \in \mathcal{Z}_p$ and $\overline{x_0} \in \mathbb{F}_p$, then define

$$x_1 := \frac{x_0 + p - \ell(x_0)}{p}.$$

We repeat the above process: if $x_j \notin \mathcal{Z}_p$, or $x_j \in \mathcal{Z}_p$ but $\overline{x_j} \notin \mathbb{F}_p$, then the sequence stops. If $x_j \in \mathcal{Z}_p$ and $\overline{x_j} \in \mathbb{F}_p$, then

$$x_{j+1} := \frac{x_j + p - \ell(x_j)}{p}.$$

Let s the least non-negative integer, or $+\infty$, such that $x_s \notin \mathcal{Z}_p$, or $x_s \in \mathcal{Z}_p$ but $\overline{x_s} \notin \mathbb{F}_p$, and set $\omega = \min(r, s)$. Define our sequence to be $x_0, \ldots, x_\omega$. Notice that the length of the sequence is $\omega + 1 \leq r + 1$. Finally, define $\mathrm{R}_p : \mathbb{C}_p \to \mathbb{C}_p$ by

$$\mathrm{R}_p(x) := \begin{cases} x - \dfrac{x}{p} + \dfrac{\alpha}{p} - \left\lceil \dfrac{\alpha}{p} \right\rceil & \text{if } |x - \alpha|_p < 1 \text{ for some } \alpha \in \mathbb{Z}, \\ x - \dfrac{1}{2} & \text{otherwise,} \end{cases}$$

where $\lceil c \rceil$ is the usual integer ceiling function for $c \in \mathbb{Q}$. Our distribution formula is

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_p\left(\frac{x+k}{n}\right) = \sum_{j=0}^{\omega} \mathrm{Log}\Gamma_p(x_j) \;-\; \log_p(n) \sum_{j=0}^{\omega} \mathrm{R}_p(x_j). \tag{13}$$

As we shall show in §5, the distribution formulas (7) and (11) are now special cases of (13) for $x \in \mathbb{Z}_p$ and $|x|_p > 1$, respectively. Also, we will see that formula (13) may be viewed as a generalization of the restricted distribution formula (12), in the sense that to be able to omit the restriction in sum in the left hand of (12), then there must appear other terms in its right hand.

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2. Locally analytic functions and the Volkenborn integral

Let $D = \{x \in \mathbb{C}_p \,\big|\, |x - y|_p < r\}$ be an open ball in $\mathbb{C}_p$ with center $y \in D$ and with positive radius r. We will call a function $f : D \to \mathbb{C}_p$ analytic on D if f can be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - y)^n$$

convergent for all $x \in D$, where $a_n \in \mathbb{C}_p$ for all $n \in \mathbb{N}_0$. Now, let A be any subset of $\mathbb{C}_p$. We will call a function $f : A \to \mathbb{C}_p$ locally analytic on A if for each $a \in A$ there is an open ball $D \subset A$ with positive radius, that contains a, such that f is analytic on D. It is easily seen that we may replace the word open for the word closed in this definition.

The following result, due to Diamond [3], will allow us to define Kashio's p-adic $\log \Gamma$-function in terms of the Volkenborn integral.

Proposition 2.1. *Let $f : \mathbb{C}_p \to \mathbb{C}_p$ be a locally analytic function on $\mathbb{C}_p$. Then, for all $b \in \mathbb{N}$, the limit*

$$F(x) := \lim_{n\to\infty} \frac{1}{bp^n} \sum_{j=0}^{bp^n-1} f(x + j) \tag{14}$$

exists and is independent of b. Moreover, $F(x)$ defines a locally analytic function on $\mathbb{C}_p$, and we have the identity

$$F'(x) = \lim_{n\to\infty} \frac{1}{bp^n} \sum_{j=0}^{bp^n-1} f'(x + j). \tag{15}$$

Proof. The existence of (14) is a special case of [3, p. 324, Corollary], and the identity (15) follows immediately from [3, Theorem 3]. □

If $g : \mathbb{Z}_p \to \mathbb{C}_p$ we say that g is Volkenborn integrable if the limit

$$\int_{\mathbb{Z}_p} g(t)dt := \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} g(j)$$

exists, and we will call it the Volkenborn integral of g [6, §V.5], [7, §55]. We can now restate Proposition 2.1 using the Volkenborn integral.

Lemma 2.2. *Let $f : \mathbb{C}_p \to \mathbb{C}_p$ be a locally analytic function on $\mathbb{C}_p$. Then the Volkenborn integral*

$$F(x) := \int_{\mathbb{Z}_p} f(x+t)dt \tag{16}$$

exists and $F(x)$ defines a locally analytic function on $\mathbb{C}_p$. Moreover, we can differentiate under the integral sign, that is

$$F'(x) = \int_{\mathbb{Z}_p} f'(x+t)dt.$$

Proof. This follows immediately from the definition of the Volkenborn integral, letting $b = 1$ in Proposition 2.1. □

Remark. The Volkenborn integral is usually defined for strictly differentiable functions [6, §V.1.1], [7, §27]. Let X be any non-empty subset of $\mathbb{C}_p$ with no isolated points and let $f : X \to \mathbb{C}_p$. We say that f is strictly differentiable at a point $a \in X$ if

$$\lim_{(x,y)\to(a,a)} \frac{f(x) - f(y)}{x - y}$$

exists, where we take the limit over $x, y \in X$ such that $x \neq y$. We say that f is strictly differentiable on X, or that $f \in C^1(X)$, if f is strictly differentiable for all $a \in X$. If $f \in C^1(\mathbb{Z}_p)$, then the Volkenborn integral

$$\int_{\mathbb{Z}_p} f(t)dt := \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} f(j)$$

of f exists [6, §V.5.1], [7, §55]. All the properties of the Volkenborn integral that we will mention from [2,6,7] are proved there for strictly differentiable functions. Since any locally analytic function on an open set $X \subset \mathbb{C}_p$ is strictly differentiable on X [7, Corollary 29.11], these properties hold for locally analytic functions on X.

