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Calculus of Variations

Cavity problems in discontinuous media

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Abstract We study cavitation type equations, $\operatorname{div}(a_{ij}(X)\nabla u) \sim \delta_0(u)$, for bounded, measurable elliptic media $a_{ij}(X)$. De Giorgi–Nash–Moser theory assures that solutions are α-Hölder continuous within its set of positivity, $\{u > 0\}$, for some exponent α strictly less than one. Notwithstanding, the key, main result proven in this paper provides a sharp Lipschitz regularity estimate for such solutions along their free boundaries, $\partial\{u > 0\}$. Such a sharp estimate implies geometric-measure constrains for the free boundary. In particular, we show that the non-coincidence $\{u > 0\}$ set has uniform positive density and that the free boundary has finite $(n - \varsigma)$-Hausdorff measure, for a universal number $0 < \varsigma \leq 1$.

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1 Introduction

Given a Lipschitz bounded domain $\Omega \subset \mathbb{R}^n$, a bounded measurable elliptic matrix $a_{ij}(X)$, i.e. a symmetric matrix with varying coefficients satisfying the (λ, Λ)-ellipticity condition

$$\lambda \mathrm{Id} \leq a_{ij}(X) \leq \Lambda \mathrm{Id}, \tag{1.1}$$

and a nonnegative boundary data $\varphi \in L^2(\partial\Omega)$, we are interested in studying local minimizers u of the discontinuous functional

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$$\mathscr{F}(u) = \int_\Omega \left\{ \frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + \chi_{\{u>0\}} \right\} dX \to \min, \tag{1.2}$$

among all competing functions $u \in H^1_\varphi(\Omega) := \{u \in H^1(\Omega) \mid \mathrm{Trace}(u) = \varphi\}$.

The variational problem set in (1.2) appears in the mathematical formulation of a great variety of models: jet flows, cavity problems, Bernoulli problems, free transmission problems, optimal designs, just to cite few. Its mathematical treatment has been extensively developed since the epic marking work of Alt and Caffarelli [1]. The program for studying minimization problems for discontinuous functionals of the form (1.2) is nowadays well established in the literature. Existence of minimizer follows by classical considerations. Any minimum is nonnegative provided the boundary data is nonnegative. Minimizers satisfy, in the distributional sense, the Euler–Lagrange equation

$$\mathrm{div}(a_{ij}(X)\nabla u) = \mu, \tag{1.3}$$

where μ is a measure supported along the free boundary. In particular a minimum of the functional $\mathscr{F}$ is a-harmonic within its positive set, i.e.,

$$\mathrm{div}(a_{ij}(X)\nabla u) = 0, \quad \text{in } \{u > 0\} \cap \Omega.$$

By pure energy considerations, one proves that minimizers grow linearly away from their free boundaries. Finally, if a_{ij} are, say, Hölder continuous, then the free boundary $\partial\{u > 0\}$ is of class $C^{1,\alpha}$ up to a possible negligible singular set. In such a scenario, the free boundary condition

$$\langle a_{ij}(\xi)\nabla u(\xi), \nabla u(\xi)\rangle = \mathrm{Const.}$$

then holds in the classical sense along the regular part of the free boundary, in particular for all $\xi \in \partial_{\mathrm{red}}\{u > 0\} \cap \Omega$.

A decisive, key step, though, required in the program for studying variational problems of the form (1.2), concerns Lipschitz estimates of minimizers. However, if no further regularity assumptions upon the coefficients $a_{ij}(X)$ is imposed, even a-harmonic functions, $\mathrm{div}(a_{ij}(X)\nabla h) = 0$, may fail to be Lipschitz continuous. That is, the universal Hölder continuity exponent granted by De Giorgi–Nash–Moser regularity theory may be strictly less than 1, even for two-dimensional problems. Such a technical constrain makes the study of local minima to (1.2) in discontinuous media rather difficult from a rigorous mathematical viewpoint.

The above discussion brings us to the main goal of this present work. Even though it is hopeless to obtain gradient bounds for minimizers of functional (1.2) in Ω, we shall prove that, any minimum is universally Lipschitz continuous along its free boundary, $\partial\{u > 0\} \cap \Omega$, see [22,23] for improved estimates that hold only along (non-physical) free boundaries, see also [17]. Such an estimate is strong enough to carry on a geometric-measure analysis near the free boundary, which in particular implies that the non-coincidence set has uniform positive density and that the free boundary has finite $(n - \varsigma)$-Hausdorff measure, for a universal number $0 < \varsigma \le 1$. We shall establish the following result:

Theorem 1.1 *Let u be a nonnegative local minimum of the functional (1.2) and $Z_0 \in \partial\{u > 0\} \cap \Omega$ be a generic interior free boundary point. Then*

$$C^{-1} r \le \sup_{B_r(Z_0)} u \le C r,$$

for a constant $C > 0$ *depending only on dimension, ellipticity constants and* $\|u\|_{L^2(\Omega)}$*. In particular, for another universal constant* $\theta > 0$,

$$\mathcal{L}^n(\{u > 0\} \cap B_r(Z_0)) \geq \theta r^n,$$

for all $0 < r \ll 1$*. Furthermore there is a universal constant* $0 < \varsigma \leq 1$ *such that*

$$\dim_{\mathcal{H}}(\partial\{u > 0\}) \leq n - \varsigma,$$

where $dim_{\mathcal{H}}(E)$ *means the Hausdorff dimension to the set* E.

