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SELF INTERACTING DARK MATTER IN LARGE SCALE STRUCTURE

TESIS PARA OPTAR AL GRADO DE MAGISTER EN CIENCIAS, MENCIÓN ASTRONOMÍA

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RESUMEN DE LA MEMORIA PARA OPTAR AL TÍTULO
DE: MAGISTER EN CIENCIAS, MENCIÓN ASTRONOMÍA
POR: Ángel Erasmo Rincón Rivero.
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SELF INTERACTING DARK MATTER IN LARGE SCALE STRUCTURES

Cold Dark matter (ΛCDM) models have been remarkably successful to explain the observed large scale structure of our universe on scales of the order of galaxy clusters (≥ 4 Mpc) and above (therefore in this work we consider large scale beyond the aforementioned limit). However, this class of models has some problems at short scales, (~ 1 Mpc or lower) dubbed "Small Scale Controversies". It is important to remark that, for purposes of our work, we consider lower than 1 Mpc as short-scales. One of small scale issues is associated to the Dark Matter halo structure: cosmological simulations that take into account only gravity and collisionless matter, predict halos and substructures with densities much higher than those derived from galactic dynamics and observations. A possible way to conciliate theory with observations is to consider self interactive dark matter (SIDM). Models with SIDM generate predictions consistent with observations on Large Scales, the domain where ΛCDM is successful, but in addition it does not conflict with observations on "small scale". In absence of a theory that incorporates self interactive dark matter, it is possible to use the so-called Effective Field Theory (EFT) framework to investigate some aspects of dark matter. The use of effective field theory techniques to study the role of dark matter during the period of structure formation in the Universe has provided a powerful parametrization of the dark matter physics at short scales. Recently, some researchers have advocated the use of the latter approach to model the large scale structure as a fluid and considering gravity by incorporating systematically non linear terms in the theoretical treatment.

In this work, we use some recent approaches [2, 12] to study analytically collisional dark matter in the form of self interactions. We derive generalized expressions of some of the equations presented in ref.[2], corresponding to corrections to the momentum equation and the effective energy equation, and discuss the implications for the behavior of dark matter and its effect on structure formation. In particular, we find that, by taking into account self interactions, some corrections terms appear both in the momentum and energy equations. These corrections arise from the non-linear effects that modify the standard equations. We show that these new terms can solve some of the "small scale" issues because the self interactive dark matter reduces the central densities of the galaxy dark matter halos.

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AUTO INTERACCIÓN DE MATERIA OSCURA EN ESTRUCTURAS A GRAN ESCALA

Modelos de materia oscura fría (ΛCDM) han explicado satisfactoriamente la estructura a gran escala de nuestro universo observable a escalas del orden de cúmulos de galaxias (≥ 4 Mpc) y superior (por lo tanto en este trabajo consideramos escalas largas más allá del límite antes mencionado). Sin embargo, esta clase de modelos tiene problemas a escalas cortas, (~ 1 Mpc o inferior) llamados “small scale issues”. Es importante comentar que, para propósitos de este trabajo, consideramos escalas cortas aquellas menores que 1 Mpc. Uno de los “small scale issues” está asociado a la estrucura del halo de matería oscura: simulaciones cosmológicas que toman en cuenta solamente gravedad y materia no colisional, predicen halos y subestructuras con densidades mucho mas altas que aquellas derivadas de la dinámica galáctica y observaciones. Una posible forma de conciliar teoría con observaciones consiste en considerar materia oscura autointeractuante (SIDM). Modelos con SIDM generan predicciones consistentes con las observaciones a gran escala, el dominio donde ΛCDM es exitoso, pero además esto no está en conflicto con las observaciones a escalas cortas. En ausencia de una teoría que incorpore materia oscura autointeractuante, es posible usar la así llamada teoría de campos efectiva “EFT” que nos permite investigar algunos aspectos de materia oscura. El uso de las técnicas de teoría efectiva para estudiar el rol de la materia oscura durante el periodo de formación de estructuras en el Universo nos ha dado una poderosa parametrización de la física de la materia oscura a escalas cortas. Recientemente, algunos investigadores se han dedicado a usar el enfoque mencionado para modelar la estructura a gran escala como un fluido y considerando gravedad, incorporando sistemáticamente términos no lineales en el tratamiento teórico.

En este trabajo, usamos el enfoque reciente [2, 12] para estudiar analíticamente materia oscura colisional en la forma de auto interacciones. Derivamos expresiones de algunas de las ecuaciones presentadas en ref.[2], correspondientes a correcciones de la ecuación de momentum y la ecuación de energía efectiva, y discutimos las implicaciones para el comportamiento de la materia oscura y su efecto en formación de estructuras. En particular, encontramos que, al tomar en cuenta auto interacciones, algunos términos correctivos aparecen tanto en la ecuación de momento como en la de energía. Estas correcciones surgen del efecto de las no linealidades que modifican las ecuaciones canónicas. Mostramos que estos nuevos términos pueden solucionar algunos de los “small scale issues” puesto que materia oscura autointeractuante reduce la densidad central de halos de materia oscura.

A mi Familia: Mi madre Irma-Elena, a mi Padre, a mis hermanos, a mi siempre amada Mamá Rosa, a mi adorada prometida Nayker, al Profesor y al sin número de familiares que, de alguna u otra manera, me han ayudado a lo largo de la travesia de mi vida. También dedico esta tesis a mis siempre recordados amigos (viejos y nuevos), los cuales recuerdo con frecuencia: a mi hermano Javier, Gabriel, marrón, la niña, Jorge, Iván, Julián-Esteban, Eli, Caro, Andres, Marcos, Jeaque, Alejandro, dani, alejo, klau, entre otros!

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Chapter 1

Introduction

Modern Cosmology is the framework to investigate the evolution of large-scale structures (LSS) in Universe [16]. In this context, we believe that our universe is very well described on large scales ($\sim$ 100 Mpc) by a Friedmann-Robertson-Walker (FRW) model. Under this model, non-linear terms are negligibly small as compared with the dominant linear terms in the deviation from a FRW. Thus, the smoothed universe model is the first approximation that will allow us to understand the large-scale structure and evolution of the universe. Despite of it, the existence of discrete objects (stars and planets) indicates that the smooth universe model is not completely accurate. The large scale structures of galaxies with its voids and filaments, needs to be explained through the agglomeration of matter over time. In the same way, we wish describe the matter/energy distribution of the Universe which allows us to understand better the process and the physics inside it.

The Universe contains different kinds of matter, and depending of the type, the required underlying physics is different. The matter in the universe is usually separated in Barionic matter ($\sim 5\%$), Dark matter ($\sim 27\%$) and Dark energy ($\sim 68\%$). The first type is basically ordinary matter, the second type is a hypothetical kind of matter that (at least so far) cannot be seen by standard astronomical methods, whereas as the nature of the dark energy is unknown, but we know it behaves differently from regular matter. Indeed, we believe that dark energy has an opposite effect to gravity pushing everything apart and thus contributing to the expansion of the Universe. Dark matter (DM) is one of the pillars of the Standard Cosmological Model, but the nature of this elusive component of the matter budget of the Universe remains unknown, despite the compelling evidence of existence at all astrophysical scales [6].

At this point is clear that to investigate the evolution of large-scale structures we need to consider the Dark Matter (DM) which is a particular kind of matter that only suffers gravitational interaction. According with the velocity of dark matter particles, it can be typified as: Cold Dark Matter (CDM) and Warm Dark Matter (WDM) particles, the first case corresponds to low velocities, and the second case corresponds to high velocities, relative to the speed of light [5].

The Dark Matter is not a new concept [60], but the scientific community was slow to accept it due it was a new radical idea. This concept was generally accepted in the early 1980s and quickly became a central element of the theory of cosmic structure formation[56]. Observational supports the existence of it introduced the concept of dark matter [5, 8, 14] and some candidates for this kind of matter have been proposed [22] as neutrinos, axions, sterile neutrinos etc.

At large scales, we know that the ΛCDM model (which take into account the existence of dark matter) fits very good the physics of the Universe, however, some issues are still unsolved. For example, the cusp-core problem and the missing satellites problem[56, 45, 46]. These are small scale issues which we want to address using a general theory.

As a possibility (that keeps the physics at large scales working good) the so-called Self Interactive Dark Matter (SIDM) [51, 55, 31, 30] was introduced. This particular kind of DM theoretically can solve the small scale issues previously commented and does not introduce modifications in the physics at large scale.

The Self Interactive Dark Matter idea was made popular by Spergel & Steinhardt [51] and consists on that cold dark matter has weak interactions with baryons but strong self-interactions [56]. The original idea introduced constrains in the cross section of DM particles, but it was ruled out by gravitational lensing. Nowadays, however, simulations show that there is a viable window of mass and cross-section where self-interacting dark matter (SIDM) can produce cored dark matter profiles and remain consistent with observational constraints (Fig.1.1). We choose only this type of DM and, as a tool to make progress [2], we consider an effective field theory model.

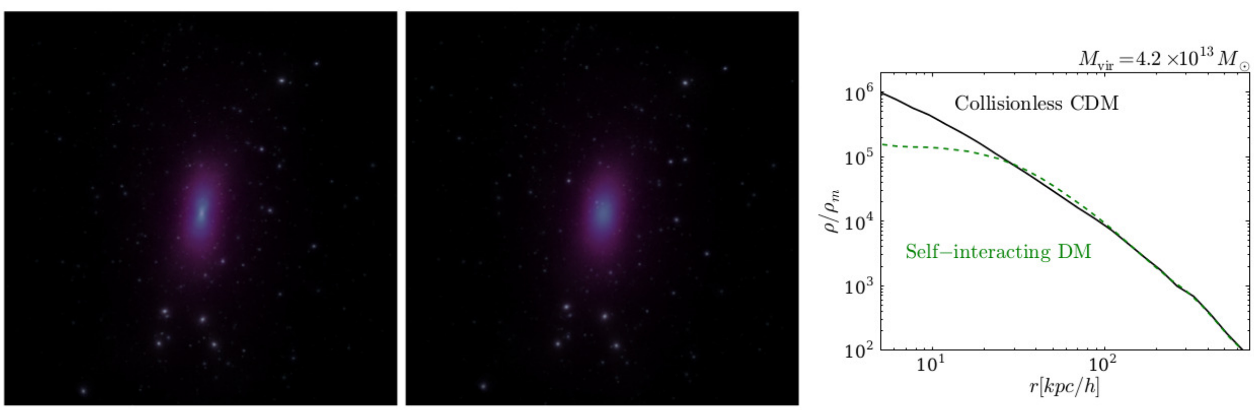

Figure 1.1: Effect of self-interacting dark matter (SIDM) on halo structure from [56]. The left panel shows a Milky Way mass CDM halo, and the middle panel shows the same halo from an SIDM simulation. The right panel compares the density profiles of a CDM and SIDM halo.

The basic idea is to get a theory which reproduces the physics at certain energy range and "remove" problems in the opposite energy range. According with [12] an "effective field theory" is a way to describe what happens at low energies (or, equivalently, long wavelengths) without having a complete picture of what is going on at higher energies. For example, in cosmology the LSS evolves from small perturbations at early times which produces galaxies and everything we see today. However, a better description is given by considering a particle point of view which is obtained by taking into account the Boltzmann equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \vec{p}} = \mathcal{C}[f], \tag{1.1}$$

where f is the distribution function, $\vec{x}$ is the position, $\vec{p}$ is the momentum, t is the time and $\mathcal{C}[f]$ is the collisional term. This equation tells us how a distribution of particles evolves in phase space. Notice that the Universe has many particles, therefore a natural choice is to make an approximation by "smoothing" the particle distribution into an effective fluid [2, 11]. The idea is not new and has some preliminary works [26], however the EFF of LSS is still in its infancy. As fluid mechanics, we have some characteristic parameters which let us re-parametrize the original problem in terms of fluid parameters. Some of these are density and velocity, but also has parameters like an effective speed of sound and viscosity. A correct EFT lets us describe what occurs even at some length scales that are, formally speaking, "non-linear", and therefore would conventionally be thought of as inaccessible to anything but numerical simulations. To summarize, basically we parametrize the problem as an effective fluid taking into account non-lineal terms.

In this work, we treat analytically the problem of LSS considering SIDM. In the same way, we use an approach explained in ref.[2] which consists on integrating the equations of motion using a windows function, to get effective equations that describe the problem at large scale (considering non-linear terms). Using the effective approach we "filter" the original equations by "integrating-out" short-wavelength fluctuations in order to split the theory in large and short quantities, which allows us to recover the physics at large scale and incorporate corrections given by small scale quantities. These non-linear terms have diverse interpretations and depend of windows function. In simple cases we use as windows function an step function. It is a good choice if we wish to stablish a hard cut-off between linear and non-linear contributions. However, recent papers [2, 12] suggest us to take a windows function a gaussian function, due it has desirable properties which let us make progress in the theory. Due to the arguments before explained, the analysis of possible effects of SIDM on observable universe should be taken as a real possibility as well as the implementation of EFF of LSS as a main tool to do it. In order to develop this thesis, the treatment made in [2] and [12] written by Baumann et. al. and Carroll et. al. was fundamental. These authors discuss and show us a modern way to apply effective field theory on large scale structures. The first cited paper was a pillar for this work because the authors consider the non-linear terms and try to find the meaning of these. The second paper allows us understand conceptual details of effective field theory of large scale structure. Other essential references are [3] where the theoretical point of view of cosmological perturbation theory is explained in detail, and [36] and [25] which are friendly textbooks to cover details on cosmological perturbations theory, where we learned why standard perturbation theory is not a good choice to investigate non linearities. This thesis aims to obtain corrections in the large scale structure theory by assuming that the dark matter can be self interactive.

This work is organized in the following way: in Chapter 2 we discuss on Dark Matter theory: the origin of the concept, why dark matter is required, some indirect ways to constrain it as well as possible candidates, and the Self Interactive Dark Matter. In Chapter 3 we introduce theory of cosmological perturbations using the Boltzmann equation at first order for the relativistic and non-relativistic cases. In Chapter 4 discuss the smoothing approach applied to EFF of LSS which is used by [2, 12]. Also, we will briefly revisit aspects of EFT due is a relevant tool to describe complicated problems in physics (and in particular in this thesis) as well as preliminary computations. In Chapter5 we show results given by the application of smoothing approach to our study case where the non-linear terms emerge naturally. Finally, in Chapter 6 we present the main conclusions of this thesis. We briefly comment the window function used for computations in the thesis in Appendix.

Chapter 2

Dark Matter and ΛCDM Cosmology

2.1 Introduction

One of the most outstanding revelations of the twentieth century related to the Universe is that ordinary baryonic matter, that is, matter made up of protons and neutrons, is not the dominant form of matter in the Universe. Instead, some strange new form of matter, dubbed "Dark Matter" (DM) fills our Universe, and it is roughly six times more abundant than ordinary matter. There is a great deal of evidence which points to the necessity of its existence [22]. General relativity dictates how dark matter acts on large scales and how the Universe may be viewed as a laboratory to study dark matter [22]. This Chapter gives us a general overview on dark matter and the state of art of dark matter. In particular, we'll discuss about modifications and new possibilities as self interactive dark matter.

2.1.1 History and inference of dark matter

In the early 1930s, J. H. Oort found that the motion of stars in the Milky Way hinted at the presence of far more galactic mass than anyone had previously predicted. By studying the Doppler shifts of stars moving near the galactic plane, Oort was able to calculate their velocities, and thus made the startling discovery that the stars would be moving fast enough to escape the gravitational pull of the luminous mass in the galaxy. Oort postulated that there must be more mass present within the Milky Way to hold these stars in their observed orbits. However, Oort noted that another possible explanation was that 85% of the light from the galactic center was obscured by dust and intervening matter or that the velocity measurements for the stars in question were wrong [22].

On the other hand, in 1933 the Swiss astronomer F. Zwicky published unambiguous evidence for dark matter in the Coma galaxy cluster [60], whereas in 1939 Babcock's rotation curve for the Andromeda Nebula indicated that much of its mass is at a large radius. Zwicky employed the virial theorem to calculate the cluster's mass considering only gravitational interactions and Newtonian gravity $2\overline{T} = -\overline{U}$, where $\overline{T}$ is the average kinetic energy and $\overline{U}$ is the average potential energy. Zwicky found that the average mass of a nebula within cluster using the virial theorem was $M_{nebula} = 4.5 \times 10^{10} M_{\odot}$, with about a thousand nebula in the cluster (i.e. $M_{nebula} \cong 4.5 \times 10^{13} M_{\odot}$) [22]. This result was given by measurements of the luminosity of the cluster using standard M/L ratios for clusters gave a mass only of 10% of this value.

In 1959 Kahn & Woltjer argued that the total mass of the Milky Way and Andromeda galaxies must be much larger than their stellar mass in order to explain why they are currently approaching each other [20].

Roughly 40 years after the discoveries of Oort, Zwicky, and others, Vera Rubin and collaborators conducted an extensive study of the rotation curves of 60 isolated galaxies. They assumed that orbits of stars within a galaxy would closely mimic the rotations of the planets within our solar system [22]. In the solar system:

$$v(R) = \sqrt{\frac{GM(R)}{R}}, \tag{2.1}$$

where $v(R)$ is the rotation speed of the object at a radius R, G is the gravitational constant, and $M(R)$ is the total mass contained within R (for the solar system essentially the sun's mass). According with the eq.(2.1) they expected that the velocity of rotation body should decrease for large distances according with $v \propto R^{-1/2}$ which is generally referred to as "Keplerian" behavior (see Fig. 2.1).

Rubin's results [48] proved that the Newtonian prediction was not verified, according with the luminosity matter. The rotation curves for the collected data, showed that the curves are "flat", meaning that the rotation speed stayed constant when the radius increased. However, the observed mass is much smaller than the one calculated to explain the "flat rotation" curve and they concluded that the "missing" mass must be non-luminous (i.e. dark matter).

Indeed, dark matter had an relevant confirmation on galactic scales given by this technique applied to spiral galaxies. Observational curves show a "flat" behavior at large scales, i.e. close and beyond the visible disk. *Theoretical rotation curves* are computed using eq.(2.1) for the circular velocity. As usual, the mass $M \equiv M(R)$ is defined as:

$$M(R) = 4\pi \int \rho(r) r^2 \mathrm{d}r, \tag{2.2}$$

and $\rho(r)$ is the mass density profile, and should be falling in $\propto 1/R$ beyond the optical disc. Notice that when $v(R)$ is approximately constant this implies the existence of a halo with $M(R) \propto R$ and $\rho \propto R^{-2}$.

Moreover, an *observational rotation curve* can be determined as follows [49]:

Consider a galaxy which has the shape of a thin circular disk. The inclination angle i defined between our line of sight to the disk and a perpendicular line to the disk, let us to fix this problem. For an elliptical galaxy the axis ratio is related to the inclination by $b/a = \cos(\mathrm{i})$. Thus, by measuring the redshift of absorption (or emission) lines of light from the disk[1], is

[1] If we put a slot along the major axis, we obtain a spectrum whose emission lines are displaced in wavelength due to the Doppler effect

possible to find the radial velocity $v_r(R) = cz(R)$ along the apparent long axis of the galaxy. Here c is the speed of light and z is the redshift. Note that z only contains the component of stars' orbital velocity which lies along the line of sight. So, the radial velocity is given by $v_r(R) = v_{\rm gal} + v(R)\sin(\mathrm{i})$, with $v_{\rm gal}$ is the radial velocity of the galaxy, and $v(R)$ is the orbital speed at a distance R from the center of the disk. In this way, the speed $v(R)$ is given in terms of observable properties by:

$$v(R) = \frac{v_r(R) - v_{\rm gal}}{\sqrt{1-(b/a)^2}} \tag{2.3}$$

For values of radii close to the center of galaxies we can approximate the inner mass by considering the rotation speed as a constant: $M = v^2R/G$. Finally, using the relation (2.3) that takes into account the observational data, we obtain the required rotational curve (see Fig.2.1).

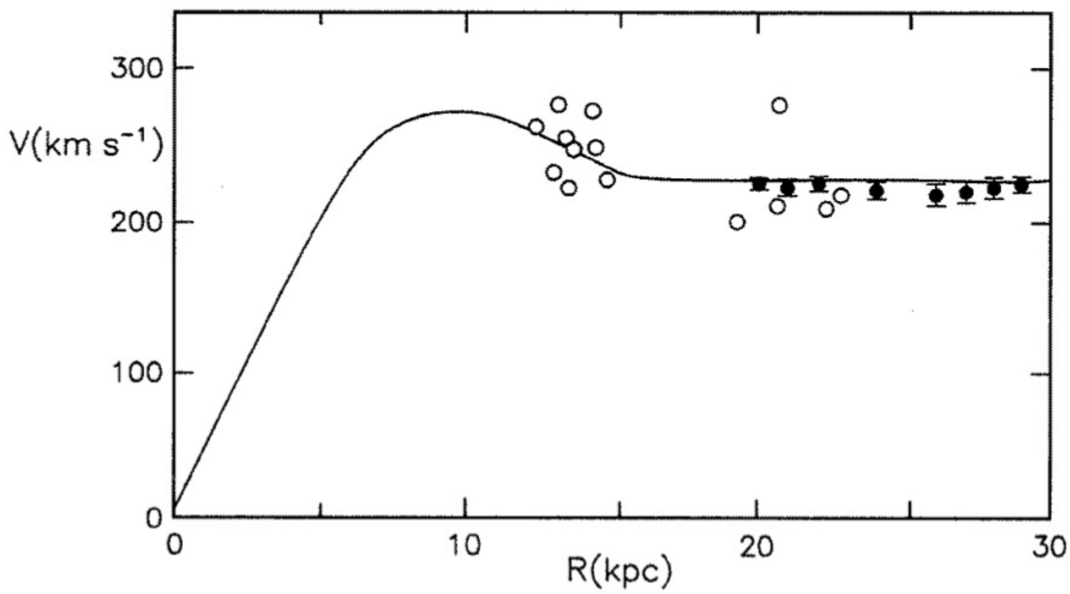

Figure 2.1: Example: orbital speed v as a function of radius in M31. Open circles show results of Rubin and Ford [48] at visible wavelengths. Solid dots with error bars show the results of Roberts and Whitehurst [47] at radio wavelengths

For elliptical galaxies is possible to infer the existence of dark matter by different ways, where we only comment briefly the stellar velocity dispersion technique: it is based on the Virial theorem which stablish that for a spherical, steady-state, static isothermal elliptical galaxy is represented by $2R \approx GM/\sigma^2$, with R is an equivalent radius. Thus, for a given R we have a dispersion, proportional to the total mass, quantified by the velocity dispersion, σ, must prevent gravitational collapse [1]. The previous relation gives a first approximate mass. More recent approaches have used sophisticated models than this one are to interpret the velocity dispersions. So, there is a "degeneracy" between the unknown anisotropy and the unknown gravitational potential. If we know the anisotropy of the orbits, it is possible to get the potential. The anisotropy is characterized by the parameter β and it is defined as:

$$\beta = 1 - \frac{\langle v_\theta^2 \rangle}{\langle v_r^2 \rangle} \tag{2.4}$$

where v_θ and v_r are the azimuthal and radial components of the velocity. Thus, an more precise result for the mass is given by:

$$M(r) = -\frac{\sigma_r^2 r}{G}\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\sigma_r^2}{\mathrm{d}\ln r} + 2\beta(r)\right). \tag{2.5}$$

This expression can be used to verify that extra matter reconcile theory with observations.

Another way to infer the presence of non-luminous matter was to use gravitational lensing (around 1970). Gravitational lensing is a result of Einstein's Theory of General Relativity: objects with high mass bend the space-time, affecting the motions of bodies around them (objects follow geodesics on this curved surface). So, the light is bent in presence of massive objects. This effect is proportional to the mass of massive object and inversely proportional to the distance to source. When an alignment with a background object is verified, it is possible to observe "arclets" that will be called Einstein's rings. Subsequent studies obtained that the "Einstein radius" (namely, the length of an arclet in radians) is given by:

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{\mathrm{d}_{LS}}{\mathrm{d}_L \mathrm{d}_S}} \tag{2.6}$$

where G is the gravitational constant, M is the mass of the lens, c is the speed of light, and d_{LS}, d_L, and d_S are the distance between the lens and source, the distance to the lens, and the distance to the source, respectively.