Perhaps the simplest non-trivial property of a function F defined by (16) is that it satisfies the difference equation [3, Theorem 4], [7, Prop. 55.5]

$$F(x+1) - F(x) = f'(x) \qquad (x \in \mathbb{C}_p). \tag{17}$$

We will also need the following "distribution" and "integration by parts" formulas.

Lemma 2.3. *Let $f : \mathbb{C}_p \to \mathbb{C}_p$ be a locally analytic function on $\mathbb{C}_p$. Then for all $N \in \mathbb{N}$*

$$\int_{\mathbb{Z}_p} f(t)dt = \frac{1}{N} \sum_{k=0}^{N-1} \int_{\mathbb{Z}_p} f(k+Nt)dt.$$

Proof. See [7, §55]. □

Lemma 2.4. *Let f and F be related as in Lemma 2.2. Then we have the identity*

$$\int_{\mathbb{Z}_p} F(x+t)dt = F(x) + (x-1)F'(x) - \int_{\mathbb{Z}_p} (x+t)f'(x+t)dt, \tag{18}$$

valid for all $x \in \mathbb{C}_p$.

Proof. See [2, Lemma 2.2]. □

3. Kashio's p-adic log-gamma function

Let $\varphi_p : \mathbb{C}_p \to \mathbb{C}_p$ be the function defined by

$$\varphi_p(x) := \begin{cases} 0 & \text{if } |x|_p < 1, \\ x\log_p(x) - x & \text{if } |x|_p \geq 1. \end{cases} \tag{19}$$

If we call χ the characteristic function of the set $B(0;1^-)^\mathsf{c} = \{x \in \mathbb{C}_p \,|\, |x|_p \geq 1\}$, we can write

$$\varphi_p(x) = x\big(\log_p(x) - 1\big)\chi(x).$$

Since the open ball $B(0;1^-)$ is also closed, its complement in $\mathbb{C}_p$ is open, so in (19) we have defined φ_p by its restriction to disjoint open sets. Now, the null function is trivially analytic on $\mathbb{C}_p$, and so is the identity function. Also, $\log_p(x)$ is locally analytic on $B(0;1^-)^\mathsf{c}$, in fact on $\mathbb{C}_p^*$ [7, Prop. 45.7]. Thus the function $x\log_p(x) - x$ is also locally analytic on the open set $B(0;1^-)^\mathsf{c}$. Hence, the function φ_p is locally analytic on $\mathbb{C}_p$. Therefore, by Lemma 2.2, the following definition makes sense.

Definition 3.1. With notation as above, define the function $\mathrm{Log}\Gamma_p : \mathbb{C}_p \to \mathbb{C}_p$ by the Volkenborn integral

$$\mathrm{Log}\Gamma_p(x) := \int_{\mathbb{Z}_p} \varphi_p(x+t)dt.$$

Hence, we can write

$$\mathrm{Log}\Gamma_p(x) = \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\chi(x+t)dt,$$

and by Lemma 2.2, $\mathrm{Log}\Gamma_p$ is locally analytic on $\mathbb{C}_p$.

Remark. This is Kashio's p-adic $\log \Gamma$-function $L\Gamma_{p,1}(x,(1))$ (see [4, eq. 5.12]). His construction is much more general. He works with a multiple p-adic Hurwitz zeta-function, and he defines his multiple p-adic $\log \Gamma$-function by means of the derivative at zero of this p-adic Hurwitz zeta-function, as in the complex case.

The simplest property of $\mathrm{Log}\Gamma_p$ is its difference equation.

Proposition 3.2. *For all $x \in \mathbb{C}_p$ we have the difference equation*

$$\mathrm{Log}\Gamma_p(x+1) - \mathrm{Log}\Gamma_p(x) = \chi(x)\log_p(x).$$

Proof. This follows from (17), noticing that

$${\varphi_p}'(x) = \chi(x)\log_p(x), \tag{20}$$

where χ is the characteristic function of the set $B(0;1^-)^{\mathrm{c}} = \left\{x \in \mathbb{C}_p \,\middle|\, |x|_p \geq 1\right\}$. $\square$

Kashio proved the above formula in [4, Lemma 5.5] but with a mistake. He claims that $\mathrm{Log}\Gamma_p(x+1) - \mathrm{Log}\Gamma_p(x) = \log_p(x)$, that is, he omits the factor $\chi(x)$.

The function $\mathrm{Log}\Gamma_p$ satisfies a Raabe-type formula and a characterization theorem similar to the ones satisfied by Diamond's $\mathrm{Log}\Gamma_{\mathrm{D}}$ and Morita's $\mathrm{Log}\Gamma_{\mathrm{M}}$ [2, p. 364].