In this paper we shall develop a more general analysis as to contemplate singular approximations of the minimization problem (1.2). Let $\beta \in L^\infty(\mathbb{R})$ be a bounded function supported in the unit interval [0, 1]. For each $\epsilon > 0$, we define the integral preserving, ϵ-perturbed potential:

$$\beta_\epsilon(t) := \frac{1}{\epsilon}\beta\left(\frac{t}{\epsilon}\right), \tag{1.4}$$

which is now supported in $[0, \epsilon]$. Such a sequence of potentials converges in the distributional sense to $\int \beta$ times the Dirac measure δ_0. Consider further

$$B_\epsilon(\xi) = \int_0^\xi \beta_\epsilon(t)dt \to \left(\int \beta(s)ds\right) \cdot \chi_{\{\xi>0\}}, \tag{1.5}$$

in the distributional sense. We now look at local minimizers u_ε to the variational problem

$$\mathscr{F}_\epsilon(u) = \int_{B_1} \left\{\frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + B_\epsilon(u)\right\} dX \to \min, \tag{1.6}$$

among all competing functions $u \in H^1_\varphi(\Omega) := \{u \in H^1(\Omega) \mid \text{Trace}(u) = \varphi\}$. There is a large literature on such a class of singularly perturbed equations, see for instance [3,7–9,15,16,19]. It is well established that the functional $\mathscr{F}$ defined in (1.2) can be recovered by letting ϵ go to zero in (1.6). For each ϵ fixed though, minimizers of the functional $\mathscr{F}_\epsilon$ is related to a number of other physical problems, such as high energy activations and the theory of flame propagation. Hence, from the applied point of view, it is more appealing to indeed study the whole family of functionals $(\mathscr{F}_\epsilon)_{0\leq\epsilon\leq1}$. We also mention that the study of minimization problem (1.6) with no continuity assumption on the coefficients is also motivated by several branch of applications, for instance in homogenization theory, composite materials, etc.

We should also mention the connections this present work has with the theory of free phase transmission problems. This class of problems appears, for instance, in the system of equations modeling an ice that melts submerged in a heated inhomogeneous medium. For problems modeled within an organized medium (say Hölder continuous coefficients), monotonicity formula [5] yields Lipschitz estimates for solutions. However, by physical interpretations of the model, it is natural to consider the problem within discontinuous media. Under such an adversity (monotonicity formula is no longer available), Lipschitz estimate along the free boundary has been an important open problem within that theory, see [2] for discussion. However, if we further assume in the model that the temperature of the ice remains constant, which is reasonable in very low temperatures, then free phase transmission problems fit into the mathematical formulation of this present article; and a Lipschitz estimate becomes available by our main result.

We conclude this Introduction by mentioning that the improved, sharp regularity estimate we establish in this work holds true in much more generality. Our approach to obtain Lipschitz estimate along the free boundary extends directly to degenerate discontinuous functionals of the form

$$\int F(X, u, \nabla u) dX \to \min.,$$

where

$$F(X, u, \xi) \sim |\xi|^{p-2} A(X)\xi \cdot \xi + f(X) \left(u^+\right)^m + Q(X) \cdot \chi_{\{u>0\}},$$

with $A(X)$ bounded, measurable elliptic matrix, $f \in L^q(\Omega)$, $q > n$, $1 \le m < p$ and Q is bounded away from zero and infinity, see [10,13]. Indeed, the proof designed herein is purely nonlinear and uses solely the Euler–Lagrange equation associated to the minimization problem (1.6). Hence, nonvariational cavitation problems, as well as parabolic versions of such models can also be tackled by our methods.

2 Preliminaries

In this section we gather some results and tools available for the analysis of minimizers to the functional (1.6) [and also to the functional (1.2)]. The results stated herein follow by methods and approaches available in the literature. We shall briefly comment on the proofs, for the readers' convenience.

Theorem 2.1 (Existence of minimizers) *For each* $\epsilon > 0$ *fixed, there exists at least one minimizer* $u_\epsilon \in H^1_\varphi(\Omega)$ *to the function* (1.6). *Furthermore* u_ε *satisfies*

$$\operatorname{div}(a_{ij}(X)\nabla u_\epsilon) = \beta_\epsilon(u_\epsilon), \quad \textit{in } \Omega, \tag{2.1}$$

in the distributional sense. Each u_ϵ *is a nonnegative function, provided the boundary data* φ *is nonnegative.*

Proof Existence of minimizer as well as the Euler–Lagrange equation associated to the functional follow by classical methods in the Calculus of Variations. Non-negativity of a minimum is obtained as follows. Suppose, for the sake of contradiction, the set $\{u_\epsilon < 0\}$ were not empty. Since $\varphi \ge 0$ on $\partial\Omega$, one sees that $\partial\{u_\epsilon < 0\} \subset \{u_\epsilon = 0\} \cap \Omega$. Since β_ε is supported in $[0, \epsilon]$, from the equation we conclude that u_ϵ satisfies the homogeneous equation $\operatorname{div}(a_{ij}(X)\nabla u_\epsilon) = 0$ in $\{u_\epsilon < 0\}$. By the maximum principle we conclude $u_\epsilon \equiv 0$ in such a set, which gives a contradiction. □

Regarding higher regularity for minimizers, it is possible to show uniform-in-ϵ L^∞ bounds and also a uniform-in-ϵ $C^{0,\alpha}$ estimate, for a universal exponent $0 < \alpha < 1$.