2.1.2 ΛCDM Cosmological inference

At cosmological scales there are additional ways to obtain the baryonic matter density in the Universe, given by $\omega_b = \Omega_b h^2$, where Ω_b is the baryon density relative to a reference critical density ρ_c and $h = H/100$ km sec^1 Mpc^1 is the reduced Hubble constant, which is used because of the large historical uncertainty in the expansion rate of the Universe. This quantity ω_b and the total matter density $\omega_m = \Omega_m h^2$ are relevant because of these quantities should be equal if the baryonic matter is the only kind of matter in the Universe.

The Cosmic Microwave Backgroung (CMB), discovered by Penzias and Wilson in 1964, was described as an excess background temperature of about 2.73 K. COBE (COsmic Background Explorer) was putted in orbit in 1989 and tested a couple of properties of CMB, first the remarkable uniformity of the CMB and second that the CMB and thus the early universe, is practically a perfect black body. COBE noted some anisotropies within the CMB and it can be split in large and small scales.

These fundamental fluctuations in the CMB are small: $30 \pm 5\mu$K, i.e. these fluctuations are uniform to 1 part in 10^5. With the passage of time other telescope called WMAP (Wilkinson Microwave Anisotropy Probe) was launched in 2001 with the mission to measure more precisely the anisotropies in the CMB.

The standard methodology, for extracting information from CMB anisotropy maps, is well established. Starting from a cosmological model with a given number of parameters (usually

6 or 7), the best-fit parameters are determined from the peak of the N-dimensional likelihood surface [5]. The increased precision of WMAP, it was possible to get the total and baryonic matter densities from WMAP [29]:

$$\Omega_m h^2 = 0.1334^{+0.0056}_{-0.0055}, \tag{2.7}$$

$$\Omega_b h^2 = 0.02260 \pm 0.00053. \tag{2.8}$$

With this result, it was evident that baryonic matter is not the only form of matter in the Universe. In fact, the dark matter density, $\Omega_{\mathrm{d}m} h^2 = 0.1123 \pm 0.0035$, is around 83% of the total mass density and corresponds to an average density of $\rho_{\mathrm{dm}} \approx 0.3$ GeV/cm$^3 \approx 5 \times 10^{-28}$ kg/m^3. Under a detailed analysis of the CMB, it is possible to have a discrimination between dark matter and ordinary matter because they act in different ways: the dark matter, unlike the baryons, they are not linked to the photons as part of the "photon- baryons fluid".

The more recent constraints of cosmological parameters are given by the Planck satellite [43] which was used from 2009 to 2013 by the European Space Agency. The actual accepted values for the matter distribution are:

$$\Omega_b h^2 = 0.02225 \pm 0.00016, \tag{2.9}$$

$$\Omega_{\mathrm{d}m} h^2 = 0.1198 \pm 0.0015, \tag{2.10}$$

which are consistent values with the dark matter.

2.2 Dark Matter as particles

The previous discussion was a brief introduction of different approaches to justify the need of Dark Matter. So, the evidence for non-baryonic dark matter is compelling at all observed astrophysical scales. Therefore it is natural to ask what is the dark matter made of? To discuss this question properly, it is necessary to comment some generalities about the standard model of particles physics.

The Standard Model (SM) is a modern theory to describe particles that consider three of the four fundamental forces in nature. This is a quantum field theory that studies the electromagnetic force, the weak nuclear force and the strong nuclear force and which mediate the dynamics of the known subatomic particles. This model does not include gravity, because energies below Plank scale gravity is unimportant at atomic level.

The SM has some problems: It does not take into account the full theory of gravitation described by general relativity. In the same way, the model does not contain any viable dark matter particle that possesses all of the required properties deduced from observational cosmology.

Until 2013, only sixteen confirmed particles in the SM were known, but nowadays we have seventeen particles due the Higgs boson was discovered in 2013 (see Fig. 2.2).

In the SM, there are six quarks: up, down, top, bottom, charm, and strange; six leptons: electron, mu, tau, and their respective neutrinos; and five force carriers: photons, gluons,

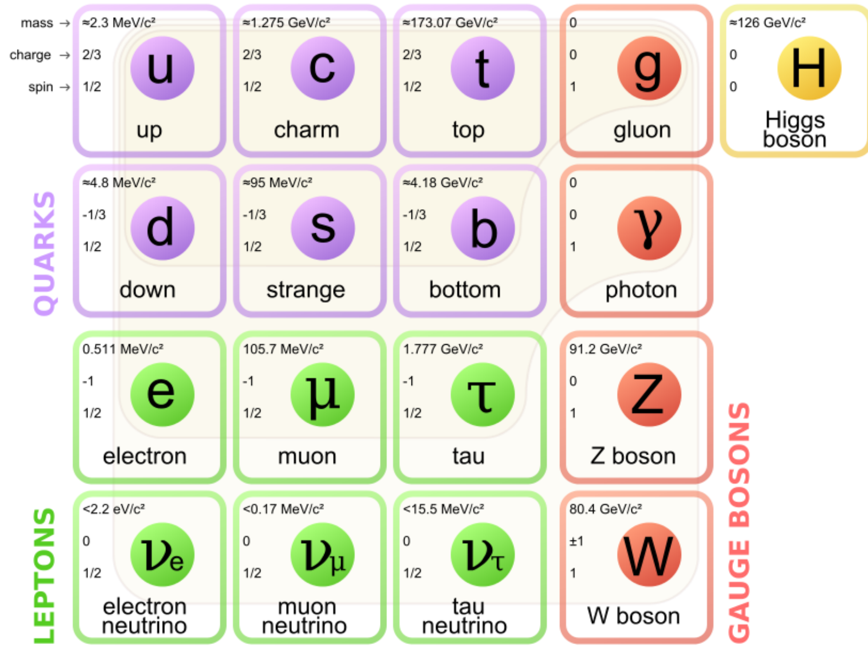

Figure 2.2: Elementary particles and gauge bosons of the Standard Model [34]

$W^{\pm}$, Z, and the Higgs boson. A standard way to classify Quarks and leptons is under the name of fermions with half integer spins and are split into three generations, where force carriers are classified as gauge bosons with integer spins. Each of these particles also have a corresponding antiparticle, denoted with a bar over the standard symbol of particle: the up antiquark's symbol is $\overline{u}$, with opposite charge.

The physics of the SM particles obey typical conservations relations, e.g.: energy and momentum laws and conservation laws for internal gauges symmetries (conservation of charge and lepton number for example). Besides the Higg boson, the SM doesn't include particles that could be considered as DM even if neutrinos are electrically neutral and weakly interactive particles in SM.

The WIMPs (Weakly Interacting Massive Particles) are massive particles that are electrically neutral, which do not interact very strongly with other matter fields. In brief, we will comment about some possible particle candidates for Dark Matter.

2.2.1 Standard Model neutrinos

During years, neutrinos were considered the best choice to dark matter but a simple calculation shows that, if we call $m_{\rm i}$ the mass of the i-th neutrino, their total relic density is predicted to be:

$$\Omega_\nu h^2 = \sum_{\rm i=1}^{3} \frac{m_{\rm i}}{93\ {\rm eV}}. \tag{2.11}$$

Some constraints can be found in Ref.[5], in particular an upper bound on the total neutrino relic density is $\Omega_\nu h^2 \lesssim 0.07$ which means that neutrinos are not abundant enough to be the dominant component of dark matter. A more precise constraint on the neutrino relic density comes from the analysis of CMB anisotropies, combined with large-scale structure data, suggesting $\Omega_\nu h^2 < 0.0067$ (95% C.L.).

Thus, the idea of considering DM as neutrino produces tension: neutrinos are relativistic and if we consider a neutrino-dominated universe, the actual theory of structure formation should be changed i.e., large structures were formed first, then these fragmented and evolved into what we see today [9]. According to observations and simulations, the structure formations should be "bottom-up" i.e. starting with stars and before galaxies and so on.

In the case that we considering only neutrinos as dark matter, we find inconsistency on cosmological density $\Omega_\nu h^2$: this density is extremely low and it suggests us that another source could exist.

2.3 Alternatives to Standard Dark Matter

Some of the most important candidates for Dark Matter are summarised below:

- Sterile neutrinos: they were proposed as dark matter candidates in 1993 by Dodelson and Widrow [17] and these are hypothetical particles, very similar to Standard Model neutrinos, however without Standard Model weak interactions (apart from mixing) [5]. Stringent cosmological and astrophysical constraints on sterile neutrinos come from the analysis of their cosmological abundance and the study of their decay products. Sterile neutrinos could be CDM, with energies between ~ 100 eV and ~ 10 keV ([50]).
- Axions: it is a particle proposed in 1977 to solve the so-called "strong-CP problem" [39]. In a nutshell, the strong force Lagrangian contains a term that can give an arbitrarily large electric dipole moment to the neutron; since no electric dipole moment for the neutron has ever been observed, Peccei and Quinn postulated that a new symmetry prevents the appearance of such a term. In the same way, they anticipated that this symmetry is slightly broken which leads to a new, very light scalar particle, the axion.

  Although this particle is extremely light (with mass in the μeV range), it can exist in sufficient numbers to act as cold dark matter.

  Since axions should couple to photons, they can be searched with precisely tuned radio frequency (RF) cavities. For example, inside the magnetic field of an RF cavity the axion can be converted into a photon which shows up as excess power in the cavity. And in a unique blend of particle and astrophysics, limits on axions have been placed through observations of red giant stars; axions, if they existed, would offer another cooling mechanism which can be constrained by studying how quickly red giant stars cool. Although the axion has never been directly observed, several experiments such as ADMX and CARRACK are continuing the search and setting new limits on axion parameters [18].

  If we consider that axions exist and SUSY is also correct, then the axino (the supersymmetric partner of the axion) is by a wide margin the LSP (lightest supersymmetric partner); neutralinos would decay into axinos through $\chi \longrightarrow \overline{a} + \gamma$. However, axinos would also act as a significant source of hot dark matter and thus could not compose the bulk of the dark matter.

- Kaluza-Klein states: Kaluza-Klein excitations of Standard Model fields, which appear in models of universal extra dimensions, have been considered as candidates for dark matter.

  The idea that our Universe could have extra spatial dimensions began in the 1920s with Theodor Kaluza and Oscar Klein; by writing down Eisntein's general theory of relativity in five dimensions. They were able to recover four dimensional gravity as well as Maxwell's equations for a vector field (and an extra scalar particle that they didn't know what to do with). It more precisely, Kaluza's theory was a purely classical extension of general relativity to five dimensions. The 5-dimensional metric contains 15 components: of it 10 components are identified with the 4-dimensional spacetime metric, 4 components with the electromagnetic vector potential, and one component with an unidentified scalar field (called the "radion" or the "dilaton").

  But, what is the connection between extra dimensions and candidates for dark matter? Theories in which extra dimensions are compactified, particles which can propagate in these extra dimensions have their quantized momenta as $p^2 \sim 1/R^2$, where p is the particle's momentum and R is the size of the extra dimension. Therefore, for each particle free to move in these extra dimensions, a set of fourier modes, called Kaluza-Klein states, appears:

  $$m^2 = \left(\frac{n}{R}\right)^2 + m_0^2 \tag{2.12}$$

  m_0 is the regular standard model mass of the particle, and n is the mode number. Each standard model particle is then associated with an infinite tower of excited Kaluza-Klein states. If translational invariance along the fifth dimension is postulated, then a new discrete symmetry called Kaluza-Klein parity exists and the **Lightest Kaluza-Klein particle** (LKP) can actually be stable and act as dark matter.

2.3.1 Self Interactive Dark Matter

Recently, different solutions have been considered in order to fix Small Scall Issues (SSI), for example Warm Dark Matter (WDM) [52] and Self Interactive Dark Matter (SIDM) [45, 46]. *Self-interacting dark matter* (SIDM) is a hypothetical form of dark matter consisting of particles with strong self-interactions [51, 55]. This type of dark matter was postulated to resolve a number of conflicts between observations and simulations on the galactic scale and smaller.

These ideas were introduced fourteen years ago, producing many articles discussing this possibility. For example [55] discusses the Spergel and Steinhardt idea: they have "proposed the concept of dark matter with strong self-interactions as a meaning to address numerous discrepancies between observations of dark matter halos on subgalactic scales and the predictions of the standard collisionless dark matter picture". In the same paper, the motivations for this scenario and some successful numerical tests are discussed. They also discuss the possibility that the dark matter interacts strongly with ordinary baryonic matter, as well as with itself. This analysis of the experimental constraints was useful for the re-evaluation of the allowed range of cross-section and mass of the original paper (see [51]). In [55] they find that simulations [59] based on the self-interaction proposal of Spergel and Steinhardt with self-interactions of strength $0.1\ \text{cm}^2/g < s < 6\ \text{cm}^2\ /g$ lend strong support for the concept, producing results that fit observations significantly better than standard collisionless cold dark matter models.

In this context, the favorite dark matter candidates (axions and neutralinos) are effectively collisionless and, hence, are at risk. The Spergel-Steinhardt proposal has stimulated the interesting possibility that dark matter consists of particles that interact through the strong force with ordinary matter. Their re-evaluation of constraints appears to indicate that the exotic hadron possibility is ruled out for a substantial range of masses near 1 GeV and cross-sections near $10^{-24}\ \text{cm}^2$, eliminating some of the most attractive possibilities. At the same time, the re-evaluation has re-opened a region encompassing larger masses and cross-sections previously thought to be ruled out.

Today, the idea is much more evolved and the dark matter theory has suffered extensions but, the main concept is the same. Thus, the possibility of new dark sector interactions beyond the usual collisionless DM paradigm are a real option. DM could have a large cross section for scattering with other DM particles and this scenario, dubbed self-interacting DM (SIDM) [31], can affect the internal structure (mass profile and shape) of DM halos compared to collisionless DM.

In turn, astrophysical observations of structure, compared to numerical N-body simulations, can prove the self-interacting nature of DM. It is worth emphasizing that tests of self-interactions can shed light on the nature of DM, even if DM is completely decoupled with respect to traditional DM searches. Usually, it is considered that the self-scattering is non-dissipative but it is possible for a sub-dominant fraction of dark matter to interact via dissipative processes [19].

There are long-standing issues on small scales that may point toward SIDM. They are commonly called Small Scale Issues (SSI) and include (a) Cusp-Core Problem (CCP), (b) Missing Satellites Problem (MSP) [45, 46], (c) To-Big-To-Fail problem [37] and (d) Plane of MW satellites. The first problem arises to the difference between the observed central density in Galaxies with respect to simulations based in ΛCDM, whereas the second is produced because the simulations predicted that between 10 and 100 times more satellite/dwarf galaxies should be orbiting within galactic halos than those actually observed. In the same way, the third problem emerge from densities of dark matter (DM) subhaloes which surround nearby dwarf spheroidal galaxies (dSphs) to be significantly lower than those of the most massive subhaloes expected around Milky Way sized galaxies in cosmological simulation, whereas as the fourth if found in ref.[38].

Recent articles[31, 30] pretend to explain some of small scale issues previously commented, using self interactive dark matter, therefore, we take these works as a motivation to investigate theoretically self interacting dark matter in large scale structure.

Chapter 3

Large Scale Structure and Perturbation Theory

3.1 Preliminaries

Understand the large-scale structures (LSS) of the Universe is one of the main goals of modern cosmology [16]. By investigate the distribution of matter at large scales new physics can be revealed. During the last three decades was established that gravitational instability plays a crucial rol in the large scale structures formation. In order to understand the required framework to this problem, we introduced the minimal information to understand it.

The Einstein field equations (in vacuum) are a set of equations that describe the interaction of gravitation as a result of space-time being curved by matter-energy. This subject has been extensively studied (see, [35, 16, 41]) and the understanding of this theory was fundamental for LSS of the universe [32].

Einstein equations are given by ten equations and can be expressed as:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{3.1}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{3.2}$$

where $T_{\mu\nu}$ is the Energy-momentum tensor, $G_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant and G is Newton's gravitational constant. The Einstein equations are computed by considering the metric and the Christoffel symbols [35, 54], but we are not interested on discussing it in detail. We only show that $R_{\mu\nu}$ can be written in terms of the Christoffel symbols as:

$$R_{\alpha\beta} = R^{\rho}{}_{\alpha\rho\beta} = \partial_\rho \Gamma^{\rho}_{\beta\alpha} - \partial_\beta \Gamma^{\rho}_{\rho\alpha} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\lambda}_{\rho\alpha} = 2\Gamma^{\rho}_{\alpha[\beta,\rho]} + 2\Gamma^{\rho}_{\lambda[\rho}\Gamma^{\lambda}_{\beta]\alpha}, \tag{3.3}$$

and the Ricci scalar R can be computed using the Ricci tensor by simple contraction. We will revisit it in Sec 3.1.1. Furthermore, note that the Christoffel symbols has the direct connection with the metric $g^{\lambda m}$ given by:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda m}\left(\frac{\partial g_{m\mu}}{\partial x^{\nu}} + \frac{\partial g_{m\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{m}}\right) = \frac{1}{2}g^{\lambda m}(g_{m\mu,\nu} + g_{m\nu,\mu} - g_{\mu\nu,m}), \tag{3.4}$$

Note that the Einstein's equations are the relativistic generalization of Poisson equation (see [41]), which is $\nabla^2\Phi = 4\pi G\rho$, where the mass density ρ is promoted in relativity to the "energy-momentum tensor" $T_{\mu\nu}$.

The simplest choice is to consider the cosmic fluid as an ideal fluid:

$$T^{\mu\nu} = (\rho + \mathcal{P})u^{\mu}u^{\nu} + \mathcal{P}g^{\mu\nu}, \tag{3.5}$$

where $\mathcal{P}$ is the pressure and u^{μ} is the 4-velocity field of the fluid. Considering a appropriate reference frame, this tensor has the form:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix}. \tag{3.6}$$

As we brief comment, the Einstein equations are the fundamental tool for studying large scale structure. In particular, our Universe is described by a Robertson-Walker metric [32] which is used into the Einstein's equations, whereas the dynamics of the expanding Universe is parametrized by the scale factor $a(t)$. In the next section we discuss about the Friedmann equations.

3.1.1 Friedmann Equations

In order to get the Friedmann's equation we need to consider three ingredients [15]:

- A Robertson-Walker line element.
- A perfect fluid.
- General Relativity with a cosmological constant.

Friedmann was the first who derived the equations [21] and it is the most used way to describe the Universe on large scales. Some literature on this topic is found in [16, 41, 40].

Starting with the FRW metric, we compute the Christoffel symbols given by 3.4:

$$\Gamma^{0}_{\mathrm{i}j} = \frac{\dot{a}}{a}g_{\mathrm{i}j}, \tag{3.7}$$

$$\Gamma^{\mathrm{i}}_{0j} = \frac{\dot{a}}{a}\delta^{\mathrm{i}}_{j}. \tag{3.8}$$

whereas the non null components to the Ricci tensor are:

$$R_{00} = -3\frac{\ddot{a}}{a}, \tag{3.9}$$

$$R_{\rm ij} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right] g_{\rm ij}. \tag{3.10}$$

The final step consists in computing the Ricci scalar R using $R = g^{\mu\nu}R_{\mu\nu}$ and get:

$$R = -6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right]. \tag{3.11}$$

Combining Eqs. (3.10), (3.11) and (3.6) into the Einstein equation we get the so-called Friedmann equations for the evolution of universe:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho, \tag{3.12}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G\mathcal{P}. \tag{3.13}$$

Note that in the previous equations we rewrite $T_{\mu\nu}$ in order to absorb the cosmological constant term $\Lambda g_{\mu\nu}$. The energy momentum tensor $T_{\mu\nu}$ satisfies the conservation law $T^{\mu\nu}{}_{;\nu}$. Furthermore, $T_{0\nu}$ gives us the first thermodynamics law:

$$\mathrm{d}(\rho a^3) + \mathcal{P}\mathrm{d}(a^3) = 0, \tag{3.14}$$

$$\therefore \mathrm{d}\left[a^3(\rho + \mathcal{P})\right] - a^3\mathrm{d}\mathcal{P} = 0. \tag{3.15}$$

If we consider a cosmological, then constant the equations look like:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{3.16}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\mathcal{P}) + \frac{\Lambda}{3}. \tag{3.17}$$

where the last equation was computed considering Rels. 3.12 and 3.13. In the same way, a good choice in this type of problem is to consider natural units thereby $c = 1$.

3.2 Dynamics of gravitational instability

The observable structure of the Universe, at the large scale level (supercluster, walls and filaments), is the product of gravitational amplification of small primordial fluctuations [3]. The most acceptable explanation consists in considering the gravitational interaction of collisionless cold dark matter (CDM) particles [8] in an expanding universe [14].

Unfortunately for us, the nature of dark matter has not yet been identified, however, all the candidates for CDM particles are light with respect to the mass of typical galaxies (see Sec. 2.2), with expected number densities of at least 10^{50} particles/Mpc^3. Under this limit, collisionless dark matter is described by the Vlasov equation for the distribution function in the phase space [27]. Some details of Vlasov equation can be consulted in Ref. [25].

The best model to describe the Universe consists of considering that CDM particles are non-relativistic (at scales much smaller than the Hubble radius) and therefor the equations of motion reduce to those of Newtonian gravity. Cold Dark Matter models work well at large scales but produce some problems on small scales. They are commonly called Small Scale Issues (SSI) and include (a) the Cusp-Core Problem (CCP) and (b) the Missing Satellites Problem (MSP) [45, 46]. The first problem consists of the difference between the observed central density in Galaxies w.r.t simulations based in ΛCDM whereas the second is produced because the simulations predict that between 10 and 100 times more satellite/dwarf galaxies should be orbiting within galactic halos than those actually observed.

The previously commented problems require us to consider new possibilities. One which consists in revisiting the hypothesis of collisional dark matter [51, 58]. There are several proposal about it. Some works include self interacting dark matter [30, 31, 51], topic that will be commented in Sec. 2.3.1. More realistic situations can be considered by taking into account the expansion of the universe. we only need to rewrite the position and momentum variable, and take into account a redefinition of the gravitational potential.

3.2.1 The Vlasov Equation

Because the universe can be considered, at zeroth order, as a very large space with a large distribution of particles, we need to start by getting an equation of motion in order to describe the evolution of these particles. It is the so-called Vlasov equation and it was used in 1915 by Jeans, which showed that the basic set of equations for galactic dynamics [27] is described by combining:

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{m}\frac{\partial f}{\partial \vec{r}} - m\frac{\partial \phi}{\partial \vec{r}}\frac{\partial f}{\partial \vec{p}} = 0, \tag{3.18}$$

with Poisson's equation. In eq. (3.18) $\vec{u} = \vec{p}/m$ is the velocity of a particle and ϕ is the gravitational potential which satisfies $\vec{F} = -m\nabla\phi$, where $\vec{F}$ is, of course, the force. The Vlasov equation was originally used to investigate the distribution function of plasma, consisting of charged particles with long-range interaction but in this case, we will show you the application of it studying the evolution of particles in the Universe.