Theorem 3.3. *The function $\mathrm{Log}\Gamma_p$ satisfies*

$$\int\limits_{\mathbb{Z}_p} \mathrm{Log}\Gamma_p(x+t)dt = (x-1){\mathrm{Log}\Gamma_p}'(x) - \mathrm{r}_p(x) \qquad (x \in \mathbb{C}_p), \tag{21}$$

where $\mathrm{r}_p : \mathbb{C}_p \to \mathbb{C}_p$ is defined by the Volkenborn integral

$$\mathrm{r}_p(x) := \int\limits_{\mathbb{Z}_p} (x+t)\chi(x+t)dt. \tag{22}$$

Moreover, $\mathrm{Log}\Gamma_p$ is the unique locally analytic function $f : \mathbb{C}_p \to \mathbb{C}_p$ satisfying the difference equation

$$f(x+1) - f(x) = \chi(x)\log_p(x)$$

and the Volkenborn integro-differential equation

$$\int\limits_{\mathbb{Z}_p} f(x+t)dt = (x-1)f'(x) - \mathrm{r}_p(x).$$

Proof. First we prove formula (21). Using (18) and (20) we have

$$\begin{aligned}\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_p(x+t)dt &= \mathrm{Log}\Gamma_p(x) + (x-1)\mathrm{Log}\Gamma_p{}'(x) \\ &\quad - \int_{\mathbb{Z}_p} (x+t)\log_p(x+t)\chi(x+t)dt \\ &= \mathrm{Log}\Gamma_p(x) + (x-1)\mathrm{Log}\Gamma_p{}'(x) \\ &\quad - \mathrm{Log}\Gamma_p(x) - \int_{\mathbb{Z}_p} (x+t)\chi(x+t)dt \\ &= (x-1)\mathrm{Log}\Gamma_p{}'(x) - \mathrm{r}_p(x).\end{aligned}$$

The uniqueness claim is proved exactly as in [2]. □

There is a reflection formula for $\mathrm{Log}\Gamma_p$.

Proposition 3.4. *For all $x \in \mathbb{C}_p$ we have the reflection formula*

$$\mathrm{Log}\Gamma_p(1-x) + \mathrm{Log}\Gamma_p(x) = 0.$$

Proof. Follows exactly as in [2, Prop. 2.5]. □

The function r_p defined by (22) can be computed explicitly.

Proposition 3.5. *For $x \in \mathbb{C}_p$ we have*

$$\mathrm{r}_p(x) = \mathrm{R}_p(x) := \begin{cases} x - \dfrac{x}{p} + \dfrac{\alpha}{p} - \left\lceil \dfrac{\alpha}{p} \right\rceil & \text{if } |x-\alpha|_p < 1 \text{ for some } \alpha \in \mathbb{Z}, \\ x - \dfrac{1}{2} & \text{otherwise,} \end{cases}$$

where $\lceil c \rceil$ is the usual integer ceiling function for $c \in \mathbb{Q}$.

Remark. One easily checks that $|x-\alpha|_p < 1$ implies that $|x|_p \leq 1$, and that $\dfrac{\alpha}{p} - \left\lceil \dfrac{\alpha}{p} \right\rceil$ only depends on α modulo p.

Proof. We begin the proof with the easy case, which is when $|x|_p > 1$. If $|t|_p \leq 1$ then $|x+t|_p = |x|_p > 1$. Thus, $\chi(x+j) = 1$ for all $j \in \mathbb{N}_0$ and

$$\begin{aligned}\mathrm{r}_p(x) &= \int_{\mathbb{Z}_p} (x+t)\chi(x+t)dt = \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} (x+j)\chi(x+j) \\ &= \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} x \; + \; \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} j = x + \lim_{n\to\infty} \frac{p^n-1}{2} = x - \frac{1}{2}.\end{aligned}$$

Now, suppose that $|x|_p \le 1$. Then

$$\begin{aligned}
\mathrm{r}_p(x) &= \int_{\mathbb{Z}_p} (x+t)\chi(x+t)dt \\
&= \lim_{n\to\infty} \frac{1}{p^n} \sum_{0\le j<p^n} (x+j)\chi(x+j) \\
&= \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0\le j<p^n \\ |x+j|_p=1}} (x+j) \\
&= \lim_{n\to\infty} \frac{1}{p^n} \sum_{0\le j<p^n} (x+j) - \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0\le j<p^n \\ |x+j|_p<1}} (x+j) \\
&= x - \frac{1}{2} - \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0\le j<p^n \\ |x+j|_p<1}} (x+j). \qquad (23)
\end{aligned}$$

In the last sum above, if there is no j such that $|x+j|_p < 1$, this sum is 0. In this case we also have $\mathrm{r}_p(x) = x - 1/2$. The remaining case is when $|x|_p \le 1$ and $|x-\alpha|_p < 1$ for some integer α, which we may choose to satisfy $0 \le \alpha \le p-1$. Then, in the sum in (23), the condition $|x+j|_p < 1$ is equivalent to $|\alpha+j|_p < 1$. Since $\alpha + j \in \mathbb{N}_0$, this is equivalent to the simpler condition $p \mid (\alpha+j)$. Hence we have

$$\begin{aligned}
\sum_{\substack{0\le j<p^n \\ |x+j|_p<1}} (x+j) &= \sum_{\substack{0\le j<p^n \\ p|(\alpha+j)}} (x-\alpha+\alpha+j) = \sum_{\substack{\alpha\le i<\alpha+p^n \\ p|i}} (x-\alpha+i) \\
&= \sum_{\frac{\alpha}{p}\le j<\frac{\alpha}{p}+p^{n-1}} (x-\alpha+pj) = \sum_{\left\lceil\frac{\alpha}{p}\right\rceil\le j<\left\lceil\frac{\alpha}{p}\right\rceil+p^{n-1}} (x-\alpha+pj) \\
&= \sum_{0\le i<p^{n-1}} (x-\alpha+pi+p\left\lceil\frac{\alpha}{p}\right\rceil) \\
&= p^{n-1}(x-\alpha) + p^n \left\lceil\frac{\alpha}{p}\right\rceil + p \sum_{0\le i<p^{n-1}} i.
\end{aligned}$$