Theorem 2.2 (Uniform Hölder regularity of minimizers) *Fixed a subdomain* $\Omega' \Subset \Omega$, *there exists a constant* $C > 0$, *depending on dimension, ellipticity constants,* $\|\varphi\|_{L^2}$ *and* Ω', *but independent of* ϵ, *such that*

$$\|u_\epsilon\|_{L^\infty(\Omega')} + [u_\epsilon]_{C^\alpha(\Omega')} < C,$$

where $0 < \alpha < 1$ *is a universal number.*

Proof The arguments to show Theorem 2.2 follow closely the ones from [2, Theorem 3.4], upon observing that for any ball $B_r(Y) \subset \Omega$, there too holds

$$\int_{B_r(Y)} B_\epsilon(u_\epsilon) dX \le C r^n,$$

for a constant C independent of ϵ. See also [20, Theorem 4.4] for a result of the same flavor. □

As a consequence of Theorem 2.2, up to a subsequence, u_ϵ converges locally uniformly in Ω to a nonnegative function u_0. By linear interpolation techniques, see for instance [19, Theorem 5.4], one verifies that u_0 is a minimizer of the functional (1.2).

The final result we state in this section gives the sharp lower bound for the grow of u_ϵ away from ϵ-level surfaces.

Theorem 2.3 (Linear growth) *Let $\Omega' \Subset \Omega$ be a given subdomain and $X_0 \in \Omega' \cap \{u_\epsilon \geq \epsilon\}$ then*

$$u_\epsilon(X_0) \geq c \cdot \operatorname{dist}(X_0, \partial\{u_\epsilon \geq \epsilon\}), \tag{2.2}$$

where c is a constant that depends on dimension and ellipticity constants, but it is independent of ϵ.

Proof The classical proof for linear growth is based on pure energy considerations, combined with a "cutting hole" argument, see for instance [19, Theorem 4.6]. Hence, the same reasoning applied here yields estimate (2.2), with minor modifications. □

3 Lipschitz regularity along the free boundary

The heart of this work lies in this section, where we prove that uniform limits of solutions to (2.1) are locally Lipschitz continuous along their free boundaries. We highlight once more that our approach is purely based on the singular partial differential equation satisfied by local minimizers; therefore it can be imported to a number of other contexts, both variational and non-variational.

Theorem 3.1 (Lipschitz regularity) *Let u_0 be a uniform limit point of solutions to*

$$\operatorname{div}(a_{ij}(X)\nabla u_\epsilon) = \beta_\epsilon(u_\epsilon) \quad in\ \Omega$$

and assume that $u_0(\xi) = 0$. Then there exists a universal constant $C > 0$, depending only on dimension, ellipticity constants, $\operatorname{dist}(\xi, \partial\Omega)$ and L^∞ bounds of the family such that

$$|u_0(X)| \leq C|X - \xi|,$$

for all point $X \in \Omega$.

Our strategy is based on a flatness improvement argument, within whom the next Lemma plays a decisive role.

Lemma 3.2 *Fixed a ball $B_r(Y) \Subset \Omega$ and given $\theta > 0$, there exists a $\delta > 0$, depending only on $B_r(Y)$, dimension, ellipticity constants and L^∞ bounds for u_ϵ, such that if*

$$div(a_{ij}(X)\nabla u_\epsilon) = \delta \cdot \beta_\epsilon(u_\epsilon)$$

and

$$\max\left\{\epsilon, \inf_{B_r(Y)} u_\epsilon\right\} \leq \delta.$$

Then

$$\sup_{B_{\frac{r}{2}}(Y)} u_\epsilon \le \theta.$$

Proof Let us suppose, for the sake of contradiction, that the Lemma fails to hold. There would then exist a sequence of functions u_{ϵ_k} satisfying

$$\mathrm{div}(a^k_{ij}(X)\nabla u_{\epsilon_k}) = \delta_k \beta_{\epsilon_k}(u_{\epsilon_k})$$

with a^k_{ij} (λ, Λ)-elliptic, $\delta_k = \mathrm{o}(1)$, and

$$\max\left\{\epsilon_k, \inf_{B_r(Y)} u_{\epsilon_k}\right\} =: \eta_k = \mathrm{o}(1),$$

but

$$\sup_{B_r/2(Y)} u_{\epsilon_k} \ge \theta_0 > 0, \tag{3.1}$$

for some $\theta_0 > 0$ fixed. Let X_k be the point where u_{ϵ_k} attains its minimum in $B_r(Y)$ and denote $\sigma := \mathrm{dist}(B_r(Y), \partial\Omega) > 0$. Define the scaled function $v_k \colon B_{\sigma\epsilon_k^{-1}} \to \mathbb{R}$, by

$$v_k(X) := \frac{u_{\epsilon_k}(X_k + \epsilon_k X)}{\eta_k}$$

One simply verifies that $v_k \ge 0$ and it solves, in the distributional sense,

$$\begin{aligned}\mathrm{div}(a^k_{ij}(X)\nabla v_k) &= \delta_k \times \left(\tfrac{\epsilon_k}{\eta_k}\beta_1(\tfrac{\eta_k}{\epsilon_k} v_k)\right) \\ &= \mathrm{o}(1),\end{aligned} \tag{3.2}$$