So, we start by considering particles of mass m that interact only by gravitational interaction. The generalization for an expanding universe is clear, by considering the scale factor $a(t)$ in the definitions shown in Section. 3.3.1. The standard equation of motion is given by:

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = Gm\sum_{\mathrm{i}} \frac{\vec{r}_{\mathrm{i}} - \vec{r}}{\|\vec{r}_{\mathrm{i}} - \vec{r}\|^3}, \tag{3.19}$$

where the summations run over all other particles at $\vec{r}_{\mathrm{i}}$. For a large number of particles, the previous equation is rewritten in terms of a smooth gravitational potential given by the particle distribution $\dot{\vec{u}} = -\nabla\phi$ where ϕ is the Newtonian potential produced by the local mass density $\rho(\vec{r})$,

$$\phi(\vec{r}) = G \int \mathrm{d}^3\vec{r}' \frac{\rho(\vec{r})}{\|\vec{r}_\mathrm{i} - \vec{r}\|}. \tag{3.20}$$

For an expanding Universe, it is convenient to take into account the equations of motion in terms of comoving coordinate, which is related to the physical coordinates by a linear relation: $\vec{r} = a(\eta)\vec{x}$. Here the function $a(\eta)$ is defined as the cosmological scale factor that depends of the conformal time η (see [3] for details), $\vec{x}$ are the comoving coordinates and $\vec{r}$ are the physical coordinates. Furthermore, there exists a relation between the conformal time and the cosmic time t given by: $\mathrm{d}t = a(\eta)\mathrm{d}\eta$.

For a statistical treatment we need to consider how the distribution function evolves. In order to do it, we will derive the Vlasov equation for a gas of non-relativistic particles. Notice that it is standard to study the evolution of particles in a universe by considering the collisionless case, i.e., using the Vlasov equation. Despite of it, for more exotic cases, it is necessary to consider the interaction between particles, thereby the equation of motion for the distribution function is given by the Boltzmann equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \vec{p}} = \mathcal{C}[f], \tag{3.21}$$

however, we will not discuss this now. To explain the Vlasov equation, we start by considering that the distribution function f depends of the phase-space variables and time, so $f \equiv f(t, \vec{x}, \vec{p})$ and we will rewrite the total derivative in terms of each parameter, thus:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \vec{p}} = 0, \tag{3.22}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla f + (-m\nabla\Phi) \cdot \frac{\partial f}{\partial \vec{p}} = 0. \tag{3.23}$$

Thus, eq.. (3.23) is the Boltzmann equation for the collisionless (Vlasov equation) and non expanding case. In the same way, for an expanding universe the Vlasov equation can be rewritten as:

$$\frac{\mathrm{d}f}{\mathrm{d}\eta} = \frac{\partial f}{\partial \eta} + \frac{\vec{p}}{ma} \cdot \nabla f + (-am\nabla\Phi) \cdot \frac{\partial f}{\partial \vec{p}} = 0. \tag{3.24}$$

Notice that for an expanding universe we redefined a couple of terms according to:

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\eta} = -a(\eta)m\nabla\Phi(\vec{x}), \tag{3.25}$$

$$\vec{p} = am\vec{u}. \tag{3.26}$$

Furthermore, the perturbations are introduced by considering:

$$\rho(\vec{x}, \eta) = \rho_0(\eta) + \rho_0(\eta)\delta(\vec{x}, \eta), \tag{3.27}$$

$$\vec{u}(\vec{x}, \eta) = \vec{u}_0(\vec{x}, \eta) + \vec{u}(\vec{x}, \eta), \tag{3.28}$$

$$\phi(\vec{x}, \eta) = \phi_0(\vec{x}, \eta) + \Phi(\vec{x}, \eta) \tag{3.29}$$

where the last terms in previous equations are the perturbations in the quantities $\{\rho, \vec{u}, \phi\}$. Eq. (3.23) or its equivalent for an expanding universe, eq. (3.24) are complicated equations, because they are non-linear partial differential equations. We emphasize that the non-linearity is introduced because the potential Φ is related to the Poisson equation which, at first order, is:

$$\nabla^2\Phi = 4\pi G\rho_0\delta, \tag{3.30}$$

$$\nabla^2\Phi = 4\pi G\left(\frac{3\mathcal{H}^2\Omega_m}{8\pi G}\right)\delta, \tag{3.31}$$

$$\nabla^2\Phi = \frac{3}{2}\mathcal{H}^2\Omega_m\delta. \tag{3.32}$$

Thus, Φ is connected with Poisson's equation by an integral of the distribution function in momentum.

3.3 Linear Perturbation Theory in Cosmology

In this section we will discuss the dynamics of small perturbations in the background fluid. We will first consider a *Newtonian* fluid given that it is a good way to understand the perturbations. After this, we will briefly discuss the relativistic situation. Both situations are well explained in Ref. [32].

Let us start commenting that the Linear Perturbation Theory (LPT) is defined when in a power series we neglect the nonlinear part of quantities in our theory. We need to consider a small correction of the original quantities as:

$$T(x) = T + xT', \tag{3.33}$$

where x is a scalar parameter supposed to be small. $T(0) \equiv T$ is called the unperturbed quantity and xT' the perturbation. In some cases $T(x)$ can be expressed as a power series in x. If this is the case, the change of the other terms will be of the same order of magnitude as the perturbation xT' itself for small $\mid x \mid$. So, eq. (3.33) can be generalized to:

$$T(x) = T + xT^{(1)} + x^2T^{(2)} + \cdots, \tag{3.34}$$

$$T(x) = T + xT^{(1)} + \sum_2^\infty x^nT^{(n)}. \tag{3.35}$$

Thus, in the more simplified case the non-linear part is very small w.r.t. the linear part, i.e.:

$$\left|\sum_2^\infty x^nT^{(n)}\right| \ll xT^{(1)}. \tag{3.36}$$

In what follows, we will consider particular applications in cosmology of it.

3.3.1 Non-relativistic perturbation theory

In this case we can split the discussion in two sub-cases: Newtonian fluid without and with expansion. The first case considering that the perturbations depend on space only whereas the second considers perturbations dependent on space and time. Let's discuss the foundations on both cases.

Newtonian fluid without expansion

The equations of motion for a perfect fluid are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \tag{3.37}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \frac{1}{\rho}\nabla \mathcal{P} + \nabla \Phi = 0 \tag{3.38}$$

$$\nabla^2 \Phi = 4\pi G \rho. \tag{3.39}$$

Here Φ in the gravitational potential, ρ is the matter density, $\mathcal{P}$ and the pressure and $\vec{u}$ is the velocity. Trivial solutions are given by $\vec{u}_0 = \vec{0}$, $\rho_0 = \text{const}$ and we choose the potential zero point that the gravitational force vanishes. We will consider perturbations around the previous static solution at linear order, i.e.:

$$T = T_0 + \delta T, \tag{3.40}$$

where we are defining $T_0 \equiv \{\rho_0, \mathcal{P}_0, \vec{u}_0, \Phi_0\}$ and $\delta T \equiv \{\delta\rho, \delta\mathcal{P}, \delta\vec{u}, \delta\Phi\}$. Notice that the pressure and density are related by the equation of state $\mathcal{P} = \omega\rho$. If we assume that there is no spatial variation in the equation of state, we have to define the adiabatic sound speed as: $c_s^2 \equiv \partial\mathcal{P}/\partial\rho$ and because there are no spatial variations $c_s^2 \equiv \delta\mathcal{P}/\delta\rho$. So, the fluid equations for perturbations at first order are:

$$\frac{\partial(\delta\rho)}{\partial t} + \rho_0 \nabla \cdot (\delta\vec{v}) = 0, \tag{3.41}$$

$$\frac{\partial(\delta\vec{v})}{\delta t} + \frac{c_s^2}{\rho_0}\nabla(\delta\rho) + \nabla(\delta\Phi) = 0, \tag{3.42}$$

$$\nabla^2(\delta\Phi) = 4\pi G(\delta\rho). \tag{3.43}$$

We can combine the previous equations in order to get a single second order differential equation for $\delta\rho$: by taking the time derivative of eq. (3.41) and taking the time derivative of eq. (3.42). So, using eq. (3.43) we finally get:

$$(\ddot{\delta\rho}) - c_s^2 \nabla^2(\delta\rho) = 4\pi G \rho_0 (\delta\rho). \tag{3.44}$$

The solution of the previous differential equation has the form:

$$\delta\rho(\vec{x}, t) = \rho_0 \delta(\vec{x}, t) = A\rho_0 \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x} - \omega t)} \tag{3.45}$$

with

$$\delta(\vec{x}, t) = \frac{\delta\rho(\vec{x}, t)}{\rho_0}. \tag{3.46}$$

As is common, ω and $\vec{k}$ satisfy the dispersion relation given by:

$$\omega^2 = c_k^2 k^2 - 4\pi G \rho_0, \tag{3.47}$$

with $k = || \vec{k} ||$. Notice that if $k \in Im$ there will be exponentially growing (and decaying) modes. Opposite to it, if $\omega \in Re$ the perturbations will oscillate as sound waves. An important cut-off in ω is established when $k = k_J$, where k_J is the Jeans wavenumber with the value:

$$k_J = \sqrt{\frac{4\pi G \rho_0}{c_s^2}} \tag{3.48}$$

Note that if $k^2 \ll k_J^2$ the density contrasts $\delta\rho$ grows (or decays) exponentially according to the dynamical timescale $\tau_{\mathrm{d}yn} \simeq (4\pi G \rho_0)^{-1/2}$. It Is possible to define the Jeans mass: the total mass contained within a sphere with radius $\lambda = \pi / k_J$. Then, the Jean mass is:

$$M_J = \frac{4\pi}{3} \left(\frac{\pi}{k_J} \right)^3, \tag{3.49}$$

$$= \sqrt{\frac{\pi^5 c_s^6}{36 G^3 \rho_0}}. \tag{3.50}$$

Let's comment that for mass $M < M_J$ the perturbations are stable against the gravitational collapse.

Newtonian fluid with expansion

In order to consider a more realistic case, we need to consider the expansion of the universe. In this sense, a Newtonian fluid with expansion is only an extension of the case previously commented by correcting the initial conditions considering a space-time dependence in each quantity. This can be made by considering small density perturbations in the universe. In this case, we are setting the initial condition as:

$$\rho_0(\vec{x}, t) = \frac{\rho_{m,0}}{a^3(t)}, \tag{3.51}$$

$$\vec{u}_0(\vec{x}, t) = \frac{\dot{a}(t)}{a(t)} \vec{x}, \tag{3.52}$$

$$\nabla \Phi_0(\vec{x}, t) = \frac{4\pi G \rho_0}{3} \vec{x}, \tag{3.53}$$

where $a(t)$ evaluated at present time is equal to one by definition. It is important to comment that the perturbative analysis used here is non relativistic and, therefore, it is only valid for perturbations on scales smaller than the size of the universe $\|\vec{x}\| < H^{-1}$. Thus, considering perturbations at first order into the eqs. (3.37), (3.38) and (3.39) we obtain:

$$\frac{\partial(\delta\rho)}{\partial t} + 3\frac{\dot{a}}{a}(\delta\rho) + \frac{\dot{a}}{a}(\vec{x} \cdot \nabla)(\delta\rho) + \rho_0 \nabla \cdot (\delta\vec{u}) = 0, \tag{3.54}$$

$$\frac{\partial(\delta\vec{u})}{\partial t} + \frac{\dot{a}}{a}(\delta\vec{u}) + \frac{\dot{a}}{a}(\vec{x} \cdot \nabla)(\delta\vec{u}) + \frac{c_s^2}{\rho_0} \nabla(\delta\rho) + \nabla(\delta\Phi) = 0, \tag{3.55}$$

$$\nabla^2(\delta\Phi) = 4\pi G(\delta\rho). \tag{3.56}$$

The previous equations were computed taking into account the standard expansion $T = T_0 + \delta T$ (with the contrast δT given by eqs. (3.51), (3.52) and (3.53) and using the expression for c_s defined in this section. Let's comment that it is possible to introduce the Fourier transform by the expression:

$$\Psi(\vec{x}, t) = \int \frac{\mathrm{d}^3 r}{(2\pi)^{3/2}} \Psi_k(t) \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}/a(t)} \tag{3.57}$$

where $r = k/a$ and Ψ represents any of the perturbations. For simplicity, we rewrite $\delta\rho$ as $\delta = \delta\rho/\rho_0$. Notice that introducing the Fourier transform it is useful to decompose the perturbed velocity field $\delta\vec{u}$ as a rotational and irrotational field. A detailed discussion of it is not required here but it is possible to deepen this in Ref. [25].

Recall that it is possible to connect the theory used in this approach with other well known results. Thus, by neglecting the expansion $\dot{a}(t)$ and identifying $k \to |\vec{k}|/a$ as the comoving wavenumber we can get the dispersion relation by considering the Fourier transform δ equation:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_0\right)\delta_k = 0 \tag{3.58}$$

with k_J defined as:

$$k_J^2 = \frac{4\pi G\rho_0}{c_s^2} a^2. \tag{3.59}$$

Just by simple inspection is easy to verify that k_J for the non relativistic and relativistic case differ by a $a(t)^2$ factor. We recall that this quantity splits gravitationally stable and unstable modes. As the previously commented case, depending on the wavenumber, the behavior changes dramatically: for short wavelength modes ($k \gg k_J$) the perturbations oscillate as a sound wave whereas for the opposite case ($k \ll k_J$) there are unstable growing modes.

3.3.2 Relativistic perturbation theory

For a more complete treatment of perturbations, general relativity needs to be considered due to the fact that the Newtonian approach breaks down when $\|\vec{x}\| > H^{-1}$. Thus for do it, we require a perturbative treatment of the Einstein and energy momentum tensor around their Friedmann Robertson Walker forms:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{3.60}$$

For simplicity, we introduce the conformal time η according to the relation $\mathrm{d}t = a\mathrm{d}\eta$ and consider a spatially flat FRW metric to get:

$$\mathrm{d}s^2 = a^2(\eta)[\mathrm{d}\eta^2 - \delta_{\mathrm{i}j}\mathrm{d}x^{\mathrm{i}}\mathrm{d}x^j]. \tag{3.61}$$

In addition, we consider a flat ($K = 0$) universe. Then, in terms of conformal time, the Friedmann equations look like:

$$\mathcal{H}^2 = \frac{1}{3}a^2(8\pi G\rho + \Lambda), \tag{3.62}$$

$$\mathcal{H}' = \frac{1}{6}a^2\left[2\Lambda - 8\pi G(\rho + 3\mathcal{P})\right]. \tag{3.63}$$

It is important to say that $\mathcal{H}$ is the conformal Hubble parameter defined as $\mathcal{H} = 1/a(\mathrm{d}a/\mathrm{d}\eta) = aH$. Recall that we are interested in studying perturbations, and in order to do it, we will consider the most general metric:

$$\mathrm{d}s^2 = a^2(\eta)\big[(1+2\psi)\mathrm{d}\eta^2 - 2B_\mathrm{i}\mathrm{d}x^\mathrm{i}\mathrm{d}\eta - \{(1-2\phi)\delta_{\mathrm{i}j} + 2E_{\mathrm{i}j}\}\mathrm{d}x^\mathrm{i}\mathrm{d}x^j\big]. \tag{3.64}$$

In the previous metric, ψ and ϕ are scalar functions, B_i is a vector function and $E_{\mathrm{i}j}$ is the tensor function. All the functions depend on η and x^i. As before, we constrain the metric by considering the conformal Newtonian gauge defined as:

$$\mathrm{d}s^2 = a^2(\eta)\big[(1+2\psi)\mathrm{d}\eta^2 - (1-2\phi)\delta_{\mathrm{i}j}\mathrm{d}x^\mathrm{i}\mathrm{d}x^j\big]. \tag{3.65}$$

In order to compute the perturbed Einstein equations we proceed in the common way: first we will calculate the connection using eq. (3.4) identifying the inverse of the perturbed metric $g^{\mu\nu}$ as:

$$g^{\mu\nu} = \frac{1}{a^2}\begin{pmatrix} 1-2\psi & 0 \\ 0 & -(1+2\phi)\delta^{\mathrm{i}j} \end{pmatrix}. \tag{3.66}$$

Thus, the Christoffel symbols for the perturbed metric are:

$$\Gamma^0_{00} = \mathcal{H} + \psi', \tag{3.67}$$
$$\Gamma^0_{0\mathrm{i}} = \partial_\mathrm{i}\psi, \tag{3.68}$$
$$\Gamma^\mathrm{i}_{00} = \delta^{\mathrm{i}j}\partial_j\psi, \tag{3.69}$$
$$\Gamma^0_{\mathrm{i}j} = \big[\mathcal{H} - \{\phi' + 2\mathcal{H}(\phi+\psi)\}\big]\delta_{\mathrm{i}j}, \tag{3.70}$$
$$\Gamma^\mathrm{i}_{j0} = \big[\mathcal{H} - \phi'\big]\delta^\mathrm{i}_j, \tag{3.71}$$
$$\Gamma^\mathrm{i}_{jk} = -\big(\delta^\mathrm{i}_j\partial_k\phi + \delta^\mathrm{i}_k\partial_j\phi\big) + \delta_{jk}\delta^{\mathrm{i}l}\partial_l\phi. \tag{3.72}$$

By considering the eq. (3.3) it is posible to get $G_{\mu\nu}$. So, neglecting higher order terms, the Ricci tensor is:

$$R_{00} = -3\mathcal{H}' + \nabla^2\psi + 3\mathcal{H}\big(\phi' + \psi'\big) + 3\phi''. \tag{3.73}$$
$$R_{0\mathrm{i}} = 2\partial_\mathrm{i}\phi' + 2\mathcal{H}\partial_\mathrm{i}\psi. \tag{3.74}$$
$$R_{\mathrm{i}j} = \big[\mathcal{H}' + 2\mathcal{H}^2 - \phi'' + \nabla^2\phi - 2(\mathcal{H}' + 2\mathcal{H}^2)(\phi+\psi) - \mathcal{H}\psi' - 5\mathcal{H}\phi'\big]\delta_{\mathrm{i}j} + \partial_\mathrm{i}\partial_j(\phi-\psi) \tag{3.75}$$

Finally, the Ricci scalar is given by:

$$R = g^{00}R_{00} + 2g^{0\mathrm{i}}R_{0\mathrm{i}} + g^{\mathrm{i}j}R_{\mathrm{i}j}. \tag{3.76}$$

Now, we are ready to compute $G_{\mu\nu}$ by simple substitution and get:

$$G_{00} = 3\mathcal{H}^2 + 2\nabla^2\phi - 6\mathcal{H}\phi', \tag{3.77}$$
$$G_{0\mathrm{i}} = 2\partial_\mathrm{i}\phi' + 2\mathcal{H}\partial_\mathrm{i}\psi, \tag{3.78}$$
$$G_{\mathrm{i}j} = -(2\mathcal{H}' + \mathcal{H}^2)\delta_{\mathrm{i}j} + \big[\nabla^2(\psi-\phi) + 2\phi'' \tag{3.79}$$
$$+ 2(2\mathcal{H}' + \mathcal{H}^2)(\phi+\psi) + 2\mathcal{H}\psi' + 4\mathcal{H}\phi'\big]\delta_{\mathrm{i}j} + \partial_\mathrm{i}\partial_j(\phi-\psi). \tag{3.80}$$

On the other hand, the perturbed energy momentum tensor can be written by considering corrections in the quantities $\{\rho, u^{\rm i}, \mathcal{P}\}$ and taking into account the trace-free anisotropic stress $\Pi^{\rm i}_j$. So, the perturbed energy momentum tensor is:

$$T^0_0 = \rho_0 + \delta\rho \equiv \rho_0(1+\delta), \tag{3.81}$$
$$T^{\rm i}_0 = q^{\rm i}, \tag{3.82}$$
$$T^{\rm i}_j = -(\mathcal{P}_0 + \delta\mathcal{P})\delta^{\rm i}_j + \Pi^{\rm i}_j. \tag{3.83}$$

Recall that $\mathcal{P}_0$ and ρ_0 are the pressure and density of the homogenous background, whereas $\Pi_{{\rm i}j}$ the trace-free anisotropic stress in the fluid. Now, by combining $G_{\mu\nu}$ and $T_{\mu\nu}$ we get equations of motion for a perturbed universe. The temporal component of the Einstein equation is:

$$G_{00} = 8\pi G T_{00} + \Lambda g_{00}, \tag{3.84}$$
$$3\mathcal{H}^2 + 2\nabla^2\phi - 6\mathcal{H}\psi' = 8\pi G a^2\rho_0(1+2\psi)(1+\delta) + \Lambda a^2(1+2\psi) \tag{3.85}$$

Notice that at zeroth order the previous equation is, of course, the Friedmann equation given by eq. (3.62). In the same way, the first order equation for the 00-component of Einstein's equation can be combined with eq. (3.62) in order to avoid the cosmological constant. Thus we obtain:

$$\nabla^2\phi = 3\mathcal{H}(\phi' + \mathcal{H}\psi) + 4\pi G a^2\rho_0\delta. \tag{3.86}$$

The mixed components are obtained as:

$$G_{0{\rm i}} = 8\pi G T_{0{\rm i}} + \Lambda g_{0{\rm i}}, \tag{3.87}$$
$$\partial_{\rm i}\phi' + \mathcal{H}\partial_{\rm i}\psi = -4\pi G a^2 q_{\rm i} \tag{3.88}$$

Defining the velocity perturbation $q^{\rm i}$ as $q^{\rm i} = (\rho_0 + \mathcal{P}_0)\partial_{\rm i}u$ and considering that the perturbations decay at infinity, is possible to compute the integral of the previous equation to obtain:

$$\partial_{\rm i}(\phi' + \mathcal{H}\psi) = \partial_{\rm i}(-4\pi G a^2(\rho_0 + \mathcal{P}_0)u), \tag{3.89}$$
$$\phi' + \mathcal{H}\psi = -4\pi G a^2(\rho_0 + \mathcal{P}_0)u, \tag{3.90}$$

It is possible to recover a modified Poisson equation with a rewritten source term. If we substitute eq.(3.90) into (3.86) we obtain:

$$\nabla^2\phi = 4\pi G a^2[\rho_0\delta - 3\mathcal{H}(\rho_0 + \mathcal{P}_0)u]. \tag{3.91}$$

Finally, by taking into account the ij-component of Einstein's equation and eliminating the zeroth order term, we obtain:

$$\left[\nabla^2(\psi - \phi) + 2\phi'' + 2(2\mathcal{H}' + \mathcal{H}^2)(\phi + \psi) + 2\mathcal{H}\psi' + 4\mathcal{H}\phi'\right]\delta_{{\rm i}j} \tag{3.92}$$
$$+ \partial_{\rm i}\partial_j(\phi - \psi) = a^2[8\pi G(\delta\mathcal{P} - 2\mathcal{P}\phi) + 2\Lambda\phi]\delta_{{\rm i}j} - 8\pi G a^2\Pi_{{\rm i}j}. \tag{3.93}$$

It is possible to decompose the previous equation into trace and trace free parts seperately. The trace-free part is:

$$\partial_{\rm i}\partial_j(\phi - \psi) - \frac{1}{3}\delta_{{\rm i}j}\nabla^2(\phi - \psi) = -8\pi G a^2\Pi_{{\rm i}j}. \tag{3.94}$$

Note that $\Pi_{ij} = 0$ implies that $\phi = \psi$ and therefore, there is only one potential. On the other hand, the trace part of Einstein's equation (combined with the second Friedmann equation) gives us:

$$\phi'' + \frac{1}{3}\nabla^2(\psi - \phi) + (2\mathcal{H}' + \mathcal{H}^2)\psi + \mathcal{H}\psi' + 2\mathcal{H}\phi' = 4\pi G a^2 \delta\mathcal{P}. \tag{3.95}$$

3.3.3 Perturbations of the Boltzmann Equations

As we commented before, the Boltzmann equation is an important tool to describe perturbations that evolve in large scale structures. The implementation of Boltzmann equation has many applications, for example for photons, Cold Dark Matter and baryons. In this opportunity we'll show how to investigate perturbations for photons at first order with a non-negligible collisional term using the Boltzmann equation [16].

Boltzmann equation for photons

In order to study a system with interactions let us consider the Boltzmann equation considering the collisional term and under a relativist approach. As we commented before, we are interested in studying the evolution of the distribution function $f \equiv f(t, \vec{x}, \vec{p})$. Thus, the evolution equation for the distribution function is $\dot{f} = \mathcal{C}[f]$. Recall that $\mathcal{C}[f]$ represents the collision terms and, therefore, if it is zero it implies that a phase space element of the particle does not change.