Replacing this in (23) we obtain

$$\begin{aligned}
\mathrm{r}_p(x) &= x - \frac{1}{2} - \lim_{n\to\infty} \frac{1}{p^n} \left(p^{n-1}(x-\alpha) + p^n \left\lceil\frac{\alpha}{p}\right\rceil + p \sum_{0\le i<p^{n-1}} i \right) \\
&= x - \frac{1}{2} - \frac{x}{p} + \frac{\alpha}{p} - \left\lceil\frac{\alpha}{p}\right\rceil - \lim_{n\to\infty} \frac{1}{p^{n-1}} \sum_{0\le i<p^{n-1}} i \\
&= x - \frac{x}{p} + \frac{\alpha}{p} - \left\lceil\frac{\alpha}{p}\right\rceil. \qquad \square
\end{aligned}$$

Corollary 3.6. *For $x \in \mathbb{Q}_p$ we have*

$$\mathrm{r}_p(x) = \mathrm{R}_p(x) = \begin{cases} x - \left\lceil \dfrac{x}{p} \right\rceil & \text{if } x \in \mathbb{Z}_p, \\ x - \dfrac{1}{2} & \text{otherwise,} \end{cases}$$

where $\left\lceil \frac{x}{p} \right\rceil$ is defined as the p-adic limit of $\left\lceil \frac{x_n}{p} \right\rceil$ as $x_n \to x$ in $\mathbb{N}_0$.

Proof. It is easily seen that the limit of $\left\lceil \frac{x_n}{p} \right\rceil$ exists and that $\left\lceil \frac{y}{p} \right\rceil = \frac{y}{p}$ if $y \in p\mathbb{Z}_p$. Now, if $x \in \mathbb{Z}_p$, then $x = \alpha + y$ where $|x - \alpha|_p < 1$ is such that $0 \le \alpha \le p - 1$ and where $y \in p\mathbb{Z}_p$. Then

$$x - \frac{x}{p} + \frac{\alpha}{p} - \left\lceil \frac{\alpha}{p} \right\rceil = x - \frac{y}{p} - \left\lceil \frac{x-y}{p} \right\rceil = x - \frac{y}{p} - \left\lceil \frac{x}{p} \right\rceil + \left\lceil \frac{y}{p} \right\rceil = x - \left\lceil \frac{x}{p} \right\rceil. \qquad \square$$

Remark. The function $\mathrm{r}_p(x)$ is a zeta value. More precisely, $\mathrm{r}_p(x) = -\zeta_{p,1}(0,(1),x)$, where $\zeta_{p,1}(s,(1),x)$ is Kashio's Hurwitz zeta function. In the complex case, the value at $s = 0$ of the Hurwitz zeta-function is $\zeta(0,x) = -x + 1/2$, which is in agreement with the p-adic case when $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$.

4. The distribution formula

Let $n = mp^r$ with $p \nmid m$, and let $g : \mathbb{C}_p \to \mathbb{C}_p$ be any p-adic function. Suppose we want to compute the sum $\sum_{k=0}^{n-1} g\left(\frac{x+k}{n}\right)$. Then we can write

$$\begin{aligned} \sum_{k=0}^{n-1} g\left(\frac{x+k}{n}\right) &= \sum_{k=0}^{mp^r-1} g\left(\frac{x+k}{mp^r}\right) = \sum_{i=0}^{m-1} \sum_{j=ip^r}^{p^r-1+ip^r} g\left(\frac{x+j}{mp^r}\right) \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{p^r-1} g\left(\frac{x+j+ip^r}{mp^r}\right) = \sum_{j=0}^{p^r-1} \sum_{i=0}^{m-1} g\left(\frac{\frac{x+j}{p^r}+i}{m}\right). \end{aligned}$$

Let $y = \frac{x+j}{p^r}$. If we can compute $\sum_{i=0}^{m-1} g\left(\frac{y+i}{m}\right)$, then we reduce the computation of the original sum to the case $n = p^r$. This can be done when $g = \mathrm{Log}\Gamma_p$.

Lemma 4.1. *Let $p \nmid m$. Then, for all $y \in \mathbb{C}_p$,*

$$\sum_{i=0}^{m-1} \mathrm{Log}\Gamma_p\left(\frac{y+i}{m}\right) = \mathrm{Log}\Gamma_p(y) - \log_p(m)\mathrm{R}_p(y).$$

Proof. Since $|m|_p = 1$,

$$\sum_{i=0}^{m-1} \mathrm{Log}\Gamma_p\Big(\frac{y+i}{m}\Big) = \sum_{i=0}^{m-1} \int\limits_{\mathbb{Z}_p} \Big(\frac{y+i}{m}+t\Big)\big(\log_p\Big(\frac{y+i}{m}+t\Big)-1\big)\chi\Big(\frac{y+i}{m}+t\Big)dt$$

$$= \frac{1}{m}\sum_{i=0}^{m-1} \int\limits_{\mathbb{Z}_p} (y+i+mt)\big(\log_p(y+i+mt)-\log_p(m)-1\big)\chi(y+i+mt)dt.$$

Using Lemma 2.3, we conclude that

$$\sum_{i=0}^{m-1} \mathrm{Log}\Gamma_p\Big(\frac{y+i}{m}\Big) = \int\limits_{\mathbb{Z}_p} (y+t)\big(\log_p(y+t)-\log_p(m)-1\big)\chi(y+t)dt$$

$$= \int\limits_{\mathbb{Z}_p} (y+t)\big(\log_p(y+t)-1\big)\chi(y+t)dt - \log_p(m)\int\limits_{\mathbb{Z}_p} (y+t)\chi(y+t)dt$$

$$= \mathrm{Log}\Gamma_p(y) - \log_p(m)\mathrm{R}_p(y). \qquad \square$$

By the above comments we obtain

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_p\left(\frac{x+k}{n}\right) = \sum_{j=0}^{p^r-1} \mathrm{Log}\Gamma_p\left(\frac{x+j}{p^r}\right) - \log_p(n)\sum_{j=0}^{p^r-1} \mathrm{R}_p\Big(\frac{x+j}{p^r}\Big). \tag{24}$$

Now we generalize a little bit to compute the two sums in the right hand of (24) working with only one function.