as $k \to \infty$, in the L^∞-topology. Also, one easily checks that $v_k(0) \le 1$. Hence, by Harnack inequality, the sequence v_k is uniform-in-k locally bounded in $B_{\sigma\epsilon_k^{-1}}(0)$. From De Giorgi, Nash, Moser regularity theory, up to a subsequence, v_k converges locally uniformly to an entire v_∞. In addition, by standard Caccioppoli energy estimates, the sequence v_k is locally bounded in H^1, uniform in k. Also by classical truncation arguments, up to a subsequence, $\nabla v_k(X) \to \nabla v_\infty(X)$ a.e. (see [20] and [21] for similar arguments). By ellipticity, passing to another subsequence, if necessary, a_{ij} converges weakly in L^2_{loc} to a (λ, Λ)-elliptic matrix b_{ij}. Summarizing we have the following convergences:

$$v_k \to v_\infty \text{ locally uniformly in } \mathbb{R}^n; \tag{3.3}$$

$$v_k \rightharpoonup v_\infty \text{ weakly in } H^1_{\mathrm{loc}}(\mathbb{R}^n); \tag{3.4}$$

$$\nabla v_k(X) \to \nabla v_\infty(X) \text{ almost everywhere in } \mathbb{R}^n; \tag{3.5}$$

$$a^k_{ij}(X) \rightharpoonup b_{ij} \text{ weakly in } L^2_{\mathrm{loc}}(\mathbb{R}^n). \tag{3.6}$$

Our next step is to the pass the limits above aiming to conclude that

$$\mathrm{div}(b_{ij}(X)\nabla v_\infty) = 0, \quad \text{in } \mathbb{R}^n. \tag{3.7}$$

This is a fairly routine procedure, but we will carry it out for the sake of the readers. Given a test function $\phi \in C^1_0(\mathbb{R}^n)$, let $k_0 \in \mathbb{N}$ be such that $B_{\sigma\epsilon_k^{-1}} \supset \mathrm{Supp}\,\phi := K$. For $k > k_0$, we define the integrals

$$\mathscr{I}_k^1 := \int_K \langle a_{ij}^k(X)\nabla v_k, \quad \nabla\phi\rangle dX;$$
$$\mathscr{I}_k^2 := \int_K \langle a_{ij}^k(X) \times (\nabla v_\infty - \nabla v_k), \quad \nabla\phi\rangle dX;$$
$$\mathscr{I}_k^3 := \int_K \langle (b_{ij} - a_{ij}^k)(X) \times \nabla v_\infty, \quad \nabla\phi\rangle dX;$$

and write

$$\int_{\mathbb{R}^n} \langle b_{ij}(X)\nabla v_\infty, \nabla\phi\rangle dX = \mathscr{I}_k^1 + \mathscr{I}_k^2 + \mathscr{I}_k^3. \tag{3.8}$$

The purpose is to show that

$$\lim_{k\to\infty} \mathscr{I}_k^1 + \mathscr{I}_k^2 + \mathscr{I}_k^3 = 0.$$

For that, let $\gamma > 0$ be a given number small, positive number. It follows straight from (3.2) that $\mathscr{I}_k^1 = \mathrm{o}(1)$ as $k \to \infty$. From (3.6), we also have straightly that $\mathscr{I}_k^3 = \mathrm{o}(1)$ as $k \to \infty$. Hence, for $k_1 \geq k_0$, we have

$$|\mathscr{I}_k^1| + |\mathscr{I}_k^3| \leq \frac{\gamma}{2}.$$

Let us now analyze the convergence of $\mathscr{I}_k^2$. It follows from (3.5) and Ergorov's theorem, that there exists a compact set $\tilde{K} \subset K$, such that

$$\int_{K\setminus\tilde{K}} |\nabla\phi| dX \leq \frac{\gamma}{5\Lambda \sup\limits_k \|\nabla v_k\|_2}$$

and $k_2 \geq k_1$ such that

$$|\nabla v_k(X) - \nabla v_\infty(X)| \leq \frac{3\gamma}{5\Lambda \|\nabla\phi\|_2 \mathscr{L}^n(K)},$$

in $\tilde{K}$, for all $k \geq k_2$. Hence, for $k \geq k_2$, we estimate, breaking it into two integrals on $\tilde{K}$ and on $K\setminus\tilde{K}$, and using Hölder inequality, finally obtain

$$|\mathscr{I}_k^2| \leq \frac{\gamma}{2}.$$

We have henceforth proven the aimed convergence which gives (3.7). Applying Liouville theorem to v_∞, we conclude that

$$v_\infty \equiv \text{Const.} < +\infty,$$

for a bounded constant, in the whole space. The corresponding limiting function u_∞ obtained from u_{ϵ_k} must therefore be identically zero. We now reach a contradiction with (3.1) for $k \gg 1$. The Lemma is proven. □

Before continuing, we remark that if u_ϵ is a solution to the original equation (2.1) and a positive number $\bar{\delta} > 0$ is given, then the zoomed-in function

$$\tilde{u}_\epsilon(X) = u_\epsilon(\sqrt{\bar{\delta}}X)$$

satisfies in the distributional sense the equation

$$\operatorname{div}(\tilde{a}_{ij}(X)\nabla\tilde{u}_\epsilon) = \bar{\delta}\beta_\epsilon(\tilde{u}_\epsilon),$$

where $\tilde{a}_{ij}(X) = a_{ij}(\sqrt{\bar{\delta}}X)$ is another $(\lambda, \Lambda)-$elliptic matrix.