Because we are interested into developing relativistic equations using the Boltzmann equation, we will consider the line element as:

$$\mathrm{d}s^2 = (1 + 2\psi)\mathrm{d}t^2 - a^2(t)(1 - 2\phi)\delta_{\mathrm{i}j}\mathrm{d}x^{\mathrm{i}}\mathrm{d}x^j. \tag{3.96}$$

In general, the evolution equation for f is given by:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^{\mathrm{i}}} \cdot \frac{\mathrm{d}x^{\mathrm{i}}}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\mathrm{d}f}{\partial \hat{p}^{\mathrm{i}}} \cdot \frac{\mathrm{d}\hat{p}^{\mathrm{i}}}{\mathrm{d}t}. \tag{3.97}$$

Note that we are splitting the momentum contribution into a scalar part given by p and into a vectorial part given by $\hat{p}^{\mathrm{i}}$. We are working with the four-momentum $p^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$, where λ parametrizes the particle's path (in this case, a photon). Thus, for the massless photon, we have $g_{\mu\nu}p^\mu p^\nu = 0 \Longrightarrow (1 + 2\psi)(p^0)^2 + p^2 = 0$. In the same way, considering that the spatial part of the momentum satisfies $p^2 = g_{\mathrm{i}j}p^{\mathrm{i}}p^j$, finally we get:

$$p^0 = \frac{p}{\sqrt{-1(1 + 2\psi)}} \simeq \frac{p}{\mathrm{i}}(1 - \psi), \tag{3.98}$$

where i is defined as $\mathrm{i} = \sqrt{-1}$. Defining the spatial component $p^{\mathrm{i}} \equiv \mathrm{d}x^{\mathrm{i}}/\mathrm{d}\lambda$ and the temporal as $p^0 \equiv \mathrm{d}t/\mathrm{d}\lambda$ we get that the velocity components are related with the momentum variables by:

$$\frac{\mathrm{d}x^{\mathrm{i}}}{\mathrm{d}t} = \frac{p^{\mathrm{i}}}{p^0}. \tag{3.99}$$

Considering isotropy in the momentum vector we can write $p^{\mathrm{i}} = C\hat{p}^{\mathrm{i}}$ to obtain:

$$p^2 = g_{\mathrm{i}j}\hat{p}^{\mathrm{i}}\hat{p}^{j}C^2 = -a^2(1-2\phi)C^2, \tag{3.100}$$
$$C = \frac{p}{\mathrm{i}a}(1+\phi). \tag{3.101}$$

Thus, by replacing C into p^{i} result:

$$p^{\mathrm{i}} = \frac{p\hat{p}^{\mathrm{i}}}{\mathrm{i}a}(1+\phi). \tag{3.102}$$

Using the eqs. (3.98) and (3.102) we finally obtain to first order:

$$\frac{\mathrm{d}x^{\mathrm{i}}}{\mathrm{d}t} = \frac{\hat{p}^{\mathrm{i}}}{a}(1+\psi+\phi). \tag{3.103}$$

Note that, at first order in perturbation theory, the last term in eq. (3.97) is negligible because is product of two first order terms and produces a second order term. In the same way, we are interested in computing $\mathrm{d}p/\mathrm{d}t$ and to do it we first compute $\mathrm{d}p^0/\mathrm{d}t$ due $p^0 \propto p$. Notice that it is possible to rewrite:

$$\frac{\mathrm{d}p^0}{\mathrm{d}\lambda} = \frac{\mathrm{d}p^0}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}\lambda} = p^0\frac{\mathrm{d}p^0}{\mathrm{d}t} = \left(\frac{p}{\mathrm{i}}(1-\psi)\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\mathrm{i}}(1-\psi)\right), \tag{3.104}$$
$$\frac{\mathrm{d}p^0}{\mathrm{d}\lambda} = \left(\frac{p}{\mathrm{i}}(1-\psi)\right)\left(\frac{\mathrm{d}p}{\mathrm{d}t}\frac{1-\psi}{\mathrm{i}} - \frac{p}{\mathrm{i}}\frac{\mathrm{d}\psi}{\mathrm{d}t}\right), \tag{3.105}$$
$$\frac{\mathrm{d}p^0}{\mathrm{d}\lambda} = -\frac{\mathrm{d}p}{\mathrm{d}t}(1-\psi)^2 p + p^2\frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{3.106}$$

Doing a Taylor expansion and rewriting the total derivative in ψ we finally get the next expression:

$$\frac{\mathrm{d}p^0}{\mathrm{d}\lambda} = -\frac{\mathrm{d}p}{\mathrm{d}t}(1-2\psi)p + p^2\left(\frac{\partial\psi}{\partial t} + \frac{\mathrm{d}x^{\mathrm{i}}}{\mathrm{d}t}\frac{\partial\psi}{\partial x^{\mathrm{i}}}\right). \tag{3.107}$$

Because we need the total derivative of the momentum norm p, it is necessary to use the geodesic equation for photons:

$$\frac{\mathrm{d}p^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta}p^\alpha p^\beta, \tag{3.108}$$
$$\frac{\mathrm{d}p^0}{\mathrm{d}\lambda} = \frac{1}{2}g^{0\nu}\left[2\partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right]p^\alpha p^\beta. \tag{3.109}$$

Note that first order the second term is

$$\partial_0 g_{\alpha\beta}p^\alpha p^\beta = p^2\left[-2\frac{\partial\psi}{\partial t} + 2H - 2\frac{\partial\phi}{\partial t}\right] \tag{3.110}$$

and

$$\partial_\beta g_{0\alpha}p^\alpha p^\beta = -2p^2\left[\frac{\partial\psi}{\partial t} + \frac{\partial\phi}{\partial x^{\mathrm{i}}}\frac{\hat{p}^{\mathrm{i}}}{a}\right] \tag{3.111}$$

Thus, combining (3.107) and (3.108) we obtain:

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = -H + \frac{\partial\phi}{\partial t} - \frac{\partial\psi}{\partial x^{\mathrm{i}}}\frac{\hat{p}^{\mathrm{i}}}{a}. \tag{3.112}$$

Finally, we can write down the Boltzmann equation for photons:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^{\mathrm{i}}}{a}\frac{\partial f}{\partial x^{\mathrm{i}}} - p\frac{\partial f}{\partial p}\left[H - \frac{\partial\phi}{\partial t} + \frac{\partial\psi}{\partial x^{\mathrm{i}}}\frac{\hat{p}^{\mathrm{i}}}{a}\right]. \tag{3.113}$$

Distribution function and collisional term

In order to make progress, we need to specify the distribution function. Starting with a perturbed temperature field $T = T_0(t)[1 + \Theta(\vec{x}, \hat{p}, t)]$, we identify $T_0(t)$ as the background photon temperature and note that T is independent of the magnitude of the momentum. Thus, the perturbed Bose-Einstein distribution is given by:

$$f \equiv f(\vec{x}, p, \hat{p}, t) = \left[\exp\left(\frac{p}{T_0(1 + \Theta(\vec{x}, \hat{p}, t))}\right) - 1\right]^{-1}, \tag{3.114}$$

and considering linear perturbations we have:

$$f \simeq f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta, \tag{3.115}$$

with $f^{(0)} = [\exp(p/T_0) - 1]^{-1}$ and satisfying the property: $T\partial f^{(0)}/\partial T = -p\partial f^{(0)}/\partial p$. So, order by order we can extract physical information relevant to understand the perturbations. At zeroth order we have no collision term and therefore $\mathrm{d}f/\mathrm{d}t = 0 \implies T_0 \propto a^{-1}$. In the same line, by considering eq. (3.115) into (3.113) we obtain:

$$\frac{\mathrm{d}f}{\mathrm{d}t}|_{\mathcal{O}(1)} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^{\mathrm{i}}}{a}\frac{\partial \Theta}{\partial x^{\mathrm{i}}} - \frac{\partial \phi}{\partial t} + \frac{\hat{p}^{\mathrm{i}}}{a}\frac{\partial \psi}{\partial x^{\mathrm{i}}}\right]. \tag{3.116}$$

We just need to briefly discuss the collisional term in the Boltzmann equation to obtain the big picture of the treatment discussed here. In this case we will consider interaction between photon and baryons by inverse Compton scattering ($\mathrm{e}^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow \mathrm{e}^-(\vec{q}\,') + \gamma(\vec{p}\,')$). An extensive discussion on the collision term can be found in Ref [16, 4] and we avoid discussing formal details of it. A way to express the collisional term is:

$$\mathcal{C}[f(\vec{p})] = \sum_{\vec{q}, \vec{q}\,', \vec{p}\,'} |A|^2[f_{\mathrm{e}}(\vec{q}\,')f(\vec{p}\,') - f_{\mathrm{e}}(\vec{q})f(\vec{p})]. \tag{3.117}$$

Skipping the steps, the collisional term for this case is given by the expression:

$$\mathcal{C}[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p}n_{\mathrm{e}}\sigma_T[\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b], \tag{3.118}$$

where n_{e} is the free electron density, σ_T is the cross section, $\vec{v}_b$ is the baryonic velocity and Θ_0 is the so-called monopole part of the perturbation defined as:

$$\Theta_0(\vec{x}, t) \equiv \frac{1}{4\pi}\int \mathrm{d}\Omega'\Theta(\hat{p}', \vec{x}, t) \tag{3.119}$$

Finally, combining eqs. (3.116) and (3.118) we obtain:

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^{\mathrm{i}}}{a}\frac{\partial \Theta}{\partial x^{\mathrm{i}}} - \frac{\partial \phi}{\partial t} + \frac{\hat{p}^{\mathrm{i}}}{a}\frac{\partial \psi}{\partial x^{\mathrm{i}}} = n_{\mathrm{e}}\sigma_T[\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b], \tag{3.120}$$

or rewriting in conformal time $\mathrm{d}t = a\mathrm{d}\eta$ to obtain:

$$\Theta' + \hat{p}^{\mathrm{i}}\frac{\partial \Theta}{\partial x^{\mathrm{i}}} - \phi' + \hat{p}^{\mathrm{i}}\frac{\partial \psi}{\partial x^{\mathrm{i}}} = n_{\mathrm{e}}\sigma_T a[\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b]. \tag{3.121}$$

Chapter 4

Smoothing Approach of LSS and Effective Theory

4.1 Introduction

So far, we have discussed the perturbations that produce large scale structures as well as the predominant matter in the Universe: Dark Matter. In the first case, we revisited details about the physics relevant to understand the Universe at these scales: the Einstein equation, the Friedmann equations and Vlasov equation, considering an expanding or non expanding Universe. In the same way, the second case explored the existence of a particular kind of matter which is only possible to verify by considering the gravitational effect of it on the environment, furthermore we described the different techniques to infer the Dark Matter existence and possible extension to Dark Matter as Self Interactive Dark Matter.

Thus, in the previous chapters we investigated the required theory to understand the Large Scale Structure, however, an exact approach is difficult to obtain here and it is necessary, in order to make progress, to develop the quantities using perturbative techniques as we have shown you in Sec. 3.3. The analysis is commonly tried by considering Linear perturbation theory, therefore we are ignoring the non-linear terms which should be important in cases where the non-linear part $\backsim$ linear part. Thus, in these cases we aren't take into account all the relevant physics that describe the Universe. So, if we are interested in generalizing the equations for a more realistic description of Universe, we necessarily need to include the non linear terms, and these introduce complications in the description.

Notice that the linear description allow us investigate a low range of energy before it is needed to take into account the non linear part, therefore, in order to extend the range of action, a new tool is necessary at this point. Nowadays, Large Scale Structure has been revisited in order to improve the description of the Universe by considering field theory techniques. Studying it under other approach (differ to Linear Perturbation Theory) could revel us an improved and more complete interpretation of the physics larger than linear scales.

The common alternatives are the so called Effective field theories (EFT), which, for a given system, can be defined as a theory of dynamics of the system at small energies compared to a given cut-off. Thus, in systems where low-energy states (relative to cut-off) are independent of states of high energy, we are in the presence of a decoupled form of the system. This difference is vital with respect to standard perturbation theory because the modes in this case are impossible to decouple.

In this chapter we will introduce the effective field technique, in particular we will focus on the smoothing approach applied at Large Scale Structure equations. In addition, we show our results, at zeroth order, of applying the previous commented technique to compute smoothed equations of motion by taking into account the first momentum of Boltzmann equation.

4.2 Standard lore of Effective Field Theory

Effective theories are those that reproduce the complete theories in some specific regime (typically the scale considered is the energy), and are used to simplify difficult problems in many different fields in physics. The basis of this approach (Effective Theory) were introduced by Kenneth Wilson because he studied how fundamental properties of a system vary, depending on the scale over which they are measured (see [57] for pioneering work). He won the nobel prize "for his theory for critical phenomena in connection with phase transitions".

These theories have the characteristic that describe the main physics of the problem because they consider only the effective degree of freedom in the system. Hence, we can investigate the low-energy sector of the theory without details of the other sector, because effective field theories can be decoupled. Note that this treatment is applied in the same way if the underlying theory is relativistic quantum field theory or condensed matter. Effective Field Theories parametrize, for a specific energy range, an underling theory. Due we are not interested in explain all the technical details of these technique, it is necessary to recall a couple of good papers about it: the review of C.P. Burgess (see [10]) and the detailed notes by Antonio Pich (see [42]).

Thus, the Effective Field Theory is required by reason of the full theory is difficult to investigate, however, in specific cases the high energy theory is known and the effective theory can be obtained in a simple form (Top-Down approach). In other cases the high energy theory may not be known, however, the effective theory can still be built by imposing symmetries and "naturalness" constraints on lagrangians (Bottom-up approach).

Many examples of effective theories have been studied, for example: multipole expansion for the electrical potential, the energy corrections for hydrogen atom because electromagnetic interaction and from the proton structure, newtonian mechanics as an inferior limit of the relativistic case, etc. In addition, maybe the most emblematic example of an effective theory is the Standard Model of particle physics (bottom-up EFT).

Bottom-up and Top-Down approaches have a complex connection with the high energy theory, however, nowadays we have ways to connect both in a more simple way by considering alternative approaches, for example, to get the effective expression only by the application of a windows function which allows us to consider the effective terms required to build the EFT. Naturally, we need windows function to satisfy specific properties in order to make progress. We will comment about this point later but for the moment, we mention it because the relevance of this alternative approach for Large Scale Structure [2].

4.3 Effective Field Theory

In the previous section we introduced the basics of Effective Field Theory: it is a theory that describes the dynamics of a given system at small energies compared to a given cut-off. Some systems have the characteristic that low energy states (wirh respect to the cut-off) are effectively independent of states at high energies. These states are the so called decoupled. In this case, it is possible to investigate the low energy sector without taking into account details of high energy sector.

The construction of an EFT follows one of two general procedures previously introduced: Top-down and Botton-Up, depending if the high energy theory is known. The standard procedure is based on an expansion of the effective action (which formally represents the EFT) in term of sum of local operators, which are constrained by symmetries and "naturalness" considerations.

The previously commented approaches differ on how the effective action is obtained. The first is obtained by eliminating degrees of freedom from the action of the high energy theory whereas the second starts by considering the most general lagrangian which is constrained.

4.3.1 Top-Down approach

This approach is used when *we know the full theory*, and it is applied by a systematic elimination of degrees of freedom associated with energies above some energy scale. A technique to decouple the high energy and low energy degree of freedom was developed by Wilson in 1970 and can be divided in two steps: (I) Identify the high energy degree of freedom and integrate out of the action. The result of this integrating is an effective action that describe non-local interactions between the low energies degree of freedom. (II) In order to obtain the local effective action (i.e., one that represent local interactions between low energies degree of freedom), we need to expand the action in terms of local operations.

Considering the two steps commented in detail (see [44]):

(I) For a field theory with an action $\mathcal{S}$ and with an energy scale E_0, we want to investigate the physics lower than some energy scale E (i.e. $E \ll E_0$). In order to split the theory, we choose a cut off Λ close or lower than E_0 and divide the fields ϕ into high and low momentum part (or quantities at large and short scale) with respect to Λ. So, the field is written as $\phi = \phi_H + \phi_L$ and each field identifies a region of the full theory. Thus, ϕ_H represents quantities with energies higher than the energy at Λ whereas ϕ_L represents the opposite situation. By integrating out the high momentum fields we "eliminate" the field ϕ_H:

$$\int \mathcal{D}\phi_L \mathcal{D}\phi_H \mathrm{e}^{\mathrm{i}\mathcal{S}[\phi_H,\phi_L]} = \int \mathcal{D}\phi_L \mathrm{e}^{\mathrm{i}\mathcal{S}_\Lambda[\phi_L]}, \tag{4.1}$$

where it is possible to identify:

$$\exp\left(\mathrm{i}\mathcal{S}_\Lambda[\phi_L]\right) \equiv \int \mathcal{D}\phi_H \exp\left(\mathrm{i}\mathcal{S}[\phi_L,\phi_H]\right), \tag{4.2}$$

and the effective lagrangian density $\mathcal{L}_{\text{eff}}$ is given by:

$$\mathcal{S}_\Lambda[\phi_L] = \int \mathrm{d}^D x \mathcal{L}_{\text{eff}}[\phi_L], \tag{4.3}$$

with D being the dimension of the space-time.

(II) Normally, the integration over heavy fields produces a non-local effective action (i.e. one action in which terms occur that consists of operators and derivatives that are not all evaluated at the same spacetime point). Thus, this point is studied by expanding the effective action in a set of local operators:

$$\mathcal{S}_\Lambda = \mathcal{S}_0(\Lambda, g^*) + \sum_{\mathrm{i}} \int \mathrm{d}^D x g^{\mathrm{i}} \mathcal{O}_{\mathrm{i}}, \tag{4.4}$$

where the sum runs over all local operators $\mathcal{O}_{\mathrm{i}}$ allowed by symmetries of the initial theory, g_{i} are the coupling constants. Assuming weak coupling, the action S_0 can be considered as the free action of the initial theory, i.e. $g^* = 0 \Longrightarrow \mathcal{S}_0(\Lambda, g^*) = \mathcal{S}_0(\Lambda)$. Note that the cut off can be used as regulator for possible divergences given by terms in calculations of the values of observable quantities. On the other hand, about the "naturalness" condition we can consider it intuitively: a "natural" EFT. This should only involve quantities that are small, but not too small, relative to the cut off. Thus, terms proportional to the cut off can not appear in eq.(4.4).

4.3.2 Bottom-Up approach

This approaches is used when *the fundamental high energy theory is not known*, we are able to build the EFT despite of it. In order to do it, we start by considering eq.(4.4) and taking into account two conditions: "naturalness" and symmetries considered important at some scale. Thus, we can determine how the terms scale with a given cut off.

In other words, we need to construct a lagrangian given by:

$$\sum_{\mathrm{i}} \mathcal{L}^{(\mathrm{i})}_{\text{low}}. \tag{4.5}$$

Note that, in this case, the couplings are unknowns. However they can be fitted with experiments. Hence, the effective theory may still be powerful since we can make predictions on observables. The desired precision tells us at what order to stop expanding relative to the cut off.

4.4 On renormalizability

To know the Lagrangian density of a quantum field theory is not enough to calculate the values of observable quantities Thus, function that represents a particular observable quantity in an infinite series in which, typically, divergent integrals appear which need to be treated with renormalization in quantum field theory. Therefore, in addition to knowing the Lagrangian density of some theory, one need to specify a renormalization scheme. So, it is a method that specifies a means of regulating divergent integrals as well as a means of subtracting the associated infinities in a systematic way. There are different methods to solve these problem, two of which are really important. The first adopts momentum cut-off regularization and a mass-dependent method of subtraction, and is used in the Wilsonian approach to constructing EFTs. The second adopt dimensional regularization and a mass-dependent method of subtraction which is called "continuum EFTs" [23]

4.4.1 Cutoff Scheme

The Wilsonian EFT has the characteristic that the cut off Λ appears explicitly into the theory which defines the limit between the low-energy physics and the high-energy physics which suggests us to use it as a way to regulate divergences into the computation of the values of observables quantities. Thus, as a pedagogical illustration, for a general term $A(p)$ integrate it on a momentum space is possible to split the integral range as:

$$\int_0^\infty \mathrm{d}^D p A(p) \equiv \left(\int_0^\Lambda + \int_\Lambda^\infty \right) \mathrm{d}^D p A(p), \tag{4.6}$$

thus, the infinite part of $A(p)$ can be rewritten into parameters of the theory by the introduction of constants which depend of the theory through the introduction of renormalization constants which are related with the high energy theory. As we commented before, this approach is based on a given cut off which plays a double role: first splitting the energy scale and second, avoiding (cutting off) the divergences.

This scheme has a good advantage: it guarantees the decoupled theorem [44](i.e., for a couple of systems with energy scales E_1 and E_2, with $E_2 > E_1$, and described by a renormalizable theory, there is *always* a condition which the effect of physics at E_2 can be included into the theory with the smaller scale E_1 changing the parameter of theory).

4.4.2 Continuum Scheme

Under this scheme, the dimensional parameter μ (analogous to the cut off Λ in the cut off approach) appears in logarithms and not in powers, therefore relevant terms are small at scales much smaller than the heavy fields which let us ignore irrelevant terms and split the theory.

Using this approach is necessary to include explicitly the heavy fields into a finite list of terms in the action $\mathcal{S}$ which tell us that it is impossible to apply the decoupling theorem. Details of this approach can be found into Georgi review [23] but the steps are: (I) Start with a dimensionally-regularized theory given by the Lagrangian density $\mathcal{L} = \mathcal{L}_H(\chi, \phi) + \mathcal{L}(\phi)$ at large energy scale, with $\mathcal{L}(\phi)$ describing the light fields and $\mathcal{L}_H(\chi, \phi)$ describing everything else. (II) When the energy scale E gets below M (mass of heavy fields), we replace the EFT by other free of heavy field ψ: $\mathcal{L} = \mathcal{L}(\phi) + \delta\mathcal{L}(\phi)$. The explicit computation of $\delta\mathcal{L}(\phi)$ is given by expanding it in local operators.

To **summarize** it, in the cut off approach the heavy fields are integrated out of the underlying high energy theory combining it with an expansion in local operators of the action. Here the cut off Λ plays a double role: it defines the high and heavy fields and avoiding the divergence of some terms. On the other hand, the continuum approach is constructed by completely removing the heavy fields from high energy theory. The role of Λ is played by the renormalization scale that splits low an high energy physics.

In the next section we will discuss our work by taking into account effective equations for large scale structure. This is obtained by computing the first moments of Boltzmann equation at zero order. Thus, we start by considering a standard astrophysical scenario and rapidly moving on the mathematical context to define some relations required to investigate the large scales structure.

4.5 A kinetic equation for large scales

In the astrophysical context, to investigate the evolution of perturbations that gave origin to Large Scale Structures, is an important topic nowadays. Revisiting the physics at large scales and using a modified version of the previous discussed effective theory, is possible to handle non-linear terms, and, therefore, to get an improved (extended) understanding of physic at large scales.

So, we start by considering a system with N particles (where N is a very large number) which can be investigated under a statistical point of view. In order to do it, we will study the Boltzmann equation which allows us to know the equation of motion for the distribution function of system. Thus, by investigating the Boltzmann equation it is possible to adopt an approach that leads directly to a macroscopic description of the dynamics.

As we need to gain comprehension about the interaction between a large number of particles, we may consider an analogy with fluid dynamics and take our system as a fluid. It is a good approach due to the fact that the fluid limit is consistent with a system with a large number of particles. The system is characterized by a six-dimensional phase space whose coordinates are the spatial coordinates $x_{\rm i}$ and the velocities $v_{\rm i}$ of the N stars, where $\rm i = 1, 2, 3$. We will focus on the evolution of the probability density in the six-dimensional phase spaces. Being the equation for the distribution function (commented before 3.21) :

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \vec{p}} = \mathcal{C}[f], \tag{4.7}$$

where f is the distribution function, $\vec{p} = m\vec{v}$ is the momentum, $\vec{F} \equiv \dot{\vec{p}}$ is the force between particles, m is the mass and $\mathcal{C}[f]$ is the collisional part of Boltzmann equation. The distribution function depends on the position, the linear momentum and the time t whereas the force depends only on the position.