Let $f : \mathbb{C}_p \setminus \{0\} \to \mathbb{C}_p$, i.e., a function possibly not defined at 0. Suppose f is locally analytic and $f(px) = f(x)$ for all $x \in \mathbb{C}_p \setminus \{0\}$. Then clearly

$$f(p^k x) = f(x) \tag{25}$$

for all $k \in \mathbb{Z}$ and $x \in \mathbb{C}_p \setminus \{0\}$. Let F be the function defined by

$$F(x) = \int\limits_{\mathbb{Z}_p} (x+t)f(x+t)\chi(x+t)dt.$$

Then F is a locally analytic function over $\mathbb{C}_p$ (same proof as for $\mathrm{Log}\Gamma_p$).

Notice that if $f(x) = 1$ then $F(x) = \mathrm{R}_p(x)$, and if $f(x) = \log_p(x) - 1$ then $F(x) = \mathrm{Log}\Gamma_p(x)$. Thus, we are going to compute $\sum_{j=0}^{p^r-1} F\Big(\frac{x+j}{p^r}\Big)$, and this would give us the value of the right hand of (24). We will do this in two complementary disjoint cases. First we need some properties of χ.

Lemma 4.2. *If* $|z|_p < 1$*, then* $\chi(y) = \chi(y+z)$ *for all* $y \in \mathbb{C}_p$*.*

Proof. If $\chi(y) = 1$ then $|y|_p \geq 1$. In this case $|y+z|_p = |y|_p \geq 1$, hence $\chi(y+z) = 1$. If $\chi(y) = 0$ then $|y|_p < 1$. In this case $|y+z|_p \leq \max\{|y|_p, |z|_p\} < 1$, hence $\chi(y+z) = 0$. $\square$

Lemma 4.3. *Let* W_p *be the set containing all* $x \in \mathbb{C}_p$ *such that* $x \notin \mathcal{Z}_p$*, or* $x \in \mathcal{Z}_p$ *and* $\overline{x} \notin \mathbb{F}_p$*. Then* W_p *is* $\mathbb{Z}_p$*-invariant, meaning that, if* $x \in W_p$ *and* $t \in \mathbb{Z}_p$*, then* $x + t \in W_p$*.*

Proof. By cases. First, $x \notin \mathcal{Z}_p$ means that $|x|_p > 1$. If $t \in \mathbb{Z}_p$, then $|t|_p \leq 1$, so that $|x+t|_p = |x|_p > 1$, and then $x + t \notin \mathcal{Z}_p$. Now, let $x \in \mathcal{Z}_p$ such that $\overline{x} \notin \mathbb{F}_p$ and let $t \in \mathbb{Z}_p$. Since $\overline{t} \in \mathbb{F}_p$, if $\overline{x+t} \in \mathbb{F}_p$, then $\overline{x+t} - \overline{t} = \overline{x} + \overline{t-t} = \overline{x} \in \mathbb{F}_p$, a contradiction. $\square$

Corollary 4.4. *Let* $x \in W_p$ *and* $t \in \mathbb{Z}_p$*. Then* $\chi(x+t) = 1$*, and* $\chi((x+j)/p^r + t) = 1$ *for all* $r \in \mathbb{N}$ *and* $j \in \mathbb{N}_0$*.*

Proof. First notice that if $y \in W_p$ then $|y|_p \geq 1$; if not, $|y|_p < 1$ and this would imply $\overline{y} = \overline{0} \in \mathbb{F}_p$. Let $t \in \mathbb{Z}_p$. If $x \in W_p$, by Lemma 4.3, $x + t \in W_p$ so that $|x+t|_p \geq 1$ and this gives $\chi(x+t) = 1$. Also, if $j \in \mathbb{N}_0$, $x + j \in W_p$, so that $|x+j|_p \geq 1$. Then, for $r \in \mathbb{N}$, $|(x+j)/p^r|_p = |(x+j)|_p p^r > 1$. In particular $(x+j)/p^r \in W_p$, so that $(x+j)/p^r + t \in W_p$, and then $|(x+j)/p^r + t|_p \geq 1$, i.e., $\chi((x+j)/p^r + t) = 1$. $\square$

Now we compute the mentioned sums involving F. Recall that for $x \in \mathcal{Z}_p$ such that $\overline{x} \in \mathbb{F}_p$ we let $\ell(x)$ be the unique natural number satisfying $1 \leq \ell(x) \leq p$ and $\overline{x} = \overline{\ell(x)}$.