We are in position to start delivering the proof of Theorem 3.1. Let u_ϵ be a bounded sequence of distributional solutions to (2.1) and u_0 a limit point in the uniform convergence topology. We assume, with no loss, that $\xi = 0$, that is $u_0(0) = 0$. Within the statement of Lemma 3.2, select

$$\theta = \frac{1}{2}.$$

Since $u_\epsilon(0) \to 0$ as $\varepsilon \to 0$, Lemma 3.2 together with the above remark, gives the existence of a positive, universal number $\delta_\star > 0$, such that if $0 < \varepsilon \leq \epsilon_0 \ll 1$, for $\tilde{u}_\epsilon(X) := u_\varepsilon(\sqrt{\delta_\star}X)$ we have

$$\sup_{B_{1/2}} \tilde{u}_\epsilon(X) \leq \frac{1}{2}.$$

Passing to the limit as $\epsilon \to 0$, we obtain

$$\sup_{B_{\frac{\sqrt{\delta_\star}}{2}}} u_0(X) \leq \frac{1}{2}.$$

Define, in the sequel, the rescaled function

$$v^1(X) := 2u_\varepsilon(\frac{\sqrt{\delta_\star}}{2}X).$$

It is simple to verify that v^1 satisfies

$$\operatorname{div}(a^1_{ij}(X)\nabla v^1(X)) = \delta_\star \beta_{2\epsilon}(v^1),$$

in the distributional sense, where $a^1_{ij}(X) = a_{ij}(\sqrt{\delta_\star}X/2)$ is another (λ, Λ)-elliptic matrix. Once more, $v^1(0) \to 0$ as $\epsilon \to 0$, hence, for $\epsilon \leq \epsilon_1 < \epsilon_0 \ll 1$, we can apply Lemma 3.2 to v^1 and deduce, after scaling the inequality back,

$$\sup_{B_{\frac{\sqrt{\delta_\star}}{4}}} u_0(X) \leq \frac{1}{4}.$$

Continuing this process inductively, we conclude that for any $k \geq 1$, that holds

$$\sup_{B_{\frac{\sqrt{\delta_\star}}{2^k}}} u_0(X) \leq \frac{1}{2^k}. \tag{3.9}$$

Finally, given $X \in B_{1/2}$ let $k \in \mathbb{N}$ be such that

$$\frac{\sqrt{\delta_\star}}{2^{k+1}} < |X| \leq \frac{\sqrt{\delta_\star}}{2^k}.$$

We estimate from (3.9)

$$\begin{aligned} u_0(X) &\leq \sup_{B_{\frac{\sqrt{\delta_\star}}{2^k}}} u_0(X) \\ &\leq \frac{1}{2^k} \\ &\leq \frac{2}{\sqrt{\delta_\star}}|X|, \end{aligned}$$

and the proof of Theorem 3.1 is concluded.

Obviously, the (improved) regularity estimate granted by Theorem 3.1 holds solely along the free boundary. For any point $Z \in \{u > 0\}$, the best estimate available drops back to $C^{0,\alpha}$, for some unknown $0 < \alpha$, strictly less than one. The question we would like to answer now is what is the minimum organization required on the medium so that solutions to the cavitation problem is locally Lipschitz continuous, up to the free boundary.

Definition 3.3 Given a large constant $K > 0$, we say that a uniform elliptic matrix $a_{ij}(X)$ satisfies (K-Lip) property if for any $0 < d < 1$ and any $h \in H^1(B_d)$ solving

$$\operatorname{div}\left(a_{ij}(X)\nabla h\right) = 0 \text{ in } B_d$$

in the distributional sense, there holds

$$\|\nabla h\|_{L^\infty(B_{d/2})} \le \frac{K}{d} \times \|h\|_{L^\infty(B_d)}.$$

It is classical that Dini continuity of the medium is enough to assure that a_{ij} satisfies (K-Lip) property, for some $K > 0$ that depends only upon dimension, ellipticity constants and the Dini-modulus of continuity of a_{ij}. Indeed under Dini continuity assumption on a_{ij}, distributional solutions are of class C^1.

Our next Corollary says that uniform limits of singularly perturbed Eq. (2.1) is Lipschitz continuous, up to the free boundary provided a_{ij} satisfies (K-Lip) property for some $K > 0$. The (by no means obvious) message being that when it comes to Lipschitz estimates, the homogeneous equation and the free boundary problem $\operatorname{div}(a_{ij}(X)\nabla u) \sim \delta_0(u)$ require the same amount of organization of the medium.