The term related with the linear momentum is associated to the gravitational acceleration and it is given by the gradient of the gravitational potential energy per unit mass, which is a solution of Poisson's equation:

$$\nabla^2\Phi = 4\pi G \int \mathrm{d}^3 p f(\vec{x}, \vec{p}, t) \tag{4.8}$$

Considering that particles could interact weakly or simply not interact, a common simplification is used ($\mathcal{C}[f] = 0$) in order to cancel the right side of eq. (4.7) and take only into account the left part of equation. By the use of this simplification, we arrive to the so called collisionless Boltzmann (or Vlasov) equation according with [28]. Notice that the collisional term gives us a way of parametrizing the particles' self-interaction and how it changes the evolution of our system.

Our goal is to get expressions that govern the dynamics of the system by investigating the Boltzmann equation. To do it, we require to evaluate the quantities for a given distribution function because, in general, getting exact solutions of the Boltzmann equation is hard. So, we will focus on getting the first moments of the Boltzmann equation. By integrating the Boltzmann equation, with respect to momentum, and using a weight function $\chi \equiv \chi(\vec{x}, \vec{p})$, we can obtain the mathematical expression of conservation laws (mass, momentum and energy) at zeroth order. Note that the χ function is parametrized as a power of velocity according with $\chi \propto v^n \;\forall\; n \in [0, 1, 2, \cdots, \infty)$.

We start by considering the following definitions:

$$\int_{\vec{p}} m f(\vec{x}, \vec{p}, t) \equiv \rho_m(\vec{x}, t), \tag{4.9}$$

$$\int_{\vec{p}} m \frac{p_\mathrm{i}}{m} f(\vec{x}, \vec{p}, t) \equiv \rho_m u_\mathrm{i}(\vec{x}, t), \tag{4.10}$$

$$\int_{\vec{p}} m \frac{p_\mathrm{i} p_j}{m^2} f(\vec{x}, \vec{p}, t) \equiv \rho_m u_\mathrm{i} u_j(\vec{x}, t) + \kappa_{\mathrm{i}j}(\vec{x}, t), \tag{4.11}$$

where the last three expressions define the particle density ρ_m, the peculiar velocity flow $\vec{u}$ and the stress tensor $\kappa_{\mathrm{i}j}$. Notice that we used the prescription $\int_{\vec{\alpha}}(\cdots) = \int \mathrm{d}^3\vec{\alpha}(\cdots)$ common in the literature. Furthermore, a standard way to truncate the hierarchy of moments of the Boltzmann equation consists in assuming that the stress tensor $\kappa_{\mathrm{i}j}$ is negligible on large scales, $\kappa_{\mathrm{i}j} \approx 0$ or by taking an ansatz that mixes previously defined quantities (see detail at Ref.[33]), for example:

$$\kappa_{\mathrm{i}j} = \mathcal{P}\delta_{\mathrm{i}j} - X\left[\nabla_\mathrm{i} u_j + \nabla_j u_\mathrm{i} - \frac{2}{3}\delta_{\mathrm{i}j}\nabla\cdot\vec{u}\right] - Y\delta_{\mathrm{i}j}\nabla\cdot\vec{u}, \tag{4.12}$$

where $\mathcal{P}$ means pressure and X, Y are viscous coefficients.

The equations (4.9), (4.10) and (4.11) are required in order to consider the first moments of the Boltzmann equations (zeroth and first momentum), however, to take into account the second momentum and decompose the stress tensor, we will require extra quantities defined as follows (according with [28]):

$$\frac{1}{3}m\langle\|\vec{c}\|^2\rangle \equiv \theta(\vec{x},t) = kT(\vec{x},t), \tag{4.13}$$

$$\frac{1}{2}\rho\langle\vec{c}\|\vec{c}\|^2\rangle \equiv \vec{q}(\vec{x},t). \tag{4.14}$$

$$\rho\langle c_{\rm i}c_j\rangle \equiv \mathcal{P}_{{\rm i}j}(\vec{x},t), \tag{4.15}$$

$$\frac{1}{2}m\left(\frac{\partial u_{\rm i}}{\partial x_j}+\frac{\partial u_j}{\partial x_{\rm i}}\right) \equiv \Lambda_{{\rm i}j}(\vec{x},t). \tag{4.16}$$

Thus, we define the temperature $\theta(\vec{x},t) \equiv kT(\vec{x},t)$ and the heat flux $\vec{q}(\vec{x},t)$ in eq. (4.13) and eq.(4.14). Furthermore, $\vec{c}(\vec{x},t) \equiv \vec{v} - \vec{u}(\vec{x},t)$ being the peculiar velocity and k being Boltzmann's constant. Note that we are defining the symbol $\langle\cdots\rangle = \int_{\vec{p}}(\cdots)f/\int_{\vec{p}}f$.

4.6 Fluid equations

In order to get the first momentum from the Boltzmann equation, we multiply the eq.(4.7) by the function χ as we previously discussed, being $\chi_\eta = m(1,\vec{v},\frac{1}{2}c_{\rm i}^2)$, integrate over $\vec{v}$ (or $\vec{p}$) the right hand side is zero (only for conservative process) and finally we get:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0, \tag{4.17}$$

$$\rho\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)\vec{u} = \frac{\rho}{m}\vec{F} - \nabla\cdot\mathsf{P}, \tag{4.18}$$

$$\rho\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)\theta = -\frac{2}{3}\nabla\cdot\vec{q} - \frac{2}{3}\mathsf{P}\cdot\mathsf{\Lambda}. \tag{4.19}$$

Here P is a dyadic [1] whose components are $\mathcal{P}_{{\rm i}j}$ whereas $\nabla\cdot\mathsf{P}$ is a vector whose ith component is $\partial\mathcal{P}_{{\rm i}j}/\partial x_j$, and $\mathsf{P}\cdot\mathsf{\Lambda}$ is a scalar $\mathcal{P}_{{\rm i}j}\Lambda_{{\rm i}j}$. At this point is clear to see that the form of the macroscopic equations is that of a fluid equation.

4.7 The zero order approximation

Starting from the fluid equations recently discussed, we want to adapt the distribution function at zeroth and first order to obtain a first order equation of motion, which will take into account self interactions. So, the distribution function can be split into a series order by order, and it is written as:

$$f = f^{(0)} + \sum_{{\rm i}=1}^{n}\lambda^{\rm i}f^{({\rm i})}, \tag{4.20}$$

[1] A dyadic is a second order tensor, formed by juxtaposing pairs of vectors and have analogous rules for matrix algebra.

where λ^{i} is a small parameter that characterizes the system. At zeroth order, we are only considering the first term in the expansion which can be written approximately as a local Maxwell-Boltzmann distribution, with slowly varying temperature, and average velocity:

$$f = f^{(0)} \equiv f^{(0)}(\vec{x}, \vec{p}, t) = J(n, m, \theta)\mathrm{e}^{-\frac{m}{2\theta}(\vec{v}-\vec{u})^2}, \tag{4.21}$$

where $J(n, m, \theta) \equiv n/\sqrt[3]{2\pi m\theta}$, n, θ and $\vec{u}$ varying slowly and depends of $\vec{x}$ and t. Just to be emphatic, eq.(4.21) is not an exact solution of Boltzmann, equation because in general the left part of the Boltzmann equation is not identically zero. We want to compute the first order equation, but to do it, we need to compute the zeroth order equations. Thus, we use the approximation showed in (4.21).

Thus, considering the fluid equations showed in Section 4.6, we only need to rewrite some terms in the last two equations. At zeroth order, the heat flux, defined in eql. (4.14) is identically equal to zero, whereas the pressure components can be computed by eq. (4.15) in terms of local hydrostatic pressure. Hence the pressure components at zeroth order are:

$$\mathcal{P}^{(0)}_{\mathrm{i}j} = \delta_{\mathrm{i}j}\mathcal{P}, \tag{4.22}$$

Notice that $\mathcal{P}$ can be defined by the relation $\mathcal{P}_{\mathrm{i}j} = \delta_{\mathrm{i}j}\mathcal{P} + \mathcal{P}'_{\mathrm{i}j}$, where $\mathcal{P}'_{\mathrm{i}j}$ represents the different elements of the diagonal part. Thus, $\mathcal{P}$ is rewritten by $3\mathcal{P} = \delta^{\mathrm{i}j}\mathcal{P}_{\mathrm{i}j}$. Furthermore, $m\delta^{\mathrm{i}j}\mathcal{P}_{\mathrm{i}j} = 3\rho\theta$, therefore $\mathcal{P}$ has the simple expression $\mathcal{P} = n\theta$. Recall that $\rho = mn$.

In the same way, it is important to comment that:

$$\nabla \cdot \mathsf{P}^{(0)} = \nabla\mathcal{P}, \tag{4.23}$$

$$\mathsf{P}^{(0)} \cdot \mathsf{\Lambda} = \mathcal{P}\sum_{\mathrm{i}=1}^{3}\Lambda_{\mathrm{ii}} = m\mathcal{P}\nabla \cdot \vec{u}. \tag{4.24}$$

Thus, by taking into account the fluid equations introduced in Section 4.6 and the previously commented relations we get:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0, \tag{4.25}$$

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)\vec{u} + \frac{1}{\rho}\nabla\mathcal{P} = \frac{\vec{F}}{m}, \tag{4.26}$$

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)\theta + \frac{1}{c_v}(\nabla \cdot \vec{u})\theta = 0. \tag{4.27}$$

where we recall that

$$\frac{2}{3}\frac{\mathsf{P} \cdot \mathsf{\Lambda}}{\rho} = \frac{1}{c_v}(\nabla \cdot \vec{u})\theta. \tag{4.28}$$

with $c_v = 3/2$. Note that the previous equations define a nonviscous flow of gas, thereby, if we want to consider viscous correction it is necessary to take into account high order corrections. As an additional remark, note that the velocity $\vec{u}$ used in eqs. (4.25), (4.26) and (4.27) is really $\langle\vec{v}\rangle = \vec{u}$.

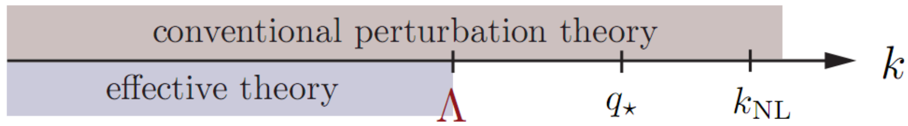

Figure 4.1: Graphic representations of relevant scales in the smoothing approach [2].

4.8 Smoothing approach of LSS

The smoothing approach has been used in theory [2, 12] and simulations [53, 13] and consists in transforming the original equations of motion into other equations, where these last equations are "smoothed". This approach is based on the cut-off Scheme of EFT previously discussed in Section 4.4.1.

Now, what is the meaning of "smooth" in this context? Basically, we make a convolution with a window function to extract or avoid a particular region of the considered space. Thus, we need to consider an appropriate window function in order to "smooth" the equations but this process will be explained in brief.

4.8.1 Smoothed fluid equations

In order to get these equations we need to integrate (4.7) over space using the appropriate window function $W_\Lambda \equiv W_\Lambda(\mid \vec{x} - \vec{x}' \mid)$ to obtain the first Boltzmann equation moments. In other words, these equations allows us to understand the theory at very large scales and to do it, it is required to define a range to study the problem. In effect, we investigate the range $\Lambda \ll q_*$ with q_* the scale of non-linearities. Note that q_* is a characteristic momentum which allows to define an effective long-wavelength theory by "integrating out" short wavelength modes below an scale $\Lambda \ll q_*$ (see Fig. 4.1 for illustration).

Integrating out short-wavelength modes corresponds to a convolution of all fields with a window function W_Λ according with [2]:

$$\rho_\ell \equiv [\rho]_\Lambda(\vec{x}) = \int_{\vec{x}'} W_\Lambda(\mid \vec{x} - \vec{x}' \mid)\rho_m(\vec{x}'), \tag{4.29}$$

$$\phi_\ell \equiv [\phi]_\Lambda(\vec{x}) = \int_{\vec{x}'} W_\Lambda(\mid \vec{x} - \vec{x}' \mid)\phi(\vec{x}'), \tag{4.30}$$

$$\rho_\ell u_\ell^{\mathrm{i}} \equiv [\rho_m v^{\mathrm{i}}]_\Lambda(\vec{x}) = \int_{\vec{x}'} W_\Lambda(\mid \vec{x} - \vec{x}' \mid)\rho(\vec{x}')u^{\mathrm{i}}(\vec{x}'), \tag{4.31}$$

$$\rho_\ell \theta_\ell \equiv [\rho_m \theta]_\Lambda(\vec{x}) = \int_{\vec{x}'} W_\Lambda(\mid \vec{x} - \vec{x}' \mid)\rho(\vec{x}')\theta(\vec{x}'). \tag{4.32}$$

Equations (4.29 - 4.32) are the smoothing of fields over domains of size Λ^{-1}. By simplicity, we will consider the object $\Pi \equiv \{\rho, \phi, \vec{q}, \rho\vec{v}, \rho\theta\}$ where each object has been well defined in previous sections. Note that we are admitting the splitting:

$$\Pi(\vec{x},t) \equiv \Pi_\ell(\vec{x},t) + \Pi_s(\vec{x},t), \tag{4.33}$$

where the subscripts (ℓ, s) mean long and short modes respectively for an arbitrary quantity $(\{\rho_m, u^{\rm i}\}$ for example). In the same way, we note that the undefined quantities i.e. those which are not contained in equations (4.29 - 4.32) are computed by taking into account a Taylor series around $\vec{x}$. So, in general we have two different terms: the first one associated with an undefined quantity at long scale (which will be rewritted in Taylor series) and the second related with small scale terms. Thus, we have: $\Pi^{\rm ud} = \Pi^{\rm ud}_\ell + \langle \Pi_s \rangle_\Lambda$, where $\Pi^{\rm ud}_\ell$ is:

$$\Pi^{\rm ud}_\ell(\vec{x},t) = \Pi^{\rm i}_\ell(\vec{x}) - \partial_j \Pi^{\rm i}_\ell(\vec{x})(\vec{x}-\vec{x}')^j + \frac{1}{2}\partial_k\partial_j\Pi^{\rm i}_\ell(\vec{x})(\vec{x}-\vec{x}')^j(\vec{x}-\vec{x}')^k + \cdots . \tag{4.34}$$

The interpretation of each term is consistent because by combining eq.(4.34) with a window function allows us to obtain terms with the correct physical meaning.

4.9 Moments under smoothing approach

We will show the implementation of this smoothing approach to LSS in the first moments of the Boltzmann equation. In ref.[2] shown the smoothing approach (under an Effective Field Theory of Large Scale Structure) for the relativistic and non-relativistic case. The authors discussed the equations of motion for the zeroth and first moment under the smoothing approach. In addition, we will extend the results to consider a particular form of the collision term of the Boltzmann equation.

4.9.1 Zero moment of Boltzmann equation

Considering equation (4.17) which is the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \tag{4.35}$$

and integrating using the windows function:

$$W_\Lambda(\vec{x}) \otimes \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u})\right] = \mathcal{F}_0, \tag{4.36}$$

$$W_\Lambda(\vec{x}) \otimes \frac{\partial \rho}{\partial t} + W_\Lambda(\vec{x}) \otimes \nabla \cdot (\rho \vec{u}) = \mathcal{F}_0 \tag{4.37}$$

where $\mathcal{F}_0$ is a function that contains long wavelength terms. We should note that the symbol $\otimes$ means convolution, thus, in general we have:

$$(f \otimes g)(t) \doteq \int_{-\infty}^{\infty} f(\eta) g(t-\eta) \mathrm{d}\eta \tag{4.38}$$

As an example, we will compute this equation in detail in order to show you how to make progress here. So, we have:

$$\mathcal{F}_0 = \int_{\vec{x}'} \left[\frac{\partial}{\partial t} (W_\Lambda \rho) - \rho \frac{\partial W_\Lambda}{\partial t} + \frac{\partial}{\partial x_{\mathrm{i}'}} (W_\Lambda \rho u_{\mathrm{i}}) - \frac{\partial W_\Lambda}{\partial x_{\mathrm{i}'}} \rho u_{\mathrm{i}} \right] \tag{4.39}$$

Notice that W_Λ does not depend explicitly on time. Using the properties of windows function given in the Appendix 6, and the Eq. 4.29 and 4.31 we are able to rewrite the previous equation as:

$$\mathcal{F}_0 = \frac{\partial}{\partial t} \int_{\vec{x}'} W_\Lambda \rho + \int_{\vec{x}'} \frac{\partial}{\partial x_{\mathrm{i}'}} (W_\Lambda \rho u_{\mathrm{i}}) - \int_{\vec{x}'} \frac{\partial W_\Lambda}{\partial x_{\mathrm{i}'}} \rho u_{\mathrm{i}}, \tag{4.40}$$

$$\mathcal{F}_0 = \frac{\partial \rho_\ell}{\partial t} + \Big[W_\Lambda \rho u_{\mathrm{i}} \Big]_{\partial\Omega} + \frac{\partial}{\partial x_{\mathrm{i}}} \int_{\vec{x}} W_\Lambda \rho u_{\mathrm{i}}, \tag{4.41}$$

$$\mathcal{F}_0 = \frac{\partial \rho_\ell}{\partial t} + \Big[W_\Lambda \rho u_{\mathrm{i}} \Big]_{\partial\Omega} + \nabla \cdot (\rho_\ell u_\ell^{\mathrm{i}}). \tag{4.42}$$

Note that the window function satisfies that $W_\Lambda(\vec{x} \to \infty) = 0$, therefore the second term is equal to zero. Thus, $\mathcal{F}_0$ has the simple form:

$$\mathcal{F}_0 = \frac{\partial \rho_\ell}{\partial t} + \nabla \cdot (\rho_\ell u_\ell^{\mathrm{i}}), \tag{4.43}$$

Finally, it is important to mention that the smoothed zero moment of Boltzmann equation does not introduce a correction given by the filtering process. Hence, the new equation can be obtained by the replacement $\Pi \longrightarrow \Pi_\ell$.

4.9.2 First moment of Boltzmann equation

As before, we want to compute some details about the first moment. It was made with some detail in [2] therefore we are not showing all the process, but we will add more details to make more pedagogical the explanation.

We start by considering the equation (4.18) and using the windows function we obtain:

$$W_\Lambda(\vec{x}) \otimes \left[\rho \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \vec{u} - \frac{\rho}{m} \vec{F} + \nabla \cdot \mathsf{P} \right] = \mathcal{F}_1. \tag{4.44}$$

According with [2], the term $\nabla \cdot \mathsf{P}$ is zero, because they considered the stress tensor $\kappa_{\mathrm{i}j}$ equal to zero as a way to introduce a cut-off in the moments of Boltzmann equation. This choice is consistent with a model of collisionless particles for Dark Matter. As a first improvement of the equations we will consider this term different from zero and we will investigate if it introduces some appreciable effect. Notice that the force is related with the potential by, $\vec{F} = -m\nabla\phi$. We split the last equation by simplicity and we get:

$$\mathcal{F}_1 = \int_{\vec{x}'} W_\Lambda \left[\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} - \frac{\rho}{m} \vec{F} + \nabla \cdot \mathsf{P} \right], \tag{4.45}$$

$$\mathcal{F}_1 = \int_{\vec{x}'} W_\Lambda \left[\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} \right] + \int_{\vec{x}'} W_\Lambda \rho \nabla \phi + \int_{\vec{x}'} W_\Lambda \nabla \cdot \mathsf{P}. \tag{4.46}$$

By simplicity, we compute each integral separately in order to be clear about the procedure, so, the first object is:

$$\mathcal{F}_1^{(0)} = \int_{\vec{x}'} W_\Lambda \left[\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} \right]. \tag{4.47}$$

Doing some manipulation we obtain:

$$\mathcal{F}_1^{(0)} = \int_{\vec{x}'} W_\Lambda \Big(\partial_t(\rho u^{\mathrm{i}}) + u^{\mathrm{i}} \partial_{j'}(\rho u^j) + \rho u^j \partial_{j'} u^{\mathrm{i}} \Big). \tag{4.48}$$

Notice that we are using index notation by simplicity: $\nabla \to \partial_{\mathrm{i}}$ and $\partial/\partial t \to \partial_t$.

$$\mathcal{F}_1^{(0)} = \int_{\vec{x}'} W_\Lambda \partial_t(\rho u^{\mathrm{i}}) + \int_{\vec{x}'} W_\Lambda u^{\mathrm{i}} \partial_{j'}(\rho u^j) + \int_{\vec{x}'} W_\Lambda \rho u^j \partial_{j'} u^{\mathrm{i}}. \tag{4.49}$$

The first term is easily computed by recalling that W_Λ does not depend on time, whereas the second term requires us to consider integration by parts. Hence we obtain:

$$\begin{aligned}
\mathcal{F}_1^{(0)} &= \partial_t \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} + \int_{\vec{x}'} W_\Lambda u^{\mathrm{i}} \partial_{j'}(\rho u^j) + \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} \partial_{\mathrm{i}'} u^j, \\
\mathcal{F}_1^{(0)} &= \partial_t \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} + \left[W_\Lambda \rho u^{\mathrm{i}} u^j + \partial_j \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} u^j - \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} \partial_{\mathrm{i}'} u^j \right] + \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} \partial_{\mathrm{i}'} u^j, \\
\mathcal{F}_1^{(0)} &= \partial_t \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} + \Big[W_\Lambda \rho u^{\mathrm{i}} u^j \Big]_{\partial\Omega} + \partial_j \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} u^j.
\end{aligned} \tag{4.50}$$

Note that the second term is equal to zero (recall that $W_\Lambda(\vec{x} \to \infty) = 0$). Thus, using the Eq.(4.31) we obtain:

$$\mathcal{F}_1^{(0)} = \partial_t(\rho_\ell u_\ell^{\mathrm{i}}) + \partial_j \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} u^j. \tag{4.51}$$

The integral in the last term requires much more computation, because of we have three quantities that require to be split into long and short modes. For simplicity, we compute it with some detail:

$$\begin{aligned}
\mathcal{E} &\equiv \int_{\vec{x}'} W_\Lambda \rho u^{\mathrm{i}} u^j, \\
\mathcal{E} &= \int_{\vec{x}'} W_\Lambda \rho (u_\ell^{\mathrm{i}} + u_s^{\mathrm{i}})(u_\ell^j + u_s^j), \\
\mathcal{E} &= \int_{\vec{x}'} W_\Lambda \rho u_\ell^{\mathrm{i}} u_\ell^j + \int_{\vec{x}'} W_\Lambda \rho u_\ell^{\mathrm{i}} u_s^j + \int_{\vec{x}'} W_\Lambda \rho u_s^{\mathrm{i}} u_\ell^j + \int_{\vec{x}'} W_\Lambda \rho u_s^{\mathrm{i}} u_s^j.
\end{aligned} \tag{4.52}$$

Just note that the second and third integral are the same under a change of index ($\mathrm{i} \to j$ and $j \to \mathrm{i}$), thus is possible to rewrite it under a symmetric object, thus:

$$\mathcal{E} = \int_{\vec{x}'} W_\Lambda \rho u_\ell^{\mathrm{i}} u_\ell^j + 2 \int_{\vec{x}'} W_\Lambda \rho u_\ell^{(\mathrm{i}} u_s^{j)} + \int_{\vec{x}'} W_\Lambda \rho u_s^{\mathrm{i}} u_s^j \tag{4.53}$$