Lemma 4.5. *Let* $r \in \mathbb{N}$ *and let* $x \in \mathcal{Z}_p$ *such that* $\overline{x} \in \mathbb{F}_p$*. Then*

$$\sum_{k=0}^{p^r-1} F\left(\frac{x+k}{p^r}\right) = F(x) + \sum_{j=0}^{p^{r-1}-1} F\left(\frac{x'+j}{p^{r-1}}\right),$$

where $x' = (x + p - \ell(x))/p$*.*

Proof. Since $|x|_p \leq 1$, then $|x+y|_p \leq 1$ for all $y \in \mathcal{Z}_p$, and in particular, $|x+k|_p \leq 1$ for all $k \in \mathbb{N}_0$. Hence, we can write

$$\sum_{0 \leq k < p^r} F\left(\frac{x+k}{p^r}\right) = \sum_{\substack{0 \leq k < p^r \\ |x+k|_p < 1}} F\left(\frac{x+k}{p^r}\right) + \sum_{\substack{0 \leq k < p^r \\ |x+k|_p = 1}} F\left(\frac{x+k}{p^r}\right). \tag{26}$$

We first compute the first sum in the right hand of (26). Notice that $|x+k|_p < 1$ if and only if $|\ell(x) + k|_p < 1$, and this occurs if and only if $p \mid (\ell(x) + k)$. Hence

$$\sum_{\substack{0 \leq k < p^r \\ |x+k|_p < 1}} F\left(\frac{x+k}{p^r}\right) = \sum_{\substack{0 \leq k < p^r \\ p \mid \ell(x)+k}} F\left(\frac{x+k}{p^r}\right) = \sum_{\substack{p-\ell(x) \leq k < p^r - \ell(x) \\ p \mid \ell(x)+k}} F\left(\frac{x+k}{p^r}\right)$$

$$= \sum_{\substack{0 \le i \le p^r - p \\ p|i}} F\left(\frac{x + p - \ell(x) + i}{p^r}\right)$$

$$= \sum_{0 \le j < p^{r-1}} F\left(\frac{x + p - \ell(x) + pj}{p^r}\right)$$

$$= \sum_{0 \le j < p^{r-1}} F\left(\frac{x' + j}{p^{r-1}}\right).$$

Now, recalling the definition of F, the second sum in the right hand of (26) is

$$\sum_{\substack{0 \le k < p^r \\ |x+k|_p = 1}} F\left(\frac{x+k}{p^r}\right) = \sum_{\substack{0 \le k < p^r \\ |x+k|_p = 1}} \int_{\mathbb{Z}_p} \left(\frac{x+k}{p^r} + t\right) f\left(\frac{x+k}{p^r} + t\right) \chi\left(\frac{x+k}{p^r} + t\right) dt.$$

Since $|x + k|_p = 1$ and $r \ge 1$, then $|(x + k)/p^r|_p = p^r > 1$. Since also $|t|_p \le 1$, then $|(x + k)/p^r + t|_p = p^r > 1$, and we deduce that $\chi((x + k)/p^r + t) = 1$. Then, using (25),

$$\sum_{\substack{0 \le k < p^r \\ |x+k|_p = 1}} F\left(\frac{x+k}{p^r}\right) = \frac{1}{p^r} \sum_{\substack{0 \le k < p^r \\ |x+k|_p = 1}} \int_{\mathbb{Z}_p} (x + k + p^r t) f(x + k + p^r t) dt$$

$$= \frac{1}{p^r} \sum_{0 \le k < p^r} \int_{\mathbb{Z}_p} (x + k + p^r t) f(x + k + p^r t) \chi(x + k) dt,$$

and using Lemma 4.2 with $y = x + k$ and $z = p^r t$, and Lemma 2.3, we finally obtain

$$\sum_{\substack{0 \le k < p^r \\ |x+k|_p = 1}} F\left(\frac{x+k}{p^r}\right) = \frac{1}{p^r} \sum_{0 \le k < p^r} \int_{\mathbb{Z}_p} (x + k + p^r t) f(x + k + p^r t) \chi(x + k + p^r t) dt$$

$$= \int_{\mathbb{Z}_p} (x + t) f(x + t) \chi(x + t) dt$$

$$= F(x).$$

The lemma follows. $\square$

Lemma 4.6. *Let $r \in \mathbb{N}$, and let $x \in \mathbb{C}_p$ such that $x \notin \mathcal{Z}_p$, or $x \in \mathcal{Z}_p$ and $\overline{x} \notin \mathbb{F}_p$. Then*

$$\sum_{k=0}^{p^r - 1} F\left(\frac{x+k}{p^r}\right) = F(x).$$

Proof. Using Corollary 4.4, equation (25) and Lemma 2.3, we obtain

$$\begin{aligned}\sum_{k=0}^{p^r-1} F\left(\frac{x+k}{p^r}\right) &= \sum_{k=0}^{p^r-1} \int_{\mathbb{Z}_p} \left(\frac{x+k}{p^r}+t\right) f\left(\frac{x+k}{p^r}+t\right) dt \\ &= \frac{1}{p^r} \sum_{k=0}^{p^r-1} \int_{\mathbb{Z}_p} (x+k+p^r t) f(x+k+p^r t) dt \\ &= \int_{\mathbb{Z}_p} (x+t) f(x+t) dt \\ &= F(x). \qquad \square\end{aligned}$$