Corollary 3.4 *Under the assumptions of Theorem 3.1, assume further that $a_{ij}(X)$ satisfies (K-Lip) property for some K. Then, given a subdomain $\Omega' \Subset \Omega$,*

$$|\nabla u_0(X)| \le C,$$

for a constant that depends only on dimension, ellipticity constants, $\operatorname{dist}(\partial\Omega', \partial\Omega)$*, L^∞ bounds of the family and K.*

Proof It follows from Theorem 3.1 and property K that u_0 is pointwise Lipschitz continuous, i.e.,

$$|\nabla u_0(\xi)| \le C(\xi).$$

We have to show that $C(\xi)$ remains bounded as ξ goes to the free boundary. For that, let ξ be a point near the free boundary $\partial\{u_0 > 0\}$ and denote by $Y \in \partial\{u_0 > 0\}$ a point such that

$$|Y - \xi| =: d = \operatorname{dist}(\xi, \partial\{u_0 > 0\}).$$

From Theorem 3.1, we can estimate

$$\sup_{B_{d/2}(\xi)} u_0(\xi) \le \sup_{B_{2d}(Y)} u_0(\xi) \le C \cdot 2d.$$

Applying (K-Lip) property to the ball $B_{d/2}(\xi)$, we obtain

$$|\nabla u_0(\xi)| \le \frac{2K}{d} \cdot 2Cd = 4C \cdot K,$$

and the proof is concluded. □

4 Lipschitz estimates for the minimization problem

Limiting functions u_0 obtained as ϵ goes to zero from a sequence u_ϵ of minimizers of functional (1.6) are minima of the discontinuous functional (1.2). Hence, limiting minima are Lipschitz continuous along their free boundaries. Nonetheless, as previously advertised in Theorem 1.1, the sharp Lipschitz regularity estimate holds indeed for *any* minima of the functional (1.2), not necessarily for limiting functions.

In this intermediate section we shall comment on how one can deliver this estimate directly from the analysis employed in the proof of Theorem 3.1. In fact, the proof of Lipschitz estimate for minima of the functional (1.2) is simpler than the proof delivered in previous section, which has been based solely on the singular equation satisfied. When a minimality property is available, the arguments can be rather simplified. For instance, strong minimum principle holds for local minima but is no longer available for a generic critical point. This is part of the reason why the arguments from previous section had be based on blow-ups and Liouville theorem.

Theorem 4.1 *Let $u_0 \geq 0$ be a minimum to*

$$\mathscr{F}(u) = \int_\Omega \left\{ \frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + \chi_{\{u>0\}} \right\} dX$$

and assume that $u_0(\xi) = 0$. Then there exists a universal constant $C > 0$, depending only on dimension, ellipticity constants, $\operatorname{dist}(\xi, \partial\Omega)$ *and its L^∞ norm such that*

$$u_0(X) \leq C|X - \xi|,$$

for all point $X \in \Omega$.

The proof follows the lines designed in Sect. 3. We obtain the corresponding flatness Lemma as follows:

Lemma 4.2 *Fixed a ball $B_r(Y) \Subset \Omega$ and given $\theta > 0$, there exists a $\delta > 0$, depending only on $B_r(Y)$, dimension, ellipticity constants and L^∞ norm of u_0, such that if u_0 is a nonnegative minimum of*

$$\mathscr{F}^\delta(u) = \int_\Omega \left\{ \frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + \delta \cdot \chi_{\{u>0\}} \right\} dX,$$

and $u_0(Y) = 0$, then

$$\sup_{B_{\frac{r}{2}}(Y)} u_0 \leq \theta.$$

Proof The proof follows by a similar tangential analysis of the proof of Lemma 3.2, but in fact in a simpler fashion. The tangential functional, obtained as $\delta \to 0$, satisfies minimum principle, hence the limiting function, from the contradiction argument, must be identically zero.

Here are some details: suppose, for the sake of contradiction, that the Lemma fails to hold. It means, for a sequence (λ, Λ)-elliptic matrices, a_{ij}^k, and a sequence of minimizers u_k of

$$\mathscr{F}^k(u) = \int_\Omega \left\{ \frac{1}{2}\langle a_{ij}^k(X)\nabla u, \nabla u\rangle + \delta_k \cdot \chi_{\{u>0\}} \right\} dX,$$

with $\delta_k = \mathrm{o}(1)$, and, say $\|u_k\|_\infty \leq 1$,

$$\sup_{B_r/2(Y)} u_k \geq \theta_0 > 0, \tag{4.1}$$

for some $\theta_0 > 0$ fixed. As in Lemma 3.2, by compactness, up to a subsequence, $u_k \to u_0$. Passing the limits we conclude u_0 is a local minimum of

$$\mathscr{F}^\infty(u) = \int \frac{1}{2}\langle b_{ij}(X)\nabla u_0, \nabla u_0\rangle dX.$$

Since, $u_0 \geq 0$ and $u_0(Y) = 0$, by the strong minimum principle, see for instance [11, Theorem 7.12], $u_0 \equiv 0$. We now reach a contradiction with (4.1) for $k \gg 1$. The Lemma is proven. □

Once we have obtained Lemma 4.2, the proof of Theorem 4.1 follows exactly as the final steps in the proof of Theorem 3.1.

5 Gradient control in two-phase problems

In this section we show that Theorem 3.1 as well as Theorem 4.1 hold for two-phase problems, provided a one-side control is a priori known. It is interesting to compare this with the program developed in [4–6], where monotonicity formula yields similar conclusion.