Note that the last integral is irreducible and it is written under the symbol $[\cdots]_\Lambda$. Furthermore, the first integral can be split by taking into account the Taylor series of it around $\vec{x}$:

$$\mathcal{E}_1 = \int_{\vec{x}'} W_\Lambda \rho u_\ell^{\mathrm{i}} u_\ell^j, \tag{4.54}$$

$$\begin{aligned}\mathcal{E}_1 = \int_{\vec{x}'} W_\Lambda \rho &\left[u_\ell^{\mathrm{i}}(\vec{x}) - \partial_f u_\ell^{\mathrm{i}}(\vec{x})(\vec{x}-\vec{x}')^f + \frac{1}{2}\partial_k\partial_f u_\ell^{\mathrm{i}}(\vec{x})(\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^k + \cdots \right] \\ &\times \left[u_\ell^j(\vec{x}) - \partial_n u_\ell^j(\vec{x})(\vec{x}-\vec{x}')^n + \frac{1}{2}\partial_o\partial_n u_\ell^j(\vec{x})(\vec{x}-\vec{x}')^n(\vec{x}-\vec{x}')^o + \cdots \right]\end{aligned} \tag{4.55}$$

Here we obtain nine terms, but only a few are important for us.

$$\begin{aligned}\mathcal{E}_1 = &\int_{\vec{x}'} W_\Lambda \rho u_\ell^{\mathrm{i}}(\vec{x}) u_\ell^j(\vec{x}) - \int_{\vec{x}'} W_\Lambda \rho u_\ell^{\mathrm{i}}(\vec{x}) \partial_n u_\ell^j(\vec{x})(\vec{x}-\vec{x}')^n \\ &+ \int_{\vec{x}'} W_\Lambda \rho u_\ell^{\mathrm{i}}(\vec{x}) \frac{1}{2}\partial_o\partial_n u_\ell^j(\vec{x})(\vec{x}-\vec{x}')^n(\vec{x}-\vec{x}')^o \\ &- \int_{\vec{x}'} W_\Lambda \rho \partial_f u_\ell^{\mathrm{i}}(\vec{x})(\vec{x}-\vec{x}')^f u_\ell^j(\vec{x}) + \int_{\vec{x}'} W_\Lambda \rho \partial_f u_\ell^{\mathrm{i}}(\vec{x})(\vec{x}-\vec{x}')^f \partial_n u_\ell^j(\vec{x})(\vec{x}-\vec{x}')^n \\ &- \int_{\vec{x}'} W_\Lambda \rho \partial_f u_\ell^{\mathrm{i}}(\vec{x})(\vec{x}-\vec{x}')^f \frac{1}{2}\partial_o\partial_n u_\ell^j(\vec{x})(\vec{x}-\vec{x}')^n(\vec{x}-\vec{x}')^o \\ &+ \int_{\vec{x}'} W_\Lambda \rho \frac{1}{2}\partial_k\partial_f u_\ell^{\mathrm{i}}(\vec{x})(\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^k u_\ell^j(\vec{x}) \\ &- \int_{\vec{x}'} W_\Lambda \rho \frac{1}{2}\partial_k\partial_f u_\ell^{\mathrm{i}}(\vec{x})(\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^k \partial_n u_\ell^j(\vec{x})(\vec{x}-\vec{x}')^n \\ &+ \int_{\vec{x}'} W_\Lambda \rho \frac{1}{2}\partial_k\partial_f u_\ell^{\mathrm{i}}(\vec{x})(\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^k \frac{1}{2}\partial_o\partial_n u_\ell^j(\vec{x})(\vec{x}-\vec{x}')^n(\vec{x}-\vec{x}')^o\end{aligned} \tag{4.56}$$

Notice that, due to the Taylor series, the quantities are evaluated at $\vec{x}$ and not at $\vec{x}'$, therefore it is possible to get these quantities out of the integral symbol and only compute the integral using the window function properties.

$$\begin{aligned}\mathcal{E}_1 = &u_\ell^{\mathrm{i}} u_\ell^j \int_{\vec{x}'} W_\Lambda \rho - u_\ell^{\mathrm{i}} \partial_n u_\ell^j \int_{\vec{x}'} W_\Lambda \rho (\vec{x}-\vec{x}')^n + \frac{1}{2} u_\ell^{\mathrm{i}} \partial_o\partial_n u_\ell^j \int_{\vec{x}'} W_\Lambda \rho (\vec{x}-\vec{x}')^n(\vec{x}-\vec{x}')^o - \\ &\partial_f u_\ell^{\mathrm{i}} u_\ell^j \int_{\vec{x}'} W_\Lambda \rho (\vec{x}-\vec{x}')^f + \partial_f u_\ell^{\mathrm{i}} \partial_n u_\ell^j \int_{\vec{x}'} W_\Lambda \rho (\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^n - \\ &\frac{1}{2}\partial_f u_\ell^{\mathrm{i}} \partial_o\partial_n u_\ell^j \int_{\vec{x}'} W_\Lambda \rho (\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^n(\vec{x}-\vec{x}')^o + \\ &\frac{1}{2}\partial_k\partial_f u_\ell^{\mathrm{i}} u_\ell^j \int_{\vec{x}'} W_\Lambda \rho (\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^k - \\ &\frac{1}{2}\partial_k\partial_f u_\ell^{\mathrm{i}} \partial_n u_\ell^j \int_{\vec{x}'} W_\Lambda \rho (\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^k(\vec{x}-\vec{x}')^n + \\ &\frac{1}{4}\partial_k\partial_f u_\ell^{\mathrm{i}} \partial_o\partial_n u_\ell^j \int_{\vec{x}'} W_\Lambda \rho (\vec{x}-\vec{x}')^f(\vec{x}-\vec{x}')^k(\vec{x}-\vec{x}')^n(\vec{x}-\vec{x}')^o.\end{aligned} \tag{4.57}$$

We are interested in investigating the modes only at Λ^{-2}, thus, integrals $\propto (\vec{x}-\vec{x}')^{\sigma}$, with $\sigma \geq 3$ are ignored. According with the properties of the window function we can rewrite it as:

$$\mathcal{E}_1 = u^{\rm i}_{\ell} u^j_{\ell} \rho_{\ell} + \frac{u^{\rm i}_{\ell}\partial_n u^j_{\ell}}{\Lambda^2}\partial_n\rho_{\ell} + \rho_{\ell}\frac{\partial^2 u^j_{\ell}}{2\Lambda^2}u^{\rm i}_{\ell} + \frac{u^j_{\ell}\partial_f u^{\rm i}_{\ell}}{\Lambda^2}\partial_f\rho_{\ell} + \frac{\partial_f u^{\rm i}_{\ell}\partial_n u^j_{\ell}}{\Lambda^2}\rho_{\ell} + \rho_{\ell}\frac{\partial^2 u^{\rm i}_{\ell}}{2\Lambda^2}u^j_{\ell} \tag{4.58}$$

grouping the terms appropriately we get:

$$\mathcal{E}_1 = u^{\rm i}_{\ell} u^j_{\ell}\rho_{\ell} + \mathcal{C}^{{\rm i}j} + \mathcal{C}^{{\rm i}j}_1, \tag{4.59}$$

where $\mathcal{C}^{{\rm i}j}$ and $\mathcal{C}^{{\rm i}j}_1$ are defined as:

$$\mathcal{C}^{{\rm i}j}_1 = \frac{u^{\rm i}_{\ell}\partial_n u^j_{\ell}}{\Lambda^2}\partial_n\rho_{\ell} + \frac{u^j_{\ell}\partial_f u^{\rm i}_{\ell}}{\Lambda^2}\partial_f\rho_{\ell} + \rho_{\ell}\frac{\partial^2 u^j_{\ell}}{2\Lambda^2}u^{\rm i}_{\ell} + \rho_{\ell}\frac{\partial^2 u^{\rm i}_{\ell}}{2\Lambda^2}u^j_{\ell}. \tag{4.60}$$

Changing the indexes (because they are dummies indexes) we can rewrite as:

$$\mathcal{C}^{{\rm i}j}_1 = \frac{\partial_f\rho_{\ell}}{\Lambda^2}\left(u^{\rm i}_{\ell}\partial_f u^j_{\ell} + u^j_{\ell}\partial_f u^{\rm i}_{\ell}\right) + \frac{\rho_{\ell}}{2\Lambda^2}\left(u^{\rm i}_{\ell}\partial^2 u^j_{\ell} + u^j_{\ell}\partial^2 u^{\rm i}_{\ell}\right). \tag{4.61}$$

Using Symmetry properties, it is possible to rewrite objects with two index using the notation of a symmetric object:

$$\mathcal{C}^{{\rm i}j}_1 = 2\frac{\partial_f\rho_{\ell}}{\Lambda^2}u^{({\rm i}}_{\ell}\partial_f u^{j)}_{\ell} + \frac{\rho_{\ell}}{\Lambda^2}u^{({\rm i}}_{\ell}\partial^2 u^{j)}_{\ell}. \tag{4.62}$$

Note that $\mathcal{C}^{{\rm i}j}$ is defined as:

$$\mathcal{C}^{{\rm i}j} = \rho_{\ell}\frac{\partial_f u^{\rm i}_{\ell}\partial_f u^j_{\ell}}{\Lambda^2} \tag{4.63}$$

On the other hand, we want to compute the object $\overline{\mathcal{E}}$ defined as:

$$\overline{\mathcal{E}} = 2\int_{\vec{x}'} W_{\Lambda}\rho u^{({\rm i}}_{\ell} u^{j)}_{s} \equiv \mathcal{E}_2 + \mathcal{E}_3 = \int_{\vec{x}'} W_{\Lambda}\rho u^{\rm i}_{\ell} u^j_{s} + \int_{\vec{x}'} W_{\Lambda}\rho u^{\rm i}_{s} u^j_{\ell}. \tag{4.64}$$

Thus we can compute the previous object as:

$$\mathcal{E}_2 = \int_{\vec{x}'} W_{\Lambda}\rho u^{\rm i}_{\ell} u^j_{s} = \int_{\vec{x}'} W_{\Lambda}\rho u^{\rm i}_{\ell}(u^j - u^j_{\ell}) = \int_{\vec{x}'} W_{\Lambda}\rho u^{\rm i}_{\ell} u^j - \int_{\vec{x}'} W_{\Lambda}\rho u^{\rm i}_{\ell} u^j_{\ell}. \tag{4.65}$$

Changing ${\rm i} \to j$ and $j \to {\rm i}$ to obtain:

$$\mathcal{E}_3 = \int_{\vec{x}'} W_{\Lambda}\rho u^j_{\ell} u^{\rm i}_{s} = \int_{\vec{x}'} W_{\Lambda}\rho u^j_{\ell}(u^{\rm i} - u^{\rm i}_{\ell}) = \int_{\vec{x}'} W_{\Lambda}\rho u^j_{\ell} u^{\rm i} - \int_{\vec{x}'} W_{\Lambda}\rho u^j_{\ell} u^{\rm i}_{\ell}. \tag{4.66}$$

Note that $\overline{\mathcal{E}}$ can be identified with a previously computed object, thus:

$$\overline{\mathcal{E}} = -\frac{\partial_f\rho_{\ell}}{\Lambda^2}\left(u^{\rm i}_{\ell}\partial_f u^j_{\ell} + u^j_{\ell}\partial_f u^{\rm i}_{\ell}\right) - \frac{\rho_{\ell}}{2\Lambda^2}\left(u^{\rm i}_{\ell}\partial^2 u^j_{\ell} + u^j_{\ell}\partial^2 u^{\rm i}_{\ell}\right), \tag{4.67}$$

where now it is clear that $\mathcal{C}_1^{\mathrm{i}j} = -\overline{\mathcal{E}}$ and finally we get the simplified equation:

$$\mathcal{E} = u_\ell^{\mathrm{i}} u_\ell^j \rho_\ell + \rho_\ell \frac{\partial_f u_\ell^{\mathrm{i}} \partial_f u_\ell^j}{\Lambda^2} + \left[\rho u_s^{\mathrm{i}} u_s^j\right]_\Lambda. \tag{4.68}$$

Thus, $\mathcal{F}_1^{(0)}$ has the form:

$$\mathcal{F}_1^{(0)} = \partial_t(\rho_\ell u_\ell^{\mathrm{i}}) + \partial_j \mathcal{E}, \tag{4.69}$$

$$\mathcal{F}_1^{(0)} = \partial_t(\rho_\ell u_\ell^{\mathrm{i}}) + \partial_j \left(u_\ell^{\mathrm{i}} u_\ell^j \rho_\ell + \rho_\ell \frac{\partial_f u_\ell^{\mathrm{i}} \partial_f u_\ell^j}{\Lambda^2} + \left[\rho u_s^{\mathrm{i}} u_s^j\right]_\Lambda \right), \tag{4.70}$$

$$\mathcal{F}_1^{(0)} = \partial_t(\rho_\ell u_\ell^{\mathrm{i}}) + \partial_j (u_\ell^{\mathrm{i}} u_\ell^j \rho_\ell) + \partial_j \mathcal{C}^{\mathrm{i}j} + \partial_j \left[\rho u_s^{\mathrm{i}} u_s^j\right]_\Lambda, \tag{4.71}$$

where the last term can be rewritten, by taking into account the continuity equation, as:

$$\mathcal{F}_1^{(0)} = u_\ell^{\mathrm{i}} \partial_t \rho_\ell + \rho_\ell \partial_t u_\ell^{\mathrm{i}} + u_\ell^j \rho_\ell \partial_j u_\ell^{\mathrm{i}} + u_\ell^{\mathrm{i}} \partial_j (u_\ell^j \rho_\ell) + \partial_j \mathcal{C}^{\mathrm{i}j} + \partial_j \left[\rho u_s^{\mathrm{i}} u_s^j\right]_\Lambda, \tag{4.72}$$

$$\mathcal{F}_1^{(0)} = u_\ell^{\mathrm{i}} [\partial_t \rho_\ell + \partial_j (u_\ell^j \rho_\ell)] + \rho_\ell \partial_t u_\ell^{\mathrm{i}} + u_\ell^j \rho_\ell \partial_j u_\ell^{\mathrm{i}} + \partial_j \mathcal{C}^{\mathrm{i}j} + \partial_j \left[\rho u_s^{\mathrm{i}} u_s^j\right]_\Lambda, \tag{4.73}$$

$$\mathcal{F}_1^{(0)} = \rho_\ell \partial_t u_\ell^{\mathrm{i}} + u_\ell^j \rho_\ell \partial_j u_\ell^{\mathrm{i}} + \partial_j \mathcal{C}^{\mathrm{i}j} + \partial_j \left[\rho u_s^{\mathrm{i}} u_s^j\right]_\Lambda. \tag{4.74}$$

On the other hand, the term $\mathcal{F}_2^{(0)}$ is easily computed by splitting ϕ and ρ into long a short modes, take into account a Taylor series and identifying known quantities. Thus:

$$\mathcal{F}_2^{(0)} = \int_{\vec{x}'} W_\Lambda \rho \partial_{\mathrm{i}'} \phi, \tag{4.75}$$

$$\mathcal{F}_2^{(0)} = \int_{\vec{x}'} W_\Lambda (\rho_\ell + \rho_s)(\partial_{\mathrm{i}'} \phi_\ell + \partial_{\mathrm{i}'} \phi_s), \tag{4.76}$$

$$\mathcal{F}_2^{(0)} = \int_{\vec{x}'} W_\Lambda \rho_\ell \partial_{\mathrm{i}'} \phi_\ell + \int_{\vec{x}'} W_\Lambda \rho_\ell \partial_{\mathrm{i}'} \phi_s + \int_{\vec{x}'} W_\Lambda \rho_s \partial_{\mathrm{i}'} \phi_\ell + \int_{\vec{x}'} W_\Lambda \rho_s \partial_{\mathrm{i}'} \phi_s. \tag{4.77}$$

Due to the practical importance of some quantities, we compute it first:

$$\int_{\vec{x}'} W_\Lambda \rho_s = \int_{\vec{x}'} W_\Lambda (\rho - \rho_\ell) = \int_{\vec{x}'} W_\Lambda \rho - \int_{\vec{x}'} W_\Lambda \rho_\ell, \tag{4.78}$$

$$\int_{\vec{x}'} W_\Lambda \rho_s = -\partial_k \rho_\ell \int_{\vec{x}'} W_\Lambda (\vec{x} - \vec{x}')^k - \frac{1}{2} \partial_n \partial_k \rho_\ell \int_{\vec{x}'} W_\Lambda (\vec{x} - \vec{x}')^n (\vec{x} - \vec{x}')^k, \tag{4.79}$$

$$\int_{\vec{x}'} W_\Lambda \rho_s = -\frac{\partial^2 \rho_\ell}{2\Lambda^2} \tag{4.80}$$

In the same way, by taking into account exactly the same procedure we obtain:

$$\int_{\vec{x}'} W_\Lambda \phi_s = -\frac{\partial^2 \phi_\ell}{2\Lambda^2}. \tag{4.81}$$

Another important integral is:

$$\int_{\vec{x}'} W_\Lambda \rho_\ell = \int_{\vec{x}'} W_\Lambda (\rho - \rho_s) = \int_{\vec{x}'} W_\Lambda \rho - \int_{\vec{x}'} W_\Lambda \rho_s, \tag{4.82}$$

$$\int_{\vec{x}'} W_\Lambda \rho_\ell = \rho_\ell + \frac{\partial^2 \rho_\ell}{2\Lambda^2}. \tag{4.83}$$

Finally, another important integral is:

$$\int_{\vec{x}'} W_\Lambda \partial_{\mathrm{i}'}\phi_\ell = \big[W_\Lambda \phi_s\big]_{\partial\Omega} - \int_{\vec{x}'} \partial_{\mathrm{i}'} W_\Lambda \phi_s = \partial_{\mathrm{i}}\Big(-\frac{\partial^2\phi_\ell}{2\Lambda^2}\Big). \tag{4.84}$$

By returning to the original problem, the integrals are easily obtained:

$$\int_{\vec{x}'} W_\Lambda \rho_\ell \partial_{\mathrm{i}'}\phi_\ell = \rho_\ell \partial_{\mathrm{i}}\phi_\ell - \frac{\partial_{\mathrm{i}}\partial_j\phi_\ell\partial_j\rho_\ell}{\Lambda^2} + \frac{\partial_{\mathrm{i}}\phi_\ell\partial^2\rho_\ell}{2\Lambda^2} + \frac{\partial_{\mathrm{i}}\partial^2\phi_\ell\rho_\ell}{2\Lambda^2}, \tag{4.85}$$

whereas the second integral is:

$$\int_{\vec{x}'} W_\Lambda \rho_\ell \partial_{\mathrm{i}'}\phi_s = \frac{\rho_\ell\partial_{\mathrm{i}}\partial^2\phi_\ell}{2\Lambda^2}. \tag{4.86}$$

The third integral basically produces:

$$\int_{\vec{x}'} W_\Lambda \rho_s \partial_{\mathrm{i}'}\phi_\ell = -\frac{\partial^2\rho_\ell\partial_{\mathrm{i}}\phi_\ell}{2\Lambda^2}. \tag{4.87}$$

Finally, the term $\mathcal{F}_2^{(0)}$ is given by:

$$\mathcal{F}_2^{(0)} = \rho_\ell\partial_{\mathrm{i}}\phi_\ell + \frac{\partial_{\mathrm{i}}\partial_j\phi_\ell\partial_j\rho_\ell}{\Lambda^2} + \Big[\rho_s\partial_{\mathrm{i}}\phi_s\Big]_\Lambda. \tag{4.88}$$

Note that $\mathcal{F}_2^{(0)}$ can be simplified by introducing the Poisson equation for the last term. Thus, we have:

$$\nabla^2\phi_s = 4\pi G\rho_s, \tag{4.89}$$

$$\rho_s = \frac{\partial^2\phi_s}{4\pi G}. \tag{4.90}$$

Putting the density ρ_s into last term in Eq.(4.88) we have:

$$\int_{\vec{x}'} W_\Lambda \frac{\partial^2_{j'}\phi_s}{4\pi G}\partial_{\mathrm{i}}\phi_s. \tag{4.91}$$

Rewriting the previous equation as:

$$\int_{\vec{x}'} W_\Lambda \frac{\partial^2_{j'}\phi_s}{4\pi G}\partial_{\mathrm{i}}\phi_s = \partial_j\int_{\vec{x}'} W_\Lambda \frac{\partial_{j'}\phi_s}{4\pi G}\partial_{\mathrm{i}'}\phi_s - \int_{\vec{x}'} W_\Lambda \frac{\partial_{j'}\phi_s}{4\pi G}\partial_{j'}\partial_{\mathrm{i}'}\phi_s, \tag{4.92}$$

and by using the fact that the last term can be written as

$$\int_{\vec{x}'} W_\Lambda \frac{\partial_{j'}\phi_s}{4\pi G}\partial_{j'}\partial_{\mathrm{i}'}\phi_s = \frac{1}{2}\partial_{\mathrm{i}}\int_{\vec{x}'} W_\Lambda \frac{\partial_{j'}\phi_s}{4\pi G}\partial_{j'}\phi_s, \tag{4.93}$$

we finally obtain that $\mathcal{F}_2^{(0)}$ is:

$$\mathcal{F}_2^{(0)} = \rho_\ell\partial_{\mathrm{i}}\phi_\ell + \frac{\partial_{\mathrm{i}}\partial_j\phi_\ell\partial_j\rho_\ell}{\Lambda^2} + \partial_j\left[\frac{\partial_{j'}\phi_s\partial_{\mathrm{i}'}\phi_s}{4\pi G} - \frac{1}{2}\frac{\partial_{j'}\phi_s\partial_{j'}\phi_s}{4\pi G}\right]_\Lambda, \tag{4.94}$$

$$\mathcal{F}_2^{(0)} = \rho_\ell\partial_{\mathrm{i}}\phi_\ell + \frac{\partial_{\mathrm{i}}\partial_j\phi_\ell\partial_j\rho_\ell}{\Lambda^2} + \partial_j\left[\frac{2\partial_{j'}\phi_s\partial_{\mathrm{i}'}\phi_s - \partial_{j'}\phi_s\partial_{j'}\phi_s}{8\pi G}\right]_\Lambda. \tag{4.95}$$

The effective equation with the long wavelength mode is given by the sum $\mathcal{F}_1^{(0)} + \mathcal{F}_2^{(0)}$:

$$\rho_\ell \big[\partial_t v_\ell^{\mathrm{i}} + v_\ell^j \nabla_j v_\ell^{\mathrm{i}}\big] + \rho_\ell \nabla_{\mathrm{i}} \phi_\ell = -\nabla_j \big[\mathcal{T}_{\mathrm{i}}^j\big]_\Lambda - \nabla_j \big[\mathcal{T}_{\mathrm{i}}^j\big]_\Lambda^{\partial^2}. \tag{4.96}$$

Note that the object $\mathcal{T}_{\mathrm{i}}^j$ is the so-called effective stress-energy tensor and it has a relativistic motivation appropriately introduced in [2]. Furthermore, the object $\mathcal{T}_{\mathrm{i}}^j$ is defined as:

$$\mathcal{T}_{\mathrm{i}}^j = \rho_m v_{\mathrm{i}}^s v_j^s - \frac{\partial_{j'} \phi_s \partial_{j'} \phi_s - 2 \partial_{j'} \phi_s \partial_{\mathrm{i}'} \phi_s}{8\pi G}, \tag{4.97}$$

whereas the high order term is defined, according with [2] as:

$$[\mathcal{T}_{\mathrm{i}}^j]^{\partial^2} = \mathcal{C}^{\mathrm{i}j} + \nabla_j \mathcal{D}^{\mathrm{i}j}, \tag{4.98}$$

where $\mathcal{D}^{\mathrm{i}j}$ is defined as:

$$\mathcal{D}^{\mathrm{i}j} = \nabla_j \left[\frac{\partial_{\mathrm{i}} \partial_j \phi_\ell \partial_j \rho_\ell}{\Lambda^2}\right] \equiv \nabla_j \mathcal{D}^{\mathrm{i}j}. \tag{4.99}$$

4.9.3 Second moment of the Boltzmann equation

The second moment of the Boltzmann equation under the smoothing approach is not a known result because in this point is truncated the hierarchy of the Boltzmann equation by considering $\kappa_{\mathrm{i}j} \approx 0$, which implies that $\mathcal{P} = 0$. Note that this consideration is consistent with the fact that dark matter is usually considered as particles that does not interacting. The computation of this momentum will be discussed in the next chapter with the first order approximation.