Now, let $x \in \mathbb{C}_p$, and fix $n = mp^r \in \mathbb{N}$. We define next a finite sequence $x_i \in \mathbb{C}_p$ of length at most $r+1$. Write $x_0 := x$. If $x_0 \notin \mathcal{Z}_p$, or $x_0 \in \mathcal{Z}_p$ but $\overline{x_0} \notin \mathbb{F}_p$, then the sequence stops. If $x_0 \in \mathcal{Z}_p$ and $\overline{x_0} \in \mathbb{F}_p$, then define

$$x_1 := \frac{x_0 + p - \ell(x_0)}{p}.$$

We repeat the above process: if $x_j \notin \mathcal{Z}_p$, or $x_j \in \mathcal{Z}_p$ but $\overline{x_j} \notin \mathbb{F}_p$, then the sequence stops. If $x_j \in \mathcal{Z}_p$ and $\overline{x_j} \in \mathbb{F}_p$, then

$$x_{j+1} := \frac{x_j + p - \ell(x_j)}{p}.$$

Let s the least non-negative integer, or $+\infty$, such that $x_s \notin \mathcal{Z}_p$, or $x_s \in \mathcal{Z}_p$ but $\overline{x_s} \notin \mathbb{F}_p$, and set $\omega = \min(r, s)$. Define our sequence to be $x_0, \ldots, x_\omega$. Notice that the length of the sequence is $\omega + 1 \leq r + 1$.

Proposition 4.7. *With notation as above,*

$$\sum_{k=0}^{p^r-1} F\left(\frac{x+k}{p^r}\right) = \sum_{j=0}^{\omega} F(x_j).$$

Proof. Follows inductively applying Lemma 4.5 and Lemma 4.6. $\square$

As a corollary we prove our distribution formula for $\mathrm{Log}\Gamma_p$.

Theorem 4.8. *With notation as above,*

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_p\left(\frac{x+k}{n}\right) = \sum_{j=0}^{\omega} \mathrm{Log}\Gamma_p(x_j) \;-\; \log_p(n) \sum_{j=0}^{\omega} \mathrm{R}_p(x_j). \tag{27}$$

Proof. Apply Proposition 4.7 for $F = \mathrm{Log}\Gamma_p$ and $F = \mathrm{R}_p$ in formula (24). $\square$

5. Relation with the functions of Diamond and Morita

We now take a look at the relation of $\mathrm{Log}\Gamma_p$ with Diamond's and Morita's functions. Let us start with Diamond's [3] function

$$\mathrm{Log}\Gamma_{\mathrm{D}}(x) := \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\,dt \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p).$$

Recall that in Lemma 4.3 we defined W_p to be the set containing the $x \in \mathbb{C}_p$ such that $x \notin \mathcal{Z}_p$, or $x \in \mathcal{Z}_p$ and $\overline{x} \notin \mathbb{F}_p$. Obviously, $W_p \subset \mathbb{C}_p \setminus \mathbb{Z}_p$.

Proposition 5.1. *For $x \in W_p$ we have*

$$\mathrm{Log}\Gamma_p(x) = \mathrm{Log}\Gamma_{\mathrm{D}}(x).$$

Proof. By Corollary 4.4, if $x \in W_p$ then $\chi(x+t) = 1$ for all $t \in \mathbb{Z}_p$, and thus

$$\mathrm{Log}\Gamma_p(x) = \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\,dt \qquad (x \in W_p),$$

i.e., $\mathrm{Log}\Gamma_p$ and $\mathrm{Log}\Gamma_{\mathrm{D}}$ are identical on W_p. $\square$

Let us prove now that the distribution formula (11) restricted to W_p, this is, for all $n \in \mathbb{N}$

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_{\mathrm{D}}\left(\frac{x+k}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{D}}(x) - \left(x - \frac{1}{2}\right)\log_p(n) \qquad (x \in W_p),$$

is a special case of Theorem 4.8. First, if $x \in W_p$, then by definition, we have that $s = 0$ in our sequence. Hence, $\omega = \min(r, s) = 0$ and (27) becomes

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_p\left(\frac{x+k}{n}\right) = \mathrm{Log}\Gamma_p(x) \;-\; \log_p(n)\mathrm{R}_p(x),$$

since $x_0 = x$. Now, it is easily seen after Lemma 4.3 that if $x \in W_p$ then all the numbers $(x+k)/n$ are also in W_p. Since $\mathrm{Log}\Gamma_p(x) = \mathrm{Log}\Gamma_{\mathrm{D}}(x)$ for $x \in W_p$, then the equation above becomes

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_{\mathrm{D}}\left(\frac{x+k}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{D}}(x) \;-\; \log_p(n)\mathrm{R}_p(x).$$

Finally, using Proposition 3.5 for $x \in W_p$, we obtain

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_{\mathrm{D}}\left(\frac{x+k}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{D}}(x) \;-\; \log_p(n)\Big(x - \frac{1}{2}\Big).$$

We now consider the relation of $\mathrm{Log}\Gamma_p$ with Morita's [5] function $\mathrm{Log}\Gamma_{\mathrm{M}}$. Recall that $\mathrm{Log}\Gamma_{\mathrm{M}}$ is defined by the Volkenborn integral

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \int\limits_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\chi_{\mathbb{Z}_p^*}(x+t)dt \qquad (x \in \mathbb{Z}_p).$$

The direct relation between $\mathrm{Log}\Gamma_p$ and $\mathrm{Log}\Gamma_{\mathrm{M}}$ is easy since $\mathrm{Log}\Gamma_{\mathrm{M}}(x)$ is actually $\mathrm{Log}\Gamma_p(x)$ restricted to $\mathbb{Z}_p$. To see this, let us go back to the function φ_p defined by (19). We have

$$\varphi_p(x) = \begin{cases} 0 & \text{if } |x|_p < 1, \\ x\log_p(x) - x & \text{if } |x|_p \geq 1, \end{cases}$$

and if we restrict ourselves to $x \in \mathbb{Z}_p$, then

$$\big\{x \in \mathbb{Z}_p \,\big|\, |x|_p < 1\big\} = p\mathbb{Z}_p \;\text{ and }\; \big\{x \in \mathbb{Z}_p \,\big|\, |x|_p \geq 1\big\} = \mathbb{Z}_p^*.$$

Hence we have

$$\mathrm{Log}\Gamma_p(x) = \int\limits_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(x+t)(x+t)\big(\log_p(x+t) - 1\big)dt \qquad (x \in \mathbb{Z}_p),$$

and this is exactly Morita's function $\mathrm{Log}\Gamma_{\mathrm{M}}$.