Let us briefly comment on such generalization, in the (simpler) minimization problem. The singular perturbed one can be treated similarly. We start by placing the negative values of u within a universally controlled slab, i.e.:

$$\inf_\Omega u \geq -\delta_\star, \tag{5.1}$$

for a universal value $\delta_\star > 0$. Such a condition is realistic for models involving very low temperatures, i.e., for physical problem near the absolute zero for thermodynamic temperature (zero Kelvin). A scaling of the problem places *any* solution into this setting. Within the proof of Lemma 4.2, one includes condition (5.1) in the compactness argument. Here is the two-phase version of Lemma 4.2:

Lemma 5.1 *Fixed a ball $B_r(Y) \Subset \Omega$ and given $\theta > 0$, there exists a $\delta > 0$, depending only on $B_r(Y)$, dimension, ellipticity constants and L^∞ norm of u, such that if u is a changing sign minimum of*

$$\mathscr{F}^{\tilde{\delta}}(u) = \int_\Omega \left\{\frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + \tilde{\delta}\cdot\chi_{\{u>0\}}\right\} dX,$$

for $\tilde{\delta} \leq \delta$, with

$$u_0(Y) = 0 \quad \text{and} \quad \inf_\Omega u \geq -\delta,$$

then

$$\sup_{B_{\frac{r}{2}}(Y)} |u| \leq \theta.$$

The proof of Lemma 5.1 follows the lines of Lemma 4.2, noticing that, by letting $\delta = \mathrm{o}(1)$ in the compactness approach, the tangential configuration is too a nonnegative minima of a functional which satisfies minimum principle.

Theorem 5.2 *Let u_0 be a sign changing minimum of the functional*

$$\mathscr{F}(u) = \int_\Omega \left\{ \frac{1}{2}\langle a_{ij}(X)\nabla u, \nabla u\rangle + \chi_{\{u>0\}} \right\} dX,$$

with $u_0(\xi) = 0$, $-1 \le u_0 \le 1$. Assume u^- is Lipschitz continuous at 0. Then u^+ (and therefore u) is too Lipschitz at 0 and

$$|\nabla u(0)| \le C|\nabla u^-(0)|.$$

Proof We can assume, with no loss, $\Omega = B_2$ and $\xi = 0$. By universal continuity estimate, Theorem 2.2, we can choose a universal number $0 < \tau_0 \ll 1$, such that the function $v\colon B_1 \to \mathbb{R}$, given by

$$v(X) := u(\tau_0 X),$$

satisfies the hypothesis of Lemma 5.1, for $\theta = \frac{1}{2}$. Selecting $0 < \tau_0 \ll |\nabla u^-(0)|^{-1}$, even smaller if necessary, we can assure

$$|\nabla v^-(0)| \le \delta_{1/2},$$

where $\delta_{1/2}$ is the number from Lemma 5.1 when we take $\theta = \frac{1}{2}$. Define in the sequel $v_2\colon B_1 \to \mathbb{R}$ by

$$v_2(X) := 2v\left(\frac{1}{2}X\right).$$

Clearly v_2 is a minimum of a functional $\mathscr{F}^{\tilde{\delta}}$ for $\tilde{\delta} \le \delta_{1/2}$, $v_2(0) = 0$, and by Lemma 5.1, it also verifies $-1 \le v_2 \le 1$. We estimate

$$\inf_{B_1} v_2 \ge -|\nabla v^-(0)| \ge -\delta_{1/2}.$$

Hence, v_2 is also within the hypothesis of Lemma 5.1. Carrying the induction process shows that

$$\sup_{B_{\tau_0 2^{-k}}} |u| \le 2^{-k}.$$

Now, given $0 < r \ll 1$, we choose $k \in \mathbb{N}$ such that $\tau_0 2^{-(k+1)} \le r \le \tau_0 2^{-k}$ and compute

$$\sup_{B_r} |u| \le \sup_{B_{\tau_0 2^{-k}}} |u| \le 2^{-k} \le \frac{2}{\tau_0} r.$$

The Theorem is proven. □

Similarly, one can use these set of ideas when a density control of the negative phase is given. For instance if

$$\mathscr{L}^n(\{u < 0\} \cap B_r) \le \delta_\star r^n, \tag{5.2}$$

where $\delta_\star \ll 1$ is universally small, then Lipschitz regularity along the free boundary holds. Indeed, as before, one could add the constrain (5.2) within the compactness approach, letting $\delta_\star = o(1)$, and the limiting configuration is too a nonnegative function. Now, within the induction procedure, condition (5.2) scales properly, in the sense that at each scale, condition (5.2) holds with the same initial constant $\delta_\star$. Compare for instance with [14].

6 Geometric estimates of the free boundary

In this section we show how the improved estimate given by Theorem 3.1 (or else Theorem 4.1) implies some geometric estimates on the free boundary. Hereafter in this section, $u_0 \geq 0$ will always denote a limit point obtained from a sequence of minimizers of the functional (1.6). We will denote by Ω_0 the non coincidence set, $\Omega_0 := \{u_0 > 0\} \cap \Omega$. Unless otherwise stated, no continuity assumption is imposed upon the medium a_{ij}.

Theorem 6.1 (Nondegeneracy) *Let $\Omega' \Subset \Omega$ be a given subdomain and $Y \in \Omega' \cap \overline{\{u_0 > 0\}}$ then*

$$\sup_{B_r(Y)} u_0 \geq c \cdot r.$$

for $r < dist(\Omega', \partial\Omega)$.