Chapter 5

First order approximation

The main goal of this work is to improve the understanding of Large Scale Structures by the implementation of the formalism of Effective Field Theory, introduced in the previous chapter. We start by fitting the large scale physics as an effective fluid according to [2] but adding a new ingredient: self interaction. We will model this effect inducing a small correction in the distribution function (only to first order), and compute the effective equation of motion for the investigated fluid.

This work can be considered as a first approach to a more realistic solution for large scales in the context of Effective Field theory in cosmology, because introduces extra ingredients never discussed simultaneously, as the effects of Self interactive Dark matter and the contribution of heat flux.

To quantify the collisional effect of Dark Matter particles into the effective equations of motion, we will parametrize the collisional term of Boltzmann's equation with a distribution function according to (4.20), where f is the exact value, whereas $f^{(0)}$ corresponds to a collisionless case. As an additional remark, many possibilities for the collisional term have been made long time ago, however, we are considering an easy solution because of we are interested on the investigation of the linear correction of distribution function of Boltzmann equation.

5.1 Introduction

Recall that in Chapter 4 we introduced the Boltzmann equation (or transport equation) according to eq.(4.7) where the right hand side takes into account the interactions. In the same way, in Chapter 4, eq.(4.20) we have a power series of the distribution function which allows us to handle the perturbations of the distribution function up to the required order.

In the context of kinetic theory for gases, it is possible to alter the collisional term in the Boltzmann equation such as each collision preserves particle number, momentum, and energy [7]. The motivation of this modification, in general, is given because the equations of macroscopic theory do not give a correct description of the phenomenon (see [7] for details).

A standard criterion for the range of validity of approximate methods for solving Boltzmann equation is given by comparing the characteristic time τ (or characteristic length L) for some relevant processes with the average time τ_c (or equivalently the mean free path L_c) between particle collisions. Thus, it is possible to decouple the approaches of how to investigate the problem into two opposite ranges of action: the first consists in systems with high density and the second with low density of particles.

For systems with high density ($\tau \gg \tau_c$) it's possible to use the assumption of local thermodynamic equilibrium and a drift velocity for the particles. After, it is necessary to correct the distribution function by taking into account the first derivatives of temperature, velocity and density. Higher corrections could be considered by the expansion for the distribution function in powers of the mean free path L_c.

For the opposite limit, i.e., when we consider the low density case, there are different techniques to solve the problem by the implementation of mean free path methods. A common procedure consists in perturbing the Boltzmann equation by considering an expansion of the distribution function terms of a series of inverse powers of the mean free path, where the first approximation consists in neglecting collisions completely.

Notice that the low density case is the relevant situation for us, because dark matter particles have very low density. Furthermore, the low density case is useful for other relevant physical situations too, for example, it's possible to apply it for ionized gases.

In this work, we make the assumption that we are considering only a one-component system. It's basically because the dark matter contribution is the most predominant matter in the Universe, so it is a good approximation to write: $f = f_{\text{DM}} + \sum f_{\text{Non-DM}} \approx f_{\text{DM}}$. As an important remark, note that the previous argument is consistent with the knowledge about dark matter, because we know that these particles are, in general, considered as free of pressure particles or with a very tiny value. Thus, for low densities, the collisions are only of secondary importance, however, we are considering this effect because of it can solve some astrophysical problems (e.g. Small Scale Issues) and allows us to incorporate self interactions into the equation of motion for large scale structure.

5.2 Collisional term at first order

In order to avoid the difficulties of the Boltzmann equation and parametrize the distribution function it is convenient to use the average time between collision τ taking into account that this parameter is small, and therefore, it is good quantity for expansion. Thus, an appropriate collisional term (see [7, 28]) is given by :

$$\mathcal{C}[f] \approx -\frac{f - f^{(0)}}{\tau}. \tag{5.1}$$

Notice that τ is, in general, a function of the velocity. So, the eq.(5.1) shows us that the collisions tend to relax the distribution function to an equilibrium value $f^{(0)}$. In the same way, this collisional term has a problem: the charge is not conserved instantaneously but if we take the average over a cycle, the problem is fixed.

We will consider a local Maxwell-Boltzmann distribution that depends of the particle mass m, the numerical density n, the temperature $\theta \equiv kT$ and the velocity $\vec{u}$. Furthermore, we shall consider that these quantities vary slowly and are functions of $\vec{x}$ and t.

5.3 Fluid equations

As we introduced in Chapter 4, the zeroth order equations are required to obtain the first order equation, thus, we can use our approach for the collisional term and consider the standard definition of *Fourier Law* ($\vec{q} = -K\nabla\theta$) plus the explicit form of the pressure tensor (ignoring higher-order terms) we have the following equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \tag{5.2}$$

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right) \vec{u} = \frac{\vec{F}}{m} - \frac{1}{\rho}\nabla\left(P - \frac{\mu}{3}\nabla \cdot \vec{u}\right) + \frac{\mu}{\rho}\nabla^2\vec{u}, \tag{5.3}$$

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right) \theta = -\frac{1}{c_v}(\nabla \cdot \vec{u})\theta + \frac{K}{\rho c_v}\nabla^2\theta. \tag{5.4}$$

where $c_v = 3/2$ is the heat capacity at constant volume. It's important to note that K in the Fourier Law is given by:

$$K \equiv \frac{m^5}{6\theta}\tau \int \mathrm{d}^3c c^4 \left(\frac{m}{2\theta}c^2 - \frac{5}{2}\right) f^{(0)} = \frac{5}{2}\theta n \tau, \tag{5.5}$$

whereas the pressure tensor, computed at first order, satisfies:

$$\mathcal{P}_{\mathrm{i}j} = \delta_{\mathrm{i}j}\mathcal{P} + \mathcal{P}'_{\mathrm{i}j} \tag{5.6}$$

with:

$$\mathcal{P} = \frac{\rho\theta}{m}, \tag{5.7}$$

$$\mathcal{P'}_{\mathrm{i}j} = -\frac{\rho m^3}{n\theta}\Lambda_{kl}\tau \int \mathrm{d}^3c c_{\mathrm{i}} c_j \left(c_k c_l - \frac{1}{3}\delta_{kl}c^2\right) f^{(0)}. \tag{5.8}$$

The object $\mathcal{P'}_{\mathrm{i}j}$ can be easily computed by noting that it is a symmetric tensor of null trace and depends linearly on the symmetric tensor $\Lambda_{\mathrm{i}j}$ (see definition at eq.(4.16)). Therefore, it should have the form:

$$\mathcal{P'}_{\mathrm{i}j} = -\frac{2\mu}{m}\left(\Lambda_{\mathrm{i}j} - \frac{1}{3}m\delta_{\mathrm{i}j}\nabla \cdot \vec{u}\right). \tag{5.9}$$

By a direct evaluation of $\mathcal{P'}_{\mathrm{i}j}$ we obtain the required constant $\mu = n\theta\tau$. Finally, we obtain:

$$\mathcal{P}_{\mathrm{i}j} = \delta_{\mathrm{i}j}\mathcal{P} - \frac{2\mu}{m}\left(\Lambda_{\mathrm{i}j} - \frac{1}{3}m\delta_{\mathrm{i}j}\nabla \cdot \vec{u}\right). \tag{5.10}$$

5.4 Smoothing approach at first order

Now, we have introduced and revisited all the necessary physics and approximation required to investigate the problem at first order. We emphasize that we only preserve the linear terms into the eqs.(5.2),(5.3) and (5.4). Thus, using the same definition made in Section (4.8.1) we obtain the long wavelength quantities by integrating-out short-wavelength modes. Thus, by the same procedure made into the Chapter 4 we can split the standard quantities into short and long modes according with eq.(4.33) and to take the Taylor series of undefined quantities around $\vec{x}$ according with eq.(4.34). Combining previous steps is easy to obtain the smoothed equation of motion at first order.

5.4.1 Zeroth-order moment

The zeroth moment equation, considering the collisional term previously discussed, doesn't give us new informations due it does not introduce a correction for this equation which can be easily verified by computing the object $\mathcal{C}[f]$ weighted by the mass m.

5.4.2 First-order moment

Equation(5.3) contains the contribution, at first order, of the collisional term where we note that part of this equation has been worked in ref. [2] but without the effect of collisions. We revisited this procedure in the past section as a way to show the standard procedure used here. In addition, as we commented this point before, we only compute the new terms produced by the introduction to the collisional term.

We define, by simplicity, the $\Omega(\vec{x},t)$ function as the new term given by our study:

$$\Omega(\vec{x},t) \equiv -\nabla\left(\mathcal{P} - \frac{\mu}{3}\nabla\cdot\vec{u}\right) + \mu\nabla^2\vec{u}. \tag{5.11}$$

We need to apply the window function in order to extract the long-wavelength modes of the equations as a way to describe the physics at large scale. As we show in previous sections (see Section 4.9.2) we are integrating out small scales combined with a Gaussian windows function to obtain the effective equations. So:

$$\Omega_1 \equiv -\int \mathrm{d}^3\vec{x}' W_\Lambda \nabla\mathcal{P} = -\int \mathrm{d}^3\vec{x}' W_\Lambda \partial_{z'}\left(\frac{\rho\theta}{m}\right). \tag{5.12}$$

Using integration by parts and the windows function properties to get:

$$\Omega_1 = -\int \mathrm{d}^3\vec{x}' W_\Lambda \nabla\mathcal{P} = -\frac{1}{m}\int \mathrm{d}^3\vec{x}' \rho(\vec{x}')\theta(\vec{x}')\Big(-\partial_z W_\Lambda\Big) = \frac{1}{m}\partial_z\big(\rho_\ell\theta_\ell\big) = \partial_z\mathcal{P}_\ell. \tag{5.13}$$

On the other hand, Ω_2 is given by:

$$\Omega_2(\vec{x},t) = \frac{\mu}{3}\int \mathrm{d}^3\vec{x}' W_\Lambda \partial_{\mathrm{i}'}\partial_{j'}u^j = \underbrace{\frac{\mu}{3}\Big(W_\Lambda\partial_{j'}u^j\Big)_{\mathrm{i}}^{f}}_{=0} - \frac{\mu}{3}\int \mathrm{d}^3\vec{x}' \partial_{\mathrm{i}'}W_\Lambda\partial_{j'}u^j. \tag{5.14}$$

Using the windows function properties we have:

$$
\begin{aligned}
\Omega_2(\vec{x},t) &= -\frac{\mu}{3}\int \mathrm{d}^3\vec{x}'\Big(-\partial_{\mathrm{i}}W_\Lambda\Big)\partial_{j'}u^j,\\
\Omega_2(\vec{x},t) &= \frac{\mu}{3}\partial_{\mathrm{i}}\int \mathrm{d}^3\vec{x}'W_\Lambda\partial_{j'}u^j = \frac{\mu}{3}\partial_{\mathrm{i}}\int_{\vec{x}'}\big(\partial_{j'}\Big(W_\Lambda u^j\Big) - u^j\partial_{j'}W_\Lambda\big),\\
\Omega_2(\vec{x},t) &= \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j\int \mathrm{d}^3\vec{x}'W_\Lambda u^j.
\end{aligned}
\tag{5.15}
$$

Note that u^j is not defined into the set of long wavelength quantities (see 4.8.1), thus, we need to split it into long and short quantities ($u^j = u^j_s + u^j_\ell$) using Taylor series:

$$
\begin{aligned}
\Omega_2(\vec{x},t) &= \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j\int_{\vec{x}'}W_\Lambda\left[u^j_\ell(\vec{x}) - \partial_r u^j_\ell(\vec{x})(\vec{x}-\vec{x}')^r + \frac{1}{2}\partial_u\partial_r u^j_\ell(\vec{x})(\vec{x}-\vec{x}')^r(\vec{x}-\vec{x}')^u\right]\\
&+ \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j\int \mathrm{d}^3\vec{x}'W_\Lambda u^j_s(\vec{x}'),\\
&= \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j u^j_\ell\int \mathrm{d}^3\vec{x}'W_\Lambda - \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j\partial_r u^j_\ell\int \mathrm{d}^3\vec{x}'\left(\frac{\partial_{r'}W_\Lambda}{\Lambda^2}\right)\\
&+ \frac{\mu}{6}\partial_{\mathrm{i}}\partial_j\partial_u\partial_r u^j_\ell\int \mathrm{d}^3\vec{x}'\left(\frac{\delta_{ru}W_\Lambda}{\Lambda^2}\right) + \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j\int \mathrm{d}^3\vec{x}'W_\Lambda u^j_s(\vec{x}'),\\
&= \left[\frac{\mu}{3}\partial_{\mathrm{i}}\partial_j u^j_\ell + \frac{\mu\partial_{\mathrm{i}}\partial_j\partial_r u^j_\ell}{3\Lambda^2}\partial_r + \frac{\mu\partial_{\mathrm{i}}\partial_j\partial^2 u^j_\ell}{6\Lambda^2}\right]\underbrace{\int \mathrm{d}^3\vec{x}'W_\Lambda}_{=1} + \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j\int \mathrm{d}^3\vec{x}'W_\Lambda u^j_s(\vec{x}'),
\end{aligned}
$$

finally we have:

$$
\Omega_2(\vec{x},t) = \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j u^j_\ell + \frac{\mu\partial_{\mathrm{i}}\partial_j\partial^2 u^j_\ell}{6\Lambda^2} + \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j\int \mathrm{d}^3\vec{x}'W_\Lambda u^j_s(\vec{x}'). \tag{5.16}
$$

The Ω_3 term can be obtained by taking into account Ω_2 inserting an appropriate kronecker delta, thus:

$$
\Omega_3 \equiv \mu\int_{\vec{x}'}W_\Lambda\nabla^2\vec{u} = \mu\int \mathrm{d}^3\vec{x}'W_\Lambda\delta^{\mathrm{e}f}\partial_{\mathrm{e}}\partial_f u^g, \tag{5.17}
$$

$$
\Omega_3 = \mu\big(\partial^2 u^g_\ell + \frac{\partial^2\partial_2 u^g_\ell}{2\Lambda^2} + \partial^2\int \mathrm{d}^3\vec{x}'W_\Lambda u^g_s(\vec{x}')\big). \tag{5.18}
$$

Thus, considering all the Omega's contributions we have

$$
\begin{aligned}
\int \mathrm{d}^3\vec{x}'W_\Lambda\Omega_{\mathrm{i}}(\vec{x}',t) =& -\partial_{\mathrm{i}}P_\ell + \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j u^j_\ell + \frac{\mu\partial_{\mathrm{i}}\partial_j\partial^2 u^j_\ell}{6\Lambda^2} + \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j\int \mathrm{d}^3\vec{x}'W_\Lambda u^j_s(\vec{x}')\\
&+ \mu\partial^2 u^{\mathrm{i}}_\ell + \mu\frac{\partial^2\partial_2 u^{\mathrm{i}}_\ell}{2\Lambda^2} + \mu\partial^2\int \mathrm{d}^3\vec{x}'W_\Lambda u^{\mathrm{i}}_s(\vec{x}').
\end{aligned}
\tag{5.19}
$$

Note that $u^{\mathrm{i}}_\ell = u^\ell_{\mathrm{i}}$. Furthermore, if we rewrite eq.(5.19) in a more convenient way:

$$
\begin{aligned}
\int_{x'}W_\Lambda\Omega_{\mathrm{i}}(\vec{x},t) =& -\partial_{\mathrm{i}}\mathcal{P}_\ell + \frac{\mu}{3}\partial_{\mathrm{i}}\partial_j u^j_\ell + \mu\partial^2 u^{\mathrm{i}}_\ell + \frac{\mu}{6\Lambda^2}\left[\partial_{\mathrm{i}}\partial_j\partial^2 u^j_\ell + 3\partial^2\partial_2 u^{\mathrm{i}}_\ell\right]\\
&+ \frac{\mu}{3}\left[\partial_{\mathrm{i}}\partial_j\left[u^j_s\right]_\Lambda + 3\partial^2\left[u^j_s\right]_\Lambda\right].
\end{aligned}
\tag{5.20}
$$

In this way, we note three different terms: the first is the standard correction for the right part of first momentum of Boltzmann equation. This term contains only long wavelength quantities. The second term depends on Λ^{-2} and contains long wavelengths quantities too. Finally, the third term depends on short wavelength quantities. This last term can be considered as a source term given by the non linearities into the theory.

Thus, the computations give us terms with long and short modes. We are interested in the long wave terms. However, it is necessary to handle the short wavelength terms. We will comment about it in the discuss 6.

At this point we only need to connect the solution given in Ref.[2] for the non-collisional case adding the previous equation because of all these terms are new and only produced by the self interaction.

5.4.3 Second-order moment

Considering eq.(5.4) and the definitions (4.29), (4.30), (4.31) and (4.32) we first compute the left hand side of this equation. We start rewriteing (5.4) in a convenient way:

$$\rho\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)\theta=-\frac{\rho}{c_v}(\nabla\cdot\vec{u})\theta+\frac{K}{c_v}\nabla^2\theta. \tag{5.21}$$

Defining $\xi(\vec{x},t)$ like:

$$\xi(\vec{x},t)\equiv-\frac{\rho}{c_v}(\nabla\cdot\vec{u})\theta+\frac{K}{c_v}\nabla^2\theta. \tag{5.22}$$

Apply the filtering process:

$$\int \mathrm{d}^3\vec{x}' W_\Lambda \xi(\vec{x},t)=\int \mathrm{d}^3\vec{x}' W_\Lambda\left(-\frac{\rho}{c_v}(\nabla\cdot\vec{u})\theta\right)+\int \mathrm{d}^3\vec{x}' W_\Lambda\left(\frac{K}{c_v}\nabla^2\theta\right). \tag{5.23}$$

$$=-\frac{1}{c_v}\underbrace{\int \mathrm{d}^3\vec{x}' W_\Lambda\left(\rho(\nabla\cdot\vec{u})\theta\right)}_{\xi_1}+\frac{K}{c_v}\underbrace{\int \mathrm{d}^3\vec{x}' W_\Lambda\left(\nabla^2\theta\right)}_{\xi_2}. \tag{5.24}$$

Using index notation the first quantity is:

$$\xi_1(\vec{x},t)=\int \mathrm{d}^3\vec{x}' W_\Lambda \rho\partial_{\mathrm{i}'}u^{\mathrm{i}}\theta. \tag{5.25}$$

Let us to separate in short and long wavelength:

$$\partial_{\mathrm{i}}u^{\mathrm{i}}=\partial_{\mathrm{i}}u^{\mathrm{i}}_\ell+\partial_{\mathrm{i}}u^{\mathrm{i}}_s, \tag{5.26}$$

$$\theta=\theta_\ell+\theta_s. \tag{5.27}$$

Changing the variable by simplicity we get:

$$\eta_\ell=\partial_{\mathrm{i}}u^{\mathrm{i}}_\ell, \qquad \eta_s=\partial_{\mathrm{i}}u^{\mathrm{i}}_s, \tag{5.28}$$

Splitting the object $\xi_1(\vec{x},t)$ using the eq.(5.26) and (5.27) and using the Taylor series we obtain the $\xi_1 = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4$. Thus:

$$\xi_1(\vec{x},t) = \int \mathrm{d}^3\vec{x}' W_\Lambda \rho \left(\partial_{\mathrm{i}'} u^{\mathrm{i}}_\ell + \partial_{\mathrm{i}'} u^{\mathrm{i}}_s\right)(\theta_\ell + \theta_s), \tag{5.29}$$

$$= \int \mathrm{d}^3\vec{x}' W_\Lambda \rho \left[\partial_{\mathrm{i}'} u^{\mathrm{i}}_\ell \theta_\ell + \partial_{\mathrm{i}'} u^{\mathrm{i}}_\ell \theta_s + \partial_{\mathrm{i}'} u^{\mathrm{i}}_s \theta_\ell + \partial_{\mathrm{i}'} u^{\mathrm{i}}_s \theta_s\right]. \tag{5.30}$$

Split step by step and suppressing the x dependence in the functions:

$$\mathcal{A}_1 = \int_{\vec{x}'} W_\Lambda \rho(\vec{x}') \eta_\ell(\vec{x}') \theta_\ell(\vec{x}'), \tag{5.31}$$

$$\begin{aligned} &= \eta_\ell \theta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho - \eta_\ell \partial_m \theta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho (\vec{x}-\vec{x}')^m + \\ &\frac{1}{2}\eta_\ell \partial_n \partial_m \theta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho (\vec{x}-\vec{x}')^m (\vec{x}-\vec{x}')^n - \theta_\ell \partial_j \eta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho (\vec{x}-\vec{x}')^j + \\ &\partial_j \eta_\ell \partial_m \theta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho (\vec{x}-\vec{x}')^j (\vec{x}-\vec{x}')^m + \frac{1}{2}\theta_\ell \partial_k \partial_j \eta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho (\vec{x}-\vec{x}')^j (\vec{x}-\vec{x}')^k. \end{aligned} \tag{5.32}$$

By taking into account the properties of windows function we have:

$$\begin{aligned} \mathcal{A}_1 = \eta_\ell \theta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho + \frac{\eta_\ell \partial_m \theta_\ell}{\Lambda^2} \partial_m \int \mathrm{d}^3\vec{x}' W_\Lambda \rho + \\ \frac{\eta_\ell \partial^2 \theta_\ell}{2\Lambda^2} \int \mathrm{d}^3\vec{x}' W_\Lambda \rho + \frac{\theta_\ell \partial_j \eta_\ell}{\Lambda^2} \partial_j \int \mathrm{d}^3\vec{x}' W_\Lambda \rho + \\ \frac{\partial_j \eta_\ell \partial_j \theta_\ell}{\Lambda^2} \int \mathrm{d}^3\vec{x}' W_\Lambda \rho + \frac{\theta_\ell \partial^2 \eta_\ell}{2\Lambda^2} \int \mathrm{d}^3\vec{x}' W_\Lambda \rho. \end{aligned} \tag{5.33}$$

After some manipulation we note that:

$$\mathcal{A}_1 = \left[\eta_\ell \theta_\ell + \frac{\eta_\ell \partial_m \theta_\ell}{\Lambda^2}\partial_m + \frac{\eta_\ell \partial^2 \theta_\ell}{2\Lambda^2} + \frac{\theta_\ell \partial_j \eta_\ell}{\Lambda^2}\partial_j + \frac{\partial_j \eta_\ell \partial_j \theta_\ell}{\Lambda^2} + \frac{\theta_\ell \partial^2 \eta_\ell}{2\Lambda^2}\right] \int \mathrm{d}^3\vec{x}' W_\Lambda \rho. \tag{5.34}$$

According with the definition (4.29) we obtain:

$$\mathcal{A}_1 = \left[\eta_\ell \theta_\ell + \frac{\eta_\ell \partial_m \theta_\ell}{\Lambda^2}\partial_m + \frac{\eta_\ell \partial^2 \theta_\ell}{2\Lambda^2} + \frac{\theta_\ell \partial_j \eta_\ell}{\Lambda^2}\partial_j + \frac{\partial_j \eta_\ell \partial_j \theta_\ell}{\Lambda^2} + \frac{\theta_\ell \partial^2 \eta_\ell}{2\Lambda^2}\right] \rho_\ell. \tag{5.35}$$