An important property of $\mathrm{Log}\Gamma_{\mathrm{M}}$ is that it has a power series expansion around 0, valid for all $x \in p\mathbb{Z}_p$. Namely, for all $x \in p\mathbb{Z}_p$ we have the identity [7, Lemma 58.1]

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \lambda_1 x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,\lambda_{n+1}}{n(n+1)}\; x^{n+1}, \tag{28}$$

where

$$\lambda_1 := \int\limits_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t)\log_p(t)dt, \quad \lambda_{n+1} := \int\limits_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t)\, t^{-n}dt \qquad (n \in \mathbb{N}).$$

Moreover, the right side of (28) defines an analytic function on the open unit ball $B(0;1^-) \subset \mathbb{C}_p$ [7, Lemma 58.2]. Hence, we extend the domain of $\mathrm{Log}\Gamma_{\mathrm{M}}$ to $B(0;1^-) \cup \mathbb{Z}_p$ by defining

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) := \lambda_1 x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,\lambda_{n+1}}{n(n+1)}\; x^{n+1} \qquad (x \in B(0;1^-)). \tag{29}$$

It can be shown that with this extended definition $\mathrm{Log}\Gamma_{\mathrm{M}}$ is no longer the Iwasawa logarithm of a p-adic function.

Proposition 5.2. *For all* $x \in \mathbb{Z}_p \cup B(0;1^-)$ *we have* $\mathrm{Log}\Gamma_p(x) = \mathrm{Log}\Gamma_{\mathrm{M}}(x)$*.*

Proof. We already proved the equality on $\mathbb{Z}_p$. To prove the equality on $B(0;1^-)$ first notice that both functions are equal on $p\mathbb{Z}_p \subset B(0;1^-)$ and that $p\mathbb{Z}_p$ has infinitely many accumulation points. Since $\mathrm{Log}\Gamma_{\mathrm{M}}(x)$ has the power series expansion (29) on $B(0;1^-)$ and $\mathrm{Log}\Gamma_p(x)$ is locally analytic on $\mathbb{C}_p$, then $\mathrm{Log}\Gamma_p(x)$ must be equal to the right side of (29) on $B(0;1^-)$. $\square$

Let us prove now that the distribution formula (7), this is, for all $n \in \mathbb{N}$ not divisible by p

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_{\mathrm{M}}\left(\frac{x+k}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{M}}(x) - \left(x - \left\lceil \frac{x}{p} \right\rceil\right) \log_p(n) \qquad (x \in \mathbb{Z}_p),$$

is a special case of Theorem 4.8. First, if $p \nmid n$, then by definition, we have that $r = 0$ in our sequence. Hence, $\omega = \min(r, s) = 0$ and (27) becomes

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_p\left(\frac{x+k}{n}\right) = \mathrm{Log}\Gamma_p(x) \; - \; \log_p(n)\mathrm{R}_p(x),$$

since $x_0 = x$. Now, it is easily seen that if $p \nmid n$ and if $x \in \mathbb{Z}_p$ then all the numbers $(x+k)/n$ are also in $\mathbb{Z}_p$. Since $\mathrm{Log}\Gamma_p(x) = \mathrm{Log}\Gamma_{\mathrm{M}}(x)$ for $x \in \mathbb{Z}_p$, then the equation above becomes

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_{\mathrm{M}}\left(\frac{x+k}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{M}}(x) \; - \; \log_p(n)\mathrm{R}_p(x).$$

Finally, using Corollary 3.6 for $x \in \mathbb{Z}_p$, we obtain

$$\sum_{k=0}^{n-1} \mathrm{Log}\Gamma_{\mathrm{M}}\left(\frac{x+k}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{M}}(x) \; - \; \log_p(n)\left(x - \left\lceil \frac{x}{p} \right\rceil\right).$$

We now take a look at the distribution formula (12), that connects the functions of Morita and Diamond. Let $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$ divisible by p. Then

$$\sum_{\substack{0 \le j < n \\ p \nmid (x+j)}} \mathrm{Log}\Gamma_{\mathrm{D}}\left(\frac{x+j}{n}\right) = \mathrm{Log}\Gamma_{\mathrm{M}}(x) - \left(x - \left\lceil \frac{x}{p} \right\rceil\right) \log_p(n).$$

By Propositions 5.1 and 5.2, and by Corollary 3.6, this formula becomes

$$\sum_{\substack{0\le j<n \\ p\nmid(x+j)}} \mathrm{Log}\Gamma_p\left(\frac{x+j}{n}\right) = \mathrm{Log}\Gamma_p(x) - \mathrm{R}_p(x)\log_p(n) \qquad (x\in\mathbb{Z}_p, p\,|\,n). \tag{30}$$

Formula (27) explains now the restriction in the sum in the left hand of (30): to be able to omit it then there must appear other terms in its right hand.

Finally, we mention that one can compute explicitly the expansions in power series of $\mathrm{Log}\Gamma_p(x)$ using those of $\mathrm{Log}\Gamma_{\mathrm{D}}(x)$ and $\mathrm{Log}\Gamma_{\mathrm{M}}(x)$, but for the sake of brevity, we omit the calculations here.

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