Proof Letting $\epsilon \to 0$ in Theorem 2.3 we conclude u_0 grow linearly away from the free boundary. Owning Lipchitz regularity along $\partial\{u_0 > 0\} \cap \Omega'$, Theorem 3.1, we can then perform a polygonal type of argument *a la* Caffarelli, see for instance [18, Lemma 4.2.7], to establish such a strong non-degeneracy estimate. □

Theorem 6.2 *Given a subdomain $\Omega' \Subset \Omega$, there exists a constant $\theta > 0$, such that if $X_0 \in \partial\Omega_0$ is a free boundary point then*

$$\mathcal{L}^n(\Omega_0 \cap B_r(X_0)) \geq \theta r^n,$$

for all $0 < r < \text{dist}(\partial\Omega', \partial\Omega)$. Furthermore there is a universal constant $0 < \varsigma \leq 1$ such that

$$\text{dim}_{\mathcal{H}}(\partial\Omega_0) \leq n - \varsigma,$$

where $dim_{\mathcal{H}}(E)$ means the Hausdorff dimension to the set E. □

Proof It follows readily from non-degeneracy property, Theorem 6.1, there exists a point $\xi_r \in \partial B_r(X_0)$ such that

$$u_0(\xi_r) \geq cr,$$

for a constant $c > 0$ depending only on the data of the problem. Now, for $0 < \mu \ll 1$, small enough, there holds

$$B_{\mu r}(\xi_r) \subset \Omega_0. \tag{6.1}$$

Indeed, one simply verifies that if

$$B_{\mu r}(\xi_r) \cap \partial\{u_0 > 0\} \neq \emptyset,$$

then from Theorem 3.1 we can estimate

$$cr \leq u_0(\xi_r) \leq \sup_{B_{\mu r}(Z_0)} u_0 \leq C\mu r$$

which is a lower bound for μ. Hence, if $\mu < c \cdot C^{-1}$, (6.1) must hold. Now, with such $\mu > 0$ fixed, we estimate

$$\mathscr{L}^n\left(B_r(X_0) \cap \Omega_0\right) \geq \mathscr{L}^n\left(B_r(X_0) \cap B_{\mu r}(\xi_r)\right) \geq \theta r^n$$

and the uniform positive density is proven.

Let us turn our attention to the Hausdorff dimension estimate. Given $\sigma = X_0$ in $\partial\{u > 0\}$, we choose

$$\sigma\prime = t\xi_r + (1-t)X_0,$$

with t close enough to 1 as to

$$B_{\frac{1}{2}\mu\cdot r}(\sigma\prime) \subset B_\mu(\xi_r) \cap B_r(\sigma) \subset B_r(\sigma)\backslash\partial\{u > 0\}.$$

We have verified $\partial\{u > 0\} \cap B_{1/2}$ is $(\mu/2)$-porous, hence by a classical result, see for instance [12, Theorem 2.1], its Hausdorff dimension is at most $n - C\mu^n$, for a dimensional constant $C > 0$. □

For problems modeled in a merely measurable medium, one should not expect an improved Hausdorff estimate for the free boundary. When diffusion is governed by the Laplace operator, then Alt-Caffarelli theory gives that $\varsigma = 1$. A natural question is what is the minimum organization of the medium as to obtain perimeter estimates of the free boundary. Next Theorem gives an answer to that issue.

Theorem 6.3 *Assume a_{ij} satisfy (K-Lip) property for some $K > 0$. Then the free boundary has local finite perimeter. In particular $dim_{\mathcal{H}}(\partial\Omega_0) = n - 1$.*

Proof Fixed a free boundary point $X_0 \in \partial\Omega_0$ and given a small, positive number μ one checks that

$$\int_{\{0<u_0<\mu\}\cap B_r(X_0)} |\nabla u_0|^2 \leq C\mu r^{n-1}. \tag{6.2}$$

This is obtained by integration by parts and Lipschitz estimate on B_r. In the sequel, we compare the left hand side of (6.2) with $|\{0 < u_0 < \mu\} \cap B_r(X_0)|$. This is done by considering a finite overlapping converging, $\{B_j\}$, of $\partial\Omega_0$ by balls of radius proportional to μ and centered on $\partial\Omega_0 \cap B_r(X_0)$. In each ball B_j, we can find subballs B_j^1, B_j^2 with the radii $\sim \mu$, such that

$$u_0 \geq \frac{3}{4}\mu \text{ in } B_j^1 \quad \text{and} \quad u_0 \leq \frac{2}{3}\mu \text{ in } B_j^2.$$

Existence of such balls is obtained by nondegeneracy property followed by Poincaré inequality. Now, for $\mu \ll r$, we have

$$B_r(X_0) \cap \{0 < u_0 < \mu\} \subset \bigcup 2B_j \subset B_{4r}(X_0).$$

Finally, if we call $A := \{0 < u_0 < \mu\}$, the above gives

$$\begin{aligned}
\int_{B_{4r}(X_0)\cap A} |\nabla u_0|^2 dX &\geq \int_{\left(\cup 2B_j\right)\cap A} |\nabla u_0|^2 dX \\
&\geq \frac{1}{m}\sum \int_{2B_j\cap A} |\nabla u_0|^2 dX \\
&\geq c\sum \mathscr{L}^n(B_j) \\
&\geq c\mathscr{L}^n(B_r(X_0) \cap A),
\end{aligned}$$

where m is the total number of balls, which can be taken universal, by Heine–Borel's Theorem. Combining the above estimate with (6.2), gives

$$\mathscr{L}^n(\{0 < u_0 < \mu\} \cap B_r(X_0)) \leq C\mu r^{n-1},$$

which implies the desired Hausdorff estimate by classical considerations. For further details, see for instance [18, Chapter 4]. □

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