The object $\mathcal{A}_2$ is defined as:

$$\begin{aligned} \mathcal{A}_2 &= \int \mathrm{d}^3\vec{x}' W_\Lambda \rho \eta_\ell \theta_s, \\ \mathcal{A}_2 &= \int \mathrm{d}^3\vec{x}' W_\Lambda \rho \theta_s \left[\eta_\ell(\vec{x}) - \partial_j \eta_\ell(\vec{x})(\vec{x}-\vec{x}')^j + \frac{1}{2}\partial_k \partial_j \eta_\ell(\vec{x})(\vec{x}-\vec{x}')^j (\vec{x}-\vec{x}')^k\right] \\ \mathcal{A}_2 &= \eta_\ell(\vec{x}) \int_{\vec{x}'} W_\Lambda \rho \theta_s + \frac{\partial_j \eta_\ell(\vec{x})}{\Lambda^2}\partial_j \int_{\vec{x}'} W_\Lambda \rho \theta_s + \frac{\partial_k \partial_j \eta_\ell(\vec{x})}{2\Lambda^2}\delta_{jk} \int_{\vec{x}'} W_\Lambda \rho \theta_s. \end{aligned} \tag{5.36}$$

Taking out the common factor and ignoring the $\vec{x}$ dependence:

$$\mathcal{A}_2 = \left[\eta_\ell + \frac{\partial_j \eta_\ell}{\Lambda^2}\partial_j + \frac{\partial^2 \eta_\ell}{2\Lambda^2}\right] \underbrace{\int \mathrm{d}^3\vec{x}' W_\Lambda \rho \theta_s}_{\mathfrak{a}_2}. \tag{5.37}$$

Rewriting θ in a more appropriate way ($\theta_s = \theta - \theta_\ell$) and using, Taylor series around $\vec{x}$:

$$\begin{aligned}
\mathfrak{a}_2 =& \partial_m\theta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho(\vec{x}-\vec{x}')^m - \frac{1}{2}\partial_n\partial_m\theta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda \rho(\vec{x}-\vec{x}')^m(\vec{x}-\vec{x}')^n, \\
\mathfrak{a}_2 =& \left[-\frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m - \frac{\partial^2\theta_\ell}{2\Lambda^2}\right] \int \mathrm{d}^3\vec{x}' W_\Lambda \rho, \\
\mathfrak{a}_2 =& -\frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m\rho_\ell - \frac{\partial^2\theta_\ell}{2\Lambda^2}\rho_\ell.
\end{aligned} \tag{5.38}$$

Putting (5.38) into (5.37) we obtain:

$$\mathcal{A}_2 = -\eta_\ell \frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m\rho_\ell - \eta_\ell\frac{\partial^2\theta_\ell}{2\Lambda^2}\rho_\ell. \tag{5.39}$$

The other term is defined by:

$$\mathcal{A}_3 = \int \mathrm{d}^3\vec{x}' W_\Lambda \rho\eta_s\theta_\ell, \tag{5.40}$$

$$\begin{aligned}
\mathcal{A}_3 =& \int_{\vec{x}'} W_\Lambda\rho\eta_s \left(\theta_\ell(\vec{x}) - \partial_m\theta_\ell(\vec{x})(\vec{x}-\vec{x}')^m + \frac{1}{2}\partial_n\partial_m\theta_\ell(\vec{x})(\vec{x}-\vec{x}')^m(\vec{x}-\vec{x}')^n\right), \\
\mathcal{A}_3 =& \theta_\ell \int \mathrm{d}^3\vec{x}' W_\Lambda\rho\eta_s + \frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m \int \mathrm{d}^3\vec{x}' W_\Lambda\rho\eta_s + \frac{\partial^2\theta_\ell}{2\Lambda^2}\int \mathrm{d}^3\vec{x}' W_\Lambda\rho\eta_s, \\
\mathcal{A}_3 =& \left[\theta_\ell + \frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m + \frac{\partial^2\theta_\ell}{2\Lambda^2}\right] \underbrace{\int \mathrm{d}^3\vec{x}' W_\Lambda\rho\eta_s}_{\mathfrak{a}_3}.
\end{aligned} \tag{5.41}$$

Note that $\mathfrak{a}_3$ is not possible to reduce in a more elegant form, thereby directly we identify this quantity as $\mathfrak{a}_3 = [\rho\eta_s]_\Lambda$. Thus, eq.(5.41) is:

$$\mathcal{A}_3 = \left[\theta_\ell + \frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m + \frac{\partial^2\theta_\ell}{2\Lambda^2}\right][\rho\eta_s]_\Lambda. \tag{5.42}$$

Finally, we will compute the $\mathcal{A}_4$ term:

$$\begin{aligned}
\mathcal{A}_4 =& \int \mathrm{d}^3\vec{x}' W_\Lambda\rho\partial_{\mathrm{i}'}u_s^{\mathrm{i}}\theta_s, \\
\mathcal{A}_4 =& \int \mathrm{d}^3\vec{x}' W_\Lambda\rho\eta_s\theta_s, \\
\mathcal{A}_4 =& [\rho\eta_s\theta_s]_\Lambda.
\end{aligned} \tag{5.43}$$

The object $\xi_1 = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4$ gives us:

$$\begin{aligned}
\xi_1 =& \left[\eta_\ell\theta_\ell + \frac{\eta_\ell\partial_m\theta_\ell}{\Lambda^2}\partial_m + \frac{\eta_\ell\partial^2\theta_\ell}{2\Lambda^2} + \frac{\theta_\ell\partial_j\eta_\ell}{\Lambda^2}\partial_j + \frac{\partial_j\eta_\ell\partial_j\theta_\ell}{\Lambda^2} + \frac{\theta_\ell\partial^2\eta_\ell}{2\Lambda^2}\right]\rho_\ell \\
&+ \eta_\ell\frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m\rho_\ell - \eta_\ell\frac{\partial^2\theta_\ell}{2\Lambda^2}\rho_\ell. + \left[\theta_\ell + \frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m + \frac{\partial^2\theta_\ell}{2\Lambda^2}\right][\rho\eta_s]_\Lambda + [\rho\eta_s\theta_s]_\Lambda,
\end{aligned} \tag{5.44}$$

$$\begin{aligned}
\xi_1 =& \eta_\ell\theta_\ell\rho_\ell + \frac{\theta_\ell\partial_j\eta_\ell}{\Lambda^2}\partial_j\rho_\ell + \frac{\partial_j\eta_\ell\partial_j\theta_\ell}{\Lambda^2}\rho_\ell + \frac{\theta_\ell\partial^2\eta_\ell}{2\Lambda^2}\rho_\ell + \theta_\ell[\rho\eta_s]_\Lambda \\
&+ \frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m[\rho\eta_s]_\Lambda + \frac{\partial^2\theta_\ell}{2\Lambda^2}[\rho\eta_s]_\Lambda + [\rho\eta_s\theta_s]_\Lambda.
\end{aligned} \tag{5.45}$$

On the other hand, we need to compute ξ_2:

$$\xi_2(\vec{x},t) = \int \mathrm{d}^3\vec{x}' W_\Lambda \left(\nabla^2\theta\right),$$
$$\xi_2(\vec{x},t) = \int \mathrm{d}^3\vec{x}' W_\Lambda \partial^2\theta,$$
$$\xi_2(\vec{x},t) = \int \mathrm{d}^3\vec{x}' W_\Lambda \partial^2\Big(\theta_\ell + \theta_s\Big). \tag{5.46}$$

Considering the Taylor expansion to get:

$$\xi_2(\vec{x},t) = \int \mathrm{d}^3\vec{x}' W_\Lambda \partial^2\theta_s + \int \mathrm{d}^3\vec{x}' W_\Lambda \partial^2\theta_\ell,$$
$$\xi_2(\vec{x},t) = \int \mathrm{d}^3\vec{x}' W_\Lambda \partial^2\theta_s + \left[\partial^2\theta_\ell + \frac{\partial_o\partial^2\theta_\ell}{\Lambda^2}\partial_o + \frac{\partial^2\partial^2\theta_\ell}{2\Lambda^2}\right]\int \mathrm{d}^3\vec{x}' W_\Lambda,$$
$$\xi_2(\vec{x},t) = [\partial^2\theta_s]_\Lambda + \partial^2\theta_\ell + \frac{\partial^2\partial^2\theta_\ell}{2\Lambda^2}, \tag{5.47}$$

Now, we want to apply the filtering process into the left part of the first-order moment, thus:

$$\Pi(\vec{x},t) \equiv \int_{\vec{x}'} W_\Lambda \rho \left(\partial_t + u^{\mathrm{i}}\partial_{\mathrm{i}}\right)\theta,$$
$$\Pi(\vec{x},t) = \int_{\vec{x}'} W_\Lambda \Big[\partial_t(\rho\theta) - \partial_t\rho\theta + \rho u^{\mathrm{i}}\partial_{\mathrm{i}}\theta\Big]. \tag{5.48}$$

The first term have a trivial expression, whereas the other two terms require to take into account the Taylor series for θ and $\partial_{\mathrm{i}}\theta$ around $\vec{x}$. Note that $\Pi \equiv \Pi_1 + \Pi_2 + \Pi_3$. The first term is:

$$\Pi_1(\vec{x},t) \equiv \int_{\vec{x}'} W_\Lambda \partial_t(\rho\theta) = \partial_t(\rho_\ell\theta_\ell). \tag{5.49}$$

The second object is given by:

$$\Pi_2(\vec{x},t) \equiv -\left[\theta_\ell + \frac{\partial_j\theta_\ell\partial_j}{\Lambda^2} + \frac{\partial^2\theta_\ell}{2\Lambda^2}\right]\int_{\vec{x}'} W_\Lambda\partial_t\rho - [\partial_t\rho\theta_s]_\Lambda,$$
$$\Pi_2(\vec{x},t) = -\theta_\ell\partial_t\rho_\ell - \frac{\partial_j\theta_\ell\partial_j}{\Lambda^2}\partial_t\rho_\ell - \frac{\partial^2\theta_\ell}{2\Lambda^2}\partial_t\rho_\ell - [\partial_t\rho\theta_s]_\Lambda. \tag{5.50}$$

In the same way, Π_3 is computed using the same procedure. So, the final object is:

$$\Pi_3(\vec{x},t) \equiv \left[\partial_{\mathrm{i}}\theta_\ell + \frac{\partial_m\partial_{\mathrm{i}}\theta_\ell\partial_m}{\Lambda^2} + \frac{\partial^2\partial_{\mathrm{i}}\theta_\ell}{2\Lambda^2}\right]\int_{\vec{x}'} W_\Lambda(\rho u^{\mathrm{i}}) + [\rho u^{\mathrm{i}}\partial_{\mathrm{i}}\theta]_\Lambda,$$
$$\Pi_3(\vec{x},t) = \partial_{\mathrm{i}}\theta_\ell\rho_\ell u_\ell^{\mathrm{i}} + \frac{\partial_m\partial_{\mathrm{i}}\theta_\ell\partial_m}{\Lambda^2}\rho_\ell u_\ell^{\mathrm{i}} + \frac{\partial^2\partial_{\mathrm{i}}\theta_\ell}{2\Lambda^2}\rho_\ell u_\ell^{\mathrm{i}} + [\rho u^{\mathrm{i}}\partial_{\mathrm{i}}\theta]_\Lambda. \tag{5.51}$$

Finally, the total contribution Π is the sum:

$$\begin{aligned}\Pi(\vec{x},t) =& \rho_\ell\partial_t\theta_\ell + \rho_\ell u_\ell^{\mathrm{i}}\partial_{\mathrm{i}}\theta_\ell + \left[-\frac{\partial_j\theta_\ell\partial_j\partial_t\rho_\ell}{\Lambda^2} + \frac{\partial_m\partial_{\mathrm{i}}\theta_\ell\partial_m\rho_\ell u_\ell^{\mathrm{i}}}{\Lambda^2} + \frac{\partial_m\partial_{\mathrm{i}}\theta_\ell\rho_\ell\partial_m u_\ell^{\mathrm{i}}}{\Lambda^2}\right] \\ & \left[-\frac{\partial^2\theta_\ell\partial_t\rho_\ell}{2\Lambda^2} + \frac{\partial^2\partial_{\mathrm{i}}\theta_\ell\rho_\ell u_\ell^{\mathrm{i}}}{2\Lambda^2}\right] - [\partial_t\rho_\ell\theta_s]_\Lambda + [\rho u^{\mathrm{i}}\partial_{\mathrm{i}}\theta_s]_\Lambda.\end{aligned} \tag{5.52}$$

At this point, we just need to combine the solutions obtained here. Thus, using ξ_1, ξ_2 and Π to obtain the effective second-order moment.

$$\int_{\vec{x}'} W_\Lambda \left[\rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\theta\right] = \int_{\vec{x}'} W_\Lambda \left[-\frac{\rho}{c_v}(\nabla\cdot\vec{u})\theta + \frac{K}{c_v}\nabla^2\theta\right] \tag{5.53}$$

In term of before defined variables we have:

$$\Pi(\vec{x},t) = -\frac{1}{c_v}\xi_1(\vec{x},t) + \frac{K}{c_v}\xi_2(\vec{x},t), \tag{5.54}$$

where we can write

$$\rho_\ell\partial_t\theta_\ell + \partial_\ell u^{\mathrm{i}}_\ell\partial_{\mathrm{i}}\theta_\ell\rho_\ell + \mathcal{G}_\ell(\Lambda) + \mathcal{G}_s(\Lambda) = -\frac{1}{c_v}\eta_\ell\theta_\ell\rho_\ell + \frac{K}{c_v}\partial^2\theta_\ell + \mathcal{R}_\ell(\Lambda) + \mathcal{R}_s(\Lambda), \tag{5.55}$$

where $\mathcal{G}_\ell(\Lambda)$, $\mathcal{G}_s(\Lambda)$, $\mathcal{R}_\ell(\Lambda)$ and $\mathcal{R}_s(\Lambda)$ are defined as:

$$\mathcal{G}_\ell(\Lambda) = -\frac{\partial_j\theta_\ell\partial_j\partial_t\rho_\ell}{\Lambda^2} + \frac{\partial_m\partial_{\mathrm{i}}\theta_\ell\partial_m\rho_\ell u^{\mathrm{i}}_\ell}{\Lambda^2} + \frac{\partial_m\partial_{\mathrm{i}}\theta_\ell\rho_\ell\partial_m u^{\mathrm{i}}_\ell}{\Lambda^2} - \frac{\partial^2\theta_\ell\partial_t\rho_\ell}{2\Lambda^2} + \frac{\partial^2\partial_{\mathrm{i}}\theta_\ell\rho_\ell u^{\mathrm{i}}_\ell}{2\Lambda^2}, \tag{5.56}$$

$$\mathcal{G}_s(\Lambda) = \Big[-\partial_t\rho_\ell\theta_s + \rho u^{\mathrm{i}}\partial_{\mathrm{i}}\theta_s\Big]_\Lambda, \tag{5.57}$$

$$\mathcal{R}_\ell(\Lambda) = -\frac{1}{c_v}\left[\frac{\theta_\ell\partial_j\eta_\ell}{\Lambda^2}\partial_j\rho_\ell + \frac{\partial_j\eta_\ell\partial_j\theta_\ell}{\Lambda^2}\rho_\ell + \frac{\theta_\ell\partial^2\eta_\ell}{2\Lambda^2}\rho_\ell\right] + \frac{K}{c_v}\frac{\partial^2\partial^2\theta_\ell}{2\Lambda^2}, \tag{5.58}$$

$$\mathcal{R}_s(\Lambda) = -\frac{1}{c_v}\left[\theta_\ell[\rho\eta_s]_\Lambda + \frac{\partial_m\theta_\ell}{\Lambda^2}\partial_m[\rho\eta_s]_\Lambda + \frac{\partial^2\theta_\ell}{2\Lambda^2}[\rho\eta_s]_\Lambda + [\rho\eta_s\theta_s]_\Lambda\right] + \frac{K}{c_v}[\partial^2\theta_s]_\Lambda. \tag{5.59}$$

Rewriting the equation and using eq.5.28 we have:

$$\boxed{\rho_\ell\partial_t\theta_\ell + \rho_\ell u^{\mathrm{i}}_\ell\partial_{\mathrm{i}}\theta_\ell = -\frac{1}{c_v}\partial_{\mathrm{i}}u^{\mathrm{i}}_\ell\theta_\ell\rho_\ell + \frac{K}{c_v}\partial^2\theta_\ell + \mathcal{R}_\ell(\Lambda) - \mathcal{G}_\ell(\Lambda) + \mathcal{R}_s(\Lambda) - \mathcal{G}_s(\Lambda).} \tag{5.60}$$

Notice that in the energy equation we identify terms with different physical meaning: first, we note that we have long wavelength terms which give us the standard energy equation. Second, we getting corrections which are given, on one hand, by the non-lineal quantities (short long wavelength) and are denoted by the subscript s. On other hand we get small corrections but these are given by long wave-length terms weight by Λ^{-2}. The eq.5.60 is a novel result never computed before. We will discuss it in the next section.

Chapter 6

Discussion and Conclusions

We know that the ΛCDM model works fine at large scales, because observations and simulations are consistent [16]. However, opposite to it, we know that at short scales ΛCDM has some problems and the reasons are unclear yet. These problems at short scale (Small Scale Issues) can be addressed by considering Self Interactive Dark Matter [31, 30]. This special kind of Dark matter does not produce appreciable effects at large scale, but modifies the small scale structure.

Thus, in order to contribute to our understanding of LSS, this thesis consists on the study the Large Scale Structure of Universe using a novel Effective Field theory approach, considering an exotic type of matter: self interacting dark matter. The use Effective theory in the cosmological context allows us parametrize the physics of dark matter for small scale without the total understanding of full theory. Thus, in this work we applied the approach of effective field theory discussed in ref.[2] to describe Large Scale Structure (LSS) as a fluid including gravity and considering non-linear terms in the theory.

It is important to emphasize that preliminary computation was made in ref.[2]), however the authors did not consider collisional particles and thereby, they neglected the pressure at zeroth order. Basically, in ref.[2] the Boltzmann equation was used to obtain the first momentums however the authors did not take into account viscous terms which can be relevant at some scales. The momentums obtained were used to get the effective equations of motion for collisionless dark matter.

Notice that fluid models with viscosity are more realistic situations and in this thesis, extra terms (obtained applying the smoothing approach) allowed us to make progress in relation to the viscosity. In this sense, we were able to identify the viscous terms with higher derivatives in our smoothing approach, therefore, with the help of observations it should be possible to constrain the values of viscous coefficients. Additionally, despite of our lack of knowledge of the explicit form of non-linear terms, we parametrized our ignorance by splitting these terms around $\vec{x}$ in terms of known quantities of large scale.

In this work, we have improved (part of) previous result [2] by extending the hierarchy of Boltzmann equation which implies to take into account an extra equation, the energy equation or the second momentum of Boltzmann equation.

We obtained a set of equations that describe LSS as a dissipative fluid as we take into account the pressure and because we are considering interaction between particles. Notice that the usual assumption consists in considering standard dark matter (non interactive), however, simulations suggest that SIDM can help to improve our understanding of LSS physics.

The DM interactions are parametrized by the collisional term in the Boltzmann equation which, in general, allows us describe the problem more realistically. In particular, we recovered the equations obtained in ref.[2] for the collisionless case and added the energy equation, never computed before, to our set. We show that, with an appropriate window function, it is possible to make progress in the theory by take into account corrections given by non-linear terms. As a mandatory comment, our set of equations are, naturally, cut-off dependent, thereby if we choose another filter function the solutions will be different (see brief comments in ref.[12]).

We computed the first momentum of Boltzmann equation for a particular collisional term, which, under the smoothing approach, introduced corrections: the equation of conservation of mass was not affected, whereas the equation of conservation of momentum and energy were significantly modified. As in ref.[1] we obtained an effective stress tensor but with additional terms for the momentum equation. Indeed, it is possible to make the identification of kinetical and gravitational parts of these new terms. The second one is given by terms proportional to the gravitational potential, whereas as the first one could be defined as the other terms. Additionally, the energy equation has small scale terms, which can be interpreted as sources terms.

About the method used in this thesis, we followed the approach of ref.[2] which consists in considering the set of equations of a fluid and take the convolution with a Gaussian function. In our case, we modified the equations by taking into account self interactions but the technique is the same. The disadvantage of our technique is that we require to fit our model with N-body simulation which allows us to get the viscous term produced by short wavelength terms. Furthermore, notice that EFF of LSS is an excellent example in which Standard Perturbation Theory breaks down. We know that the non-linear fluctuations are the starting point of all our observable Universe, therefore we need a tool which be compatible with this fact.

Our results together with the idea of effective field theory and its application in the astrophysical context have an important value nowadays. This approach allowed description of the underlying physics considering only the relevant terms of interest. In addition, the idea of an effective fluid let us to identify terms in our treatment with physical quantities as the viscosity. Because dark matter is usually considered collisionless, investigations that take into account viscous terms are innovative. Thus, despite of the simplifications adopted in this work, our equations with interaction are novel.

Other important point to discuss is related with the interpretation of short wavelength terms. To analyse the effects of small-scale non-linearities on the long-wavelength universe, we considered the energy equation given (5.60) where we identified three parts in the equation: the first one represents the effective energy equation only for long-wavelength modes, whereas the second term describes the contribution of long-wavelength weighed by the cut off which is a model dependent effect. Finally, we have found irreducible short-wavelength terms which we interpreted as source terms as commented briefly before. So, this non-linearities couple the long and short-wavelength sectors, with fluctuations providing sources for the formation and evolution of large-scale structures. Explicitly:

$$\rho_\ell \partial_t \theta_\ell + \rho_\ell u_\ell^{\rm i} \partial_{\rm i} \theta_\ell = -\frac{1}{c_v} \partial_{\rm i} u_\ell^{\rm i} \theta_\ell \rho_\ell + \frac{K}{c_v} \partial^2 \theta_\ell + \underbrace{\mathcal{R}_\ell(\Lambda) - \mathcal{G}_\ell(\Lambda)}_{\text{second term}} + \underbrace{\mathcal{R}_s(\Lambda) - \mathcal{G}_s(\Lambda)}_{\text{third term}}. \tag{6.1}$$

Note that high order derivatives in the equation of momentum and in the equation of energy (which fits with viscous terms) are "indirectly induced" by the small-scale non-linearities which can be understood by comparing our effective equation with the standard's energy equation in fluid theory. This implication is not trivial because high order derivatives emerge as a way to parametrize our ignorance for small scale terms.

Finally, as a general remark, we should note that at a non-linear level, different scales are coupled, and these non-linearities affect even large scale perturbations that are mildly non linear and so potentially treatable in a perturbative way. Additionally, although the effective equations computed in this thesis fit as an effective fluid description for self interactive dark matter, they can be extended to all matter including baryons which trace the dark matter. It is important to realize that the full description of problem was made under the microscopical assumption which means that we are considering a classical gas of particles, where we have smoothed the relevant quantities (the Boltzmann equation is the key equation in this approach). We want to emphasize that the effective field theory approach to large scale structure is complementary to N-body simulations by providing an elegant fluid description. This allows intuitive for non-linear effects, as well as provides computational efficiency, since the numerical calculations required to measure the fluid parameters are expected to be computationally less expensive than a full scale simulation.

Appendix

Window functions are used in the astrophysical context by long time (see paper [24]). First, it was a very important tool in numerical simulations (Smoothed-particle Hydrodynamics) as a way to take into account the important quantities for a given problem. In physics, the windows function are a very good tool to solve problems, for example, the filter is used in regularization and cosmology.

In order to make progress in our theory, we select a windows function defined by:

$$W_\Lambda(\mathbf{x}) \equiv \left(\frac{\Lambda}{\sqrt{2\pi}}\right)^3 \mathrm{e}^{\frac{1}{2}\Lambda^2\mathbf{x}^2} \tag{6.2}$$

This filtering function satisfies the following properties:

$$W_\Lambda(\vec{x}-\vec{x}')^{\mathrm{i}} = -\frac{\partial_{\mathrm{i}} W_\Lambda}{\Lambda^2}, \tag{6.3}$$

$$W_\Lambda(\vec{x}-\vec{x}')^{\mathrm{i}}(\vec{x}-\vec{x}')^{j} = \frac{\delta_{\mathrm{i}j} W_\Lambda}{\Lambda^2}. \tag{6.4}$$

Thus, is possible to use other filtering function, however the Gaussian function is easy to compute and, by thus reason, we using Gaussian functions.

Chapter 7

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